Metamaterials and Double Negative (DNG) Media: General Properties of Unbounded Media and Guided Wave Propagation

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Abstract—Complex media and metamaterials suggest promising applications in electromagnetics. These media can generate new effects in several propagation canonical problems and has attracted great interest from the electromagnetics research community. In this present work, electromagnetic waves are studied in a media with negative permeability and negative permittivity, called DNG media or DNG metamaterials. This media has some new proprieties like negative refraction and the appearance of backward waves. First an unbounded DPS-DNG interface is studied. Due to the fact that a negative medium cannot have negative energy density, that would appear in a lossless case, dispersion is considered with the Lorentz Dispersion Model. Domains of existence of TE and TM modes in metamaterials are also analyzed for a less considered with the Lorentz Dispersion Model. Domains of existence of super slow modes and mode bifurcation.

Index Terms—Backward Waves, Double Negative Media, Metamaterials, Microwaves, Negative Refraction, Planar Waveguides, Photonics

I. INTRODUCTION

METAMATERIALS brought new problems never before deemed possible in the electromagnetic panorama. The restrict class of DNG media, with its negative permeability and permittivity, caught the attention of the electromagnetic community.

The start of history of metamaterials and complex media, begins with the concept of “artificial” materials in 1898, when Sir Jagadis Chunder Bose developed the first microwave experiment on twisted structures. Currently, these elements immersed in a host medium are denominated by artificial chiral medium [1], [2]. In 1914, Karl Ferdinand Lindman studied wave interaction with compilations of randomly oriented small wire helices, in order to create an artificial chiral media [3]. In 1948, Winston Kock, made lightweight microwave lenses by combination of conducting spheres, strips periodically and disks. These metamaterials, built for lower frequencies, can be designed for higher frequencies by length scaling [4]. Materials, which exhibited reversed physical characteristics were first described theoretically by Victor Veselago in 1967. When he investigated the plane wave propagation in a material which permittivity and permeability were simultaneous negative, he demonstrated that for a monochromatic uniform plane wave in this kind of media, the direction of the Poynting vector is antiparallel to the direction of phase velocity [5]. During the late 1990s, Pendry and his colleagues at Imperial College began to produce structures with these kind of properties. Pendry was interested in developing materials with negative permeability. Also, he created an array of closely spaced, thin, conducting elements, such as metal hoops [7]. In 1999, he described how he adjusted the array’s properties and he developed an array with negative permeability. This structure consisted of periodic array of split-ring resonators (SRRs) that expressed negative effective permeability over a narrow frequency band [6].

This is possible if the magnetic field of incident wave is normal to the plane of the structure. Veselago medium is probably the most famous class of metamaterials in the present wave in complex electromagnetic media. Veselago medium has been known by several names, as negative-index media, negative refraction media, backward wave media (BW media), double-negative media (DNG), media with simultaneously negative permittivity and permeability and even left-handed media (LHM).

Following up on this work, in the year 2000, Smith et al. reported the experimental demonstration of functioning electromagnetic metamaterials, by horizontally stacking, periodically, split-ring resonators and thin wire structures. Later, a method was provided in 2002 to realize negative index metamaterials using artificial lumped-element loaded transmission lines in microstrip technology. At microwave frequencies, the first real invisibility cloak was realized in 2006. However, only a very small object was imperfectly hidden [4], [5], [8], [9], [10]. A core of researchers, composed mainly by Carlos Paiva, António Topa, Sérgio Matos and João Canto, contributed greatly to the study of this topic with several important works and results [11], [12], [13], [14], [15].

II. ELECTROMAGNETICS OF DNG METAMATERIALS

If we classify media by two electromagnetic constitutive parameters, the electrical permittivity \( \varepsilon \) and the magnetic permeability \( \mu \), we have a new kind of media classification. Since complex numbers allow the specification of magnitude and phase, the permittivity and the permeability are often treated as complex functions of the frequency of the applied field, [16],

\[
\varepsilon = \varepsilon' + i\varepsilon'' \quad (1)
\]

with \( \varepsilon', \varepsilon'' \in \mathbb{R} \). And:

\[
\mu = \mu' + i\mu'' \quad (2)
\]

with \( \mu', \mu'' \in \mathbb{R} \).

The following figure shows this media classification in a diagram whose axis are formed by \( \varepsilon' = \Re(\varepsilon) \) and \( \mu' = \Re(\mu) \).

![Material Classification](image)

Figure 1. Material Classification by permittivity and permeability.
Energy propagates through the positive \( z \) condition of passive medium for an electromagnetic wave whose \( \kappa \) needs to be always positive otherwise we wouldn’t have the \( \omega \) frequency, as function of frequency,
\[
n = \sqrt{\mu \varepsilon}
\]
(7)

With,
\[
\eta = \frac{k}{\omega \varepsilon \mu} = \frac{\omega \mu_0 \mu}{k} = \zeta \eta_0
\]
(8)

where
\[
\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}
\]
(9)

Leading us to
\[
\zeta = \frac{n}{\eta_0} = \sqrt{\frac{\mu}{\varepsilon}}
\]
(10)

With \( \eta_0 \) being the vacuum wave impedance. Note that the normalized wave impedance can be re-written to
\[
\zeta = \frac{n}{\varepsilon} = \frac{\mu}{n}
\]
(11)

We have to define a complex refraction index, since the polarization does not respond instantaneously to an applied field,
\[
n = n’ + in”
\]
(12)

with \( n’ \) being the refractive index indicating the phase velocity coefficient and \( n” \) the extinction coefficient. The extinction coefficient shows the amount of absorption loss when the electromagnetic wave propagates through the material. Both \( n’ \) and \( n” \) are dependent of the frequency [16].

Re-writing the complex amplitude equations for both the electric and magnetic field using (12), we get,
\[
E = \hat{x} E_0 \exp[i(kz - \omega t)]
\]
(3)
\[
H = \hat{y} H_0 \exp[i(kz - \omega t)]
\]
(4)

Where the complex wave number, \( k \), is expressed as,
\[
k = k \hat{z}
\]
(5)

And the vacuum wave-number \( k_0 \), is given by,
\[
k_0 = \omega \sqrt{\varepsilon_0 \mu_0} = \frac{\omega}{c}
\]
(6)

where \( c \) is the speed of light and \( \omega \) the angular frequency. From (5) and (6), it’s possible to write the refraction index, as function of frequency,
\[
n = \sqrt{\mu \varepsilon}
\]
(7)

In order to obtain valid roots to (7) and (10), since in general \( \varepsilon, \mu \in \mathbb{C} \), we can re-write (15), for a DPS or DNG media, has
\[
\mathcal{S}_L = \frac{1}{2} \Re \{ E \times H^* \} = \frac{\varepsilon_0^2}{2\eta_0} \Re \{ \frac{1}{\zeta} \} \exp[-2\Im(k)z]
\]
(15)

A. Backward Waves

From the Maxwell’s equations, in the differential form, we can easily assimilated the left hand rule [5]
\[
\nabla \times E = -\frac{\partial D}{\partial t}
\]
(17)
\[
\nabla \times H = \frac{\partial B}{\partial t}
\]
(18)

to describe the response of the medium to the applied fields, we have the following two constitutive relations,
\[
B = \mu_0 \mu H
\]
(19)
\[
D = \varepsilon_0 \varepsilon E
\]
(20)

The relations between operators, in the time harmonic regime, can be derived,
\[
\frac{\partial}{\partial t} \rightarrow -i \omega
\]
(21)

leading to
\[
\nabla \rightarrow i k
\]
(22)

In order to calculate the sign of the Poynting vector, \( \mathcal{S} \), and of the wave vector, \( k \), we use the following relations for the spatial orientation of the electric and magnetic field vectors, \( E \) and \( H \),
\[
k \times E = \omega \mu_0 \mu H
\]
(23)
\[
k \times H = -\omega \varepsilon_0 \varepsilon E
\]
(24)

Has shown on the following figure,

In a DNG medium, the trihedral [\( E_0, H_0, \Re(k) \)] is left handed. In the other way, has expected, the trihedral for a DPS medium, \( [E_0, H_0, \Re(k)] \) is right handed. But, for both a DPS or a DNG medium, the trihedral \( [E_0, H_0, S_\omega] \) is always right handed. From Figure 2 we can see that in the DPS medium, both vectors, \( k \) and \( S \), have the same direction, creating a forward wave, meaning that the electromagnetic wave and the electromagnetic energy have the same direction. Moreover, we can see that the vectors \( k \) and \( S \) have opposite orientation in the DNG medium. This results in a backward wave, which means that the electromagnetic wave and the electromagnetic energy have opposite directions.

Based on (12) and (5), we can define the phase velocity, \( v_p \), of an electromagnetic wave as,
\[
v_p = \frac{\omega}{\Re(k)} = \frac{\omega}{\eta_0 \Re(n)} = \frac{c}{n'}
\]
(25)

In order to obtain valid roots of (7) and (10), since in general \( \varepsilon, \mu \in \mathbb{C} \), we can re-write (15), for a DPS or DNG media, has

![Trihedral vectors for the i) DPS and ii) DNG media.](image-url)
\begin{equation}
\Re \left( \frac{1}{\zeta} \right) = \Re \left( \frac{1}{\mu} \right) = \Re \left( \frac{n' + in''}{\mu + iv''} \right) = \frac{n'\mu' + n''\mu''}{(\mu')^2 + (\mu'')^2} \tag{26}
\end{equation}

Since this is a passive medium, we have \( \varepsilon'' > 0, \; \mu'' > 0 \) and \( n'' > 0 \). Furthermore, being a DNG medium, we also have \( \varepsilon' < 0 \) and \( \mu' < 0 \) and, has we shall see later, \( n' > 0 \). Knowing this, we must verify the following condition

\begin{equation}
\frac{n'\mu' + n''\mu''}{(\mu')^2 + (\mu'')^2} > 0 \Rightarrow \hat{z} \cdot \mathbf{S}_0 > 0 \tag{27}
\end{equation}

The same result obtained in (27), is applied to the DPS medium since \( \varepsilon' > 0 \) and \( \mu' > 0 \), when multiplied, have the same signal than \( \varepsilon' < 0 \) than \( \mu' < 0 \).

### B. Dispersion

From the laws of Electromagnetism, we know that for a certain isotropic medium, defined by \( \varepsilon \) and \( \mu \),

\begin{equation}
\langle W_e \rangle = \frac{1}{4} \varepsilon \varepsilon_0 \mathbf{E} \cdot \mathbf{E}^* \tag{28}
\end{equation}

\begin{equation}
\langle W_m \rangle = \frac{1}{4} \mu \mu_0 \mathbf{H} \cdot \mathbf{H}^* \tag{29}
\end{equation}

Where \( \langle W_e \rangle \) is the time-average electric density and \( \langle W_m \rangle \) the time average magnetic density. If both \( \varepsilon \) and \( \mu \) are negative, like in DNG media, we have fundamental problems since we can’t have negative energy density. This proves that a DNG medium is necessarily dispersive due to the definition of electromagnetic energy. Therefore both equations, (28) and (29), are invalid and need to be corrected to a lossless dispersive media [17],

\begin{equation}
\langle W_e \rangle = \int_{-\infty}^{t} \frac{\partial \mathbf{D}}{\partial t'} \mathbf{E} dt' \tag{30}
\end{equation}

\begin{equation}
\langle W_m \rangle = \int_{-\infty}^{t} \frac{\partial \mathbf{B}}{\partial t'} \mathbf{H} dt' \tag{31}
\end{equation}

Both the electric and the magnetic fields are almost monochromatic, with \( \mathbf{E}(t) = \mathbf{E} \exp(-\omega t) \) and \( \mathbf{H}(t) = \mathbf{H} \exp(-\omega t) \) respectively, where the period variation is very slow \( T = 2\pi/\omega \), meaning that \( v << \omega \).

### C. Group Velocity and Phase Velocity

The expression for the time-average energy density of a monochromatic plane wave is defined by:

\begin{equation}
\langle W \rangle = \frac{1}{4} \left[ \varepsilon_0 \frac{\partial (\varepsilon \omega)}{\partial \omega} |\mathbf{E}|^2 + \mu_0 \frac{\partial (\mu \omega)}{\partial \omega} |\mathbf{H}|^2 \right] \tag{32}
\end{equation}

If we consider

\[ A = \mu \varepsilon + \mu \mu \left[ \frac{\partial (\omega \varepsilon)}{\partial \omega} + \omega \frac{\partial (\omega \mu)}{\partial \omega} \right] = 2 \mu \varepsilon + \omega \left[ \mu \frac{\partial (\omega \varepsilon)}{\partial \omega} + \varepsilon \frac{\partial (\omega \mu)}{\partial \omega} \right] \tag{33} \]

As seen in the previous section, in a DNG medium we have \( \mu < 0 \) and \( \varepsilon < 0 \), leading us to \( A = 0 \). From (11) we have,

\begin{equation}
k^2 = \omega^2 \mu \varepsilon_0 \varepsilon \tag{34}
\end{equation}

Applying the first derivative in order of the \( \omega \) to \( k^2 \), we get

\begin{equation}
\frac{\partial (k^2)}{\partial \omega} = \mu_0 \varepsilon_0 \omega \left[ 2\mu + \mu \omega \frac{\partial (\omega \varepsilon)}{\partial \omega} + \varepsilon \frac{\partial (\omega \mu)}{\partial \omega} \right] = \mu_0 \varepsilon_0 \omega A \tag{35}
\end{equation}

Which enables us to write (35) has

\begin{equation}
\frac{\partial (k^2)}{\partial \omega} = 2k \frac{\partial (k)}{\partial \omega} = 2nk_0 \frac{\partial (k)}{\partial \omega} = n \frac{\omega}{c} \frac{\partial (k)}{\partial \omega} \tag{36}
\end{equation}

The Phase Velocity, \( v_p \), is given by

\begin{equation}
v_p = \frac{\omega}{\Re (k)} = \frac{\omega}{k_0 \Re (n)} = \frac{c}{n} \tag{37}
\end{equation}

and the Group Velocity, \( v_G \), is given by

\begin{equation}
v_g = \frac{\partial \omega}{\partial k} \tag{38}
\end{equation}

Replacing (37) and (38) on (35), we have

\begin{equation}
\frac{\partial (k^2)}{\partial \omega} < 0 \tag{40}
\end{equation}

Since we cannot have negative energy density, we must conclude that \( v_G \) and \( v_p \) have different signs (and opposite directions).

For a non-dispersive media, one has \( v_G = v_p \). On the DNG media, that has to be dispersive, one can prove that both group velocity and phase velocity, not only have different signs, but also has different values.

From (37) and (38) we get,

\begin{equation}
\frac{1}{v_g} = \frac{1}{v_p} + \frac{1}{c} \frac{\partial n}{\partial \omega} \tag{41}
\end{equation}

Due to last parcel, the group velocity is only equal to the phase velocity when the refraction index is frequency independent.

### III. Guided Wave Propagation in DNG Media

In this section we will study guided wave propagation with DNG media is studied for a DPS-DNG interface, Figure 3. The modal characterization of this planar structures is analyzed as several numerical simulations are exposed.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3.png}
\caption{Planar interface between a DPS medium and a DNG medium.}
\end{figure}

For the Transverse Electric (TE) waves, we have only \( E_y, H_z \) and \( H_x \) components, while in the Transverse Magnetic (TM) only \( H_y, E_x \) and \( E_z \) propagate. The transverse wave number of the medium \( i \), given by \( k_0^2 = k_0^2 n_1^2 - \beta^2 \). To maintain wave guiding on the interface, the fields must be evanescent and decay with distance away from the boundary surface. Because of this requirement, the propagation
constant is in the range of $k_0n_i < \beta$, the propagation constant $h_i$ is complex in both the regions and it’s given by,

$$h_i = \mp i \alpha_i$$  \hspace{1cm} (42)

Where $\alpha_i$ is the attenuation constant given by,

$$\alpha_i^2 = \beta^2 + k_0^2 n_i^2$$  \hspace{1cm} (43)

The sign of $h_i$ is chosen to represent the field decaying with the distance from the interface DPS and DNG. Assuming the interface on $x = 0$, the field’s are given by,

$$E_y(x) = \begin{cases} E_0 \exp[-\alpha_1 x] & , x > 0 \\ E_0 \exp[\alpha_2 x] & , x < 0 \end{cases}$$  \hspace{1cm} (44)

With $\alpha_1, \alpha_2 > 0$. To calculate the magnetic field, we use the Faraday’s Law from the Maxwell Equations:

$$\nabla \times \mathbf{E} = i \omega \mu \mathbf{H}$$  \hspace{1cm} (45)

From (45) we obtain the expressions for the magnetic field in both regions,

$$H_z(x) = \begin{cases} \frac{\mu_0 \alpha_1}{i \omega \mu_1} \exp[-\alpha_1 x] & , x > 0 \\ -\frac{\mu_0 \alpha_2}{i \omega \mu_2} \exp[\alpha_2 x] & , x < 0 \end{cases}$$  \hspace{1cm} (46)

For the boundary conditions, at $x = 0$, and to guarantee the continuity of the magnetic field at the interface, we need to assure $H_z(0)^- = H_z(0)^+$. This leads to the modal for the TE mode, given by

$$\alpha_2 \mu_1 + \alpha_1 \mu_2 = 0$$  \hspace{1cm} (47)

To obtain the modal equation for the TM mode, we can apply similar computation to the wave equation and field expressions for the TM modes, using the following equation from the Maxwell Equations,

$$\nabla \times \mathbf{H} = -i \omega \varepsilon \mathbf{E}$$  \hspace{1cm} (48)

We get,

$$\alpha_2 \varepsilon_1 + \alpha_1 \varepsilon_2 = 0$$  \hspace{1cm} (49)

With the modal equations (47) and (49), we can conclude if there is propagation along the interface between a DPS medium and a DNG medium. Since we have $\alpha_1, \alpha_2 > 0$ and $\mu_1, \mu_2 > 0$, from (47), for the TE mode:

$$\alpha_1 = -\frac{\mu_1}{\mu_2} \alpha_2 > 0 \implies \mu_2 < 0$$  \hspace{1cm} (50)

And, from equation (49) we get for the TM mode:

$$\alpha_1 = -\frac{\varepsilon_1}{\varepsilon_2} \alpha_2 > 0 \implies \varepsilon_2 < 0$$  \hspace{1cm} (51)

From implications (50) and (51) we can conclude that there is propagation on an interface between a DPS medium and a DNG medium, with $\varepsilon_2, \mu_2 < 0$.

### A. Surface Mode Propagation

To analyze the solutions of the modal equations, and due to the fact that a negative medium cannot have negative energy density, dispersion has to be considered. We use the Lorentz Dispersive Model (LDM), which permittivity and permeability are frequency dependent,

$$\varepsilon_{r,L}(\omega) = 1 + \frac{\omega_{pm}^2}{\omega_{pL}^2 + \omega^2}$$  \hspace{1cm} (52)

$$\mu_{r,L}(\omega) = 1 + \frac{\omega_{pm}^2}{\omega_{pL}^2 + \omega_{pm}^2 - \omega^2}$$  \hspace{1cm} (53)

This model is going to be used to describe the parameters, dependent of the frequency, on the DNG medium. We remove the index $x$ and the index $m$, and consider $\Gamma_L = \Gamma_x = \Gamma_m$. For the simulation we will use the following parameters, where $\varepsilon_{1,\mu}$ and $\mu_{1,\mu}$ are used to describe the DPS medium,

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{pe}$</td>
<td>$2 \pi \times 7 \times 10^9$ rad.s$^{-1}$</td>
</tr>
<tr>
<td>$\omega_{pm}$</td>
<td>$2 \pi \times 6 \times 10^9$ rad.s$^{-1}$</td>
</tr>
<tr>
<td>$\omega_{pL}$</td>
<td>$2 \pi \times 2.5 \times 10^9$ rad.s$^{-1}$</td>
</tr>
<tr>
<td>$\omega_{pm}$</td>
<td>$2 \pi \times 2.5 \times 10^9$ rad.s$^{-1}$</td>
</tr>
<tr>
<td>$\Gamma_L$</td>
<td>$0.05 \times \omega_{pe}$</td>
</tr>
<tr>
<td>$\varepsilon_{1,\mu}$</td>
<td>1</td>
</tr>
<tr>
<td>$\mu_{1,\mu}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table I: PARAMETERS USED IN THE LORENTZ DISPERSIVE MODEL, ON THE DPS-DNG INTERFACE

Which leads to the following result,

![Figure 4](image-url)  

**Figure 4.** Lossless Lorentz Dispersive model for $\varepsilon_{r,L}$ and $\mu_{r,L}$

We identify the three regions in Figure 4: DNG region when $\varepsilon < 0$ and $\mu < 0$, ENG region when $\varepsilon < 0$ and $\mu > 0$ and DPS when $\varepsilon > 0$ and $\mu > 0$.

To observe the above three regions, we analyze the relative refraction index, using the lossless LDM on the DPS-DNG interface,

$$n_L = \frac{n}{\sqrt{\varepsilon_{r,L}}}$$  \hspace{1cm} (54)

By simulation, seen in Figure 5, we can denote that $n$ varies with the frequency. This effect is known as dispersion and only occurs when a media is present, in vacuum all the frequencies travel at the same speed, $c$, the speed of light. We also can identify the three already mentioned regions: the DNG region with negative refraction index and the DPS with positive refraction. The ENG region, since
the permittivity is negative and the permeability is positive, has purely imaginary refraction index.

In Figure 6 we can observe the already mentioned three regions. Where the DNG region has a negative refraction index and the DPS region has positive refraction index. These were the expected results, but from the ENG region, since it has a negative real refraction index, we can conclude that a DNG medium is always a Negative Refraction Index (NRI) medium but a NRI medium does not mean a DNG medium if we consider losses and dispersion.

From (43) and (47), we get the dispersion relation for the TE mode,

\[
\beta(\omega) = \sqrt{\frac{(\epsilon_2(\omega)/\mu_1)^2 \mu_1 \epsilon_1 - \mu_2(\omega) \epsilon_2(\omega)}{1 - (\epsilon_2(\omega)/\mu_1)^2}} k_0 \quad (55)
\]

For the TM mode, using 43 and 49, we get,

\[
\beta(\omega) = \sqrt{\frac{(\epsilon_2(\omega)/\mu_1)^2 \mu_1 \epsilon_1 - \mu_2(\omega) \epsilon_2(\omega)}{1 - (\epsilon_2(\omega)/\mu_1)^2}} k_0 \quad (56)
\]

Obtaining the graphic simulation for the dispersion relation,

Figure 5. Relative refraction index between a DPS and a DNG medium using the lossless LDM

Since the dispersion is intrinsically linked to losses, the latter must be included,

Figure 6. Relative refraction index between a DPS and a DNG medium using the lossy LDM

We calculate the frequency dependent attenuation constants \(\alpha_1\) and \(\alpha_2\) using the lossy LDM for a DPS-DNG interface. For the TM mode, from (43), (55) and (55), we have

\[
\alpha_1 = \sqrt{\frac{\epsilon_1 \mu_1 - \mu_2(\omega) \epsilon_2(\omega)}{1 - (\epsilon_2(\omega)/\mu_1)^2}} k_0 \quad (57)
\]

and

\[
\alpha_2 = \sqrt{\frac{(\epsilon_2(\omega)/\mu_1)^2 (\mu_1 \epsilon_1 - \mu_2(\omega) \epsilon_2(\omega))}{1 - (\epsilon_2(\omega)/\mu_1)^2}} k_0 \quad (58)
\]

For the TE mode, with similar calculation,

\[
\alpha_1 = \sqrt{\frac{\epsilon_1 \mu_1 - \mu_2(\omega) \epsilon_2(\omega)}{1 - (\epsilon_2(\omega)/\mu_1)^2}} k_0 \quad (59)
\]

And,

\[
\alpha_2 = \sqrt{\frac{(\mu_2(\omega)/\epsilon_1)^2 (\mu_1 \epsilon_1 - \mu_2(\omega) \epsilon_2(\omega))}{1 - (\mu_2(\omega)/\epsilon_1)^2}} k_0 \quad (60)
\]

And its graphical representation,

Figure 7. Lossy dispersion relation \(\beta\) for the TE mode

Figure 8. Lossy dispersion relation \(\beta\) for the TM mode

Figure 9. Attenuation constants \(\alpha_1\) and \(\alpha_2\) for the TE modes for the lossy LDM
B. Existence of Solution Bands

The key issue for the practical implementation of a DNG metamaterial has to do with necessity to assure that the resonances of ε(ω) and μ(ω) are close enough to each other. An acceptable model lossless model is studied in the present section.

Considering the interface between air and a metamaterial at x = 0, with the metamaterial medium characterized by the following constitutive relations,

\[ D = \varepsilon_0 \varepsilon(\omega) E \]  \hspace{1cm} (61)  
\[ B = \mu_0 \mu(\omega) H \]  \hspace{1cm} (62)

With [18],

\[ \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \]  \hspace{1cm} (63)  
\[ \mu(\omega) = 1 - F \frac{\omega_p^2}{\omega^2} \]  \hspace{1cm} (64)

Assuming,

\[ X = \frac{\omega_p^2}{\omega^2} \]  \hspace{1cm} (65)  
\[ a = \frac{\omega_p^2}{\omega^2} \]  \hspace{1cm} (66)

We get,

\[ \varepsilon(X) = 1 - X \]  \hspace{1cm} (66)  
\[ \mu(X) = 1 - F \frac{1}{1 - a X} \]  \hspace{1cm} (67)

Where F , 0 < F < 1, is filling factor for the model and a the square of the relation between the magnetic resonance frequency, ω₀, and the plasma frequency, ωₚ.

The modal refraction index, \( \bar{n} \), for the TE modes is given by,

\[ \text{TE} \rightarrow \bar{n}^2 = \left( \frac{\beta}{k_0} \right)^2 = \left| \mu(\omega) \right| \left| \varepsilon(\omega) \right| / \left( \mu(\omega)^2 - 1 \right) \]  \hspace{1cm} (68)

And for the TM modes,

\[ \text{TM} \rightarrow \bar{n}^2 = \left( \frac{\beta}{k_0} \right)^2 = \left| \varepsilon(\omega) \right| / \left| \mu(\omega) \right| - 1 \]  \hspace{1cm} (69)

The TE modes only occurs in regions where \( \mu(X) < 0 \) and the TM modes only occur in regions where \( \varepsilon(X) < 0 \). In both cases, the modal refraction index has to be \( \bar{n} > 1 \).

There is always one TE mode propagating except when \( a = 0.36 \) and \( F = 0.56 \), where there aren’t any TE modes propagating. This is a forward TE mode since it propagates in the DPS region.

There are two TM modes propagating when \( F \neq 0.56 \) and one TM mode propagating when \( F = 0.56 \). The forward TM mode appears in the DPS region and the backward TM mode propagates in the DNG region.
IV. PROPAGATION ON A DNG SLAB

In this chapter we shall study the propagation of electromagnetic waves on a DNG slab waveguide. This structure is composed by a slab of DNG media, limited on the $x$ axis, which is between two semi-infinite DPS media, like the following figure,

![Figure 15. A DNG slab waveguide immersed on a DPS media](image)

**A. Modal Equations**

For the TE modes, the homogenous wave equation is given by,

$$\frac{\partial^2 E}{\partial x^2} + (k_0^2 n_1^2 + \beta^2)E = 0 \quad (69)$$

In the DNG medium, the solutions for this equation can take the following form,

$$E_y(x) = E_{0e} \cos(h_2 x) + E_{0o} \sin(h_2 x) \quad (70)$$

With the transverse propagation constant $h_2$,

$$h_2 = \omega^2 \mu_2 \varepsilon_2 + \beta^2 \quad (71)$$

We can easily see that there are two kind of solutions in (70): An odd solution for $\sin(h_2 x)$ term and an even solution for $\cos(h_2 x)$ term.

We want the electric field decaying with the distance to $x$ axis, so the evanescent of the electric field can be represented, in the DPS media, by,

$$E_y(x) = \begin{cases} 
    E_{01} \exp(-\alpha_1 x) & , x \geq d \\
    E_{02} \exp(\alpha_1 x) & , x \leq -d 
\end{cases} \quad (72)$$

With the transverse wave number given by,

$$h_i = i \alpha_i \quad (73)$$

And the attenuation constant, $\alpha_i$, is defined by,

$$\alpha_i^2 = \beta^2 + k_0^2 n_i^2 \quad (74)$$

Placing the attenuation constant, we can establish for the evanescent fields the following expressions,

$$E_y(x) = \begin{cases} 
    E_{01} \exp(-\alpha_1 x) & , x \geq d \\
    E_{02} \exp(\alpha_1 x) & , x \leq -d 
\end{cases} \quad (75)$$

For the odd mode, the electric fields has the relations,

$$E_y(x) = \begin{cases} 
    E_{01} \exp(-\alpha_1 (x-d)) & , x \geq d \\
    E_{02} \exp[\alpha_1 (x+d)] \exp(-j\beta z) & , x \leq -d 
\end{cases} \quad (76)$$

To obtain the expression for the magnetic field, we use the Maxwell–Faraday equation,

$$\nabla \times E = i\omega \mu H \quad (77)$$

Leading us to

$$H_z(x) = -\frac{i}{\omega \mu} \begin{cases} 
    -E_{01} \alpha_1 \exp[-\alpha_1 (x-d)] & , x \geq d \\
    E_{02} h_2 \cos(h_2 x) & , |x| \leq d \\
    E_{02} \alpha_1 \exp[\alpha_1 (x+d)] & , x \leq -d 
\end{cases} \quad (78)$$

To assure the continuity of the fields, we apply the boundary conditions at the interface $x = d$, and obtain,

$$E_y(x = d^-) = E_y(x = d^+) \quad (79)$$

$$H_z(x = d^-) = H_z(x = d^+) \quad (80)$$

After some algebraic manipulation, we can achieve the results for both the odd and even. So, we get the TE modal equations,

$$\begin{cases} 
    \alpha_1 = -h_2 \frac{n_1}{n_2} \cot(h_2 d) & (Odd Modes) \\
    \alpha_1 = h_2 \frac{n_1}{n_2} \tan(h_2 d) & (Even Modes) 
\end{cases} \quad (81)$$

Using the same procedure to obtain the TM modes we get:

$$\begin{cases} 
    \alpha_1 = -h_2 \frac{n_1}{n_2} \cot(h_2 d) & (Odd Modes) \\
    \alpha_1 = h_2 \frac{n_1}{n_2} \tan(h_2 d) & (Even Modes) 
\end{cases} \quad (82)$$

We can simplify the modal equations by making the following substitutions,

$$a = \alpha_1 d \quad (83)$$

$$b = h_2 d \quad (84)$$

The relation between the normalized propagation’s constants is given by:

$$a^2 + b^2 = V^2 \quad (85)$$

Where $V$, the normalized frequency, is given by,

$$V = k_0 d \sqrt{\varepsilon_2 \mu_2 - \varepsilon_1 \mu_1} \quad (86)$$

The intersection of the curve (85) with the modal equations will represent the modal solutions for these modes in the slab.
B. Surface Mode Propagation

In this section we shall analyze the surface modes on the DNG slab. From (71), \( h_2 \) is real when \( \beta < \omega \sqrt{\varepsilon_1 \mu_2} \), and has imaginary values when \( \beta > \omega \sqrt{\varepsilon_1 \mu_2} \). If we assume either imaginary or real values of \( h_2 \), in the analysis of the DNG slab, we still maintain the surface mode conditions where the wave diminishes with the distance away from the slab. If we consider the relations \( \tan(i \chi) = i \tanh(x) \) and \( \cot(i \chi) = -i \coth(x) \) and assuming that \( B = -ib \), the above equations can be re-written to

\[
\begin{array}{l}
\{ \\
\begin{aligned}
a &= -B \frac{\mu_1}{\mu_2} \coth(B) & \text{(Assymmetric mode)} \\
a &= -B \frac{\mu_1}{\mu_2} \tanh(B) & \text{(Symmetric mode)} \\
\end{aligned}
\end{array}
\]  

(87)

The numerical solutions of the equations can be found, both in plane \((b, a)\) or \((B, a)\), through the intersection of the corresponding modal curves and the ones representing equations (85) and (88).

First we will analyze the DPS situation with \( \varepsilon_1 = 1, \mu_1 = 1, \varepsilon_2 = 2 \) and \( \mu_2 = 2 \).

Figure 16. The modal solutions (red dots) for a conventional DPS dielectric slab with \( \varepsilon_1 = 1, \mu_1 = 1, \varepsilon_2 = 2 \) and \( \mu_2 = 2 \).

The horizontal positive semi-axis represents the transverse wave number, \( b \), while the negative semi-axis represents the imaginary part, previously called \( B \). As expected, in a DPS slab, there are only modal solutions with real \( b \).

However, if we consider a DNG slab, there are solutions with imaginary \( b \). This result can be seen in Figure 17.

Figure 17. The modal solutions (red dots) for a DNG slab with \( \varepsilon_1 = 1, \mu_1 = 1, \varepsilon_2 = 2 \) and \( \mu_2 = 2 \).

These modes are called super-slow modes since the phase velocity, given by \( v_p = \omega/\beta \), is lower than the speed of light in the outer medium.

Since we have a DNG medium in the slab, the sign of the right hand side of the TE modal equations changes due to \( \mu_2 < 0 \). This inversion in the sign of the modal equations causes a change in the slope of the branches of tangent and cotangent function and, for a given range of frequencies, we also have some slow-modes that can have more than one solution for the same \( h_2d \), as we can see on Figure 18. Has seen before, the electric field \( E_y \) of the even slow modes varies with a cosine function. On the other hand, the electric field \( E_y \) of the odd slow modes varies with the a sine function. But, if we have an imaginary wave number, both trigonometric functions become hyperbolic functions. When the inner medium is more dense than the outer medium, there are slow modes and, given a DNG metamaterial inner medium, also originates super slow modes.

The representation of the dispersion diagram for the TE modes of the DNG dielectric slab, with \( \varepsilon_1 = 1, \mu_1 = 1, \varepsilon_2 = -1.5, \mu_2 = -1.5 \), is shown in Figure 19. The dashed lines, defined by \( k_0d \) as a function of \( \beta d \), represent the transition limits. The first one, with a higher slope, defined by

\[
k_0d = \frac{\beta d}{\sqrt{\varepsilon_1 \mu_1}}
\]

(91)

represents the cutoff condition of the surface modes on the slab, where \( h_1d = 0 \). The second dashed line, with lower slope, given by

\[
k_0d = \frac{\beta d}{\sqrt{\varepsilon_2 \mu_2}}
\]

(92)

expresses the transition between the super slow modes and the slow modes.
Two degenerated modes are excited in the dielectric slab, a conventional mode and a limited conventional mode has we can verify in Figure 19.

The odd super-slow mode is the fundamental mode, which is the first one since it’s excited from null frequency, and becomes a slow mode when \( V = \mu_1/|\mu_2| \) and propagates until \( V = \pi/2 \).

Using the expression of the relation for the outer medium,

\[
\alpha_1^2 = \beta^2 - \omega^2 \varepsilon_1 \mu_1
\]

and ensuring \( a \geq 0 \) for the electric slab, we get the following inequality,

\[
B^2 + V^2 \geq 0
\]

(93)

Where, from expression (86), we obtain

\[
V^2 < 0
\]

(94)

Although the expression (93) is only verified when we have a super-slow mode, otherwise we would get \(-b^2 + |v^2| \geq 0\), which is always untrue. With this result, while still considering the situation where the outer medium is more dense than the slab’s inner medium, we have from (85),

\[
a^2 + b^2 < 0
\]

(95)

We can conclude, from equations (94) and (95), that the following relation must be verified in order to have propagation in the slab,

\[
b^2 < 0
\]

(96)

Only in the presence of super-slow modes, these conditions are valid. This result can be verified on Figure 22, where we verify (93), (95) and (96). As expressed in (94), the propagation in a less dense medium is only possible if we are in the presence of super-slow modes. This is only satisfied using a DNG dielectric slab.

In the case of a less dense inner medium, we can have two different dispersion diagram cases:

- One where \( |\mu_2| > \mu_1 \), shown in Figure 22;
- One where \( |\mu_2| < \mu_1 \), shown in Figure 23.
In the first case, Figure 22, there is propagation of an even and an odd super-slow mode. Both these modes have a null cutoff frequency and a longitudinal wave number, $\beta$, that tends to the same value while increasing the frequency.

In the second case, Figure 23, only the even super slow mode propagates and it’s limited by frequency. While on this frequency band, there are always two solutions that tend to the same value. The point where the double-solutions intersect represents the limit from which there is no surface mode propagation on the DNG slab. We can also conclude that there are propagation conditions for super slow modes, for all values of $k_0$.

V. CONCLUSIONS

This work starts with a media classification based on the permeability and permittivity. After that we studied electromagnetic properties of a DNG metamaterial. A DNG medium presents negative permeability and negative permittivity. These lead to negative refraction index and to existence of waves that have wave vector and energy flow propagating in an antiparallel direction, these waves are known as Backward Waves. If both the permeability and the permittivity are negative, we have fundamental problems since we can’t have negative energy density, this proves that a DNG medium is necessarily dispersive due to the definition of electromagnetic energy. Therefore a study to the dispersion and to the Lorentz Dispersive Model is done. Since the dispersion is intrinsically connected to losses, we also concluded that, not only the group velocity has different value of the phase velocity, but also has opposite direction in a DNG media.

Then we studied and analyzed and unbounded interface between a DNG medium and a DPS medium. The Maxwell’s equations are applied to the plane waves propagating in the isotropic and unbounded DNG medium and some results, like the homogeneous wave equations, are shown. The constitutive relations for this medium were calculated and then a modal analysis is done, revealing that this interface allows both TE and TM surface modes wave propagation. These propagation modes are not possible in an ordinary DPS wave guiding structure. The dispersion in the DNG metamaterial is included recurring to the Lorentz Dispersive Model. In the lossless case, when $\mu_2/\mu_1 = -1$ and $\varepsilon_2/\varepsilon_1 = -1$, for TE mode and TM mode, respectively, there is no physical solution since the dispersion relation tends to infinitive. However, when considering a dispersive lossy model, physical solutions arise. The electric field, as function of frequency and distance, for TE modes, is shown, revealing that the electric field in the DNG medium is greatly affected by the variation of frequency. In both cases, the DNG and the DPS, the electric fields decays with considerable attenuation with the distance.

For a more practical view of the DNG-DPS interface, a less theoretical problem is studied. The problem is formulated assuring that resonances of both the permeability and permittivity are close enough, which is one of the main problems when creating these new materials. Some simulations are obtained for various values of filling factor, leading, in specific cases, to two TM modes, one forward in the DPS zone and one backward in the DNG zone.

Finally an analysis to the guided propagation on a DNG slab is done. This has special interest since the construction of guided waveguides may be one of the greatest applications to DNG metamaterials. This structure also revealed the possibility of having surface wave mode propagation. However, the most important result is the propagation of super-slow modes, a consequence of the phase velocity in the inner medium (DNG medium) being smaller than in the outer medium (DPS medium). The surface slow modes reveal a double solution for a restricted frequency domain. The existence of these super-slow modes allows the propagation on the DNG slab even when a less dense medium for the slab is used, compared to the outer medium, $\mu_1 > \mu_2$. This phenomenon can only be verifed on double negative materials. Analyzing the dispersion relation diagrams when the slab medium is less dense than the outer medium, we could see that, for a medium with $\mu_2 < \mu_1$, there are two super-slow modes. For a medium with $\mu_2 > \mu_1$, in a limited frequency band, only one super-slow mode propagates, however, in this band, there are always two possible modal solutions.

REFERENCES


