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Energy Use, Pollution and Economic Growth

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Resumo

Nesta tese é proposto o estudo, através de modelos simples, das interacções entre crescimento económico, consumo energético e poluição. Assume-se que a economia cresce propocionalmente ao sector energético. A análise foca-se principalmente no impacto negativo da poluição e na sua influencia na escolha de tecnologias para a produção e transformação de energia útil. Para cada modelo a trajectória do consumo óptimo é calculada utilizando teoria de controlo óptimo. Apresentam-se aplicações para funções de produção AK e Cobb-Douglas. Demonstra-se que o uso de tecnologias poluentes permite um crescimento económico mais elevado mas eventualmente leva a uma paragem deste, devido às emissões poluentes. Por outro lado, a utilização de tecnologia limpa permite crescimento ilimitado. Conclui-se que, quando ambas as tecnologias estão disponíveis, se a economia se encontrar num nível de desenvolvimento baixo, é óptimo investir apenas na tecnologia mais produtiva, mesmo se for poluente. No entanto, mostra-se também que, a dado momento os impactos negativos da poluição fazem com que seja óptimo investir na tecnologia menos produtiva, mas limpa. Demonstra-se que esta troca ocorre sempre. Para este modelo, a curva da poluição em relação ao redimento global da economia apresenta uma forma que pode ser identificada com uma curva ambiental de Kuznets (EKC).

Palavras-chave: crescimento económico, energia, poluição, EKC

Abstract

This thesis encompasses a comprehensive study, through simple models, of the interactions between economic growth, energy consumption and pollution. It is assumed that the economy grows proportionally to the energy sector. The study focus mainly on the negative impact from pollution and its consequences regarding the choice of the technology for useful energy production and transformation. The optimal path of consumption is investigated, for every case, using optimal control theory. Applications to AK and Cobb-Douglas production functions are presented. It is shown that the use of a polluting technology may allow for higher economic growth but it eventually leads to a halt due to increasing emissions. On the other hand, a clean technology allows for unbounded growth. It is concluded that when both technologies are available, if the economy is in a low stage of development, it is optimal to invest only in the most productive technology, even if it is polluting. However, at some point, given the negative impacts of pollution, it will be optimal to switch to the less productive but clean technology. It is shown that this switch always occurs. For this model the shape of the pollution to income relationship may be identified with an Environmental Kuznets Curve.

Keywords: economic growth, energy, pollution, EKC

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Chapter 1

Introduction

Energy is indispensable for every life form, action or process. It is hard, even impossible, to present a concrete definition for it, but physics, particularly thermodynamics, provides the tools to study energy transformation and its interactions with matter.

The hypothesis that economic growth originates from and is limited by energy goes back to the 19th century. Herbert Spencer, in 1862, stated that the evolution of societies depends on their ability to collect energy to produce goods[1]. Ostwald, in 1907, developed these ideas saying that it is the degree of efficiency of energy conversion that defines society's stage of economic development[1]. This notion arises from the acknowledgment that heat engines and, more recently, transistors, powered by energy, are like amplifiers of human muscle and brain capacity. As such they allow for substantial increases in wealth creation, through work performance and information processing[2].

Over the past century, industrialization and economic growth have been characterized by increasing energy consumption and improvements in useful energy production technologies[3]. Modern economies strongly depend on a continuous flow of energy, in several different forms like fossil fuels for transportation and electricity from non-renewable and renewable sources[3]. One can argue that, as the economy evolves, it shifts its production structure away from energy intensive industries to less energy intensive service activities[3]. Nonetheless, even then, energy plays an indirect but crucial role in the economy.

The experience from the effects of resource scarcity and energy conservation measures adopted following the dramatic oil price hikes in 1973/74 and 1979/80 suggests that constraints on energy consumption can adversely affect economic output[3]. Also, the strong correlation between the global consumption of commercial energy with the best available reconstruction of the world's economic product during the twentieth century is striking. Growth rates of these variables coincide almost perfectly, showing an approximately sixteen-fold increase in 100 years[4].

However the relationship between energy use and economic growth is far from simple, displaying some predictable regularities but with different trends for each developmental stage and with many national specificities[4]. Empirical studies have revealed that the existence and directionality of causal relationships between energy use and economic growth depends on the period studied, used methodology, and aggregation of energy flows[3].

From the theoretical view point, economic activity is described as a closed loop between abstract production, abstract consumption and investment. A self-organized system capable of growth where every product is produced from other products within the system. Georgescu-Roegen argued that this model economy is an immaterial perpetual motion machine, in violation of both of the laws of thermodynamics[5]. Conventional economic theory attributes only marginal importance to energy as a factor of production, based on the reasoning that energy's share in total factor cost is small compared to the cost shares of capital and labour. From this argument it follows that reducing energy consumption wouldn't impact significantly total economic output[3].

The topic of causal relationship between energy consumption and economic growth has been well-studied in the energy economics literature. From the empirical view point, there are four hypotheses: *growth hypothesis* (unidirectional causality running from energy to growth, meaning that growth requires energy); *conservation*

hypothesis (unidirectional causality running from growth to energy, implying that a country is not fully dependent on energy for growth); *feedback hypothesis* (bidirectional causality between energy and growth); *neutrality hypothesis* (energy and growth are neutral with respect to each other)[6].

In addition to its influence on economic growth, the debate around energy consumption is of the utmost importance due to resource availability and environmental pollution. The fact that the main sources of useful energy are non-renewable poses a great challenge to modern economies. And, maybe even more important than that, the energy sector is responsible for a significant impact on the biosphere, not only through CO_2 emissions and consequent climate change, but also through other atmospheric pollutants. In this context, the Environmental Kuznets Curve hypothesis as gained importance, since it states that environmental pollution and economic growth are correlated by an inverted U-shape, i.e., polluting emissions rise in the initial stage of economic development but once a given level of wealth is attained they begin to decline[7]. This conjecture has a great political importance since governments must decide whether to take action to abate existing pollution and prevent future environmental degradation. Particularly for the energy sector it is very relevant given its dimensions and connections to the other economic sectors. A decrease in this sector's emissions can be done through a decrease in total consumption (e.g. higher energy efficiency) or a shift in technology (e.g. from non-renewable to renewable resources).

This thesis will focus only on the relation between energy use and pollution, disregarding the aspects related to resource scarcity. Chapter 2 presents a literature review concerning the relation between pollution and economic growth. The concepts of flow and stock pollutants are introduced, and the Environmental Kuznets Curve hypothesis is further analyzed.

Chapter 3 comprises the theoretical models developed and chapter 4 its applications for specific functional forms.

Considering the empirical evidences that support the feedback hypothesis for the causality relation between energy consumption and growth, particularly for Portugal[8], this thesis is based on the assumption that the economy lies within a balanced growth path, where every sector grows proportionally to one another. This means that one could just study one particular sector and draw conclusions for the overall economy. Particularly, we propose a one-sector growth model that describes an economy driven by the energy sector. The production function describes the process of production and transformation of useful energy, that supplies the rest of the economy. If the models were to be calibrated and applied to empirical data it would suffice to have information concerning the energy sector.

The economic impact of different choices of energy profile will be seen explicitly throughout the work. The designations used for the energy profiles are *Simple Energy Profile*, where only one technology is used, and *Mixed Energy Profile* where two technologies are available. Energy profiles are also characterized by the productivities of the technologies used and whether these are polluting or clean technologies. This framework was inspired by the work developed in the simulator for *CityOn*, a computer game created by the portuguese company Biodroid in collaboration with IN+ Center for Innovation, Technology and Policy Research of Instituto Superior Técnico (available online at *www.cityon.pt*). In the game's futuristic city, electricity is the only existing form of energy and it is used as currency. To produce it, it is necessary to build power plants that can be either pollutant and based on fossil fuels, or clean and based on renewable resources. The player will act as a social planner that decides which power plants to build, when to add new ones and also when to upgrade the existing ones in order to improve efficiency and reduce pollution. The player's performance is evaluated based on the population's satisfaction, which will increase with electricity available to consume and decrease with high pollution levels.

Contrary to CityOn, this thesis is not restricted to electricity production, capital from the whole energy sector will be identified with capital in the traditional economic sense and the neoclassical tools and approaches will be used[9]. Capital productivity corresponds to the productivity associated with the technology used by the energy sector (e.g. power plants) and pollution associated with production to the correspondent polluting emissions. Also, there will exist a conversion efficiency relating total energy consumed and useful consumption. This is a very important factor, as it represents the overall energetic efficiency of the country.

The final goal is to attain aggregated models that are simple enough to be treated analytically but that contain sufficient information so that they can be interpreted within a realistic context. The interest in this top-down approach lies in the possibility of capturing general trends and patterns in a comprehensive manner[10]. For each model, optimal control theory will be used to investigate what the optimal consumption and capital stock paths are for a utility function that increases with useful energy consumption and decreases

with pollution.

Initially, the models consider *Simple Energy Profiles*, first for clean technology (section 3.1), then for polluting technology (section 3.2). Afterwards the study is extended to *Mixed Energy Profiles*. For the first one the composition of the energy profile cannot be changed (section 3.3). For the others, the fraction of investment in each technology is added as a control variable. First with two different but clean technologies (section 3.4), then with a clean technology and a polluting technology (section 3.5).

For each model some applications are presented with the very well known and studied AK and Cobb-Douglas production functions. Those functions were chosen in view of the large literature available. Nonetheless, it is important to keep in mind that for useful energy production and transformation different functional forms may be more accurate. Particularly, functions that consider the specific characteristics of different technologies.

Although it was chosen to interpret the calculations as if they were restricted to the energy sector and that the overall economy grows proportionally, it is important to refer that one can choose to make the opposite reasoning, where each model corresponds to a traditional economic growth model, capital is the aggregate productive capital for the overall economy and output is identifiable with gross domestic product (GDP).

Chapter 2

Pollution and Economic Growth

The second law of thermodynamics states that there are irreversibilities associated with most real processes meaning that economic and human activity is coupled with the emission of particles and heat. Large quantities of emissions change the molecular composition of the biosphere and the energy flows through it. If these changes are so big and occur so rapidly that adaptation deficits of the living species and their societies occur, they are perceived as environmental pollution[2].

Conventionally pollution is defined as unwanted alterations of the physical, chemical or biological characteristics of the air, soil and water, that could affect negatively the life of humans or other species, the industrial processes, the life conditions or the cultural patrimony. It can also be defined as something that can jeopardize and deteriorate natural resources. The costs and consequences of pollution are difficult to account, as they can be direct or indirect, immediate or delayed, local or widespread.

There are several different forms of pollution, that can be categorized as follows: air pollution, light pollution, littering, noise pollution, soil contamination, radioactive contamination, thermal pollution, water pollution and visual pollution.

This chapter introduces the concepts of stock and flow pollutant, and describes the role of pollution throughout the following chapters. A literature review concerning the relation between pollution and economic growth is presented and the Environmental Kuznets Curve analyzed.

2.1 Stock vs. Flow Pollutants

It is common in economics to distinguish between quantities considered as stocks and others considered as flows. A stock variable represents a quantity of something that has been accumulated over the past, until the moment of measurement. On the other hand, a flow variable will be measured as a quantity over an interval of time. Stocks and flows are the basic building blocks of system dynamics models. A flow changes a stock over time.

In this context, pollutants can be distinguished between *flow pollutants* and *stock pollutants* according to whether they accumulate or not.

A flow pollutant would be one that just has a negative impact in the moment it is emitted. Only noise pollution really complies with this definition, it only affects its surroundings while it is being produced.

A stock pollutant is one for which the environment has only some absorptive capacity, it will accumulate in the ecosystems. Even if the source of pollution stops emitting, the negative consequences of these pollutants will still be felt in the future. If the environment's absorptive capacity is so low that the stock of pollutant doesn't decrease over time, they're classified as *persistent pollutants*. Examples of persistent pollutants are non-biodegradable plastics and heavy metals.

Most of the pollutants are stock pollutants, for which the environment has some absorptive capacity, and that can be converted into less harmful substances or diluted to non-harmful concentrations. Depending on the considered time scale stock pollutants can be considered mathematically as flow or persistent pollutants.

For instance, the atmospheric lifetime of sulphur dioxide (SO_2) is 1-4 days, that of nitrogen oxides (NO_x) is 2-5 days, and for carbon monoxide (CO) is 1-3 months. Suspended particulate matter (SPM) is washed out by rain and snowfalls, thus have also a short lifetime. So, from a long-run point of view, these pollutants can be treated as flow pollutants[11].

On the other hand, carbon dioxide (CO_2), although it is absorbed by plants and oceans, has a long lifetime of about 125 years so, typically, in economic growth models it is considered as a stock pollutant[11].

Since economic growth depends on several different activities and industrial processes it is important to include both stock and flow pollutants in any study about its environmental impact. However due to its mathematical tractability, flow pollutants have been preferred in theoretical models.

Looking specifically to the energy sector, there are several flow pollutants associated with the use of fossil fuels, like SPM, and CO_2 emissions, that, as referred before, should be treated as a stock pollutant. Not only because its lifetime is long but also because its impacts are determined mainly by the total stock (concentration) of the gas in the atmosphere rather than by its current flow (emissions).

However if the impacts of CO_2 emissions are climate change, present emissions will only have a negative effect in the future. When considering a representative agent, i.e., the typical consumer that makes choices in view of improving his overall welfare, his present choice to pollute, in order to consume more, will not affect his well-being. However, if this representative agent is well informed about long-term consequences of CO_2 emissions he may make the more responsible choice and opt for a non-polluting technology, even if that means a lower level of consumption. To introduce this kind of reasoning in the calculations, we chose to treat CO_2 emissions as a flow pollutant, for which present emissions affect present utility.

Given this argument and the mathematical advantages, the present work considers that overall pollution is proportional to production and it is treated as a flow pollutant. This means that the representative agent's utility function is only affected by the current level of emissions, and that is what will affect the consumption path.

One must keep in mind that when choosing that approximation, the optimal solution may lead to a constant flow of pollution that is optimal for the agent governed by the considered utility function, but that could lead to an increasing stock of CO_2, with unaccounted negative consequences. For that reason this approach should be used with awareness regarding its scope and limitations. Also, the choice of the parameters must be done taking these observations into account.

2.2 The Environmental Kuznets Curve

In 1991, Grossman and Krueger published a study concerning the potential impacts of NAFTA (North American Free Trade Agreement). The year after the Shafik and Bandyopadhyay study was published, followed by a survey done by Panayotou. All of these investigations reached the same conclusion: it appeared out of cross-country analyses that the connection between some pollution indicators and income per capita could be described as an inverted-U curve. Panayotou, in 1993, first coined it the Environmental Kuznets Curve (EKC)[12].

The EKC hypothesis states that environmental damage first increases with income, then declines, implicating that environmental degradation and per capita income are correlated by an inverted U-shape. The reasoning behind this premise is that, at low levels of development, few resources are consumed and there aren't large amounts of waste, thus the impacts on the ecosystem are small. As the economy expands, it becomes industrialized, the agriculture and the resource extraction become more intensive, even exceeding the natural rate of regeneration. Also, waste and pollution generation build up. At this moment, the priority for the population is to increase income and consumption, no attention is paid to the environment, so the tendency is for degradation to accentuate. However, as income continues to rise, the welfare gains from consumption diminish, and the preferences tend to change. More importance is given to the biosphere, and people are more willing to pay for cleaner production technologies. Simultaneously, at high levels of development, industries normally shift to producing services and information-intensive goods. Therefore, pollution levels diminish and so does the environmental deterioration[7].

The EKC conjecture is far from being consensual, most economists believe that the inverted-U curve only captures the 'net effect' of income upon the ecosystem, in which income growth is used as a global variable representing a variety of underlying influences, whose separate effects are obscured. On that account,

assuming that economic growth will eventually lead to improved environmental quality is not automatic[13]. This is the political significance of the studies concerning the EKC, since governments must decide whether or not to take action to protect the environment, like taxes on pollution or incentives to switch to cleaner technologies.

Many authors, consider the EKC premise as a "stylized" fact and try to present structural explanations for it: better efficiency in production, consumption and pollution abatement; institutional development; improvements in market and institutional efficiency; strengthened public awareness of the negative effects of pollution on health; increasing willingness-to-pay for pollution abatement; and an economic structure that, from one dominated by industrial sectors, becomes dominated by tertiary sectors, etc[7].

There is a large quantity of empirical studies regarding the existence or non-existence of an EKC, contrasting to a small amount of theoretical studies[14].

Most empirical studies agree that there is an EKC for sulphur dioxide (SO_2), suspended particulate matter (SPM), oxides of nitrogen (NO_x), carbon monoxide (CO), and for some kinds of river pollution. Figure 2.1 contains the empirical results for the PIR (pollution to income relation) of some pollutants[11]. These are all stock pollutants with short lifetimes, so can be considered flow pollutants from a long-run point of view. Municipal waste is a stock pollutant as it is deposited and accumulates, and its estimated PIR is monotonically rising (see figure 2.1). However for CO_2 the evidence is mixed. Based on the table in figure 2.1, the majority of researchers finds a monotonically rising PIR or an EKC with a turning point that lies far outside the income sample range[11].

These econometric studies are based on reduced-form models described by equations such as (2.1)[14], and seem to be sensitive to the sample of countries chosen and to the time period considered:

$$f_{it} = \alpha_{it} + \beta_1 x_{it} + \beta_2 x_{it}^2 + \beta_3 x_{it}^3 + \beta_4 z_{it} + \epsilon_{it} \tag{2.1}$$

where f_{it} is the environmental indicator, x_{it} is the income, z_{it} relates to other variables significant to the environment and ϵ_{it} is the error. The subscript i stands for country, t for time and α is a constant. Through the determination of the parameters β, it is possible to visualize the shape of the relationship between the environmental indicator and income:

(i) $\beta_1 = \beta_2 = \beta_3 = 0$, flat pattern or no relationship between x and f;

(ii) $\beta_1 > 0$ and $\beta_2 = \beta_3 = 0$, monotonic increasing relationship between x and f;

(iii) $\beta_1 < 0$ and $\beta_2 = \beta_3 = 0$, monotonic decreasing relationship between x and f;

(iv) $\beta_1 > 0$, $\beta_2 < 0$ and $\beta_3 = 0$, inverted-U-shaped relationship, i.e., EKC;

(v) $\beta_1 < 0$, $\beta_2 > 0$ and $\beta_3 = 0$, U-shaped relationship

(vi) $\beta_1 > 0$, $\beta_2 < 0$ and $\beta_3 > 0$, cubic polynomial or N-shaped figure

(vii) $\beta_1 < 0$, $\beta_2 > 0$ and $\beta_3 < 0$, opposite to the N-shaped curve

So, the EKC is only one of the possible outcomes of the empirical model. Another result that has caught the attention of some economists is the N-shaped figure, that has been interpreted as the *rebound effect*. This term is used to describe a phenomenon associated with an increase of the economy's efficiency. At first it may reduce the pollution levels, since it is possible to consume the same using less resources. But it eventually translates in an increase in consumption, that corresponds to increasing pollution. Implying that the pollution to income relation (PIR) is an N-shaped figure. This effect has been of particular significance for energy policy, when discussing the impacts of improvement in energy efficiency[15].

Following Kijima[14], the main criticisms concerning the econometric studies are:

- Reduced-form relationships reflect correlation rather than a causal mechanism;
- Specific functional forms to estimate the relationship are assumed;
- More sophisticated techniques of curve fitting should be investigated;

	flow pollutants					stock pollutants	
	SO_2	SPM	NO_x	CO	RP	waste	CO_2
Grossman and Krueger 1993	$\frown$	$\sim$					
Selden and Song 1994	$\frown$	$\frown$	$\frown$				
Shafik 1994	$\frown$	$\frown$			$\sim$	$\nearrow$	$\nearrow$
Grossman 1995	$\sim$	$\frown$	$\frown$	$\frown$			
Grossman and Krueger 1995	$\sim$	$\frown$			$\frown$		
Holtz-Eakin and Selden 1995							$\nearrow$
Panayotou 1995	$\frown$	$\frown$	$\frown$				
Carson et al. 1997	$\frown$	$\frown$	$\frown$	$\frown$			$\frown$
Cole et al. 1997	$\frown$	$\frown$	$\frown$	$\frown$	$\frown$	$\nearrow$	$\nearrow$
Lim 1997	$\frown$	$\frown$	$\frown$		$\frown$		$\nearrow$
Moomaw and Unruh 1997							$\sim$
Panayotou 1997	$\sim$						
Roberts and Grimes 1997, 192							$\nearrow$
Kaufmann et al. 1998	$\sim$						
Schmalensee et al. 1998							$\frown$
Scruggs 1998	$\frown$	$\frown$					
Torras and Boyce 1998	$\sim$	$\frown$			$\frown$		
Wu 1998		$\frown$					
Agras and Chapman 1999							$\frown$
Barrett and Graddy 2000	$\sim$	$\frown$			$\frown$		
Cavlovic et al. 2000	$\frown$	$\frown$	$\frown$	$\frown$	$\frown$		$\nearrow$
Cole 2000, 112	$\frown$		$\frown$				
Dinda et al. 2000	$\frown$	$\sim$					
Hettige et al. 2000					$\frown$		
List and Gerking 2000	$\frown$		$\frown$				
Perrings and Ansuategi 2000	$\frown$						$\nearrow$
Halkos and Tsionas 2001							$\nearrow$
Heil and Selden 2001							$\nearrow$

Note. SPM = suspended particulate matter. RP = river pollution. $\frown$ = EKC. $\nearrow$ = The PIR is monotonically rising or the EKC has an out-of-sample turning point. $\sim$ = The PIR is N-shaped (first rising, then falling, and finally rising again) with both turning points inside the sample range.
The result of Selden and Song (1994) for CO is not reported because it is insignificant.

Figure 2.1: Empirical results for the PIR of some pollutants [11]

- The fitted function should be decided based on theoretical research;

Within the scope of this thesis, it is relevant to dedicate more attention to the theoretical approach to the EKC relationship. Generally, theoretical models are based in the following factors[14]:

- Behavioral changes and preferences (e.g., McConnel[16], Andreoni and Levinson[17] and Lieb[18]);
- Institutional changes (e.g., Jones and Manuelli[19], Egli and Steger[20] and Kijima et al.[21]);
- Technological and organizational changes (e.g., Stokey[22] and Tahvonen and Salo[23]);

Also, Pearson[24] stated that the factors that influence environmental quality can be separated between *supply side* and *demand side*.

Factors from the supply side:

- Levels of population and economic activity;
- Structures of production and consumption;
- Efficiencies;
- Use of new or different fuels/materials;
- External influences;

Factors from the demand side:

- Price of environmental quality;
- Preferences;
- Information and its acquisition;

Each model may approach differently each of these factors but they all feature a trade-off involving pollution/environmental quality, e.g., a utility function that decreases with pollution, a production function that is influenced by pollution, etc.

A common feature to all theoretical models is that the EKC is only attained for very specific cases, after a number of assumptions that change according to the perspective taken. For instance, Andreoni and Levinson[17] present a model with a linear utility function affected by consumption and pollution, that results from it:

$$U = C - zP \tag{2.2}$$

Abatement effort depends on resources (E) spent in it and on consumption:

$$P = C - C^{\alpha}E^{\beta} \tag{2.3}$$

Andreoni and Levinson demonstrate that if $\alpha + \beta > 1$, that is, an abatement curve satisfying increasing returns to scale, pollution and income exhibit an EKC. A few years later, Di Vita [25] developed a similar model that would lift that restriction and arrive to the same results. However another major restriction is forced: the marginal disutility caused by pollution depends on the level of income (M):

$$U = C - z(M)P \tag{2.4}$$

And the function $z(M)$ must satisfy a U-shaped relation to income. Assuming that the marginal disutility depends on income is imposing *a priori* what one wants to demonstrate. In that sense, this improvement between models is questionable.

The previous models bring to our attention two important features of the theoretical models about the EKC: the source of pollution and the abatement function. Andreoni and Levinson and Di Vita both consider that pollution originates from consumption. Many other authors follow this tendency like McConnell[16] and Lieb[18], in static models, John and Pecchenino[26] , Prieur[27] and Egli and Steger[20], in dynamic models.

This choice is highly debatable, although there is pollution associated with consumption, namely domestic waste or CO_2 emissions from automobiles, there is much more "hidden" pollution from production. If these embodied emissions are unaccounted for, the supposition that pollution is originated from consumption is an underestimation of it. But, even if the embodied emissions were included there's another issue that can't be ignored. If pollution only arises from consumption, this means that the abatement efforts are free from pollution, implying that one can reduce it without environmental costs.

There are also models that consider pollution proportional to the stock of capital (e.g. Selden and Song[28], Rubbio and Garcia[29]) which is also questionable, since capital in itself doesn't pollute. This is why, many authors choose to admit that pollution increases with production (e.g. Dinda[30], Hartman and Kwon[31], Kelly[32] and Smulders[33], Brock and Taylor[34]). This is the approach chosen in this thesis.

Another aspect that needs to be discussed is the abatement function. The previous models, like most models (e.g., Selden and Song[28], Dinda[30], Hartman and Kwon[31], Kelly[32]), incorporate the trade-off between consumption and environment trough abatement efforts. The representative agent must chose between consuming or abating pollution. Some authors (e.g., John and Pecchenino[26], Prieur[27], Wang and Chuang[35]) prefer the action of maintaining/improving environmental quality, and instead of using increasing pollution, consider diminishing environmental quality. Considering pollution or environmental quality is equivalent, however there are many different approaches.

Andreoni and Levinson[17] and Di Vita[25] assume a specific form for the abatement that depends on the level of consumption and resources spent in it. This quantity is determined by static optimization. Andreoni and Levinson go further and state that abatement presents increasing returns to scale. The authors argue that this is a valid hypothesis and provide some empirical evidence to support it. While it may be valid for some kind of pollutants, can one make the leap to pollution in the overall economy?

An example of a model that doesn't make any restriction on abatement is McConnell's. From the maximization of utility with respect to consumption, comes the solution to how much output is devoted to abatement. The main weakness of this model is its static nature, that doesn't consider the evolution of consumption and abatement in the course of time. Also, the results depend on the choice of the utility function, in this particular case the inverted-U curve of pollution related to output just appears if the mixed derivative of utility (U_{CP})[1] changes from large positive to large negative. The authors don't present arguments to support this assumption. Lieb[18] extends this model and considers a corner solution at low income, for which there is no abatement. But the author also concludes that 'the relationship between the income elasticity of demand for environmental quality and the EKC is ambiguous'.

There are dynamic models that follow the same line of thinking, when writing down the dynamics of capital stock, the investment will be total production minus consumption and abatement expenditures, this is the case of models like Selden and Song[28] and Kelly[32].

Amongst the dynamic models, another method is to consider two types of capital, one associated with physical capital and the other with abatement technology. The optimization process would lead to the optimal investment pattern between the two stocks. For instance, Rubio and Garcia[29] follow this procedure, assuming an utility function with logarithmic preferences. This model considers that pollution increases with physical capital and decreases with technological capital:

$$P = K^{\lambda} Z^{-\gamma} \tag{2.5}$$

where $\lambda, \gamma > 0$ stand for the elasticities of the pollution function. The authors also consider that pollution, besides the impact on utility, affects production negatively. They conclude that if the ratio of physical to technological capital is initially low and the technological capital is efficient enough, the optimal investment pattern supports an inverted-U pattern for the relation between pollution and income. The curve appears because during a first stage, it is optimal to invest only in physical capital.

A very similar model is the one from Dinda[30], with the additional feature that the environmental stock is also considered, and it is an input to production. The author demonstrates that for the chosen functions, the optimal solution leads to a steady-state, where the ratio of allocation of capital between the two sectors is constant. The EKC exists dynamically off the steady state, the shift from insufficient to sufficient investment for upgrading the environment is what makes pollution to curve down, corresponding to the inverted U-shape.

Finally, Hartman and Kwon[31] also present a dynamic model where a fraction of physical capital can be allocated to pollution abatement. This model is worth mentioning because it has the particular characteristic

[1] Subscript denotes partial derivative

that there are two stocks of capital: physical and human. Both used to produce final output, but only the first one contributing to pollution. The authors infer that the model is consistent with an environmental Kuznets curve for realistic parameter values. And that in the long run it is optimal for human capital to grow more rapidly than physical capital, output, and consumption while pollution declines.

The main criticism to these works would be the choice of the abatement function that strongly influences results. The difficulty lies in deciding on a function to model the abatement process for the overall economy. Some models circumvent this problem ignoring the possibility of abatement but introducing cleaner technologies for production, e.g. Smulders[33] and Tahvonen and Salo[23].

Smulders models the transition to cleaner production processes as a change in *general purpose technology* (GPT). Technology changes along two dimensions: firms improve the quality of their products incrementally and pollution-saving or pollution-using inventions arise in clusters at discrete times. Both types of innovation require expenditures and firms choose the type of innovation that yields highest profits. Firms are heterogeneous in terms of pollution output ratios, prices and output levels. The author assumes two important simplifications: only one type of innovation is undertaken at each moment; and at most two types of firms are active simultaneously. The model is divided in four phases: "green phase", "confidence phase" (divided in "adoption subphase" and "improvement subphase"), "alarm phase" and "cleaning-up phase" (divided in "adoption subphase" and "improvement subphase"). The paper analyzes the dynamics of each phase, considering that households maximize the utility function $U = \ln(C) - H$, where C is consumption and H is harm from emissions. It is shown that the gradual adoption of new GPT, which lead to intrasectoral shifts from clean to dirty firms or vice versa, predicts a pattern of pollution over time that is consistent with the EKC hypothesis. However, it is also seen that it cannot be expected an unambiguous correlation between changes in pollution and innovation, since both variables are endogenous and determined by several factors acting simultaneously. The model doesn't necessarily predict an EKC, the adoption of the cleaner GPT requires sufficient incentives, otherwise no technology shift takes place and the pollution tax has only a level effect.

Lastly, the model presented by Tahvonen and Salo[23] adresses specifically the energy use and its impact in the overall economy. Their model differs from all the others because pollution doesn't interfere with overall welfare. The factor that influences the switch between technologies is the cost of each one and the scarcity of non-renewable resources. In the model there are two state variables: capital stock and stock of the nonrenewable resource. The results show declining prices and increasing consumption of non-renewables resources endogenously. It describes the transition between energy forms as a smooth shift in the emphasis from renewables to non-renewables and back to renewables. The authors suggest that an inverted-U relation between carbon emissions and income level may follow even without environmental policy.

The focus of this thesis is the environmental impact of using different technologies in the energy sector. Since the theoretical models that investigate the EKC hypothesis analyze the trade-off between consumption and environment preservation, this work is integrated in this research field. For the models presented in the following chapters, pollution abatement or efforts to improve environmental quality won't be considered. The approach chosen, is similar to Smulders[33] and Tahvonen and Salo[23], where a change in the production technology will have an impact in the pollution levels, that in turn influences utility.

The study and analysis of different theoretical models, that relate economic growth, utility optimization and pollution, is an essential part of the present dissertation, since it allows for a better understanding of the important issues and the consequences of making different assumptions or following different approaches. The assessment of the strengths and weaknesses of each model lead to a more critical evaluation of the present work.

Chapter 3

Energy Technology in Economic Growth - Theoretical Models

Throughout this chapter the theoretical models developed will be presented from the most simple to the most elaborated. For each case some general observations will be made but a more thorough analysis is left to the following chapter (chapter 4) where some applications to specific production functions are presented.

As was mentioned in the introduction (chapter 1) the economic models presented are based on the assumption that the energy sector is the main economic driver, and that the overall economy will grow proportionally to it. However the following sections may also be interpreted as traditional economic growth models.

According to the technology used for useful energy production and transformation, its production will generate pollution, it is assumed that this is the only source of pollution.

The models allow for a explicit analysis of the impact of changes in the energy profile of a country. In the following sections, the evolution of the economy in this context will be studied for different assumptions and situations. For each case the optimal trajectory of consumption and capital stocks is investigated, using the techniques from optimal control theory, that are summarized in appendix A.

3.1 Simple Energy Profile: One Stock of Non-Polluting Capital

This section presents the simple case of an economy with only one stock of non-polluting capital. The situation is described as follows:

- Only one kind of producing useful energy technology, non-polluting;
- The stock of producing useful energy capital is the only state variable K;
- Known initial stock of capital $K(0) = K_0$;
- Generic production function $f(K)$: $f'(K) > 0$;
- Budget constraint:
$$Y = C + I \tag{3.1}$$
where C is consumption, I is investment in the stock of capital and Y is total production;
- Consumption is the control parameter;
- Utility increases with consumption $U_C > 0$ and is concave $U_{CC} < 0$

Capital stock depreciates at a rate δ, and increases according to the investment made. The budget constraint imposes that for a given output higher levels of consumption correspond to less investment in the capital stock, and vice-versa. These statements can be translated into the equation governing the dynamics of the capital stock:

$$\dot{K} = f - C - \delta K \tag{3.2}$$

The dynamic optimization problem is formulated like the one in equation (A.19), from appendix A, subject to equation (3.2), and the same initial and transversality conditions (equations (A.21) and (A.18)).

Like described in appendix A, one starts by writing the current value hamiltonian:

$$\begin{aligned} H^c &= U + \mu\dot{K} \\ &= U + \mu(f - C - \delta K) \end{aligned} \tag{3.3}$$

The first-order necessary condition for a maximum with respect to consumption will be:

$$\begin{aligned} H^c_C &= 0 \\ \iff U_C &= \mu \end{aligned} \tag{3.4}$$

The Euler equation is:

$$\begin{aligned} H^c_K &= \rho\mu - \dot{\mu} \\ \iff \dot{\mu} &= -(f' - \delta - \rho)\mu \end{aligned} \tag{3.5}$$

Taking the time derivative of equation (3.4) it is possible to relate $\dot{C}$ with $\dot{\mu}$:

$$U_{CC}\dot{C} = \dot{\mu} \tag{3.6}$$

Substituting in the Euler equation (3.5) results in the equation for the dynamics of consumption that together with the transition equation for the stock of capital (3.2) make the following ODE system:

$$\dot{K} = f(K) - C - \delta K \tag{3.7}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f'(K) - \delta - \rho) \tag{3.8}$$

Rearranging the terms in equation (3.8) we obtain the Ramsey rule[9]:

$$f'(K) - \delta = \rho - C\left[\frac{U_{CC}}{U_C}\right]\frac{\dot{C}}{C} \tag{3.9}$$

This rule states that an optimizing agent would choose consumption in a way so as to equate the net marginal productivity, $f'(K) - \delta$, to the rate of time preference, ρ, plus the rate of decrease of the marginal utility of consumption, U_C, due to growing consumption, C, which is the equivalent of saying that in the optimum the representative agent is indifferent at the margin between consuming and saving.[9]

The solution of this dynamic system will yield the optimal trajectory of consumption and stock of capital. Since $U_C > 0$ and $U_{CC} < 0$, one realizes that consumption has a positive growth rate, that will be higher for higher values of the marginal productivity of capital, and lower for higher values of depreciation and pure time preference rates. A higher pure time preference rate indicates that the representative agent is less willing to trade present consumption, for future consumption, which is congruent with a lower growth rate of consumption.

So far no restrictions were made on the functional form of the production function, in section 4.1 this model will be further analyzed for an AK production function that presents constant returns to scale ($f''(K) = 0$) and for a Cobb-Douglas production function with decreasing returns to scale ($f''(K) < 0$).

3.2 Simple Energy Profile: One Stock of Polluting Capital

In the previous section only consumption influences utility, at this moment pollution is introduced. The situation is described as follows:

- Only one kind of producing energy technology, polluting;
- The stock of producing useful energy capital is the only state variable K
- Known initial stock of capital $K(0) = K_0$;
- Generic production function $f(K)$: $f'(K) > 0$;
- The pollution emitted doesn't accumulate (flow pollutant);
- Emissions increase with production ($\gamma_P > 0$):

$$P \propto Y \Rightarrow P = \gamma_P f(K) \tag{3.10}$$

- Budget constraint:

$$Y = C + I \tag{3.11}$$

  where C is consumption, and I is the investment in the stock of capital;
- Consumption is the only control parameter;
- Utility increases with consumption $U_C > 0$ and is concave $U_{CC} < 0$
- Utility decreases with pollution $U_P < 0$ and is concave $U_{PP} < 0$
- Utility is separable between the two components $U_{PC} = U_{CP} = 0$

The equation governing the dynamics of capital will be the same as in the previous section, equation (3.2) and the dynamic optimization problem is also identical: equation (A.19), subject to equations (3.2), (A.21) and (A.18)). The only difference lies in the fact that utility also depends on the level of pollution.

The current value hamiltonian is defined like previously:

$$H^c = U + \mu(f - C - \delta K) \tag{3.12}$$

The first-order necessary condition for a maximum with respect to consumption will be the same:

$$\begin{aligned} H^c_C &= 0 \\ \iff U_C &= \mu \end{aligned} \tag{3.13}$$

Until this point the fact that there are emissions associated with production hasn't altered the calculations. The difference appears in the Euler equation, since the utility function depends on pollution that, in turn, depends on the stock of capital:

$$\begin{aligned} H^c_K &= \rho\mu - \dot{\mu} \\ \iff \dot{\mu} &= -(f' - \delta - \rho)\mu - U_K \end{aligned} \tag{3.14}$$

From equation (3.10), that relates pollution with total production: $U_K = \gamma_P U_P f'$, so equation (3.14) becomes:

$$\dot{\mu} = -(f' - \delta - \rho)\mu - \gamma_P U_P f' \tag{3.15}$$

As before, from the previous equations it is possible to write the dynamic system for consumption and capital:

$$\dot{K} = f(K) - C - \delta K \tag{3.16}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}\left((1 + \frac{\gamma_P U_P}{U_C})f'(K) - \delta - \rho\right) \tag{3.17}$$

Since utility decreases with pollution ($U_P < 0$), increases with consumption ($U_C > 0$) and the term that relates pollution with production is positive ($\gamma_P > 0$), the negative impact of pollution in utility will appear as a factor that diminishes the net marginal productivity of capital:

$$\underbrace{(1 + \frac{\gamma_P U_P}{U_C})f'}_{\text{with pollution}} < \underbrace{f'}_{\text{without pollution}} \tag{3.18}$$

This term will depend on the marginal contributions of pollution and consumption to utility and represents the relative value of pollution. According to the preferences of the representative agent the optimal trajectory of consumption and capital stock accumulation will differ. As it will become clearer for particular production functions, this term will act as a limitation to growth.

Also, it is interesting to realize that rearranging the terms in equation (3.17) leads to a modified Ramsey rule, where the left side of the equation is altered by the term associated with pollution:

$$(1 + \frac{\gamma_P U_P}{U_C})f'(K) - \delta = \rho - C\left[\frac{U_{CC}}{U_C}\right]\frac{\dot{C}}{C} \tag{3.19}$$

This equation indicates that, for the same parameters, due to pollution, the equilibrium for which the agent is indifferent between consuming and saving will shift. The difficulty in interpreting this equation arises from the fact that the term associated with the role of pollution depends simultaneously on the level of consumption (through U_C) and the level of pollution (through U_P), that in turn depends on the production function.

The dynamics for the capital stock are the same as for the simple energy profile with and without pollution, therefore the difference between the two cases lies in the equation for the consumption path. which becomes more evident through the comparison of the Ramsey rules for each case, equation (3.19) for this case and equation (3.9) for the case without pollution, in section 3.1.

For a better comparison between the two cases, in section 4.2, this model will be studied for AK and Cobb-Douglas production functions. These two applications will reveal explicitly the consequences of introducing pollution.

3.3 Mixed Energy Profile: Predetermined Stocks of Polluting and Non-Polluting Capital

In this situation there will exist two stocks of capital, one polluting and the other clean, this corresponds to a mixed energy profile, e.g. a country that has some coal power plants and some hydropower plants. The main assumption is that the proportion of each type of capital is given exogenously, described as follows:

- Two kinds of producing useful energy technology;
- Two stocks of capital, thus two state variables $K_1 = a_1 K, K_2 = a_2 K, a_1 + a_2 = 1$;
- Known initial stocks of capital: $K_1(0) = K_{1,0}, K_2(0) = K_{2,0}$;
- Total production is the sum of each production $Y = f_1 + f_2$;
- Generic production functions $f_1(K_1), f_2(K_2)$: $f_1'(K_1) > 0, f_2'(K_2) > 0$;
- There's no pollution associated with K_1;

- Pollution associated with K_2 doesn't accumulate (flow pollutant);
- Emissions increase with production of 2:

$$P \propto Y_2 \Rightarrow P = \gamma_P f_2(K_2) \tag{3.20}$$

- Budget constraint:

$$Y = C + I = C + I_1 + I_2 \tag{3.21}$$

  where C is consumption, and I is the total investment, and I_i the investment made in the ith stock of capital;
- Consumption is the only control parameter;
- Utility increases with consumption $U_C > 0$ and is concave $U_{CC} < 0$
- Utility decreases with pollution $U_P < 0$ and is concave $U_{PP} < 0$
- Utility is separable between the two components $U_{PC} = U_{CP} = 0$

The depreciation rate of capital i is δ_i, so the equations governing its dynamics are:

$$\dot{K}_1 = I_1 - \delta_1 K_1 \tag{3.22}$$
$$\dot{K}_2 = I_2 - \delta_2 K_2 \tag{3.23}$$

Given that the fraction of each stock of capital is known (a_i), total capital K can be introduced as a new variable:

$$K := K_1 + K_2 \tag{3.24}$$

Using the budget constraint (3.21), equation (3.22) can be rewritten for total capital:

$$\dot{K} = f_1 + f_2 - C - (\delta_1 a_1 + \delta_2 a_2)K \tag{3.25}$$

The dynamic optimization problem is identical to the previous one, described by equation (A.19), subject to equation (3.25), to a similar initial condition (A.21), if one denominates $K(0) = K_{1,0} + K_{2,0} = K_0$, and transversality condition (A.18). The current value hamiltonian can be written as:

$$\begin{aligned} H^c &= U + \mu\dot{K} \\ &= U + \mu[f_1 + f_2 - C - (\delta_1 a_1 + \delta_2 a_2)K] \\ &= U + \mu(f - C - \delta K) \end{aligned} \tag{3.26}$$

This is very similar with the situation where only polluting capital existed (equation (A.13) in section 3.2), if the parameter total depreciation were introduced as $\delta = \delta_1 a_1 + \delta_2 a_2$ and total production as $f = f_1 + f_2$, the current value hamiltonian would be described the same way, as it is seen in equation (3.26)

As expected, the condition for a maximum with respect to consumption is stated as before (equation (3.13)), and the difference appears only in the Euler equation:

$$\begin{aligned} H^c_K &= \rho\mu - \dot{\mu} \\ \iff U_K + \mu(a_1 f_1' + a_2 f_2' - \delta) &= \rho\mu - \dot{\mu} \end{aligned} \tag{3.27}$$

From equation (3.20): $U_K = \gamma_P U_P a_2 f_2'$, re-substituting δ, equation (3.27) becomes:

$$\dot{\mu} = [\rho + (a_1(\delta_1 - f_1') + a_2(\delta_2 - f_2')]\mu - \gamma_P U_P a_2 f_2' \tag{3.28}$$

Using equation (3.28) and the condition for a maximum with respect to consumption, yields the ODE that describes the evolution of consumption, that together with the transition equation form the dynamical system to be solved:

$$\dot{K} = f_1 + f_2 - C - (\delta_1 a_1 + \delta_2 a_2)K \tag{3.29}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}\{(a_1[f_1' - \delta_1] + a_2[(1 + \frac{\gamma_P U_P}{U_C})f_2' - \delta_2] - \rho\} \tag{3.30}$$

Like in the previous case, the net marginal productivity of the polluting capital is decreased but this time there's also the marginal productivity of the clean technology. Hinting that the introduction of clean technology will allow for higher consumption growth, in comparison to the previous case. It is possible once again to write a modified Ramsey rule, for which the "global" net marginal productivity is a weighted sum of the net marginal productivities from the previous cases:

$$a_1[f_1' - \delta_1] + a_2[(1 + \frac{\gamma_P U_P}{U_C})f_2' - \delta_2] = \rho - C\left[\frac{U_{CC}}{U_C}\right]\frac{\dot{C}}{C} \tag{3.31}$$

Note that $a_1 + a_2 = 1$, so the maximum value that the left side of the equation can take is to be equal to the net marginal productivity for the case with a clean stock of capital (equation (3.9) from section 3.1). On the other extreme ($a_1 = 0$), this equation resumes to the one for only one polluting stock of capital (equation (3.19) from section 3.2). So, once again the effect of pollution contributes to a lower net marginal productivity. The fact that a_1 and a_2 are fixed is a major restriction that binds the social planner to always use both technologies, in the same way. This condition will be lifted in the next section.

3.4 Mixed Energy Profile: Variable Stocks of Non-Polluting Capital

It has been mentioned before that a mixed energy profile with constant fractions of each technology is very restrictive. Also, empirical knowledge shows that choosing how much to invest in different technologies is a natural control. Governments have the power to decide how to invest in the energy sector, e.g. build new fossil fueled power plants, oil refineries or renewable energy sources. So, this section will consider two stocks of capital, corresponding to different clean technologies, and the fraction of investment in each technology is introduced as an extra control variable.

The following statements describe the model:

- Two kinds of producing useful energy technology;
- There will be two state variables K_1 and K_2;
- Known initial stocks of capital: $K_1(0) = K_{1,0}$, $K_2(0) = K_{2,0}$;
- Total production is the sum of each production $Y = f_1 + f_2$;
- Generic production functions $f_1(K_1), f_2(K_2)$: $f_1'(K_1) > 0, f_2'(K_2) > 0$;
- Both technologies are clean, so there's no pollution;
- Budget constraint:
$$Y = C + I = C + I_1 + I_2 \tag{3.32}$$
where C is consumption, and I is total investment;
- Investment in each technology is:
$$I_1 = \alpha I \tag{3.33}$$
$$I_2 = (1 - \alpha)I \tag{3.34}$$

- There are two control parameters: consumption (C) and the fraction of investment applied to technology 1 ($\alpha \in [0,1]$);
- Utility increases with consumption $U_C > 0$ and is concave $U_{CC} < 0$

The depreciation rate for each type of capital is δ_1 and δ_2. The transition equations will be:

$$\dot{K}_1 = \alpha(f_1 + f_2 - C) - \delta_1 K_1 \tag{3.35}$$
$$\dot{K}_2 = (1-\alpha)(f_1 + f_2 - C) - \delta_2 K_2 \tag{3.36}$$

The dynamic optimization problem, for two control variables, is rewritten:

$$\max_{C,\alpha} W = \int_0^{+\infty} U(C) e^{-\rho t} dt \tag{3.37}$$

subject to equations (3.35) and (3.36), and the initial and transversality conditions:

$$K_i(0) = K_{i,0} > 0 \tag{3.38}$$
$$\lim_{t \to +\infty} [\mu_i(t) e^{-\rho t} K_i(t)] = 0, \text{ with } i = 1, 2 \tag{3.39}$$

The current value hamiltonian is written as:

$$\begin{aligned} H^c &= U + \mu_1 \dot{K}_1 + \mu_2 \dot{K}_2 \\ &= U + \mu_2(f_1 + f_2 - C) - \mu_1 \delta_1 K_1 - \mu_2 \delta_2 K_2 + (\mu_1 - \mu_2)(f_1 + f_2 - C)\alpha \end{aligned} \tag{3.40}$$

The condition for a maximum with respect to consumption will be:

$$\begin{aligned} H^c_C &= 0 \\ \iff U_C &= (\mu_1 - \mu_2)\alpha + \mu_2 \end{aligned} \tag{3.41}$$

The Euler equations are:

$$\begin{aligned} H^c_{K_1} &= \rho \mu_1 - \dot{\mu_1} \\ \iff \dot{\mu_1} &= (\rho + \delta_1)\mu_1 - (\alpha \mu_1 + (1-\alpha)\mu_2) f_1' \end{aligned} \tag{3.42}$$

$$\begin{aligned} H^c_{K_2} &= \rho \mu_2 - \dot{\mu_2} \\ \iff \dot{\mu_2} &= (\rho + \delta_2)\mu_2 - (\alpha \mu_1 + (1-\alpha)\mu_2) f_2' \end{aligned} \tag{3.43}$$

Notice that these equations for the shadow prices are very similar to equation (3.5), from section 3.1, for one stock of non-polluting capital, but the term multiplying the marginal productivity of capital f' is not just the shadow price of that stock of capital, it is a term depending on both prices, weighted by the fraction of investment in each technology: $\{\alpha \mu_1 + (1-\alpha)\mu_2\}$.

The hamiltonian is linear in the control variable α, and can be rewritten as:

$$H^c = \Psi + \alpha \Phi \tag{3.44}$$

Where

$$\begin{cases} \Psi = U + \mu_2(f_1 + f_2 - C) - \mu_1 \delta_1 K_1 - \mu_2 \delta_2 K_2 \\ \Phi = (\mu_1 - \mu_2)(f_1 + f_2 - C) \end{cases} \tag{3.45}$$

Φ is called the *switching function*, and from the Pontryagin maximum principle it follows that the Hamiltonian will be maximized for the extreme values of the fraction of investment in technology 1, according to the sign of the switching function:

$$\alpha = \begin{cases} 1 & \text{if } \Phi > 0 \\ 0 & \text{if } \Phi < 0 \end{cases} \tag{3.46}$$

When the optimal control is piece-wise constant, it is called a *bang-bang control*. In this context it implies that the optimal solution is to invest only in one technology at a time. The switch between the two possible values of α will be when the switching function Φ is zero. If this function only vanishes instantaneously, this moment is called the *switching time*, and the fraction of investment is only undefined for that instant.

If the switching function Φ vanishes over some time interval, the maximum principle doesn't specify the value of the optimal control α, so it is necessary to use alternative methods. The condition resulting from $\Phi = 0$ is called *singularity condition* and the optimal control problem is called a *singular control*. The approach used will be to determine the control for which the singularity condition holds over that time interval.

For clarity, the bang-bang and singular control situations are analyzed separately.

Bang-bang control $\alpha = 1$

If it was known *a priori* that at a given moment the shadow price of technology 1 μ_1 was higher than the one of technology 2 μ_2, it is straightforward to see that the switching function Φ is positive. Therefore, from condition (3.46) the optimal control is $\alpha = 1$, that is, investing only in technology 1.

This implication is coherent with the interpretation of the shadow price. From the budget constraint (3.32) it is known that one unit of capital costs exactly one unit as well (1 euro for 1 euro; 1 kWh for 1 kWh), the shadow price represents the gain from investing that unit, weighted by the constraints of the dynamics of that stock of capital. In this sense, if the stock of capital 1 is more valuable than the stock of capital 2, one would invest in the first one.

Since only investment in K_1 is undertaken, it is only possible to write the Euler equation for μ_1 (equation (3.42)). Using this equation and the condition for a maximum with respect to consumption (3.41) it is possible to write an equation for the dynamics of consumption that, with the transition equations for each stock of capital (equations (3.35) and (3.36)) form the following dynamical system:

$$\dot{K}_1 = f_1 + f_2 - C - \delta_1 K_1 \tag{3.47}$$

$$\dot{K}_2 = -\delta_2 K_2 \tag{3.48}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{3.49}$$

Equation (3.48) for the stock of capital 2 is straightforward to solve, corresponding to the exponential decay of this stock due to depreciation.

Since the time-path for K_2 has an independent closed form solution, the system formed by equations (3.47) and (3.49) becomes mathematically similar to the one studied in section 3.1, formed by equations (3.7) and (3.8). For a given production and utility functions it can be solved and further analyzed.

For this case, that has no pollution, the traditional Ramsey rule is, once again, retrieved from equation (3.49), with an identical form to the one in equation (3.9) from section 3.1.

Bang-bang control $\alpha = 0$

If, on the other hand, at a given instant the shadow price of technology 1 is lower than the one of technology 2, it is easy to see that the switching function Φ is negative. Therefore, from condition (3.46) the optimal control is $\alpha = 0$, that is, investing only in technology 2.

This time, only investment in K_2 is undertaken so the Euler equation for its shadow price μ_2 can be written. Once again, this equation, the condition for a maximum with respect to consumption (3.41) and

the transition equations for capital (3.35) and (3.36) yield the dynamical system for both stocks of capital and consumption:

$$\dot{K}_1 = -\delta_1 K_1 \tag{3.50}$$

$$\dot{K}_2 = f_1 + f_2 - C - \delta_2 K_2 \tag{3.51}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f_2' - \delta_2 - \rho) \tag{3.52}$$

This system is perfectly equivalent to the one obtained when investment is directed to technology 1 (equations (3.47), (3.48) and (3.49)).

Given the difficulty in determining the initial values of the shadow prices, the proposed solution is to analyze each case separately followed by a comparison of the utility in each path. Since there is no pollution, the utility function will only depend on the consumption level. The obtained differential equations for consumption were:

$$\text{For } \alpha = 1: \quad \dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{3.53}$$

$$\text{For } \alpha = 0: \quad \dot{C} = -\frac{U_C}{U_{CC}}(f_2' - \delta_2 - \rho) \tag{3.54}$$

From the comparison between equations (3.53) and (3.54) becomes clear that the technology that yields the higher utility will be the one that has higher net marginal productivity ($f_i' - \delta_i$). It can be concluded that one will invest only in the most productive technology, which is in accordance with economic intuition. Depending on the choice of a production function the switch from investing in one technology to the other may or not occur. In chapter 4, section 4.4, an application with the AK production function is analyzed.

Singular Control $0 < \alpha < 1$

The previous sections allowed for the characterization of the system for which the fraction of the investment is a bang-bang control. But, as mentioned before, the switching function may vanish for a finite time-interval, for which the optimal control is a singular control, and takes intermediate values ($0 < \alpha < 1$). Recall that the switching function is:

$$\Phi = (\mu_1 - \mu_2)(f_1 + f_2 - C) \tag{3.55}$$

The singularity conditions that make this function zero are equal shadow prices or zero investment. The last option is discarded because it is pointless to determine the optimal distribution of investment, when no investment is undertaken. Also, no investment results in decreasing consumption, a kind of solution of low interest within the context of this work, where solutions with consumption growth are sought.

If the shadow prices are equal, $\mu_1 = \mu_2 = \mu$, the Hamiltonian (equation (3.40)) becomes independent of the control variable α.

The condition for a maximum with respect to consumption will become:

$$U_C = \mu \tag{3.56}$$

Since investment in both stocks of capital is made, both Euler equations can be written:

$$\dot{\mu_1} = (\rho + \delta_1)\mu - \mu f_1'$$
$$\dot{\mu_2} = (\rho + \delta_2)\mu - \mu f_2'$$

For the singularity condition to remain valid for a finite time-interval it is necessary that its time derivative is also zero. Another way to state this is that if the shadow prices are equal at a given instant, they will only

remain equal if their rates of change are equal, so $\dot{\mu_1} = \dot{\mu_2}$. Using the Euler equations, a new form for the singularity condition is attained:

$$f_1' - \delta_1 = f_2' - \delta_2 \tag{3.57}$$

Note that this condition no longer depends on the shadow prices. It states that the optimal distribution of investment will be the one that makes the net marginal productivities of both stocks of capital equal.

For given production functions, condition (3.57) corresponds to a relation between the stocks of capital, so the dynamical system for K_1, K_2, C and α is described by that equation and the following:

$$\dot{K_1} = \alpha(f_1 + f_2 - C) - \delta_1 K_1 \tag{3.58}$$

$$\dot{K_2} = (1-\alpha)(f_1 + f_2 - C) - \delta_2 K_2 \tag{3.59}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{3.60}$$

$$\tag{3.61}$$

Making the simplification that the depreciation rates are equal $\delta_1 = \delta_2 = \delta$, and defining total capital and total production as

$$K := K_1 + K_2 \tag{3.62}$$

$$f(K) := f(K_1) + f(K_2) \tag{3.63}$$

it is possible to rewrite the problem:

$$\dot{K} = f(K) - C - \delta K \tag{3.64}$$

$$\dot{C} = -\frac{U_C}{2U_{CC}}(f'(K) - \delta - \rho) \tag{3.65}$$

Equations (3.64) and (3.65) form a dynamic system equivalent to the one that resulted from a single stock of non-polluting capital, described in section 3.1 by equations (3.7) and (3.8). The solution of this system yields the optimal path of consumption and total stock of capital. From the singularity condition (3.57) it is possible to determine the optimal path of each capital stock. Finally the optimal distribution of investment is given by:

$$\alpha = \frac{\dot{K_1} + \delta K_1}{f_1 + f_2 - C} \tag{3.66}$$

The optimal solution for this control problem may turn out to be just a singular control, a bang-bang control with or without switching or a mix between the two. However, it is important to notice that the condition for a maximum (3.41) imposes that consumption will be continuous for any switch that occurs. This is straightforward to show by writing this condition for each case:

$$\text{Bang-bang control } (\alpha = 1) : U_C = \mu_1 \tag{3.67}$$

$$\text{Bang-bang control } (\alpha = 0) : U_C = \mu_2 \tag{3.68}$$

$$\text{Singular control } (\mu_1 = \mu_2) : U_C = \mu_2 = \mu_1 \tag{3.69}$$

Given that U_C is a function of consumption C and that the switching instant is characterized by equal shadow prices of capital ($\mu_1 = \mu_2$) the value of consumption immediately after the switch must be equal to the one immediately before.

It will be shown in chapter 4, section 4.4, that the AK production function yields very particular results, with simple calculations and interpretation.

3.5 Mixed Energy Profile: Variable Stocks of Polluting and Non-Polluting Capital

This section is an extension of the previous one. There will be two stocks of capital, corresponding to different technologies and the distribution of investment will be a control parameter. This time, one of the technologies will be polluting, having a negative effect on utility. Summarizing:

- Two kinds of producing useful energy technology;
- There will be two state variables K_1 and K_2;
- Known initial stocks of capital: $K_1(0) = K_{1,0}$, $K_2(0) = K_{2,0}$;
- Total production is the sum of each production $Y = f_1 + f_2$;
- Generic production functions $f_1(K_1), f_2(K_2)$: $f_1'(K_1) > 0, f_2'(K_2) > 0$;
- There's no pollution associated with K_1;
- Pollution associated with K_2 doesn't accumulate (flow pollutant);
- Emissions increase with production of 2:

$$P \propto Y_2 \Rightarrow P = \gamma_P f_2(K_2) \tag{3.70}$$

- Budget constraint:

$$Y = C + I = C + I_1 + I_2 \tag{3.71}$$

  where C is consumption, and I is total investment;

- Investment in each technology is:

$$I_1 = \alpha I \tag{3.72}$$
$$I_2 = (1 - \alpha) I \tag{3.73}$$

- There are two control parameters: consumption (C) and the fraction of investment applied to technology 1 ($\alpha \in [0, 1]$);
- Utility increases with consumption $U_C > 0$ and is concave $U_{CC} < 0$
- Utility decreases with pollution $U_P < 0$ and is concave $U_{PP} < 0$
- Utility is separable between the two components $U_{PC} = U_{CP} = 0$

The equations for the dynamics of each stock of capital will be identical to the previous ones (3.35) and (3.36). The dynamic optimization problem is identical to the previous one (equation (3.37)) with the difference that utility also depends on pollution. The initial and transversality conditions are the same as in equations (3.38) and (3.39).

The current value hamiltonian will have the same form:

$$H^c = U + \mu_2(f_1 + f_2 - C) - \mu_1\delta_1 K_1 - \mu_2\delta_2 K_2 + (\mu_1 - \mu_2)(f_1 + f_2 - C)\alpha \tag{3.74}$$

And the condition for a maximum with respect to consumption will be the same as in equation (3.41).

The Euler equation for the non-polluting capital is the same as in equation (3.42) but, due to pollution, the Euler equation for the stock of capital 2 is different from the equation (3.43):

$$
\begin{aligned}
& H^c_{K_1} = \rho\mu_1 - \dot{\mu_1} \\
\iff & \dot{\mu_1} = (\rho + \delta_1)\mu_1 - (\alpha\mu_1 + (1-\alpha)\mu_2)f_1'
\end{aligned} \tag{3.75}
$$

$$
\begin{aligned}
& H^c_{K_2} = \rho\mu_2 - \dot{\mu_2} \\
\iff & \dot{\mu_2} = (\rho + \delta_2)\mu_2 - (\alpha\mu_1 + (1-\alpha)\mu_2)f_2' - U_{K_2}
\end{aligned} \tag{3.76}
$$

Using equation (3.70): $U_{K_2} = \gamma_P U_P f_2'$.

Every aspect regarding the linearity of the hamiltonian in the control variable α appears again with the same switching function. The bang-bang control and singular control will be studied in the following sections.

Bang-bang control $\alpha = 1$

Just like before, it will be assumed that at some instant it is know that the shadow price of technology 1 is higher than the one of technology 2, so that $\alpha = 1$, from equation (3.46). Investment is all directed towards the clean technology.

The resulting dynamical system is exactly the same as for the previous section, equations (3.47), (3.48) and (3.49):

$$\dot{K_1} = f_1 + f_2 - C - \delta_1 K_1 \tag{3.77}$$

$$\dot{K_2} = -\delta_2 K_2 \tag{3.78}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{3.79}$$

This time, the stock of K_2 is polluting, so the fact that no investment is directed towards it implies that pollution will be falling over time.

Bang-bang control $\alpha = 0$

If the shadow price of technology 1 is lower than the one of technology 2, then $\alpha = 0$, from equation (3.46), which means that investment is only directed to polluting technology.

Given that the Euler equation for μ_2 is altered due to pollution, so will the resulting dynamical system, that is given by:

$$\dot{K_1} = -\delta_1 K_1 \tag{3.80}$$

$$\dot{K_2} = f_1 + f_2 - C - \delta_2 K_2 \tag{3.81}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}((1 + \frac{\gamma_P U_P}{U_C})f_2' - \delta_2 - \rho) \tag{3.82}$$

For this case, the stock of clean capital will be declining exponentially over time, with the depreciation rate. The system formed by equations (3.81) and (3.82) is mathematically similar to the one studied in section 3.2, given by equations (3.16) and (3.17), where the negative impact of pollution acts as a factor that diminishes the net marginal productivity of capital.

Once again, the difficulty of determining the initial values of the shadow prices arises but, this time, the comparison of utility between the two possible bang-bang solutions is not so simple either. Utility will depend not only on consumption, but also on pollution levels. And the consumption path it self is affected by pollution, as can be seen by the comparison of the obtained differential equations for consumption:

$$\text{For } \alpha = 1 : \dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{3.83}$$

$$\text{For } \alpha = 0 : \dot{C} = -\frac{U_C}{U_{CC}}((1 + \frac{\gamma_P U_P}{U_C})f_2' - \delta_2 - \rho) \tag{3.84}$$

In chapter 4, section 4.5, this issue will be a analyzed for specific initial conditions. It will be showed that for low levels of pollution and consumption, the comparison between the two bang-bang solutions resumes to the comparison of the marginal productivities of capital and, just like in section 3.4, one would invest in the more productive capital.

Singular Control

To conclude the analysis, it is necessary to look also at the singular control. Once again, the singularity condition will correspond to equal shadow prices: $\mu_1 = \mu_2 = \mu$.

The condition for a maximum with respect to consumption will be the same as in equation (3.56). Using the Euler equations and the same reasoning as before, that if the shadow prices are equal at a given instant, they will only remain equal if their rates of change are equal ($\dot{\mu_1} = \dot{\mu_2}$), the singularity condition may be written as:

$$f_1' - \delta_1 = (1 + \frac{\gamma_P U_P}{U_C})f_2' - \delta_2 \tag{3.85}$$

As expected, this situation is equivalent to the one in section 3.4, equation (3.57), meaning that one will invest in both technologies, in a way that the net marginal productivity of the clean technology is equal to the net marginal productivity of polluting capital, that is diminished by the term corresponding to the negative effect of pollution. Using equation (3.85), the final dynamical system for K_1, K_2, C and α is completely characterized:

$$\dot{K_1} = \alpha(f_1 + f_2 - C) - \delta_1 K_1 \tag{3.86}$$

$$\dot{K_2} = (1 - \alpha)(f_1 + f_2 - C) - \delta_2 K_2 \tag{3.87}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{3.88}$$

The optimal distribution of investment is given by:

$$\alpha = \frac{\dot{K_1} + \delta K_1}{f_1 + f_2 - C} \tag{3.89}$$

In this case, it isn't useful to re-write the problem for total capital, because the effect for pollution depends only on the stock of capital of technology 2.

The main difference between this case and the one without pollution lies in the fact that the singularity condition (3.85) will impose a relation between the two stocks of capital, the consumption level and the pollution level. Without pollution, the singularity condition (3.57) only imposed a relation between the capital stocks.

Once again, the optimal solution for this control problem may turn out to be just a singular control, a bang-bang control with or without switching or a mix between the two, and just like before, consumption must be continuous throughout the switches. In chapter 4, section 4.5, is presented the analysis for AK production functions. Particularly, the evolution of pollution over time is analyzed and interpreted.

Stock of Capital	**Ramsey Rule**
Non-polluting	$f' - \delta = \rho - \left[\frac{CU_{CC}}{U_C}\right]\frac{\dot{C}}{C}$
Polluting	$(1 + \frac{\gamma_P U_P}{U_C})f' - \delta = \rho - \left[\frac{CU_{CC}}{U_C}\right]\frac{\dot{C}}{C}$
Fixed fraction of non-polluting (a_1) and polluting (a_2)	$a_1(f_1' - \delta_1) + a_2\left[(1 + \frac{\gamma_P U_P}{U_C})f_2' - \delta_2\right] = \rho - \left[\frac{CU_{CC}}{U_C}\right]\frac{\dot{C}}{C}$

Table 3.1: Ramsey rule for the different capital stocks: polluting, non-polluting and both

3.6 Overview

This chapter laid the theoretical foundations for the applications and analysis that follow in the next chapter.

Section 3.1 presents the simplest model, with one stock of clean capital, for which the conventional result from the Ramsey growth model was retrieved. Equations (3.7) and (3.8) form the obtained dynamical system.

The following section (section 3.2) presents also a simple energy profile, but for a polluting technology. The resulting dynamical system is formed by equations (3.16) and (3.17).

The comparison between these two models revealed that the existence of pollution will decrease the net marginal productivity of capital. And that the term responsible for this diminution represents the value of pollution and it is proportional to the ratio between the marginal disutility from pollution and the marginal utility from consumption.

Section 3.3 presented the first mixed energy profile, with clean and polluting technologies, used in fixed fractions. The resulting dynamical system, formed by equations (3.29) and (3.30), turned out to be a generalization of the previous models. Again, pollution appeared as a term diminishing the net marginal productivity but the simultaneous use of clean technology allowed for an attenuation of this effect.

For the first three sections a generalization of the Ramsey rule was determined that allows for a better comparison between models. The rules are presented in table 3.1.

Section 3.4 considered a mixed energy profile with only clean technology, and the fraction of investment in one of the stocks was introduced as a control variable. Since the hamiltonian was linear in this new control variable the concepts of bang-bang and singular control were introduced. It was determined that, if the net marginal productivities of the capital stocks are equal, the control will be such that they will remain equal (singular control). If this is not the case, then the optimal control is bang-bang, meaning that investment will be only in the more productive technology.

Section 3.5 extended the previous model to a mixed energy profile with clean and polluting technologies. The analysis was very similar but, as in sections 3.2 and 3.3, the introduction of pollution affected the net marginal productivity of capital stock.

In the following chapter these models will be applied to a particular functional form of utility and specific production functions.

Chapter 4

Energy Technology in Economic Growth - Applications

The previous chapter laid the foundations for the analysis of different macroeconomic models from simpler to more complex assumptions. This one will present the applications of these models to specific functional forms for production and utility. When it comes to a choice of a production function, the returns to scale are a factor that strongly influences the results and subsequent interpretation. When it comes to the energy sector, the marginal productivity of capital represents mainly the overall efficiency of power plants and the question if there are decreasing, increasing or constant returns to scale arises.

If one looks at the historical evolution of the energy sectors worldwide, production has become more and more efficient, so, should one have increasing returns to scale? While it may be a good description for an energy sector that it is improving overtime, there's a thermodynamic limit to the efficiency of any engine, so eventually this increasing returns to scale would converge to constant returns to scale. So our analysis will exclude this kind of production functions.

The hypothesis of decreasing returns to scale is more interesting, because it can replicate the physical limitations of building more and more power plants. For instance, if one thinks of coal power plants or nuclear power plants, the first facilities would be built in strategic locations, where it would be cheaper to construct, easier to bring fuel and to channel out produced electricity. But as demand increases, more plants are needed, so the next generation of power plants would be more expensive. These extra costs may not be significant enough to justify decreasing returns to scale when it comes to fossil fuels. But if one thinks of renewable energy the circumstances are the same with the extra complication that renewables depend greatly on weather conditions. At first the locations with best conditions would be chosen, e.g., for photovoltaic, the sunnier places, but as these facilities reach their maximum capacity, one must go for the second best location, and so on. So with an increasing stock of renewable capital, for the same technology, the electricity production will present decreasing returns to scale.

On the other hand, consider a country that already has a sizable stock of polluting capital and no clean capital and its energetic goal is to shift production from fossil fuels to renewables. If the polluting power plants were to be gradually decommissioned, it seems logic that the less productive ones would be shutdown first. So, as the stock of capital diminishes the returns to scale increase, that is, the global production efficiency is higher. This presents a situation where diminishing returns to scale exist but only when the stock of capital is decreasing. Meaning that one could chose to model polluting capital with a production function with this property but only if one is looking at this stage of the energy sector. Also, in this situation, since the stock of renewables is small, it could be said that this economy is far from the point where no more "best locations" are available, thus constant returns to scale seem accurate.

Finally, it becomes clear that for the study to be complete one must look at least to decreasing and constant returns to scale. The AK production function is an example of constant returns to scale, where if the production factor is increased by a given factor, total production increases by the same amount; mathematically this means that the function is homogeneous of degree 1[9]. Its functional form is $f = AK$,

where A is a positive constant that represents the state of technology, it can also be interpreted as the global efficiency of the energy sector.

The Cobb-Douglas production function is a very well known functional form of a neo-classical production function: $Y = AK^{\upsilon}$, where υ measures the elasticity of the production factor. That is, how much production varies if the production factor is altered. If $\upsilon = 1$ it reduces to the AK model, with constant returns to scale. If $\upsilon > 1$ it presents increasing returns to scale. And finally, if $\upsilon < 1$ it presents diminishing returns to scale[9].

This chapter will follow the same structure as the previous one, applying those models to AK and the first two also to Cobb-Douglas production function, with diminishing returns. Some numerical solutions are presented. For every case the same functional form for utility will be used.

Utility is a measure of the level of satisfaction resulting from the consumption of a certain good or service. A common measure of sustainability of a country is to have non-decreasing utility per capita. In the following sections, there will be two factors contributing to utility: useful consumption, contributing positively, and pollution, contributing negatively. As previously mentioned, pollution is associated with the energy sector, e.g. power plants, oil refineries, etc, and not directly with consumption. It will be assumed that the utility function is perfectly separable between the two inputs, indicating that the cross-derivatives are null, $U_{C^U P} = U_{PC^U} = 0$[1].

Also, it will be assumed that the considered functional form of utility presents constant intertemporal elasticity of substitution (CIES) for both components of the function (consumption and pollution). The intertemporal elasticity of substitution reflects the individuals' willingness to accept deviations from a uniform pattern of consumption (or pollution) over time. If the utility presents CIES then this preference doesn't change overtime, that is, the willingness to abdicate from consumption at time t_1 in order to consume later, at time t_2, is the same whether this choice is made earlier or later in time. This reasoning applies also for the negative utility (*desutility*) arising from pollution. Also, the higher the IES is, the slower the decline of the marginal utility in response to consumption growth, and therefore the more willing are individuals to accept variations in their consumption. The intertemporal elasticity of substitution (IES) is defined as:

$$IES = \left[\frac{C^U(t_1)/C^U(t_2)}{-U_{C^U}[C^U(t_1)]/U_{C^U}(C^U(t_2))} . \frac{d[U_{C^U}(C^U(t_1))/U_{C^U}[C^U(t_2)]]}{d[C^U(t_1)/C^U(t_2)]} \right]^{-1} \tag{4.1}$$

The final assumption regarding the utility function is that the marginal rate of substitution (MRS) between consumption and pollution will increase with increasing useful consumption. This means that to compensate for a given increase in pollution and maintain the same utility level, the increase in useful consumption is higher for higher levels of useful consumption. Another way to interpret this property is to think that if the consumption level is high, the consumers are more sensible to pollution, so the compensation for an increase in it must be large. The marginal rate of substitution is defined as:

$$MRS = \frac{dC^U}{dP} = -\frac{U_P}{U_{C^U}} \tag{4.2}$$

Given these properties, the utility function[36][37] will be defined as:

$$U(C^U, P) = \frac{(C^U)^{1-\sigma} - 1}{1-\sigma} - \kappa \frac{P^{\phi}}{\phi} \tag{4.3}$$

This functional form presents the characteristics mentioned before, it is separable between pollution and consumption. It has constant intertemporal elasticity of substitution for consumption $IES_{C^U} = 1/\sigma$ and for pollution $IES_P = 1/(1-\phi)$, and the marginal rate of substitution increases with increasing consumption:

$$MRS_{C^U} = k{C^U}^{\sigma} P^{\phi-1} > 0 \tag{4.4}$$

The marginal rate of substitution corresponds to the slope of the indifference curves, represented in figure 4.1. This graph shows the utility isoclines and it illustrates that for higher levels of consumption, in order to maintain the same utility, it is necessary a higher increase in consumption to compensate a certain increase in pollution.

[1]Subscripts indicate partial derivatives

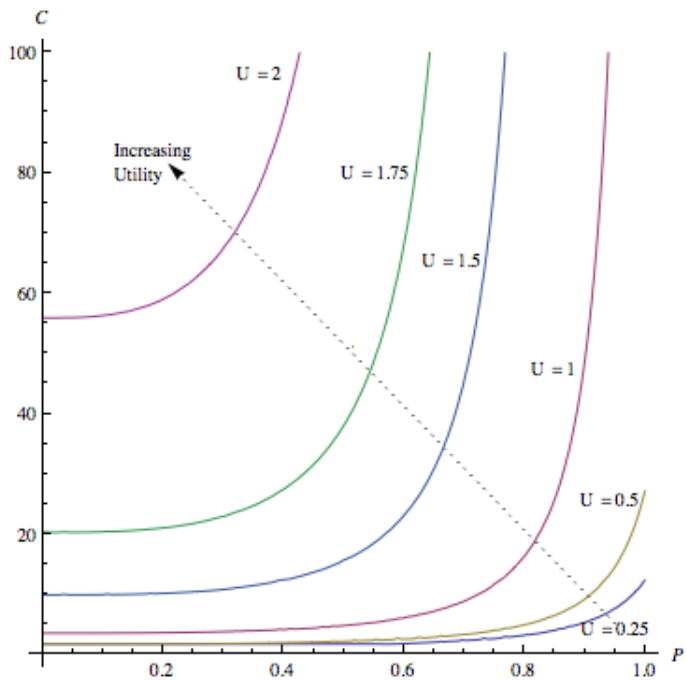

Figure 4.1: Utility isoclines (indifference curves), x-axis: pollution, y-axis: consumption

To determine constraints for the given parameters it is necessary to analyze the characteristics of the utility function. For higher levels of consumption, an increase in its level will correspond to a smaller increase in utility, which means that utility is concave with respect to C^U, so $U_{C^U C^U} < 0$ indicating that $\sigma > 0$. The utility function will also be concave with respect to pollution[38]. This means that, if the pollution levels are low, an increase in polluting emissions will have a small effect on utility, and vice-versa, so $U_{PP} < 0$ indicating that $\phi > 1$. Since pollution contributes negatively to utility, $U_P < 0$ implying $k > 0$.

Summarizing:

$$k > 0 \tag{4.5}$$

$$\sigma > 0 \tag{4.6}$$

$$\phi > 1 \tag{4.7}$$

Note that for higher values of σ, the lower the intertemporal elasticity of substitution for consumption, which means that the individuals will prefer a smoother pattern of consumption, and vice-versa. The same applies to pollution, but it is important to notice that the IES for pollution is negative, for higher values of ϕ, the less negative IES will be, indicating that the agents will prefer also a smoother pattern of pollution. Based on typical values used on the literature[36] these parameters were fixed at $\sigma = 1.4$ and $\phi = 2$.

The parameter κ appears as a weighting factor, through which pollution can exogenously be given more significance. It represents the conversion of *desutility* from pollution into utility from consumption, for the simulations that follow it was assumed that $\kappa = 1$ which indicates that the relative importance of each input to utility is measured solely through its intertemporal elasticities.

Every sector in the economy consumes output from the energy sector, converting it in consumption goods, this conversion is done with a certain energetic efficiency ϵ. What will contribute to the overall utility is the useful consumption of energy, that means that a higher energetic efficiency, for the same total consumption of energy, yields higher values of utility. Useful consumption C^U and total consumption C are related by $C^U = \epsilon C$, the utility function can be rewritten:

$$U(C, P) = \frac{(\epsilon C)^{1-\sigma} - 1}{1 - \sigma} - \kappa \frac{P^{\phi}}{\phi} \tag{4.8}$$

Due to the linear relation between useful and total consumption, the path for which total consumption is optimal will also correspond to an optimal path for useful consumption. For qualitative analysis this parameter will be taken into account, however the results presented next consider $\epsilon = 1$.

The following sections present the applications of the theoretical models developed in chapter 3. For further reference, some expressions concerning the partial derivatives of utility with respect to total consumption are presented next:

$$-\frac{U_C}{U_{CC}} = \frac{C}{\sigma} \tag{4.9}$$

$$\frac{U_P}{U_C} = -\frac{\kappa P^{\phi-1}}{\epsilon^{1-\sigma} C^{-\sigma}} \tag{4.10}$$

$$U_{PP} = -\kappa(\phi - 1)P^{\phi-2} \tag{4.11}$$

4.1 Simple Energy Profile: One Stock of Non-Polluting Capital

Recall the final system of ODE, equations (3.7) and (3.8), from section 3.1:

$$\dot{K} = f(K) - C - \delta K \tag{4.12}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f'(K) - \delta - \rho) \tag{4.13}$$

AK Production Function

The production function is:

$$f(K) = AK \tag{4.14}$$

Introducing the production function and the partial derivatives of the utility function in equations (4.12) and (4.13) yields:

$$\dot{K} = (A - \delta)K - C \tag{4.15}$$

$$\dot{C} = \frac{A - \delta - \rho}{\sigma} C \tag{4.16}$$

Using the initial condition, equation (A.21), this autonomous ODE system can be solved to yield:

$$K(t) = K_0 \exp\left[\frac{A - \delta - \rho}{\sigma} t\right] \tag{4.17}$$

$$C(t) = C_0 \exp\left[\frac{A - \delta - \rho}{\sigma} t\right] \tag{4.18}$$

This solution corresponds to a balanced growth path, where capital stock and consumption grow at the same rate. From the transversality condition (equation (A.18)) comes that initial consumption relates to the initial stock of capital through the following relation:

$$C_0 = \left[(A - \delta) - \frac{(A - \delta - \rho)}{\sigma}\right] K_0 \tag{4.19}$$

This relation also implies that $\left[(A - \delta) - \left(\frac{A-\delta-\rho}{\sigma}\right)\right] > 0$, which is guaranteed for $\sigma > 1$.

Provided that $(A - \delta - \rho)$ is positive, this economy shows long-run unbounded growth even without increasing technological progress (A is constant). This conclusion can also be drawn from the analysis of

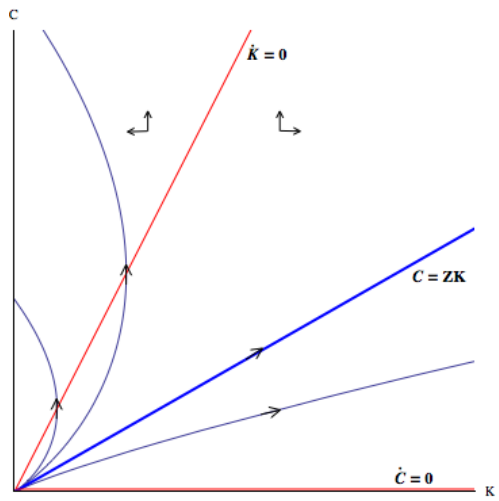

Figure 4.2: Phase diagram of the model with one stock of clean capital and AK production function

the original ODE system, without solving it explicitly. Its eigenvalues are $\lambda_1 = A - \delta$ and $\lambda_2 = \frac{(A-\delta-\rho)}{\sigma}$, if they're both positive, it means that the point where the capital stock is zero is an unstable node, indicating that no matter how small the stock of capital gets, it will always grow, diverging from the node. So, the capital stock will be zero only if its initial value is zero, staying there forever. But this situation is rejected *a priori* by the initial condition (A.21).

The phase diagram, in figure 4.2, illustrates the situation. The loci $\dot{K} = 0$ and $\dot{C} = 0$ divide the phase space in two regions, and the arrows show the direction of motion in each region. The optimal solution s also represented as $C = ZK$, with Z constant given by equation (4.19).

Finally, the Ramsey rule, from equation (3.9), can be rewritten, and the traditional result for an AK production function is obtained[9]:

$$A - \delta = \rho + \sigma \frac{\dot{C}}{C} \tag{4.20}$$

The interpretation of the parameters is straightforward, a higher net marginal productivity for capital, given by higher marginal productivity A or lower depreciation rate δ will yield higher long-run growth of consumption (and capital stock). If the pure rate of time preference ρ is higher it indicates that the representative agent gives less importance to the future, preferring to consume in the present, this will lead to a lower growth rate. Finally, the role of σ must be analyzed, a higher value means that a smoother path for consumption is preferred, which is consistent with a lower growth rate. Trough the Ramsey rule it is also possible to interpret the restriction that $(A - \delta - \rho)$ must be positive. If this requirement is not met, then the growth rate of consumption will be negative, so at most the net marginal productivity of capital must equal the pure rate of time preference, corresponding to a constant level of consumption.

Despite the fact that this model is extremely simplified, it serves the purpose of showing that using a production function with constant returns to scale implies unbounded growth, even without increasing technological progress. These results will act as benchmarks for the following AK model applications.

Cobb-Douglas Production Function

To study the hypothesis of a production function with diminishing returns to scale, the Cobb-Douglas functional form will be considered:

$$f(K) = AK^{\upsilon} \tag{4.21}$$

where A is a positive constant that, like before, represents the state of technology, and can be related to the overall efficiency of the energy sector, and $\upsilon < 1$.

Introducing the production function and the partial derivatives of the utility function in equations (4.12) and (4.13), one gets:

$$\dot{K} = AK^{\upsilon} - \delta K - C \tag{4.22}$$

$$\dot{C} = \frac{(\upsilon AK^{\upsilon-1} - \delta - \rho)}{\sigma} C \tag{4.23}$$

Equations (4.22) and (4.23) form a non-linear system of ODEs that can only be solved numerically. It is pertinent to analyze the phase diagram of the system and draw some conclusions. For that it is necessary to determine the locus along which the capital stock doesn't change overtime ($\dot{K} = 0$):

$$C|_{K=const} = (AK^{\upsilon-1} - \delta)K \tag{4.24}$$

Given that the $\upsilon < 1$, equation (4.24) corresponds to a concave line, $\left.\frac{\partial^2 C}{\partial K^2}\right|_{K=const} < 0$, with a maximum point, given by:

$$K = \left(\frac{\upsilon A}{\delta}\right)^{\frac{1}{1-\upsilon}} \tag{4.25}$$

The same condition for consumption ($\dot{C} = 0$) yields the following equations:

$$C = 0 \tag{4.26}$$

$$K|_{C=const} = \left(\frac{\upsilon A}{\delta + \rho}\right)^{\frac{1}{1-\upsilon}} \tag{4.27}$$

The locus corresponding to equation (4.26) is an horizontal line where consumption is null and equation (4.27) it is a vertical line corresponding to a constant value of capital.

The points of intersection between the loci are given by:

$$\begin{cases} (K^*, C^*)_a = \{0, 0\} \\ (K^*, C^*)_b = \left\{ \left(\frac{\upsilon A}{\delta+\rho}\right)^{\frac{1}{1-\upsilon}}, A\left(\frac{\upsilon A}{\delta+\rho}\right)^{\frac{\upsilon}{1-\upsilon}} - \delta\left(\frac{\upsilon A}{\delta+\rho}\right)^{\frac{1}{1-\upsilon}} \right\} \\ (K^*, C^*)_c = \left\{ \left(\frac{A}{\delta}\right)^{\frac{1}{1-\upsilon}}, 0 \right\} \end{cases} \tag{4.28}$$

In figure 4.3 the phase diagram is represented, the loci divide the space in four regions, the arrows in each one indicate the direction of motion in the region. This system is saddle-path stable, with an unstable manifold running from regions 2 and 4, and a stable manifold running from regions 1 and 3. If the optimal trajectory was located in region 2, consumption would increase and capital would decrease, at some point, capital would reach zero, forcing consumption to jump to zero. In region 4, capital would increase and tend to a constant positive value, violating the transversality condition[39]. As it is shown by the arrows in figure 4.3, any trajectory, other than the stable manifold, will eventually cross over the loci to regions 2 or 4. Therefore, every trajectory is excluded except the stable manifold that converges to the steady-state $(K^*, C^*)_b$.

The comparison between the expression for the steady-state $(K^*, C^*)_b$ and equation (4.25) for the maximum, reveals that the steady-state lies always to the left of this extremum point. This means that higher steady-state values of capital will correspond also to higher steady-state values of consumption. Also, from the analysis of the expression $(K^*, C^*)_b$, in equation (4.28), it can be concluded that a higher marginal productivity of capital A will lead to a higher stock of capital and consumption level. The contrary effect will be seen for higher values of depreciation and time preference rates. These parameters preserve the same interpretation as in the previous section, for AK production function. The only new parameter is the elasticity υ representing the diminishing returns to scale. When it approaches unit, the steady state tends to

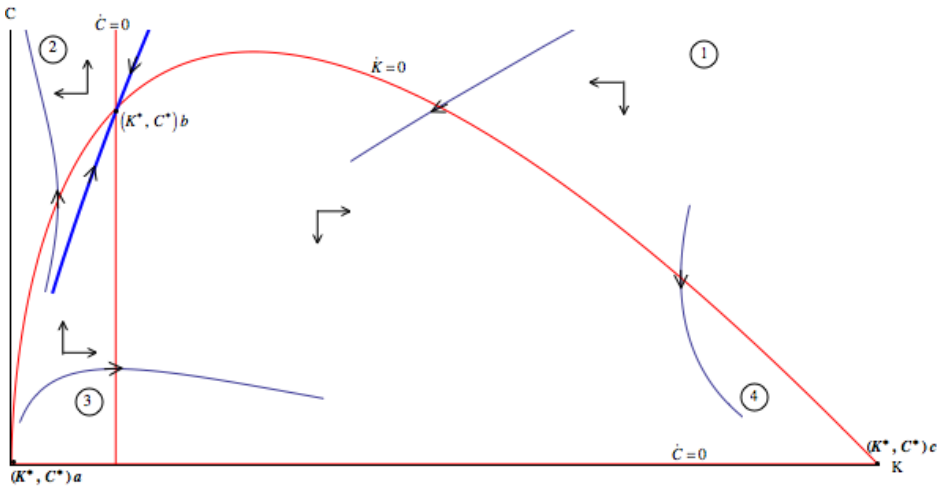

Figure 4.3: Phase diagram of the model with one stock of clean capital and Cobb-Douglas production function

infinity, that is, the diminishing returns to scale become constant returns to scale, and this case reduces to the previous one. If on the other hand, υ is very small, then the system will converge to very small values of capital and consumption.

This result corresponds to the classical Ramsey problem with a neoclassical production function. In contrast with the previous section, this model doesn't show long-run unlimited growth, the diminishing returns to capital are sufficient to make the optimal trajectory tend to a steady-state point.

4.2 Simple Energy Profile: One Stock of Polluting Capital

Recall the final system of ODE, equations (3.16) and (3.17) from section 3.2:

$$\dot{K} = f(K) - C - \delta K \tag{4.29}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}((1 + \frac{U_P \gamma_P}{U_C})f'(K) - \delta - \rho) \tag{4.30}$$

AK Production Function

The production function is:

$$f(K) = AK \tag{4.31}$$

So, according to equation 3.10, pollution will be defined as:

$$P = \gamma_P AK \tag{4.32}$$

Introducing the production function and the partial derivatives of the utility function in equations (4.29) and (4.30) yields:

$$\dot{K} = (A - \delta)K - C \tag{4.33}$$

$$\dot{C} = -\frac{\kappa(A\gamma_P)^{\phi}\epsilon^{\sigma-1}}{\sigma}(K)^{\phi-1}(C)^{\sigma+1} + \frac{(A - \delta - \rho)}{\sigma}C \tag{4.34}$$

Note that, even with AK production function, the introduction of pollution resulted in a non-linear dynamical system. Like in section 3.1 for the Cobb-Douglas production function, it is useful to analyze the phase diagram, represented in figure 4.4. The locus $\dot{K} = 0$ is a straight line corresponding to:

$$C = (A - \delta)K \tag{4.35}$$

The locus $\dot{C} = 0$ corresponds to the following conditions:

$$C = 0 \tag{4.36}$$

$$C = \left(\frac{A - \delta - \rho}{\kappa (A\gamma_P)^{\phi} \epsilon^{\sigma - 1}} \right)^{1/\sigma} (K)^{\frac{1-\phi}{\sigma}} \tag{4.37}$$

These two loci divide the space in four regions, and the arrows, in figure 4.4, show the direction of motion in each region. Like in section 3.1 for the Cobb-Douglas production function, this system is saddle-path stable, with an unstable manifold running from regions 2 and 4, and a stable manifold running from regions 1 and 3. For the same reasons presented previously, every trajectory except the stable manifold is rejected[39].

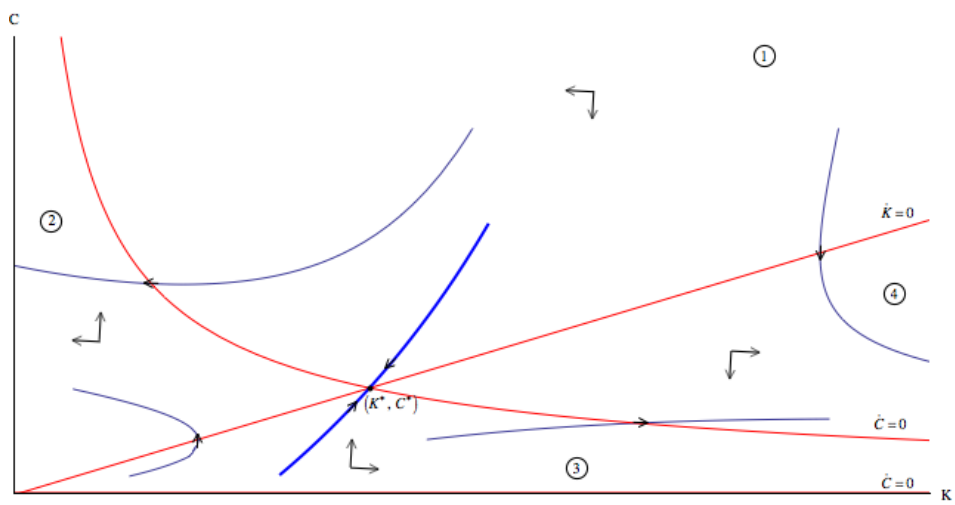

Figure 4.4: Phase diagram of the model with one stock of polluting capital and AK production function

For the steady-state, the values of capital and consumption are given by expressions (4.38) and (4.39).

$$K^* = \left(\frac{A - \delta - \rho}{\kappa (A\gamma_P)^{\phi} \epsilon^{\sigma - 1} (A - \delta)^{\sigma}} \right)^{\frac{1}{\sigma + \phi - 1}} \tag{4.38}$$

$$C^* = (A - \delta) \left(\frac{A - \delta - \rho}{\kappa (A\gamma_P)^{\phi} \epsilon^{\sigma - 1} (A - \delta)^{\sigma}} \right)^{\frac{1}{\sigma + \phi - 1}} \tag{4.39}$$

As expected the steady-state values will depend on all the parameters of the problem. It is necessary that $A > \delta + \rho$, otherwise the capital could take negative values.

Recall the definition of pollution from equation (4.32), and the dependence of utility (equation (4.8)) in it. If the contribution of pollution is higher, in terms of $k(A\gamma_P)^{\phi}$, it is straightforward to see from expressions (4.38) and (4.39) that consumption and capital will converge to lower values.

If the efficiency of conversion ϵ is increased, the steady state value of total consumption and capital is lower, because it is possible to attain the same useful consumption for less total consumption and capital. This is expressed directly in the functional form of utility (equation (4.8)), that with for higher ϵ, less total consumption is needed to achieve the same welfare. Also, with increasing efficiency, useful consumption is

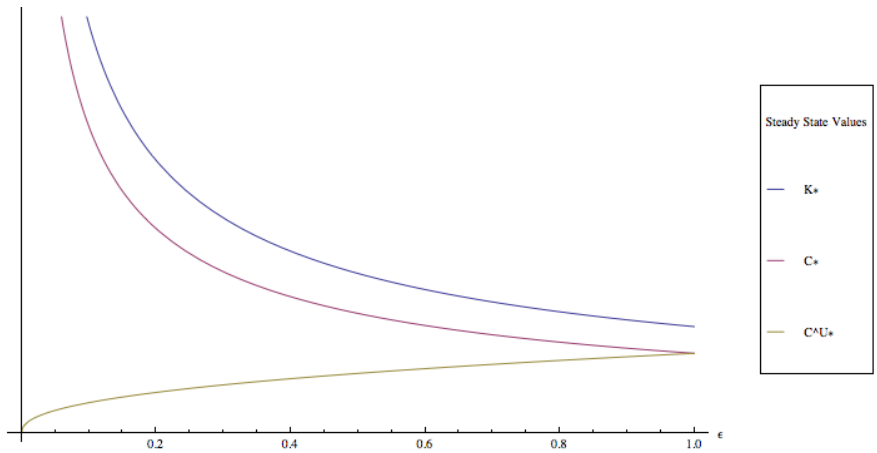

Figure 4.5: Variation of the steady state values of capital, consumption and useful consumption with respect to the efficiency of conversion ϵ

also increasing. In figure 4.5 it is represented the graph of the steady state values of capital, consumption and useful consumption with respect to the efficiency of conversion ϵ, that illustrates the previous statements.

Conditions (4.6) and (4.7), guarantee that the exponent $\frac{1}{\sigma+\phi-1}$ is always positive, rearranging the terms gives:

$$\frac{1}{\sigma+\phi-1} = \frac{1}{\phi-(1-\sigma)} > 0 \tag{4.40}$$

The denominator in this expression is a relation for the balance between the impacts of consumption and pollution in the overall utility. The explicit functional form given by equation (4.8) reveals that $1-\sigma$ is the factor that weighs the contribution of consumption to utility. The same applies to ϕ but with respect to pollution. For the same conditions, utility will be lower for higher values of ϕ, and will be higher for higher values of $1-\sigma$.

In figure 4.6, there is a representation of the steady state values of consumption and capital in relation with ϕ. It is interesting to notice that when pollution has more impact, that is when ϕ increases, these values increase, but not indefinitely, in fact the limit when ϕ tends to infinity is known:

$$\lim_{\phi\to\infty} K^* = (A\gamma_P)^{-1}$$
$$\lim_{\phi\to\infty} C^* = (A-\delta)(A\gamma_P)^{-1}$$

A higher exponent means that pollution has more impact on utility so in order to achieve the same steady-state value of utility, the optimal solution is to increase consumption. On the other hand, if $(1-\sigma)$ increases, consumption has a higher impact on utility, so to achieve the same welfare, less consumption is needed, that's why the steady state values decrease with $(1-\sigma)$, as represented in figure 4.7.

Finally in figure 4.8 it is represented the pollution to income relation, that increases linearly until a maximum level, corresponding to the total output of the steady state. It is important to note that over time pollution will be constant, which means that even though economic growth comes to a halt the model presents long-term polluting emissions. This is a consequence of considering a flow pollutant, instead of a stock pollutant. For the representative agent, the attained level of pollution is optimal but for the environment it may not be.

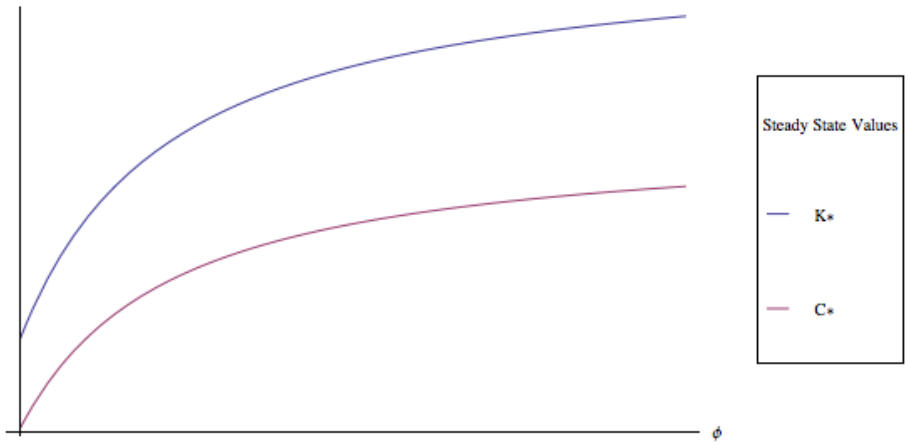

Figure 4.6: Variation of K^* and C^* with respect to the weighing factor of the effect of pollution in utility (ϕ)

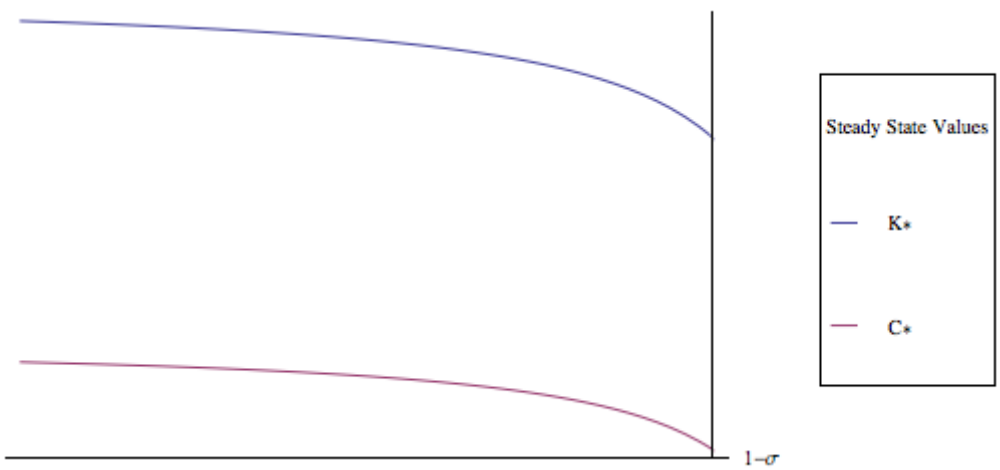

Figure 4.7: Variation of K^* and C^* with respect to the weighing factor of the effect of consumption in utility $(1 - \sigma)$

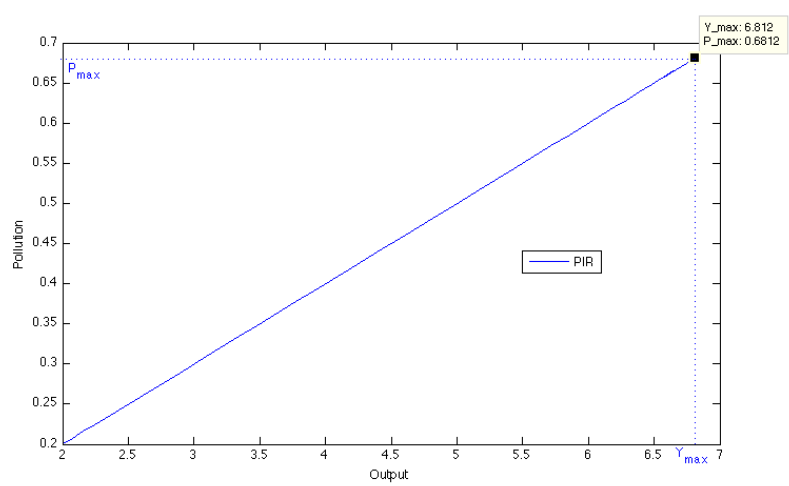

Figure 4.8: Pollution to income relation (PIR)

Cobb-Douglas Production Function

Just like in the section 4.1, the Cobb-Douglas production function will also be studied. Previously, the introduction of diminishing returns to scale turned a system that has exponential growth into a saddle-path stable system. For the case with polluting technology, the system's optimal growth was already bounded, and presented also saddle-path stability. The introduction of a Cobb-Douglas production function will largely complicate the calculations and conclusions won't be as general as before because it is not possible to write an explicit form for the steady-state values. However, it is important to introduce the analysis made, so that it becomes clear that the limitations to growth due to pollution will weigh considerably more than the diminishing returns.

The production function is:

$$f(K) = AK^{\upsilon} \tag{4.41}$$

So pollution will be defined as:

$$P = \gamma_P AK^{\upsilon} \tag{4.42}$$

From equation (4.42) it is noticeable that the diminishing returns to capital propagate to the pollution function. This is an unavoidable consequence of imposing pollution proportional to total output, but one could question if having a less productive capital means that less pollution is produced. Exploiting further the relation between pollution, output and technology is beyond the scope of this work, but this observation should be kept in mind.

Introducing the production function and the partial derivatives of the utility function in equations (4.29) and (4.30) yields:

$$\dot{K} = AK^{\upsilon} - \delta K - C \tag{4.43}$$

$$\dot{C} = -\frac{\kappa(A\gamma_P)^{\phi}\upsilon\epsilon^{\sigma-1}}{\sigma}(K)^{\upsilon\phi-1}(C)^{\sigma+1} + \frac{(\upsilon AK^{\upsilon-1} - \delta - \rho)}{\sigma}C \tag{4.44}$$

Equation (4.43) is identical to equation (4.23) from section 4.1, where the Cobb-Douglas production function was applied for the case without pollution. On the other hand,equation (4.44), presents some similarities with equation (4.34), from the previous section 4.2. The terms associated with pollution are the same (first term of the equation) but the introduction of a non-linear production function added a dependency in K for the second term of the equation. In the previous case, the second term was only dependent on C, this results from the fact that an AK production function is equivalent to having $\upsilon = 1$. Once again, this is a non-linear ODE system, and the main interest lies in interpreting the phase diagram.

The locus along which the capital stock doesn't change overtime ($\dot{K} = 0$) is given by:

$$C|_{K=const} = (AK^{\upsilon-1} - \delta)K \tag{4.45}$$

And the same condition for consumption ($\dot{C} = 0$) yields the two equations:

$$C = 0 \tag{4.46}$$

$$C = \left(\frac{A\upsilon K^{\upsilon-1} - \delta - \rho}{\kappa(A\gamma_P)^{\phi}\upsilon\epsilon^{1-\sigma}}\right)^{1/\sigma}(K)^{\frac{1-\upsilon\phi}{\sigma}} \tag{4.47}$$

The locus defined by equation (4.45) has exactly the same properties as the one in section 4.1, described by equation (4.24).

Regarding the locus corresponding to equation (4.47), making the assumption that $\upsilon\phi > 1$, it can be stated that consumption tends to infinity if capital stock tends to zero and that the locus is convex $\frac{\partial C}{\partial K} < 0$. Finally, the capital stock for which consumption is zero may also be determined:

$$K|_{C=0} = \left(\frac{A\upsilon}{\delta + \rho}\right)^{\frac{1}{1-\upsilon}} \tag{4.48}$$

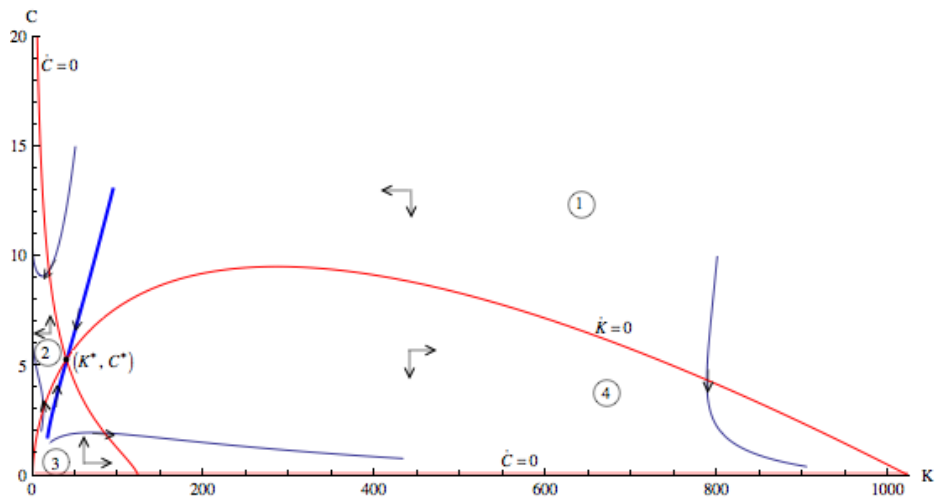

Figure 4.9: Phase diagram of the model with one stock of polluting capital and Cobb-Douglas production function

Using these observations, the phase diagram maybe constructed and it is represented in figure 4.9.

Like in section 4.1, there are three points of intersection between the loci, two of them being the same: $(K^*, C^*)_a$ and $(K^*, C^*)_c$. Once again, the system is saddle-path stable, the space is divided in four, from regions 2 and 4 runs an unstable manifold, and from regions 1 and 3 runs a stable manifold. The continuity and transversality conditions assure that the optimal trajectory exists in the stable regions, converging to the steady state: $(K^*, C^*)_b$, where both consumption and capital are non-null. For this case it isn't possible to write down an explicit expression for this point, but it is defined by the following equations, and it can be calculated for given parameter values:

$$K^* : (AK^{\upsilon-1} - \delta)K = \left(\frac{A\upsilon K^{\upsilon-1} - \delta - \rho}{\kappa(A\gamma_P)^{\phi}\upsilon\epsilon^{\sigma-1}}\right)^{1/\sigma} (K)^{\frac{1-\upsilon\phi}{\sigma}} \tag{4.49}$$

$$C^* = (AK^{*\upsilon-1} - \delta)K^* \tag{4.50}$$

It is pertinent to wonder how does this steady state relate to the ones found previously. In figure 4.10 it is represented the phase diagram of the current case and the one with pollution but using AK production function, from the previous section. The subscript AK stands for the point determined using AK production function and the subscript CD for the one using Cobb-Douglas production function.

Since it is not possible to have an explicit expression for the steady state and compare the points directly, one can only compare the shape of each curve and attempt to draw some conclusions.

The loci associated with $\dot{K} = 0$ are given by:

$$C_{AK}|_{K=const} = (A - \delta)K \tag{4.51}$$

$$C_{CD}|_{K=const} = (AK^{\upsilon-1} - \delta)K \tag{4.52}$$

These lines have two intersection points, corresponding to zero capital stock and $K = 1$. For $0 < K < 1$, the locus associated with Cobb-Douglas production function will correspond to higher values of consumption, if the capital stock is higher than unity then the opposite occurs. In figure 4.10 the second situation is more visible.

If one makes the assumption that $K > 1$, then it is possible to conclude that for a given stock of capital, the AK model will correspond to higher consumption values. But the steady-state will also depend on the other locus, associated with $\dot{C} = 0$, that are given by:

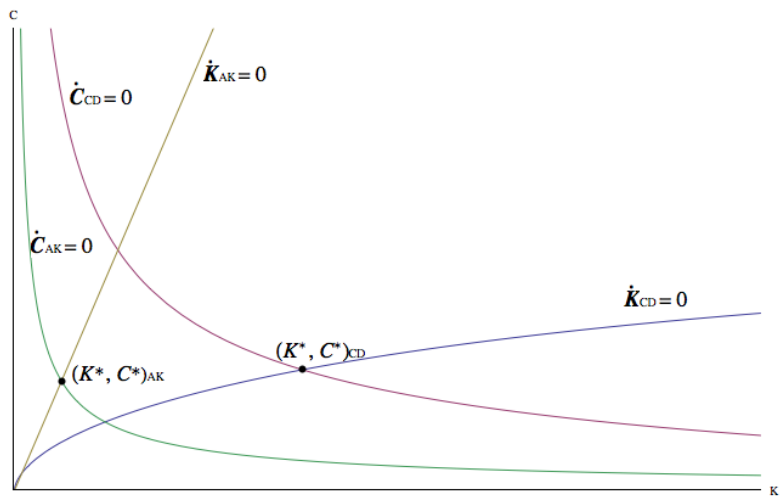

Figure 4.10: Comparison of the phase diagrams of the models with one stock of polluting capital for Cobb-Douglas and AK production functions

$$C_{AK} = \left(\frac{A - \delta - \rho}{\kappa (A\gamma_P)^{\phi} \epsilon^{\sigma - 1}} \right)^{1/\sigma} (K)^{\frac{1-\phi}{\sigma}} \tag{4.53}$$

$$C_{CD} = \left(\frac{A\upsilon K^{\upsilon - 1} - \delta - \rho}{\kappa (A\gamma_P)^{\phi} \upsilon \epsilon^{\sigma - 1}} \right)^{1/\sigma} (K)^{\frac{1-\upsilon\phi}{\sigma}} \tag{4.54}$$

For small δ and ρ, the previous expressions can be rewritten:

$$C_{AK} \simeq \left(\frac{A}{\kappa (A\gamma_P)^{\phi} \epsilon^{\sigma - 1}} \right)^{1/\sigma} (K)^{\frac{1-\phi}{\sigma}} \tag{4.55}$$

$$C_{CD} \simeq \left(\frac{A}{\kappa (A\gamma_P)^{\phi} \epsilon^{\sigma - 1}} \right)^{1/\sigma} (K)^{\frac{\upsilon(1-\phi)}{\sigma}} \tag{4.56}$$

The direct comparison of the previous expressions indicates that, for a given level of consumption, the steady state value of capital will be higher for the Cobb-Douglas production function. This outcome has to do with the introduction of diminishing returns into the production function, which will affect the amount of pollution produced. For the AK production function, if the stock of capital is doubled, so will production and with it pollution, however for the Cobb-Douglas production doubling the stock of capital doesn't implicate doubling production nor pollution. This relationship is what lies beneath the fact that the Cobb-Douglas production function allows for higher values of capital, for the same consumption. Another way to see this mechanism is to take into account the fact that to achieve the same production levels with the Cobb-Douglas a bigger stock of capital is needed, once again, due to diminishing returns to scale.

So it turns out that depending on the parameter values, the choice of a production function with or without constant returns to capital, may affect differently the optimal trajectory. It can be concluded that the most important effect arises from the impact of pollution on growth, independent of the choice of a production function. This observation is strongly supported by the comparison between the results of a Cobb-Douglas production function with and without pollution (section 4.1), as it is explained bellow.

The locus associated with $\dot{K} = 0$ will be the same for both cases, given by equation (4.45), and the loci for $\dot{C} = 0$ are:

$$K_{CDNP} = \left(\frac{\delta + \rho}{\upsilon A}\right)^{\frac{1}{\upsilon - 1}} \tag{4.57}$$

$$C_{CDP} = \left(\frac{A\upsilon K^{\upsilon-1} - \delta - \rho}{\kappa (A\gamma_P)^{\phi} \upsilon \epsilon^{\sigma}}\right)^{1/\sigma} (K)^{\frac{1-\upsilon\phi}{\sigma}} \tag{4.58}$$

where $CDNP$ stands for "Cobb-Douglas without pollution" and CDP stands for "Cobb-Douglas with pollution". The steady state for the case without pollution will always correspond to a higher value of capital stock and, since it is located before the maximum of the locus $\dot{K} = 0$, also to a higher value of consumption. One way to see this, is to substitute the expression for K given by equation (4.57) in equation (4.58). It results that the value of consumption associated with that stock of capital is null: $C_{CDP}\left\{\left(\frac{\delta+\rho}{\upsilon A}\right)^{\frac{1}{\upsilon-1}}\right\} = 0$. In figure 4.11, the three loci are represented, illustrating this comparison. It can be concluded that the effect of pollution is dominant over the effect of diminishing returns, forcing the system to stabilize at lower levels of capital and consumption.

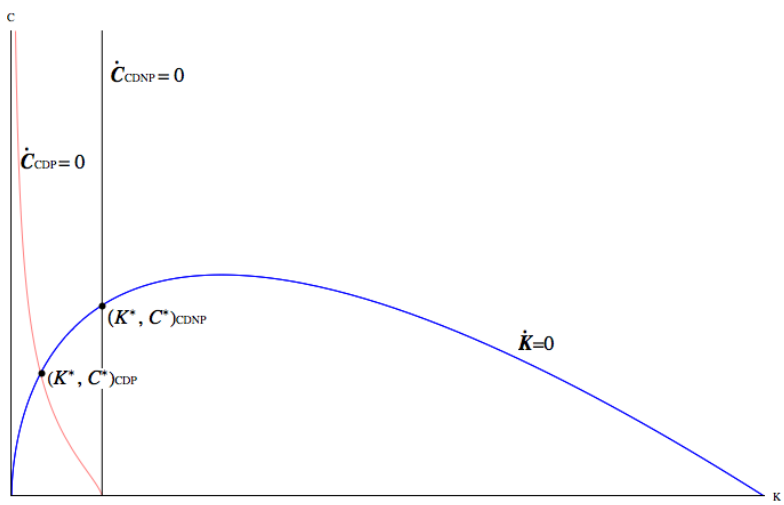

Figure 4.11: Comparison of the phase diagrams of the models with Cobb-Douglas production function with and without pollution

Finally it is important to mention that the pollution to income relation will be similar to the one presented in the previous section (figure 4.8)), which means that this model also presents long-term constant polluting emissions.

In the following sections the analysis of the models will be restricted to AK production function.

4.3 Mixed Energy Profile: Predetermined Stocks of Polluting and Non-Polluting Capital

In chapter 3, section 3.3, the equations governing the dynamical system formed by capital stock and consumption were determined for a predetermined mixed energy profile, equations (3.29) and (3.30), that are presented next:

$$\dot{K} = f_1 + f_2 - C - (\delta_1 a_1 + \delta_2 a_2)K \tag{4.59}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}\{(a_1[f_1' - \delta_1] + a_2[(1 + \frac{U_P \gamma_P}{U_C})f_2' - \delta_2] - \rho\} \tag{4.60}$$

AK Production Function

Considering that both technologies are characterized by AK production functions and that the fraction of each stock is known:

$$f_1 = A_1 a_1 K \tag{4.61}$$

$$f_2 = A_2 a_2 K \tag{4.62}$$

Only one of the technologies is pollutant, so:

$$P = \gamma_P A_2 a_2 K \tag{4.63}$$

Substituting in equations (4.59) and (4.60):

$$\dot{K} = [(A_1 a_1 + A_2 a_2) - (\delta_1 a_1 + \delta_2 a_2)]K - C \tag{4.64}$$

$$\dot{C} = -\frac{\kappa(a_2 A_2 \gamma_P)^{\phi} \epsilon^{\sigma-1}}{\sigma}(K)^{\phi-1}(\epsilon C)^{\sigma+1} + \frac{[(A_1 a_1 + A_2 a_2) - (\delta_1 a_1 + \delta_2 a_2) - \rho]}{\sigma}C \tag{4.65}$$

The system formed by equations (4.64) and (4.64) is a non-linear system of ODEs that presents saddle-path stability, and the expression for the steady-state is given by:

$$K^* = \left(\frac{M - \rho}{\kappa(a_2 A_2 \gamma_P)^{\phi}(\epsilon M)^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}} \tag{4.66}$$

$$C^* = M\left(\frac{M - \rho}{\kappa(a_2 A_2 \gamma_P)^{\phi}(\epsilon M)^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}} \tag{4.67}$$

where $M = (A_1 a_1 + A_2 a_2) - (\delta_1 a_1 + \delta_2 a_2)$.

This case is a generalization of the models presented in sections 4.1 and 4.2, for AK technology. If there's no clean technology stock, $a_1 = 0$, the system would be exactly the same as for AK technology with one stock of polluting capital (section 4.2). The introduction of a non-polluting productive capital allows for higher steady-state values of consumption and capital stock, in fact, if all production is clean, $a_1 = 1$, the system resumes to the one for AK technology with one stock of clean capital (section 4.1), for which these values tend to infinity. In figure 4.12, it is represented the phase diagram for the three situations: $\{a_1 = 0\}$, $\{0 < a_1 < 1\}$ and $\{a_1 \to 1\}$. The steady state for the present case (K^{*2}, C^{*2}) corresponds to higher values of capital stock and consumption than when only polluting technology is used (K^{*1}, C^{*1}).

The steady-states of capital and consumption will decrease with increasing fraction a_2 of polluting capital, in figure 4.13 this variation is illustrated. The minimum values that the steady-states can take correspond to the case where only polluting technology is used, a result determined in section 4.2, presented in equations (4.38) and (4.39) and rewritten here:

$$K^* = \left(\frac{A_2 - \delta_2 - \rho}{\kappa(a_2 A_2 \gamma_P)^{\phi}\epsilon^{\sigma-1}(A_2 - \delta_2)^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}} \tag{4.68}$$

$$C^* = (A_2 - \delta_2)\left(\frac{A_2 - \delta_2 - \rho}{\kappa(a_2 A_2 \gamma_P)^{\phi}\epsilon^{\sigma-1}(A_2 - \delta_2)^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}} \tag{4.69}$$

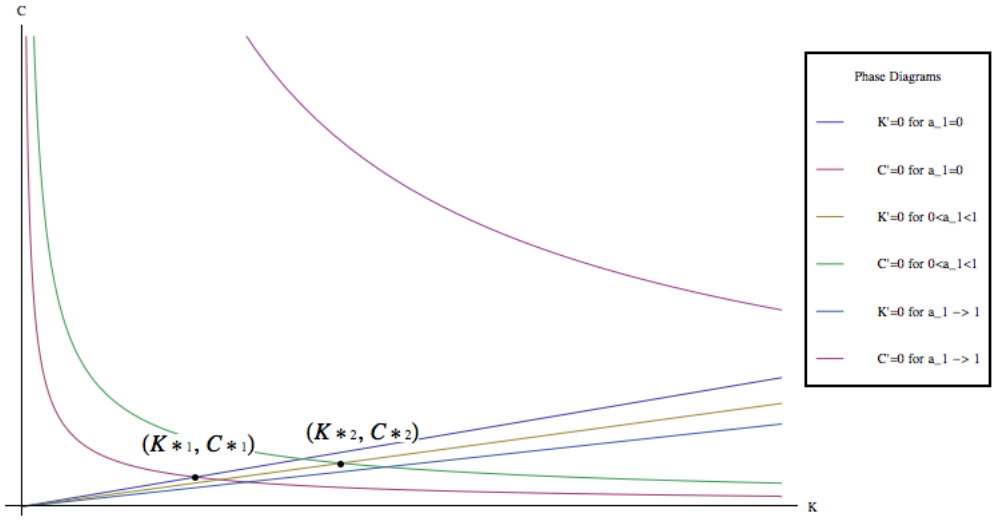

Figure 4.12: Different phase diagrams for different values of a_1

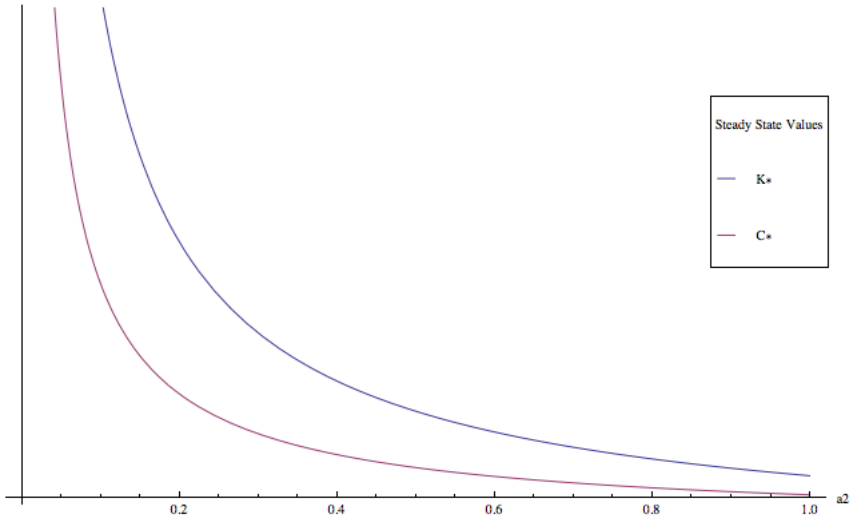

Figure 4.13: Variation of K^* and C^* with respect to $a2$

Considering the extensive analysis made for the simple energy profiles, the intuition behind this model is very straightforward, as it presents itself like a combination of those. The results correspond to the results of each model (with and without pollution) pondered by the fraction of stock devoted to each technology. Given the similarity of interpretation and the additional difficulty in calculations an application for Cobb-Douglas production function is not presented. In the next section models with the fraction of investment in each technology will be presented.

4.4 Mixed Energy Profile: Variable Stocks of Non-Polluting Capital

In chapter 3, section 3.4, the model with two stocks of non-polluting capital using consumption and distribution of investment as control variables was analyzed. The optimization problem turned-out to be linear in the second control variable α resulting in either a bang-bang solution, where α takes extreme values, or a singular solution, for which the Pontryagin's maximum principle doesn't specify the optimal distribution of investment.

As for the bang-bang control, it means that the optimal control is either $\alpha = 1$ or $\alpha = 0$ indicating that investment is made in only one of the two stocks of capital. The analysis of the dynamical system for each case revealed that the comparison between the two situations resumed to the relation between the net marginal productivities of each capital. This conclusion arose from the fact that, given that there's no pollution, utility is only derived from consumption. Equations (3.53) and (3.54), from section 3.4, describe the dynamics of consumption for each case, and are rewritten here:

$$\text{For } \alpha = 1: \qquad \dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{4.70}$$

$$\text{For } \alpha = 0: \qquad \dot{C} = -\frac{U_C}{U_{CC}}(f_2' - \delta_2 - \rho) \tag{4.71}$$

From the comparison of these equations one concludes that for the bang-bang control, the optimal solution will be to invest in the most productive technology.

Also in chapter 3, section 3.4, it was determined that the singularity condition corresponded to equal shadow prices and from the Euler equations it could be rewritten as the equality of net marginal productivities:

$$f_1' - \delta_1 = f_2' - \delta_2 \tag{4.72}$$

If this condition is only valid instantaneously it corresponds to the switching instant, from one bang-bang control to the other. If, on the other hand, it remains valid for a finite time interval, the optimal control is singular, which means that one must use the singularity condition to determine the relation between each stock of capital and solve the dynamical system. This process was described thoroughly in section 3.4 of chapter 3.

In this section, the AK production function will be analyzed.

AK Production Function

The AK production function is very simple to analyze, given the functional form:

$$f_1 = A_1 K_1 \tag{4.73}$$

$$f_2 = A_2 K_2 \tag{4.74}$$

The marginal productivities are constant, thus so are the net marginal productivities ($f_i' - \delta_i$). If these are equal ($A_1 - \delta_1 = A_2 - \delta_2$), the system lies in the singular case, but it is a trivial situation, for which any value of α is optimal. It is indifferent to invest in one or the other stock of capital.

On the other hand, if the marginal productivities are different, the singular condition never occurs meaning that there's only investment in one of the stocks. Rewriting equations (4.70) and (4.71), for this production

function and the specific form of utility, confirms the previous conclusion that the investment will be towards the most productive technology.

$$\dot{C}_1 = \frac{(A_1 - \delta_1 - \rho)}{\sigma} C \tag{4.75}$$

$$\dot{C}_2 = \frac{(A_2 - \delta_2 - \rho)}{\sigma} C \tag{4.76}$$

Since there's no pollution, utility depends solely on the consumption level, from the previous equations, it becomes clear that if the net marginal productivity of 1 is higher, $A_1 - \delta_1 > A_2 - \delta_2$, the associated consumption will be also higher, thus leading to higher utility levels, implying that the optimal choice is to invest only in K_1, and vice-versa

This result is in agreement with economic intuition, according to which it is best to invest in a stock of capital with higher returns, that is, higher net marginal productivity. For the specific case of the AK model, the evolution of the stock of capital doesn't enter the calculations because the marginal productivities of stock are constant.

For a situation where investment affects the net marginal productivity of capital the calculations will be more complicated. In the next section pollution is introduced in a similar model.

4.5 Mixed Energy Profile: Variable Stocks of Polluting and Non-Polluting Capital

The approach for this case is very similar to the one used in the previous section, with the additional complication of pollution. So there will exist two stocks of capital, one clean and the other polluting, and the control variables will be consumption and distribution of investment. In section 3.5, from chapter 3, it was shown that this optimization problem is also linear in the second control variable α. Resulting in either a bang-bang solution, where α takes extreme values, or a singular solution, for which the Pontryagin's maximum principle doesn't specify the optimal distribution of investment, just like in section 4.4.

From section 3.5, recall the dynamical system resulting from investing only in clean technology $\alpha = 1$ (equations (3.77), (3.78) and (3.79)):

$$\dot{K}_1 = f_1 + f_2 - C - \delta_1 K_1 \tag{4.77}$$

$$\dot{K}_2 = -\delta_2 K_2 \tag{4.78}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}(f_1' - \delta_1 - \rho) \tag{4.79}$$

Also from section 3.5, the dynamic system when investment is directed towards polluting technology $\alpha = 0$ (equations (3.80), (3.81) and (3.82)) :

$$\dot{K}_1 = -\delta_1 K_1 \tag{4.80}$$

$$\dot{K}_2 = f_1 + f_2 - C - \delta_2 K_2 \tag{4.81}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}\left[\left(1 + \frac{\gamma_P U_P}{U_C}\right) f_2' - \delta_2 - \rho\right] \tag{4.82}$$

Also in chapter 3, section 3.5, the singularity condition, given by equation (3.85), corresponded to equal shadow prices, corresponding to equal net marginal productivities:

$$f_1' - \delta_1 = \left(1 + \frac{\gamma_P U_P}{U_C}\right) f_2' - \delta_2 \tag{4.83}$$

Just like in the previous section, if this condition is only valid instantaneously it corresponds to the switching instant, from one bang-bang control to the other. If, on the other hand, it remains valid for a finite time interval, the optimal control is singular, which means that one must use the singularity condition to determine the relation between each stock of capital, consumption and pollution to solve the dynamical system.

In this section the model will be applied for AK production function.

AK Production Function

Considering that both technologies are characterized by AK production functions:

$$f_1 = A_1 K_1 \tag{4.84}$$
$$f_2 = A_2 K_2 \tag{4.85}$$

It has been demonstrated in section 4.4 that, without pollution, the optimal solution is to invest in the technology with higher net marginal productivity. It has also been shown, in section 3.2, that pollution decreases the net marginal productivity of the polluting capital (see expression (3.18)).

So, if *a priori* the net marginal productivity of the polluting technology is lower ($A_1 - \delta_1 > A_2 - \delta_2$), the negative effects of pollution will make it even lower ($A_2 - \delta_2 > (1 + \frac{\gamma_P U_P}{U_C})A_2 - \delta_2$). So, it is always optimal to just invest in clean technology.

This result is in agreement with the empirical knowledge that one uses polluting technologies because these are more productive. Otherwise, one would choose clean technologies without hesitation.

From this point on the analysis will focus on the case where polluting technology is more productive ($A_2 > A_1$).

The starting point of this economy is characterized by a small amount of polluting capital and no clean capital:

$$K_1(0) = 0 \tag{4.86}$$
$$K_2(0) = K_0 \tag{4.87}$$

Like it was described in the beginning of this chapter, the utility function is concave with respect to consumption ($U_{CC} < 0$). This means that the marginal utility derived from consumption (U_C) decreases with increasing consumption. So for an economy with a small initial stock of capital it can be said that the level of consumption is also low thus the marginal utility derived from consumption is high. This is equivalent to saying that when the level of development of an economy is low, a small increase in consumption corresponds to a significant increase in the overall welfare. On the other hand, if this initial stock of capital is small then the associated polluting emissions are also insignificant. Utility is also concave with respect to pollution ($U_{PP} < 0$), but the utility associated with pollution is negative ($U_P < 0$), so for little emissions, there exists a higher marginal utility, that is, less negative. This can also be interpreted as follows, when emissions are low people give less importance to pollution and are willing to accept increasing pollution in order to consume more. Following the previous reasoning, if the initial stock of capital is small enough the following approximation holds:

$$\text{For } K_2 \text{ small:} \qquad \frac{\gamma_P U_P}{U_C} \simeq 0 \tag{4.88}$$

For this approximation, equation (4.82) for consumption when investment is directed towards the polluting technology, reduces to the equation (4.71) corresponding to a clean technology, studied in section 4.4 for an AK production function. It was shown that, for this case, the choice that maximizes utility is to invest in the technology with higher productivity, which here is the polluting one, K_2. This reasoning allows the determination of $\alpha = 0$ for the initial instant, that is, one begins by investing only in the polluting technology.

Since the initial stock of capital was zero, it will remain non-existing. The dynamical system is governed by the following equations:

$$\dot{K}_2 = (A_2 - \delta)K_2 - C \tag{4.89}$$

$$\dot{C} = -\frac{U_C}{U_{CC}}((1 + \frac{\gamma_P U_P}{U_C})A_2 - \delta - \rho) \tag{4.90}$$

Assuming that both production functions are AK, indicates that the net marginal productivity of the clean technology is constant $(A_1 - \delta)$. However, as the stock of polluting capital increases, the level of consumption and emissions also rise indicating that its net marginal productivity will be decreasing, indicating that, at some *switching point*, the singularity condition (4.91) will be fulfilled:

$$A_1 - \delta_1 = (1 + \frac{\gamma_P U_P}{U_C})A_2 - \delta_2 \tag{4.91}$$

After the switching point investment will be also directed to clean technology, meaning that α is no longer zero. If condition (4.91) is met only for an instant, then investment shifts completely to clean capital ($\alpha = 1$) and the optimal control after the switch is bang-bang. If, on the other hand, it is satisfied for a time interval investment is divided between both stocks ($0 < \alpha < 1$) and the optimal control is singular.

To determine the conditions for each case it is necessary to further analyze the singularity condition. Particularly for AK production function, for the utility function described by equation (4.3) and assuming equal depreciation rates ($\delta_1 = \delta_2 := \delta$) it represents a relation between the stock of polluting capital and consumption:

$$K_2 = \left\{ \left(\frac{1}{\kappa \gamma_P^{\phi} A_2^{\phi-1}} \right) \left(1 - \frac{A_1}{A_2} \right) C^{-\sigma} \right\}^{\frac{1}{\phi-1}} \tag{4.92}$$

Given that consumption presents a positive growth rate, expression (4.92) corresponds to a decreasing polluting capital stock decreasing exponentially with rate $\frac{(A_1-\rho-\delta)}{\phi-1}$.

On the other hand, the bang-bang control also corresponds to a polluting capital stock decreasing exponentially, but with the depreciation rate δ (equation (3.78)). From these observations it becomes clear that the optimal control is singular or bang-bang depending on the parameter values:

$$\begin{cases} \text{If } \frac{(A_1-\delta-\rho)}{\phi-1} < \delta & \Rightarrow \text{Singular control} \\ \text{If } \frac{(A_1-\delta-\rho)}{\phi-1} \geq \delta & \Rightarrow \text{Bang-bang control} \end{cases}$$

After the switching instant the optimal trajectory is described by the following dynamical system:

$$\dot{K}_1 = (A_1 - \delta)K_1 + (A_2 - \delta + \Delta)K_2 - C \tag{4.93}$$

$$\dot{K}_2 = -\Delta K_2 \tag{4.94}$$

$$\dot{C} = \frac{(A_1 - \delta - \rho)}{\sigma} C \tag{4.95}$$

The fraction of investment in clean technology is given by:

$$\alpha(t) = \frac{A_1 K_1 + (A_2 - \delta + \Delta)K_2 - C}{A_1 K_1 + A_2 K_2 - C} \tag{4.96}$$

where

$$\Delta := \begin{cases} \frac{(A_1-\delta-\rho)}{\phi-1} & \text{for singular control} \\ \delta & \text{for bang-bang control} \end{cases}$$

At the switching instant (t_T), clean capital is still zero, there's a given amount of polluting capital K_T and a certain level of consumption C_T.

Equation (4.94) corresponds to a stock of polluting capital decreasing exponentially in time, from K_T, and equation (4.95) corresponds to a level of consumption growing exponentially from C_T. These two differential equations are straightforward to solve, using K_T and C_T as the integration constants. Substituting these solutions into equation (4.93) allows for its analytical solution as well:

$$K_1(t) = \frac{(A_2 - \delta + \Delta)K_T}{A_1}\left(e^{(A_1-\delta)(t-t_T)} - e^{-\Delta(t-t_T)}\right) \tag{4.97}$$
$$+ \frac{C_T}{\frac{(A_1-\delta-\rho)}{\sigma} - (A_1 - \delta)}\left(e^{(A_1-\delta)(t-t_T)} - e^{\frac{(A_1-\delta-\rho)}{\sigma}(t-t_T)}\right)$$
$$K_2(t) = K_T e^{-\Delta(t-t_T)} \tag{4.98}$$
$$C(t) = C_T e^{\frac{(A_1-\delta-\rho)}{\sigma}(t-t_T)} \tag{4.99}$$

From the Euler equations (3.75) and (3.76), and the condition for a maximum with respect to consumption (3.41) it is possible to determine the shadow price of capital:

$$\mu_1(t) = \epsilon^{1-\sigma} C^{-\sigma} \tag{4.100}$$

This expression and equation (4.97) are used to evaluate the transversality condition $\lim_{t\to+\infty}[K_1(t)\mu_1(t)e^{-\rho t}] = 0$, from which it is possible to obtain a relation between K_T and C_T:

$$C_T = \frac{A_2 - \delta + \Delta}{A_1} K_T \left(A_1 - \delta - \frac{A_1 - \delta - \rho}{\sigma}\right) \tag{4.101}$$

As shown in section 3.4, consumption is continuous, therefore it is possible to state that the trajectory from the initial period will lead to values of consumption and capital that satisfy the previous relation.

Also, from the singularity condition, it is possible to determine another condition for the switching point:

$$C_T = \left\{\left(\frac{1}{\kappa\gamma_P^{\phi} A_2^{\phi-1}}\right)\left(1 - \frac{A_1}{A_2}\right) K_T^{1-\phi}\right\}^{\frac{1}{\sigma}} \tag{4.102}$$

The combination of the previous conditions allows for the determination of the values for consumption and capital at the switching instant. This point is used as the "initial" value for the numerical calculation of the policy function described by the differential equation that results from the division of equation (4.90) by (4.89). Once the policy function is known, from the initial value of capital (K_0) the initial value of consumption (C_0) is also determined, allowing for the numerical resolution of equations (4.89) and (4.90), that yield the time-paths of capital and consumption for the initial period.

The switching moment corresponds to the intersecting of these time-paths with the transition values of capital and consumption (K_T and C_T). (see figures 4.14 and 4.22)

The procedure is the same whether the control is singular or bang-bang. Simulations for each case are presented next.

Summarizing, for an economy in an early stage of development with a small amount of polluting capital and no clean capital, initially, it is optimal to invest only in the most productive polluting technology. But as one invests in this stock, the value of pollution rises, decreasing its net marginal productivity. At some point (t_T), for a stock of capital (K_T) and a level of consumption (C_T) the net marginal productivities of both stocks are equal. There's a singular condition that governs the evolution of the capital stocks for which this equality is fulfilled. Depending on the choice of parameters, this condition is valid only for an instant and investment is totally directed to clean technology (bang-bang control) or it is fulfilled indefinitely and investment is divided between the two technologies (singular control). Both situations correspond to a diminishing stock of polluting capital.

Simulations

The model described above will be applied for specific parameters reasonable for a real economy. The depreciation rate will be set at 5% per year, corresponding to a stock of physical capital with a mean lifetime of 20 years, and the rate of time of preference at 2% per year, typical values for macroeconomic models[9]. The pollution parameter γ_P represents the intensity of this economy, measuring how much pollution is emitted for each unit of physical capital producing for a year. The value of $\gamma_P = 0.1$kg/100€ it's in the order of magnitude for flow pollutants[2].

Two cases were simulated, one corresponding to the bang-bang control where the marginal productivities of capital were fixed at $A_1 = 1.5$ y^{-1}, $A_2 = 2$ y^{-1}, the other to the singular control for which $A_1 = 0.1$ y^{-1}, $A_2 = 0.3$ y^{-1}. The computation of the equivalent annual rates of return r for each value reveals that the first set of parameters is highly unrealistic for a real economy, hinting that the singular solution is more plausible. Table 4.1 presents a summary of the parameters used for both simulations.

Parameter	**Bang-Bang**	**Singular**
A_1	1.5 y^{-1}	0.1 y^{-1}
A_2	2 y^{-1}	0.3 y^{-1}
r_1	326 %	16 %
r_2	603 %	28 %
δ	0.05 y^{-1}	
ρ	0.02 y^{-1}	
γ_P	0.1 kg/100€	
κ	1 $\frac{(€/\text{y})^{1-\sigma}}{(\text{kg/y})^{\phi}}$	
σ	1.4	
ϕ	2	

Table 4.1: Parameter Values Used for Simulations

From a qualitative perspective both cases are very similar, the main difference lying in the fact that for the singular control investment in polluting capital doesn't drop to zero after the switching point, as can be seen in figures 4.19 and 4.27.

For both cases, the clean capital will increase exponentially without bound since there aren't any constraints to its growth, and so will consumption, as shown in figures 4.15, 4.16 , 4.23 and 4.24. And polluting capital will tend exponentially to zero, at rate Δ, fulfilling the transversality condition (3.39), and with it so will polluting emissions, as shown in figures 4.18 and 4.26. However if that rate is too small, it could take a long time to bring down pollution to lower levels.

However, quantitatively the singular case presents more satisfactory results. Comparing the evolution of consumption over time, represented in figures 4.15 and 4.23, reveals that for the singular case the turning point occurs much later in time, after about 25 years. This result is much more realistic than the previous one that placed the shift after just 1 year of economic development. The marginal productivities considered initially ($A_1 = 1.5y^{-1}$, $A_2 = 2y^{-1}$) were unreasonable, corresponding to annual rates of return of 326% and 603%, respectively. Whereas for the second case, these rates are 16% and 28%, on the upper side but still attainable. These observations are supported by the analysis of the evolution of polluting capital during the initial period. For the first case, represented in figure 4.14, the studied economy growths with an average growth rate of 134%, which is, as expected, implausible. On the other hand, for the second case, represented in figure 4.22, the average growth rate was estimated in 15%, a typical value for an economy in an initial expansion state, e.g. China. These results hint that the singular control is more likely for real data, which is in agreement with the economic intuition that it is not viable, from one moment to the other, to switch all investment from one technology to other.

Plotting pollution with respect to output gives rise to a curve that rises sharply during the initial period and after achieving a maximum value (for the bang-bang case $P_{max} = 0.48$, for singular case $P_{max} = 1.45$), decreases slowly throughout time, this trend is shown in figures 4.21 and 4.29. The shape of it could be

[2]For Portugal, annual emissions of SO_2 are in the order of 10^8kg, dividing this value by the GDP provides an order of magnitude for the economy's intensity.

interpreted as an Environmental Kuznets Curve since first emissions increase and after a turning point the tendency is reversed. The interest of this observation lies in the simplicity of the model that gives rise to the curve. It is interesting to note that this curve is not smooth nor symmetric, pollution attains its maximum very rapidly and decreases with a much lower rate, and instead of being an inverted-U, it is an inverted-V. Also, its decreasing side is convex, which is a direct consequence of considering that the polluting stock decays exponentially.

Bang-Bang Control

$$
\begin{aligned}
K_0 &= 1.000 \times 100€ \\
C_0 &= 0.6151 \times 100€/y \\
K_T &= 3.9703 \times 100€ \\
C_T &= 2.2687 \times 100€/y \\
t_T &= 1.028\ y \\
\left|\frac{U_P}{U_C}\right|(t_T) &= 2.63 \times 100€/kg \text{ (value of pollution)}
\end{aligned}
$$

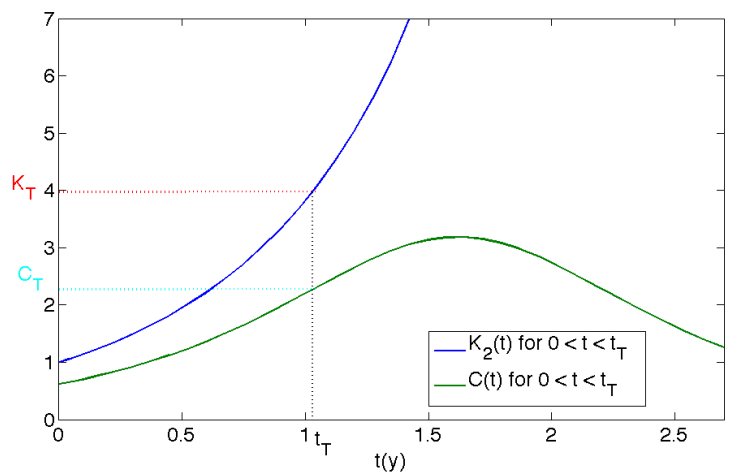

Figure 4.14: Determination of transition time (t_T)

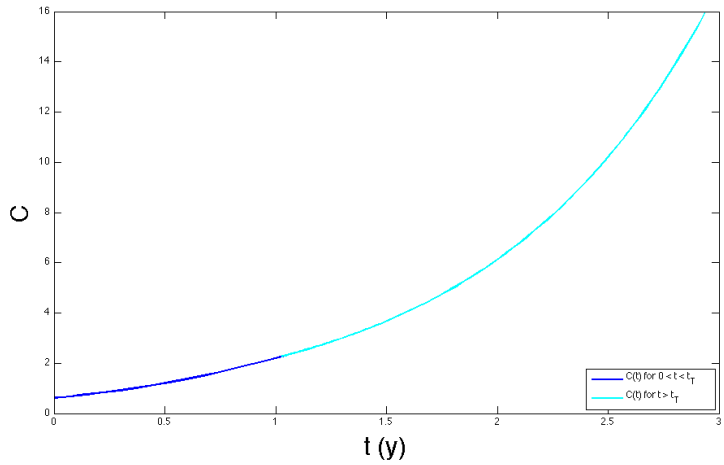

Figure 4.15: Consumption over time

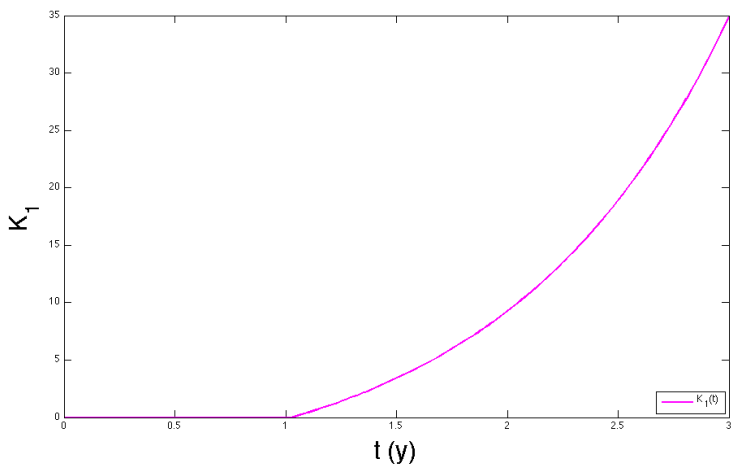

Figure 4.16: Clean capital over time

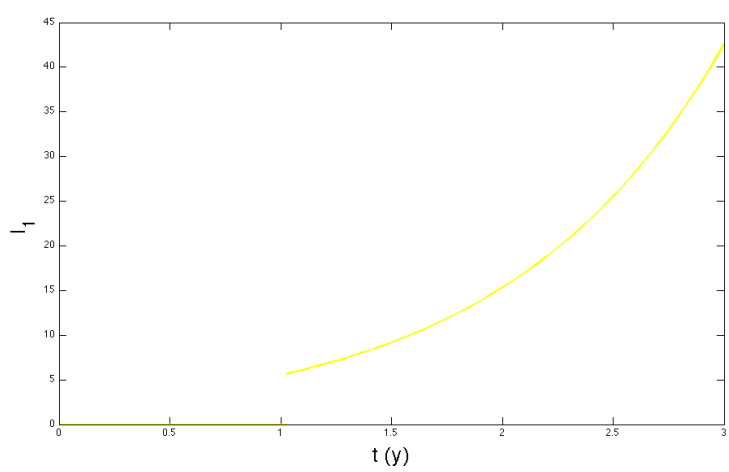

Figure 4.17: Investment in clean capital

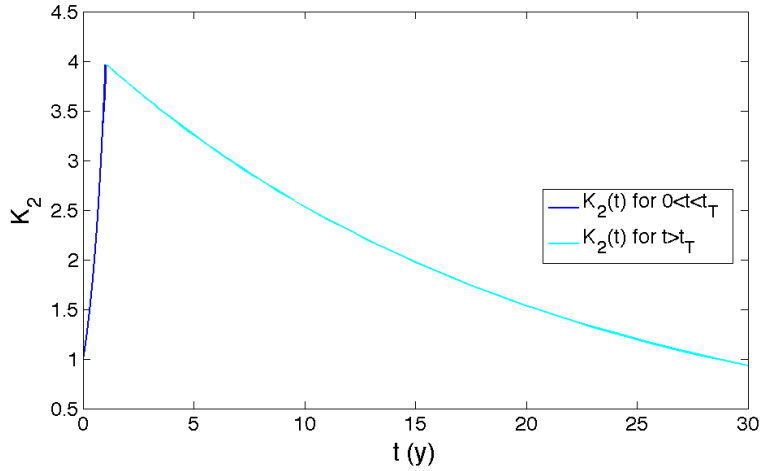

Figure 4.18: Polluting capital over time

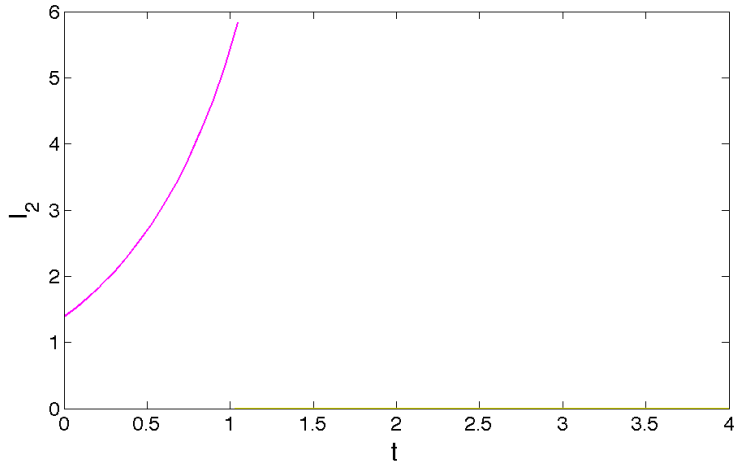

Figure 4.19: Investment in polluting capital

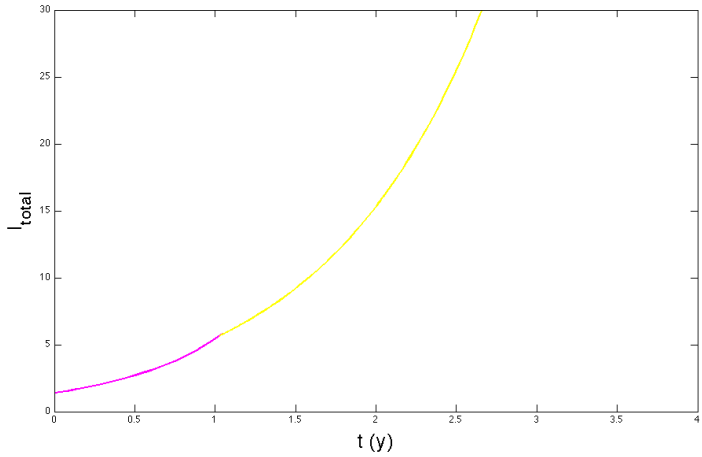

Figure 4.20: Total investment

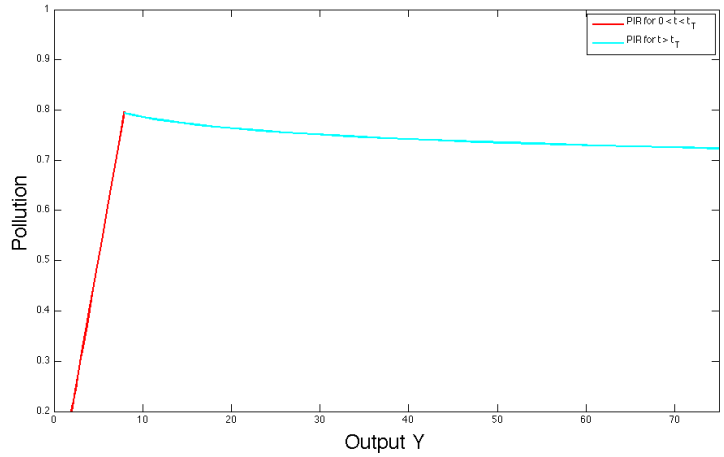

Figure 4.21: EKC

Singular Control

$$
\begin{aligned}
K_0 &= 1.0000 \times 100€ \\
C_0 &= 0.0893 \times 100€/y \\
K_T &= 47.7296 \times 100€ \\
C_T &= 3.0001 \times 100€/y \\
t_T &= 24.45\ y \\
\left|\frac{U_P}{U_C}\right|(t_T) &= 7.30 \times 100€/kg \text{ (value of pollution)}
\end{aligned}
$$

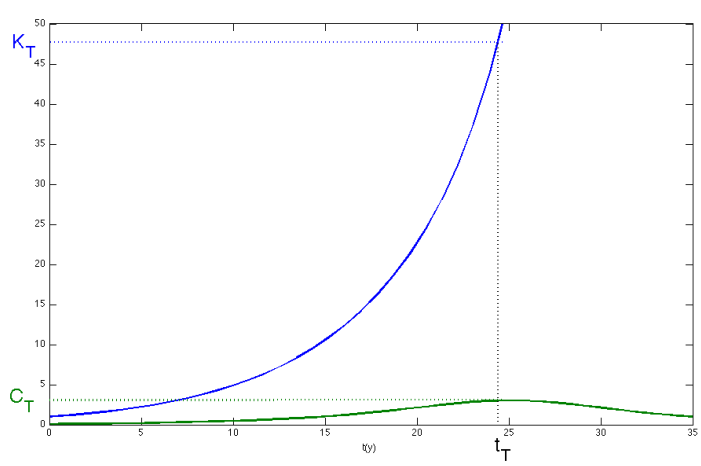

Figure 4.22: Determination of transition time (t_T)

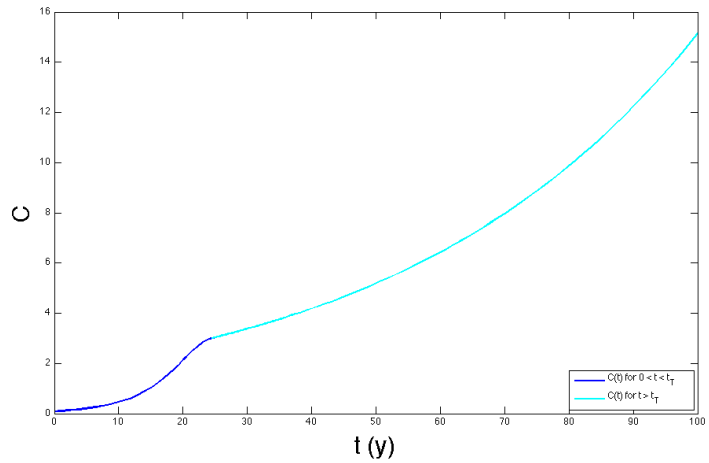

Figure 4.23: Consumption over time

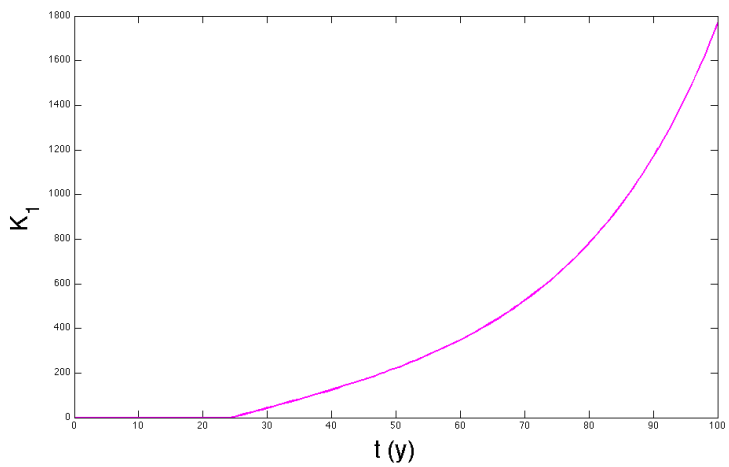

Figure 4.24: Clean capital over time

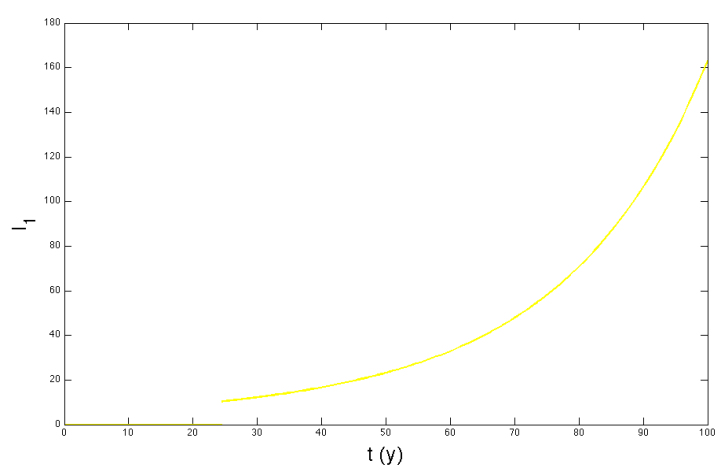

Figure 4.25: Investment in clean capital

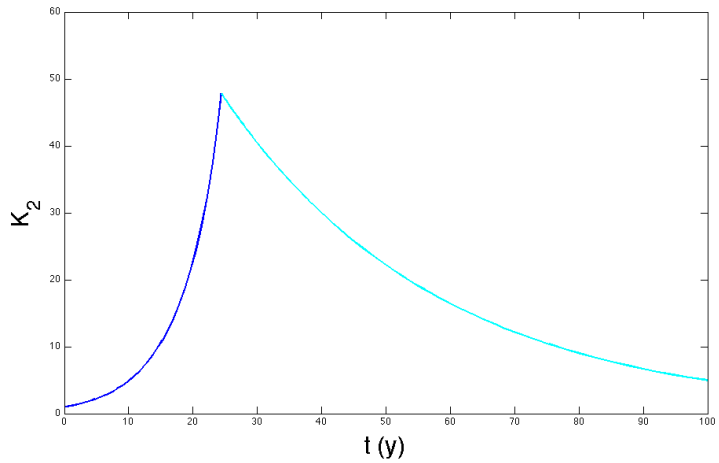

Figure 4.26: Polluting capital over time

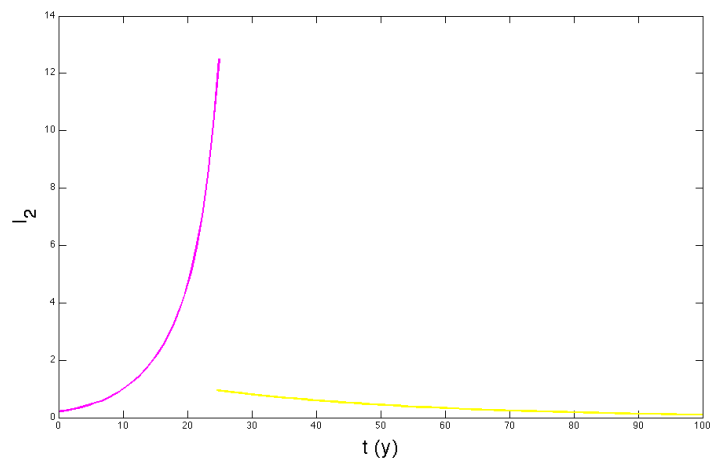

Figure 4.27: Investment in polluting capital

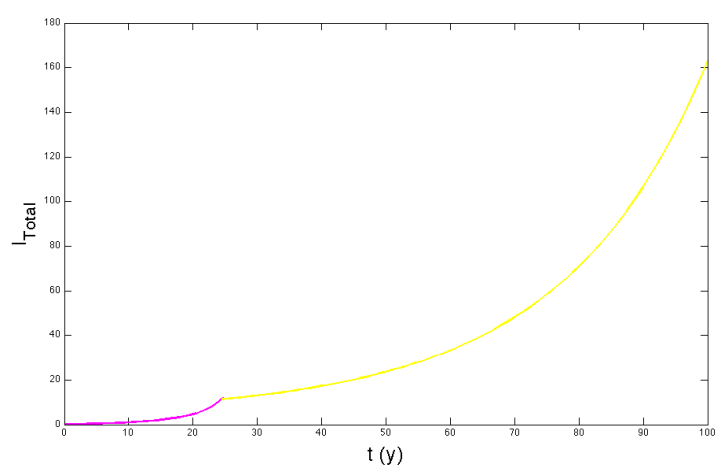

Figure 4.28: Total investment

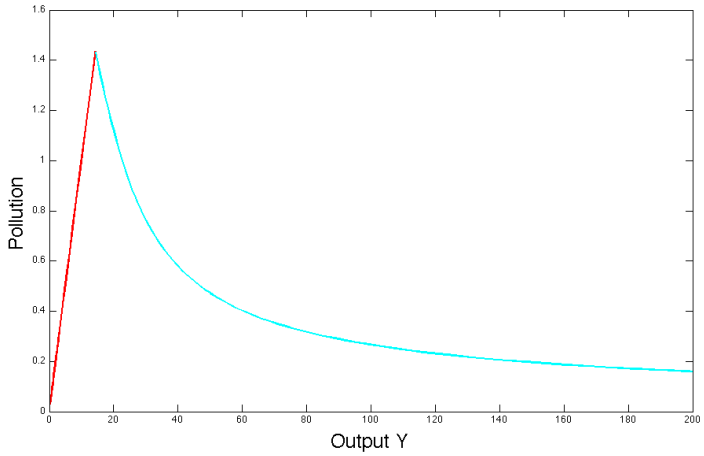

Figure 4.29: EKC

4.6 Overview

This chapter presented the applications of the models presented in chapter 3 to specific function forms for utility and production. The beginning of the chapter presents an analysis of the properties of the chosen utility function and the interpretation of the parameters involved.

In section 4.1, the model for one stock of clean capital was studied for AK and Cobb-Douglas production functions. For the first one, that presents constant returns to scale, unbounded exponential growth is attainable. However, for the other production function, with decreasing returns to scale, even without pollution, consumption and capital stock attain a constant steady-state.

Section 4.2 includes the same production functions as before but for one stock of polluting capital. The AK production function no longer presents unbounded growth, the analysis of the phase diagram of the system revealed that consumption and capital converged to a steady-state. The influence of the different parameters in the steady-state values was evaluated and interpreted. A similar result was obtained for Cobb-Douglas production function, however the explicit determination of the steady state wasn't possible. The comparison between the results of the previous models showed that pollution has a much stronger effect on the optimal trajectory than the decreasing returns to capital, for that reason the applications that followed were restricted to the AK production function.

In section 4.3 the mixed energy profile, with predetermined fractions of clean and polluting technology, was studied. As noticed before, this model corresponds to an extension of the two previous ones, therefore the results are very similar. Due to pollution, consumption and capital tend to a steady-state, but as a result of the use of clean capital these values are higher than the ones for the case with just polluting technology.

The mixed energy profile with two clean technologies with different productivities and variable fractions of investment was studied in section 4.4. For the AK model the results are very straightforward since one will invest only in the most productive technology.

The introduction of pollution in section 4.5 makes the study more interesting. The assumption that the economy's initial state of development is low allowed for the approximation that, at first, pollution plays a negligible role and therefore initial investment is directed towards polluting technology. The condition for the switch, when the net marginal productivities are equal, was analyzed for two sets of parameters. One case corresponded to the bang-bang control, for which investment switches instantaneously from polluting to clean technology. The other, to the singular control, for which investment is divided between both technologies. Considering the interpretation of the marginal productivity of capital it seems more likely that a real economy would be described by the second case. The pollution to income relation was obtained and can be identified with an EKC.

Chapter 5

Conclusions

The present era has been marked by an increasing concern with the environment, namely CO_2 emissions and pollution associated with industrial processes. During the 90's, many economists and scientists argued that beyond a certain point of economic development the negative impacts of production would decrease, stating that development and economic growth itself were the solutions to environmental degradation. This hypothesis is known as the Environmental Kuznets Curve, according to which the pollution to income relation (PIR) presents an inverted-U shape. Even though some empirical studies revealed this correlation, the increasing literature on the subject shows the lack of consensus. Also the theoretical reasoning behind the EKC seems feeble, since most models only obtain the EKC for very particular situations and assumptions. See chapter 2, section 2.2.

Nevertheless the study of this relations is of the essence, particularly when one's interest is mainly the energetic sector, which is an important driver of modern economies and it is strongly connected with atmospheric pollution.

In this thesis, simple aggregated models, for which the production function represented useful energy generation and transformation, were presented. The study considered two general technologies: one clean but less productive that can be identified, for instance, with photovoltaic systems, wind power or any other renewable technology; the other more productive but polluting, corresponding to fossil fueled technology (power plants, transportation, etc).

Several different panoramas were analyzed: the case where only one of the technologies is available; the case where both are used but with fixed fractions and finally a case where investment in each is variable. For each situation, the method consisted of considering an optimizing agent that seeks for the path of consumption that maximizes overall welfare.

Table 5.1 presents a summary of the results obtained for simple and mixed energy profiles, with different production functions: whether the system attained a steady-state, how pollution evolved over time and the pollution to income relation.

The utility function is the tool used to measure well-being and it is positively affected by consumption and negatively by pollution. Techniques from optimal control theory were used to perform the calculations. As it is expected for this kind of aggregated models, results turned out to be straightforward and easily interpreted and contextualized. A summary of these results is presented next, and some conclusions are drawn.

When only clean technology is used, provided that the production function presents constant returns to scale (AK production function), there's no limit to growth, so the optimal path will be an exponential increase of consumption and capital stock (first line on table 5.1). However, for Cobb-Douglas production function, with decreasing returns to scale, even without pollution, consumption and capital stock attain a constant steady-state (second line on table 5.1).

On the other hand, when only polluting technology is available, consumption and capital stock reach a steady-state, for AK production function (third line on table 5.1) and for Cobb-Douglas production function (fourth line on table 5.1). This is the point for which the optimal trade-off between consumption and pollution is achieved. At this point, in order to obtain even a slight increase in consumption it is necessary to increase the capital stock, which in turn produces more pollution, resulting in lower overall utility. If, on the other

Energy Profile	Stocks	Prod. Function	Steady State	P(t)	PIR
Simple	NP	AK	$(K, C) \to \infty$ (unbounded growth)	-	-
	NP	CD	$(K^*, C^*) = \left\{ \left(\frac{\delta+\rho}{\upsilon A}\right)^{\frac{1}{\upsilon-1}}, A\left(\frac{\delta+\rho}{\upsilon A}\right)^{\upsilon} - \delta\left(\frac{\delta+\rho}{\upsilon A}\right) \right\}$	-	-
	P	AK	$(K^*, C^*) = \left\{ \left(\frac{A-\delta-\rho}{\kappa(A\gamma_P)^{\phi}(\epsilon(A-\delta))^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}}, (A-\delta)\left(\frac{A-\delta-\rho}{\kappa(A\gamma_P)^{\phi}(\epsilon(A-\delta))^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}} \right\}$	$\nearrow\!\!\longrightarrow$	$\nearrow^{\bullet}$
	P	CD	$K^* : (AK^{\upsilon-1} - \delta)K = \left(\frac{A\upsilon K^{\upsilon-1}-\delta-\rho}{\kappa(A\gamma_P)^{\phi}\upsilon\epsilon^{\sigma}}\right)^{1/\sigma}(K)^{\frac{1-\upsilon\phi}{\sigma}}, C^* = (AK^{*\upsilon-1} - \delta)K^*$	$\nearrow\!\!\longrightarrow$	$\nearrow^{\bullet}$
Mixed	Fixed, NP and P	AK	$(K^*, C^*) = \left(\frac{M-\rho}{\kappa(a_2A_2\gamma_P)^{\phi}(\epsilon M)^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}}, M\left(\frac{M-\rho}{\kappa(a_2A_2\gamma_P)^{\phi}(\epsilon M)^{\sigma}}\right)^{\frac{1}{\sigma+\phi-1}}, M = (A_1a_1 + A_2a_2) - (\delta_1a_1 + \delta_2a_2)$	$\nearrow\!\!\longrightarrow$	$\nearrow^{\bullet}$
	Variable, NP	AK	$(K, C) \to \infty$ (unbounded growth)	-	-
	Variable, NP and P	AK	$(K, C) \to \infty$ (unbounded growth)	$\nearrow\!\searrow$	$\nearrow\!\searrow$

Table 5.1: Summary of the main results obtained; NP - Non-polluting capital; P - Polluting; AK - AK production function; CD - Cobb-Douglas production function; $\nearrow\!\!\longrightarrow$ Pollution increases over time until attaining a constant level; $\nearrow\!\searrow$ Pollution increases for a given period of time, decreasing after a turning point; $\nearrow^{\bullet}$ PIR is increasing towards a steady-state; $\nearrow\!\searrow$ PIR increases at first, then decreases;

hand, one decided to diminish pollution, as a consequence consumption is lower and utility is not maximized. This point is saddle path stable, which indicates that the system will always tend to it. It is important to note that even though economic growth comes to a halt, the PIR doesn't present an EKC, pollution will grow until a given level, where it stabilizes. The comparison between the results of the previous models showed that pollution has a much stronger effect on the optimal trajectory than the decreasing returns to capital, for that reason the applications that followed only used AK production functions.

Afterwards, the combination of the two technologies was studied. At first, from a statical perspective, which means that the economy is endowed with a given fraction of each stock, and it evolves in a way that these fractions remain unchanged (fifth line on table 5.1). Calculations demonstrate that this case is mathematically identical to the previous one, given that it also leads the system towards a steady-state. However the introduction of clean technology allows for higher steady-state values of consumption, for the same level of pollution. Once again, EKC is not verified.

Since empirically it is known that an energy profile is not static and investment in different technologies is a factor controlled by decision makers, an extra control variable is introduced: investment in each capital stock. Suggesting that the social planner can choose how each stock of capital evolves.

For a first assessment, pollution was withdrawn, so that the optimal control problem resumed to the first case but with the choice of investment between two different clean technologies (sixth line on table 5.1). It turned out that the solution for the optimal investment was a mix between *bang-bang* and *singular* controls, as explained in section 3.4.

The *bang-bang* solution corresponds to investing only in one technology at a time. One would invest in the technology with higher net marginal productivity. If it presents diminishing returns to capital, net marginal productivity will decrease with the growth of the stock of capital. So, at some point the net marginal productivities would be equal. At this moment one arrives to the *singular* control condition, that determines a condition for which the capital stocks evolve in a way that their net marginal productivities remain equal. If for given production functions the singularity condition just corresponds to an instant, this moment is denominated the *switching point*, where investment switches from one technology to the other.

Without pollution this problem just resumes to the choice of the better, most profitable technology, since there are no disadvantages, whereas when pollution is introduced the study gains more interest (seventh line on table 5.1). It was considered that the assessment begins for an economy in an early stage of development. This corresponds to an initial small amount of polluting capital and no clean capital. It was demonstrated that if the polluting technology is more productive then it is optimal to start by just investing in it. However, it was also determined that the ratio between marginal desutility from pollution and marginal utility from consumption decreases the net marginal productivity of the polluting technology, as shown by equation (3.18) in section 3.2. So, as one invests in it, pollution will increase, and its net marginal productivity will in turn, diminish.

This phenomenon occurs even if the polluting technology presents constant returns to capital, as it is the case of AK production function. Therefore, at some point both marginal productivities will be equal.

Once again, there will exist a singular condition that governs the evolution of the capital stocks for which the net marginal productivities are equal. Depending on the choice of parameters, this condition is valid only for an instant and investment totally for clean technology (bang-bang control) or it is fulfilled indefinitely and investment is divided between the two technologies (singular control). Both situations correspond to a diminishing stock of polluting capital and the obtained pollution to income relation (PIR) may be identified with an EKC, that at first is increasing, reaches a turning point, from which it decreases. This curve is an inverted-V shape, rather than a smooth inverted-U shape.

Finally, the results showed that, if there are no drawbacks associated with the use of clean technology, long-term, unbounded growth is possible.

This thesis can be considered a starting point for more elaborate models that study the intricacies of the relations between economic growth, energy production and pollution. The proposed approach, inspired by the computer game *CityOn*, can be further developed and enhanced.

From the technological perspective, one could use production functions that describe more accurately the production and transformation of useful energy, or consider technological growth over time. Also, adding stock pollutants to the model could be a way to extend the study's scope. Other possible extensions are the introduction of natural resources as inputs of production and the subsequent analysis of the scarcity issues, or the consideration of investment costs associated with a technological shift.

Nevertheless, one shouldn't underestimate the power of models that, as simplistic as the ones presented here, can provide insight into very complex systems. The fact that most calculations can be made analytically and interpreted graphically allows for a comprehensive and contextualized understanding of the problem. It can be concluded that the purpose of determining and understanding general patterns and trends regarding energy use, economic growth and pollution, using aggregated models and analytical calculations was successfully achieved.

Appendix A

Optimal Control Theory

Physicists are very familiar with the calculus of variations, where a functional is minimized or maximized, e.g., to obtain the equations of motion of a system, one uses the principe of least action, a very well known variational principle. The Lagrangean and Hamiltonian formulations of classical mechanics follow from this principle.

Optimal control theory is an extension of the calculus of variations, developed by L. Pontryagin, in the 1950's. The *maximum principle of optimal control* provides solutions to problems in which the constraints involve the derivatives of some of the state variables.[9]

Pontryagin's maximum principle presents a necessary condition that must hold on an optimal trajectory. Consider a dynamical system defined by a system of ordinary differential equations:[40]

$$\dot{x} = g(x, c, t) \tag{A.1}$$

where $x \in \mathbb{R}^n$ is the state variable of the system and $c \in \mathbb{R}^m$ is a set of controls and g is a vector valued function. Ordinary differential equations (ODEs) like the one in equation (A.1) may describe all sorts of systems, physical (e.g., x as generalized coordinates), biological, economics (e.g., x as stocks of capital), and so on. From the applications point of view it is necessary to act on these systems, given a choice of initial and final conditions in the phase space. However, in most cases, initial and final conditions aren't in the same trajectory, so it is necessary to act on the system using the controls, $c \in \mathbb{R}^m$. This operation is described by the introduction of a temporal dependency: $c(t) \in \mathbb{R}^m$ for $t \geq 0$. When the variation of the controls with time allows the system to attain the final conditions, then one states that these conditions are accessible and the control problem has a solution. Optimal control theory provides maximum conditions for the determination of the time evolution of the controls.

The typical problem consists in finding the control function $c(t)$ such that the dynamical system (A.1), with initial condition $x(0)$, arrives at $x(T)$, minimizing or maximizing some functional:

$$\max_{c(t)} J = \int_0^T g_0(x(t), c(t), t)dt \tag{A.2}$$

Introducing a new scalar variable $x_0 = J$, the functional can be rewritten in a way similar to equation (A.1):

$$\dot{J} = \dot{x}_0 = g_0(x, c, t) \tag{A.3}$$

Using this form, and redefining the state variable $x = (x_1, ..., x_n)$ as $x = (x_0, x_1, ..., x_n)$, the phase space becomes $(n + 1)$ dimensional, and the system can be written as:

$$\begin{cases} \dot{x}_0 = g_0(x, c, t) \\ \dot{x}_1 = g_1(x, c, t) \\ \vdots \\ \dot{x}_n = g_n(x, c, t) \end{cases} \tag{A.4}$$

To solve the dynamic optimization problem it is necessary to introduce co-state variables λ, in physics known as conjugate momenta and in economics as shadow prices. Each co-state variable is associated with a constraint of the dynamic problem, i.e., the differential equations (A.4) that govern the system's dynamics. The Hamiltonian is defined as:

$$H(\lambda, x, c, t) = \sum_{j=0}^{n} \lambda_j g_j(x, c, t) \tag{A.5}$$

Pontryagin's theorem states that if $c(t)$ and $x(t)$ represent the optimal control and state trajectory, then there exists a co-state variable $\lambda(t)$ that satisfies[41][42]:

$$\dot{x}_i = H_{\lambda_i} \qquad i = 0, ..n \tag{A.6}$$

$$\dot{\lambda}_i = -H_{x_i} \qquad i = 0, ..n \tag{A.7}$$

and for all $0 \leq t \leq T$ and feasible controls $\nu(t)$:

$$H(x(t), \nu(t), \lambda(t)) \leq H(x(t), c(t), \lambda(t)), \tag{A.8}$$

i.e. the optimal control $c(t)$ is the value of $\nu(t)$ that maximizes $H(x(t), \nu(t), \lambda(t))$. Equations (A.7) are known as Euler equations and are formally identical to Hamilton's equations from classical mechanics. Furthermore, $\lambda_0 = 0$ or $\lambda_0 = 1$, and the following transversality conditions are satisfied:

$$\lambda_i(T) \geq 0 \tag{A.9}$$

$$\lambda_i(T) x_i(T) = 0 \qquad i = 1, ...n \tag{A.10}$$

These conditions can be easily transposed to an infinite planning horizon: $T \to \infty$; the main difference lies in the transversality condition that becomes the limit as time tends to infinity:

$$\lim_{t \to \infty} [\lambda_i(t) x_i(t)] = 0 \qquad i = 1, ..., n \tag{A.11}$$

The model presented in the following chapters will consider the economic perspective. Typically, the state variables represent each stock of capital ($K(t) = (K_1(t), ..., K_n(t))$) and the most common control parameter is the level of consumption ($C(t)$). However, for more elaborate models, other controls can be added, like investment in R&D, pollution abatement, amongst others. The functional to be maximized is a discounted utility function that depends on consumption, but may also depend on the capital stocks and time, that measures the welfare of the population affected by the model.

The economic interpretation of the Hamiltonian considers that at each instant, the representative agent consumes $C(t)$ and owns an amount of capital $K(t)$, which affect utility in different ways. Consumption will contribute directly to overall welfare, and if this function also depends explicitly on the capital stock, $K(t)$ will also have a direct impact on utility. On the other hand, consumption influences the change in capital stock through the transition equations (A.1). So, for a given value of the shadow price, λ, the Hamiltonian expresses the total contribution to utility from the choice of $C(t)$.

In this context, the transversality condition may be interpreted as a statement that the value of the capital stock must tend to zero, which means that nothing valuable is left over in the end.

The utility function traditionally considered in neoclassical economics can be written as:

$$g_0(x(t), c(t)) = e^{-\rho t} U[K(t), C(t)] \tag{A.12}$$

where $\rho > 0$ is the constant discount rate. This formulation explicitly states that future utility weights less than present utility.

Writing the discount factor explicitly has the direct consequence that the utility function does not depend explicitly on time. Substituting in the Hamiltonian yields:

$$H = e^{-\rho t} U[K(t), C(t)] + \sum_{j=1}^{n} \lambda_j g_j \tag{A.13}$$

The shadow prices $\lambda(t)$ represent the value of the capital stocks at time t in units of utility from time-zero, so in present value. But, it may be convenient to restructure the problem in terms of current-value prices, which means that the price is measured in utility from time-t. So, the Hamiltonian is rewritten:

$$H^c = He^{\rho t} = U[K(t), C(t)] + \sum_{j=1}^{n} \mu_j g_j \tag{A.14}$$

where H^c is the current-value Hamiltonian and $\mu(t) \equiv \lambda(t)e^{\rho t}$ the current-value shadow prices.

From Pontryagin's maximum principle, one obtains the first-order conditions, i.e., a condition for a maximum with respect to the control and the equations for the shadow prices, as written in equation (A.7). These can be expressed in terms of the current-value Hamiltonian and current-value prices:

$$H^c_C = 0 \tag{A.15}$$

$$H^c_K = \rho\mu - \dot{\mu} \tag{A.16}$$

And the transversality condition becomes:

$$\mu(T)e^{-\rho T}K(T) = 0 \text{ for finite-time horizons} \tag{A.17}$$

$$\lim_{t\to\infty} [\mu(t)e^{-\rho t}K(t)] = 0 \text{ for infinite-time horizons} \tag{A.18}$$

Equation (A.16) has a very intuitive interpretation: μ is the price of capital in terms of current utility, H^c_K is the marginal contribution of capital to utility (like the dividend received by the agent), $\dot{\mu}$ is the change in the price of the asset (like the capital gain) and ρ is the rate of return on consumption. That equation states that, at the optimum, the agent is indifferent between consumption and investment because the overall rate of return to capital $\frac{H^c_K + \dot{\mu}}{\mu}$, equals the return to consumption, ρ[9].

Finally, the procedure to solve the typical optimal control problem, for infinite-time horizons:

$$\max_C W = \int_0^{+\infty} U(C)e^{-\rho t}dt \tag{A.19}$$

subject to

$$\dot{K} = g(K, C) \tag{A.20}$$

$$K(0) = K_0 > 0 \tag{A.21}$$

$$\lim_{t\to+\infty} [\mu(t)e^{-\rho t}K(t)] = 0 \tag{A.22}$$

can be summarized in the following steps:

1) Formulate the current-value Hamiltonian, as defined in equation (A.14);

2) Write down the condition for a maximum with respect to the control variable (this case, consumption) (A.15);

3) Write down the Euler equations (A.16);

4) Evaluate the transversality condition (A.18) (for finite-time horizons, (A.17));

Appendix B

Glossary

Capital Stock K The equipment and structures used to produce goods and services;

Consumption C Amount of total output that is consumed by the representative agent and contributes to increasing utility;

Depreciation Rate δ Rate at which physical capital depreciates; its inverse corresponds to the mean lifetime of capital;

Flow Pollutant Pollutant that only has negative effects while it is emitted; it does not accumulate in the ecosystem;

Intertemporal Elasticity of Substitution IES Parameter that reflects willingness to accept deviations from a uniform pattern of consumption (or pollution) over time;

Investment I Amount of total output that is invested to increase the capital stock K;

Marginal Productivity $f'(K)$ Measure of how production changes with variations of the stock of capital;

Marginal Rate of Substitution MRS Measure of how an increase in consumption may act as compensation for an increase in pollution, and vice-versa;

Persistent Pollutant Pollutant that accumulates in the ecosystem, for which the environment has no absorptive capacity;

Pollution Abatement Amount of output devoted to reduce existing pollution;

Production Function $f(K)$ Function that describes how output is produced from the existing stock of physical capital;

Rate of Time Preference ρ Rate at which future utility is discounted;

Returns to Scale Term to designate how production changes with variations of the stock of capital, depending on the production function there can exist constant, increasing or decreasing returns to scale;

Stock Pollutant Pollutant that accumulates in the ecosystem, for which the environment has some absorptive capacity; its negative impact is felt even after emissions have stopped;

Total Production Y Output produced with the existing capital stock K, through the production function $f(K)$, that can be either consumed or invested;

Utility Function $U(C, P)$ Function that measures the level of satisfaction resulting from the consumption of a certain bundles of goods or services;

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