Numerical assessment of pulse propagation in optical fibers: Linear and Nonlinear Regime

Dissertation submitted for obtaining the degree of Master in Electrical and Computer Engineering

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My sister Maria, who was extremely dedicated, and helped me in the translation of my thesis and corrected my mistakes.

Rita, who helped me in the process of formatting my dissertation and showed me confidence during this time.

Nuno, my “fiber’s companion”, who helped me solving some problems in this period.

All my friends, but especially Pedro, Duarte and Miguel, my thesis companions with whom I shared many nights working.

Lastly, I want to dedicate this thesis to my grandfather who would be very proud of me in this special moment.
ABSTRACT

The principal idea of this dissertation is to study two different regimes in optical fiber: the linear and the nonlinear. Although, an analytical study for both of them was made, the main purpose was to use some mathematical instruments, FFT and SSFM, in order to illustrate with figures the analytical study done.

In the linear regime, the dispersion effects, for GVD and the high-order effects were studied, and perturbation known as Chirp parameter introduced. In order to face the dispersion effect, a solution that deals perfectly with GVD in certain conditions was presented.

In the nonlinear regime, two kinds of solitons were distinguished: the bright and the grey. The bright solitons are in fact the ones that interest in telecommunication, so a study was done introducing some perturbations and testing how they reacted during their propagation inside the optical fiber. The interaction between solitons and how this perturbation evolves in optical propagation was also described.

Keywords: Propagation, Linear Regime, Nonlinear Regime, GVD, high-order effects, Chirp’s parameter, pulse broadening, soliton
RESUMO

O foco desta dissertação é estudar dois diferentes regimes de operação numa fibra óptica: o regime linear e o regime não linear. Embora, tenha sido feito um estudo analítico para ambos, o objectivo principal desta dissertação foi o uso de alguns instrumentos matemáticos, FFT e SSFM, a fim de ilustrar com números (gráficos) o estudo analítico realizado.

No regime linear estudaram-se os efeitos dois efeitos dispersivos: a DVG e os efeitos dispersivos de ordem superior. Também se estudou outra perturbação como o parâmetro Chirp, devido à fonte espectral. Por último, para enfrentar o efeito de DVG foi apresentado uma solução que compensa perfeitamente o efeito da DVG em determinadas condições, designada por DCF.

No regime não linear estudaram-se dois tipos de solitões: o claro e o escuro. Apesar destes dois tipos de solitões o que tem maior relevo no âmbito das comunicações são os solitões claros, comumente designados por solitões. No âmbito deste estudo introduziram-se algumas perturbações para perceber como os solitões reagem durante a sua propagação no interior da fibra óptica.

Palavras-chave: Propagação, Regime Linear, Regime Não Linear, Dispersão da velocidade de grupo, efeitos de ordem superior, o parâmetro de Chirp, propagação de impulso, solitão
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<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$v_{l=1,2}$</td>
<td>Light velocity in a medium</td>
</tr>
<tr>
<td>$c$</td>
<td>Velocity of light in vacuum</td>
</tr>
<tr>
<td>$n_{l=1,2}$</td>
<td>Refraction index (1- corresponds to the core, 2-corresponds to</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$\theta_{l=1,2}$</td>
<td>Light incidence angle</td>
</tr>
<tr>
<td>$E$</td>
<td>Electric field vector</td>
</tr>
<tr>
<td>$H$</td>
<td>Magnetic field vector</td>
</tr>
<tr>
<td>$D$</td>
<td>Electric flux density</td>
</tr>
<tr>
<td>$B$</td>
<td>Magnetic flux density</td>
</tr>
<tr>
<td>$P$</td>
<td>Induced electric polarization</td>
</tr>
<tr>
<td>$M$</td>
<td>Induced magnetic polarization</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>Vacuum permittivity</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>Vacuum permeability</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Dielectric contrast</td>
</tr>
<tr>
<td>$E_0$</td>
<td>Amplitude of the electric field</td>
</tr>
<tr>
<td>$F(r)$</td>
<td>Modal function</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Propagation constant</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Frequency</td>
</tr>
<tr>
<td>$a$</td>
<td>Radius core</td>
</tr>
<tr>
<td>$k_0$</td>
<td>Vacuum propagation constant</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wavelength</td>
</tr>
<tr>
<td>$h$</td>
<td>Refractive index of the core</td>
</tr>
<tr>
<td>$q$</td>
<td>Refractive index of the cladding</td>
</tr>
<tr>
<td>$u$</td>
<td>Normalized refractive index of the core</td>
</tr>
<tr>
<td>$w$</td>
<td>Normalized refractive index of the cladding</td>
</tr>
<tr>
<td>$\bar{\eta}$</td>
<td>Modal refractive index</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Normalized frequency</td>
</tr>
<tr>
<td>$b$</td>
<td>Normalized modal refractive index</td>
</tr>
<tr>
<td>$\nu_c$</td>
<td>Cutoff normalized frequency</td>
</tr>
</tbody>
</table>
$\lambda_c$  
Cutoff wavelength  

$J_m(u)$  
Bessel function of the first kind  

$K_m(u)$  
Bessel function of the second kind  

$B(0,t)$  
Longitudinal variation of the field  

$A(0,t)$  
Pulse envelope the entrance of the fiber  

$\omega_0$  
Frequency carrier  

$\tilde{B}(z,\omega)$  
Spectral component of the pulse  

$\beta_l$  
Linear parameter of the propagation constant  

$\beta_{NL}$  
Nonlinear parameter of the propagation constant  

$\alpha$  
Fiber loss parameter  

$v_g$  
Group velocity  

$\beta_i$  
Inverse of the group velocity  

$\beta_2$  
Group velocity dispersion coefficient  

$\beta_3$  
High order dispersion coefficient  

$L_D$  
Dispersion length  

$\tau_0$  
Characteristic time of pulse duration  

$C$  
Chirp parameter  

$\sigma_0$  
Pulse width at the entrance of the fiber  

$\mu$  
Pulse  

$\zeta$  
Normalized distance  

$\tau$  
Normalized time  

$D$  
Dispersion parameter  

$S$  
Dispersion slope  

$\lambda_D$  
Dispersion wavelength  

$M$  
Figure of merit  

$\phi_{NL}$  
Nonlinear phase  

$\gamma$  
Nonlinear parameter  

$P_i$  
Input power  

$L$  
Effective length  

$A_{eff}$  
Effective core area
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0$</td>
<td>Peak power of the pulse</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Amplitude of the pulse</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>Location in the nonlinear equation</td>
</tr>
<tr>
<td>$B$</td>
<td>Depth of the dip</td>
</tr>
<tr>
<td>$\hat{D}$</td>
<td>Differential operator</td>
</tr>
<tr>
<td>$\hat{N}$</td>
<td>Nonlinear operator</td>
</tr>
<tr>
<td>$\hat{F}_T$</td>
<td>Fourier transform operator</td>
</tr>
<tr>
<td>$h$</td>
<td>Step of the SSFM</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Frequency in the nonlinear equation</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>Position in the nonlinear equation</td>
</tr>
<tr>
<td>$\phi_s$</td>
<td>Phase in the nonlinear equation</td>
</tr>
<tr>
<td>$z_0$</td>
<td>Soliton period</td>
</tr>
<tr>
<td>$B$</td>
<td>Bit rate</td>
</tr>
<tr>
<td>$T_B$</td>
<td>Time between two consecutive solitons</td>
</tr>
<tr>
<td>$q_0$</td>
<td>Initial separation</td>
</tr>
<tr>
<td>$r$</td>
<td>Relative amplitude</td>
</tr>
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</table>
# LIST OF ACRONYMS

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>SSFM</td>
<td>Split-Step Fourier Method</td>
</tr>
<tr>
<td>GVD</td>
<td>Group Velocity Dispersion</td>
</tr>
<tr>
<td>DCF</td>
<td>Dispersion-Compensating Fiber</td>
</tr>
<tr>
<td>TAT</td>
<td>Trans-Atlantic Telephone cable</td>
</tr>
<tr>
<td>LED</td>
<td>Light Emitting Diode</td>
</tr>
<tr>
<td>LD</td>
<td>Laser Diode</td>
</tr>
<tr>
<td>GaAlAs</td>
<td>Gallium-Aluminum-Arsenide</td>
</tr>
<tr>
<td>InGaAsP</td>
<td>Indium Gallium Arsenide Phosphide</td>
</tr>
<tr>
<td>BER</td>
<td>Bit Error Ratio</td>
</tr>
<tr>
<td>TIR</td>
<td>Total Internal Reflection</td>
</tr>
<tr>
<td>STL</td>
<td>Standard Telecommunication Laboratories</td>
</tr>
<tr>
<td>MIT</td>
<td>Massachusetts Institute of Technology</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>American Telephone and Telegraph</td>
</tr>
<tr>
<td>ABC</td>
<td>American Broadcasting Company</td>
</tr>
<tr>
<td>ITT</td>
<td>International Telephone &amp; Telegraph</td>
</tr>
<tr>
<td>NTT</td>
<td>Nippon Telephone &amp; Telegraph Corporation</td>
</tr>
<tr>
<td>EDFA</td>
<td>Erbium Doped Fiber Amplifier</td>
</tr>
<tr>
<td>WDM</td>
<td>Wavelength Division Multiplexing</td>
</tr>
<tr>
<td>FTTC</td>
<td>Fibre To The Curb</td>
</tr>
<tr>
<td>FTTH</td>
<td>Fiber To The Home</td>
</tr>
<tr>
<td>PDH</td>
<td>Plesiochronous Digital Hierarchy</td>
</tr>
<tr>
<td>SDH</td>
<td>Synchronous Digital Hierarchy</td>
</tr>
<tr>
<td>ATM</td>
<td>Asynchronous Transfer Mode</td>
</tr>
<tr>
<td>RMS</td>
<td>Root-Mean-Square</td>
</tr>
<tr>
<td>SPM</td>
<td>Self-Phase Modulation</td>
</tr>
<tr>
<td>SMF</td>
<td>Single Mode Fiber</td>
</tr>
<tr>
<td>Acronym</td>
<td>Definition</td>
</tr>
<tr>
<td>---------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>ISI</td>
<td>Inter-Symbol Interference</td>
</tr>
<tr>
<td>AC</td>
<td>Alternating Current</td>
</tr>
<tr>
<td>DC</td>
<td>Direct Current</td>
</tr>
<tr>
<td>TOD</td>
<td>Third Order Dispersion</td>
</tr>
<tr>
<td>DDF</td>
<td>Dispersion Decreasing Fiber</td>
</tr>
<tr>
<td>IFFT</td>
<td>Inverse Fast Fourier Transform</td>
</tr>
</tbody>
</table>
1. Introduction
1.1 The basis of the optical fiber

To understand how optical fibers work, it is important to understand how the optical region is one of the regions that have good transmission conditions in the spectrum. This region is limited between the 50\text{nm} and 100\mu\text{m}. It is important to notice how the spectrum is organized, so the next figure will show exactly the wavelengths and the correspondent regions.

In the Figure 1.1 it is possible to see the visible region which goes from the 400\text{nm} (blue region) to the 700\text{nm} (red region). The transmission using optical fibers is well known as guided transmission. There exist other kinds of mediums known as non-guided mediums with atmosphere being a nice example. The difference between a guided and a non-guided medium is that the guided medium has to respect the boundary conditions. For instance, an example of a guided medium could be a communication system using a cable, and a non-guided medium could be a communication system using antennas. The basis of an optical system is represented in the next figure, and it is similar to other well known systems such as coaxial cables. It is composed essentially by an emitter, a receptor and a cable (the optical fiber).
Figure 1.2 - Principal elements of a link using optical fiber

As it can be seen in Figure 1.2, there exist other elements such as regenerators, optical-electric converters, amplifiers, fotodetectors and other constituents. The size of the optical fibers can achieve ranges of just some meters or many kilometers. They can be used underwater (like the TAT-9 and TAT-10), aerials and buried in the soil. As it is impossible to make an optical fiber cable with many kilometers it is important to use joints. The joints are responsible for linking the optical fibers cables to make a unique cable. When linking optical fibers in the ocean this process is done in the boat and then they send it back to the ocean. There is another mechanism to link optical fibers called connectors. These mechanisms have great advantage when comparing with the joints because they allow the link to be opened and the problem detected in the optical fiber.

Besides the optical fiber, other indispensable constituent is the optical source, which is used to send power through the optical fiber. One important characteristic that this source has to have is that it needs to be compatible with the dimensions of the core, because most of the optical power needs to travel along the core. There exist two types of lasers (LED and LD) and they have different uses. Both of these optical sources have the property of the modulation of the light being easy to implement just trough the variation of the electric current to the desired debit, which lead to an optical signal that codifies the data signal. It depends on what region we are projecting the optical system in, but if the wavelength of $800-900\text{nm}$ is considered the material constituents of the optical sources would be GaAlAs; on the other hand if the region of the wavelengths of $1100-1600\text{nm}$ is considered the optical source is going to be composed by the InGaAsP.

After emitting the light in the optical fiber it will suffer attenuation and it will be proportional to the distance that the light travels. At the end of the fiber, the light faces
the receptor. The fotodetector receives the optical signal and converts it into electric current. This component has to be very sensitive because the light that it detects has very low levels of power and it needs to be very coherent with the dimensions of the core.

All these systems are rated with a parameter called bit-error rate, or BER. This rate measures the probability of a bit being wrong. This value is lower when distortion is lower too. In digital transmission, the number of bit errors is the number of received bits of a data stream over a communication channel that have been altered due to noise, interference, distortion or bit synchronization errors [2].

1.2 Historical Context

Fiber optics is used nowadays with any question but its path started in the 19th century. In fact, guiding light by refraction, the principle that makes fiber optics possible, was demonstrated by Daniel Colladon and Jacques Babinet. Years later, John Tyndall included a demonstration of it in his public lectures. Tyndall also wrote about the property of total internal reflection (TIR) in an introductory book about the nature of light in 1870: "When the light passes from air into water, the refracted ray is bent towards the perpendicular... When the ray passes from water to air it is bent from the perpendicular... If the angle which the ray in water encloses with the perpendicular to the surface be greater than 48 degrees, the ray will not quit the water at all: it will be totally reflected at the surface.... The angle which marks the limit where total reflection begins is called the limiting angle of the medium. For water this angle is 48° 27', for flint glass it is 38° 41', while for diamond it is 23° 42'.

Erro! A origem da referência não foi encontrada.. Practical applications started to appear years later such as close internal illumination during dentistry among others. Scientists as Clarence Hansell and John Logie Baird also contributed for the development of the optical fiber principals but the principals and the ideas of making a system based in optical fibers started in the 60s due to some British needs. As we know, Britain is an island so their needs were different from the USA. STL (Standard Telecommunication Laboratories) decided not to continue studying the millimeter waveguides (an industry that was growing during this decade), because it was too expensive to justify its use. As the amount of data being transmitted due to communications was growing, it was really important to have a medium that could serve these needs.

During this époque, Karbowiak, an STL engineer, became responsible for the study of waveguides. He wrote that hollow waveguides might be “capable of attenuation as low as one decibel per mile, but engineering difficulties associate with beam structure are
likely to render this scheme commercially impractical” [3]. One of the major problems of these solutions was that laser communication was still young and the investigator did not have the necessary technology to focus on waveguides solutions. Karbowiak stated this fact and wrote “The only thing left is optical fibers”.

With the millimeter waveguide gone, Karbowiak decided to focus his attention in optics. He stated that one of the big problems of the two solutions of waveguides was how they guided waves, so he started studying the propagation modes. This issue was important in a time where computers were huge and slow, so the single-mode waveguide would be the ideal solution because of its mathematical simplicity. A multimode transmission is messy and more difficult to study, so this choice was very natural. Of course this was not pacific because this single-modes waveguides are not so easy to make. At this point, Karbowiak decided to talk with Charles K. Kao and both realized that the right track to follow was the single-mode waveguide for light, which became the next big problem: to make a practical one. Karbowiak focused his attention on the single-mode dielectric waveguide made by a thin plastic rod. He realized that this solution was impracticable due to reduced dimensions to support. This fight to find the dielectric microwave guide was not being held only by Karbowiak, Kao and Hockham but also by some French scientists, who had already had a similar idea. Their names were Jean-Claude Simon and Eric Spitz who worked for CSF. Similarly to as the other three scientists, this French group started studying the simplest type of dielectric waveguide, a thin rod suspended in air, like an unclad optical fiber. In this solution “the energy travels along outside of the rod avoiding absorption by the material”[3]. With this principle they assumed that transmission loss would be very small. But again they faced the problem of the small dimensions, so small that it would be impractical to make this waveguides for optical wavelengths. Luckily, they realized that the “clad” could be something other than air. The only necessary condition was that this surrounding material could not conduct electricity and its refractive index must be smaller than the core dielectric index. Another important thing that they needed to consider about was that surrounding the waveguide with some material would change the diameter for single-mode operations. The larger the difference, the smaller the core must be to transmit only a single mode, however, the smaller the difference between the refractive indexes of the core and the surrounding material the larger the diameter for single-mode transmission [3]. Given this situation an important issue was solved, because the bigger the target gets the easier it becomes to “send” light into the waveguide. But once more, there were other problems that started to appear. The next one was that light should travel in transparent material instead of air. The major
problem was that the engineering of materials were not so developed, so this issue threatened to grow transmission loss. The next battle of this group of engineers was to find a material that propagated light and had low loss. Karbowiak started to focus his attention on this matter and decided to arrange a theory that guided him to a fiber that reaches attenuation of a few decibels per kilometer at the infrared wavelengths of 1 to 10 micrometers [3]. At this time, Kao and Hockham were mastered at the waveguide theory, and both decided to run to clad optical fibers after Karbowiak left. This new theory killed Karbowiak’s idea of making a thin-film guide, because this “theory showed that a cladding would keep light from leaking out at bends in a single-mode waveguide”[3]. The trouble was that material transparency was needed. A window of opportunity appeared when they decided to work for British Post Office, which needed some development in the area of telecommunications. After making their calculus, they stated that they needed something that had a maximum loss of 20 decibels per kilometer, so that they could recover the entire sent signal in the receptor. The major problem was that the available system permitted 20 decibels over a distance of 20 meters. At this point, they turned their attention into different matters. Hockham studied the problematic of the tiny fluctuations that appeared in the real waveguides and made light leak out, like the behavior of miniature antennas. Kao chose to dedicate his attention in the search of a material clear enough to obtain the fascinating value of 20 decibels per kilometer. He noticed that people knew a few of glass properties. During this investigation, he realized that glass had some major properties such as surface reflection, scattering of light by atoms in the glass and the absorption of light by atoms in the material. After hearing some specialists about this problem, Professor Harold Rawson said that removing all the impurities would be the solution to reach Kao’s goal. Only in 1965 they sent an article to the Proceedings of the Institution of Electrics Engineers [3] in order to convince people their ideas were good and could be an important medium to develop. As these ideas needed some time to be assimilated, Kao decided to talk about his new inventions in the Institution of Electrical Engineers in January of 1966 by saying: “When these methods are perfected, it will be possible to transmit very large quantities of information (telephone, television, data, etc) between say, Americas and Europe, along a single undersea cable [3]. The major problem that Kao, Hockham and Spitz faced was to convince the scientific world that optical fibers would work in a time when the scientific community believed that satellites were the future.

Purifying Glass
Between 1965 and 1970, Kao understood that the only way possible was to create a glass as pure as he could. In this period, F.F. Roberts, who was an employer in Post Office, decided to help Kao. He delegated a work to George Newns to purifying glass. Meanwhile, as with the great majority of new developments one of the first consumers was the military army, and in this specific case the Americans and the British. This door was opened by the British Ministry of Defense who was seeking for new mediums of communication. This was a very important help just for two main reasons: firstly, guaranteed access to necessary equipments to continue this search, and secondly the fact that functionally and easy accessibility were more urgent than perfection, in a war situation. This last factor was enough to convince not only the army but also the scientific community, who could be looking at their potential first client.

Japanese Response

Kao and his team decided to search for some more support such in Germany. But the decisive answer came from Japan. There were two men both professor, named Zen-ichi Kiyasu and Jun-ichi Nishizawa who decided to listen to Kao’s ideas. They decided to implement a graded-index fiber despite the step-index fiber. With this idea they could reduce some losses that appeared in the system they had. Kao had to focus his attention in making pure glass. First, he decided to measure the attenuation in the fiber (it is a very difficult problem). Then he improved his knowledge about the basic component of glass: the silica (SiO\textsubscript{2}). Finally, in 1969 and after many experiences, he would obtain an encouraging result: 5 decibels per kilometer of losses, bearing the good news he managed to convince the skeptic ones that optical fibers were possible and needed to be studied. Still, the problem was only half solved: this result did not show how to make ultra transparent optical fibers. This was the next battle they needed to fight! In the 70’s, the Japanese encouraged by the last results focused on the graded index fiber which, according to them, could avoid pulse spreading by a factor of 100 to 1000; the effect meant that graded index-fiber would have almost as much transmission capacity as single-mode fiber over distances of a few miles. In 1969, Nippon Sheet Glass obtained losses of around 100 decibels per kilometer, which was a big improvement, considering the 1000 decibels per kilometer by the start of that decade.

Purification of glass, the first optical laboratory system

In order to achieve a telecommunication system, an emitter, a receptor as well as light detectors and electronic amplifiers are needed. Kao asked to a radio engineer named Chown to build a system. In 1969 he had an operating system and showed his
technology to the Physical Society. This was the starting and decisive point to the optical fibers development. As Kao said “fiber communication was ready to be moved into the development phase”[3]. Despite this magnificent discovery, this decade was more interesting in other mediums of communication such as satellites and millimeter waveguides. And so, the optical fibers were left behind. But an investigator of the Corning Sullivan Park Research Center, named Bob Mauer, understood that Kao’s idea just needed the purification of glass to become practical. He delegated the task of purifying silica to Frank Hyde, a young organic chemist. Knowing some properties of silica and understanding flame hydrolysis he was able to obtain the purest silica anyone had ever made. With this hydrolysis he could obtain just parts per billion of impurities. With this purification they decided to make a fiber in which the core was constituted by titanium doped silica and the cladding is constituted by this new pure silica. Mauer wanted a person working only in the development of optical fibers and he found Keck. When Keck was working in the project, he suddenly pointed the laser to the core and he realized that something new had occurred. As he said “…and when the laser spot hit the core, all of a sudden I got this flash of light”[3]. He was in the presence of the clearest glass ever made and after some new measurements they had targeted the value of 16 decibels per kilometer. He, Bob Mauer and Schultz, another scientist involved in this process had obtained the optical fibers jackpot, so they decided to make a patent of their new achievement. Bob Mauer showed his invention to the community and realized that “…everyone else remained far, far behind”[3]. Although there were only a few people listening to him, The Post Office invited him to show his creation and measure it with the most well known criteria of this time. In spite of how difficult it was to do the measurements, he could state that “attenuation was an astonishingly 15 decibels per kilometer” [3].

New challenger appeared in this race

Maurer’s obstacle occurred when doing these measurements, a small piece of the fiber broke and gave opportunity to Dyott and Newns to understand how this fiber was made. With this occurrence, Dyott’s team received a great prize and they entered the race to create optical fibers. Of course knowing all the constituents did not mean making them, so even though this step was helpful they still did not know how to make that pure silica. Encouraged by these results and after passing that Post Office’s exam this fiber was undoubtedly the major contribution to this new technology. During the 70’s, many scientists started their investigations, and Kao’s dream was now becoming real. In fact, others were studying solutions to improve semiconductors lasers and how to send light to the core. In 1972, Maurer’s team showed the world a fiber with the
value of 4 decibels per kilometer and they predicted that they could achieve total attenuation of 2 decibels per kilometer in the region of 800 nanometers.

Semiconductor Laser

An optical system needs a beam of light. As it is known, semiconductors are mixtures of some properties of conductors and some of isolators. The major idea is that only a few electrons that are in the valence band can reach the conduction band. Some of them leave this layer leaving a hole, and others are captured to fill the holes. The way to control the number of holes or the number of electrons captured is by adding some impurities to this crystal, and with that, it is possible to obtain either the n-type or p-type semi-conductor. When an electrical voltage is applied to this components two outcomes are possible: the first boring one is that the electrons and the holes go to their terminals and stay there, if a positive voltage is applied to the n-type and a negative voltage to the p-type; the second one is that when the positive voltage is changed to the p-type and the negative voltage to the n-type, then electrons and holes will go to their opposite poles leading to a current in that transference. Due to this movement and exactly between the n and p material, a recombination can happen, electrons filling holes, which can lead to two different results depending on the material used: if silicon is used, it is possible to obtain energy released as heat, but other materials such as gallium arsenide are used, it is possible to obtain energy released as light. After understanding this mechanism, the birth of semiconductor was imminent. Meanwhile, a group of MIT researchers made lasers from light emitting semiconductors but were not so convinced about their results because they could not explain how this new discovery was not violating the second law of the thermodynamics. Bob Hall, a semiconductor expert of General Electric was impressed with this new invention and started to develop his own semiconductor laser. He got, in 1964, a semiconductor laser that could fire a single pulse at room temperature when 25 amperes flowed through an area of 0.02 square millimeter for 50 billionths of a second[3].
Figure 1.3 - A simple cube of Gallium Arsenide was the basis of the first semiconductor laser

The major problem of these components was their reduced lifetime. In Russia, there were other scientists developing the same technology. A trace that was being followed by this Russian group, who Zhores Alferov was the leader, was to make heterojunctions, just because this would increase the odds of recombination between holes and electrons. Although this seemed to be the right track to follow it was not sufficient to the telecommunications sector. They needed semiconductors lasers that generated a steady beam they could modulate with a signal\cite{3}. One of the major problems of this laser was that they needed high currents at the environment temperature. In Bell Labs the investigation in this area started to grow and Hayashi, in 1967, studied single-heterojunction lasers intensively and made one with lower thresholds currents. But that was not enough; they needed to double heterojunctions lasers. Knowing that Alferov’s group was way ahead in the technology of semiconductors laser, Hayashi and Panish started to work enthusiastically. Alferov had obtained a double heterojunction laser with low threshold current at room temperature but it was not operating continuously. Hayashi needed to do the same but operating in continuous wave. In 1\textsuperscript{st} of June Hayashi made what he wanted.
This double heterojunction was really a nice shot since it increased laser efficiency as well as reduced the threshold current. Meanwhile, Alferov made an extra confinement by covering parts of the laser chip with silica, which brought him an improvement of its quality. At Bell Labs, the researchers concluded that other real problems such as the stripe geometry which conducted to a poor beam quality had to be solved. With that, they realized that mode control was an important advance, and stripe-geometry lasers helped focus light into single-mode fibers [3]. Alferov’s group was the first to invent the semiconductor laser emitting at room temperature after covering their wafer with a thin insulating layer of silicon dioxide. The next task was to create a laser that lasted longer than just minutes or hours. Bell Labs’ scientists understood this limitation and started to cooperate directly with the people responsible for optical fibers. In fact, the lasers they were developing were to work with Corning's low-loss fiber. AT&T needed lasers that lasted for many years and Bell Labs could only offer tones they estimated lasted only a few years. During this time, the industry of measuring lasers’ lifetime started to grow exponentially. With that, investigators tried to understand the reason why the laser for it stopped working. Studies with polarized light showed that the more strain in the crystal, the faster the laser died, and also revealed some problems with the gallium substrate. With this amount of studies and the technology developing Bell Labs finally discovered a laser that lasted 5 to 10 years for a batch of 90 diode lasers.

The decade of the 70’s

The decade of the 70s was very profitable for the optical fibers. Corn and Bell Labs decided to work together purifying and improving their last discoveries. They could use their patents without paying anything to their creators. During this decade there was another thing that helped in this growth the presentation for Queen Elizabeth II. Murray
Ramson was in charge of that, which consisted of video communications, and the fiber worked flawlessly. A wave of enthusiasm took care of the world’s opinion, and Martin Chown said: “with future improvements in components, the channel capacity is expected to be around 10000”[3]. Although this industry was growing there were other aspects that everyone needed to think about and one of them was to connect the fibers. That was not so obvious and the processes known at that time could cause the fibers to lose efficiency increasing the losses. The Defense Advanced Research Projects Agency and the Naval Research Laboratory took care of this problem and started to improve these techniques. At this time, Bell Labs had made fibers with losses of 4 decibels per kilometer at 900 nanometers and 2 decibels per kilometer at 1060 nanometers. In the years between 1975 and 1983 it was shown that optical fibers were the future and all the investments made in this research would generate long term benefits. As the optical fiber’s industry became more commercial, it was needed to test it in the “normal” environment. Using graded index fiber, the pulse spreading could be reduced and they could send more information in that fiber. Graded-index fiber could easily carry tens of millions of bits per second over that distance, and perhaps a few hundred million [3]. Bell Labs decided to implement a communication system working with optical fibers in Atlanta in 1975. After a year of experiments they realized that the system was working satisfactorily better than they first expected. Of course this run was not occurring only in America. In Europe, their systems were sending signals of 8.4 million bits per second over a distance of about 13 kilometers.

A help came from Japan

After some development, a Japanese scientist named Horyguchi understood that due to scattering it was impossible to drop fiber losses. Moreover, he learned the higher the wavelength the lower the limit of scattering. He realized that the material dispersion, one of the major causes of the losses, dropped when it was operated in a wavelength of about 1.2 and 1.3 micrometers. It was important, once more, to measure these losses, so he decided to create a system that could do that between ranges of 0.4 to 2.5 micrometers. He also realized the water that fibers contained would absorb most of the light, so he concluded that if they could lower the value to 100 parts per billion of water they would be able to make a very clean fiber. This statement was very important and on March 27 he had a low-water fiber that had 0.47 decibels per kilometer at 1.2 micrometers. With this progress they had hit the jackpot; they opened a new window for fiber optics communication.
Figure 1.5 - The operating windows of optical fibers

It is important to notice that the third window as well as the fourth window has low levels of losses as it can be depicted on the Figure 1.5. Another advantage about this window is that it is possible to use EDFA, i.e., they are programmed to work in third window wavelength. Despite the conclusion, it is important to remember that the second window also shows an advantage comparing with these ones. It shows low levels of dispersion, i.e., it can be considered that the group velocity dispersion is null (as it is demonstrated in this dissertation).

Two more windows

The response for that problem was answered by J. Jim Hsieh, who made a quaternary compound with gallium, indium, arsenic and phosphorus. In 1976, he was able to get room-temperature lasers emitting steadily at 1.21 and 1.25 micrometers. During the summer of 1976, there was a big help that came across from Japan. Two scientists named Osanai and Horiguchi understood that at exactly 1.39 micrometers hydrogen-oxygen bonds absorbed very strongly, even though they realized they needed other wavelength to continue the developing of the optical fibers. They realized that the number was 1.5 micrometers (3rd window) and they obtained fibers with only 0.46 decibels per kilometer of loss. They opened another window for the optical fibers. This second new generation of fibers had a great advantage comparing to the old ones. As there were lower losses and the pulse spreading was reduced, they could send signals over a greater distance. And this change was what avoided installing repeaters in unfriendly environments; instead they could be installed in buildings or other places easier to access and avoiding bad environment conditions. During this time, Bell Labs set new goals for the optical fibers: the usage of submarine cables linking Europe and America and single-mode fibers (it is important not to forget that the systems were
operating with graded index fibers) and the adjustment of the operation window to 1.1 and 1.3 micrometers. These new thoughts about single-mode fibers lead them to other problems such as to how could light be transferred between linked optical fibers. It appeared that some noise was showing at the junctions, and this problem was due to the coherence of light.

Single-mode fibers instead

Epworth soon realized that the right track to follow was the single-mode fibers which avoided an undesirable effect: the modal noise. This 1.3 micrometer window was, in fact, one of the great reasons for this new fiber to work. At this wavelength the signals could travel tens of kilometers and as the material dispersion was lower they increased the capacity. Another interesting aspect was that the core increased too, which made it easier to send light along fiber. In Japan, they could obtain a single-mode fiber with losses of about 0.2 decibels per kilometer (the common losses today). Will Hicks decided to focus his attention on the study of these revive fibers. He realized that one fiber could simultaneously carry signals at many wavelengths, an idea called wavelength-division multiplexing (WDM). This was a really interesting property because it could increase the information that a single-mode fiber could carry. By this time, Corning and Bell Labs were able to drop losses from 0.2 decibels per kilometer at 1.55 micrometers to 0.16 decibels per kilometer (the low value of attenuation that it is possible to obtain due to some physics limitations such as the dispersion of Rayleigh). The problem once more was to convert these new systems into specific applications. Telephone companies decided to make a realistic bet by changing their systems to the gallium-arsenide lasers and graded-index fibers. Although this concerned the first systems, there was a company that started to make lasers work with single-mode fibers, named Lasertron. This company was property of Hsieh. As they were a small company competing with giants and moreover trying to make them face the new technology, they needed a big company to offer them some support. The answer was Chinese. 1980 was an important year for optical fibers. In the Winter Olympics in Lake Placid the world saw fiber optics working, and ABC television thanked, because they could add more advertising to their broadcast which meant more revenues. The amount of data and the reduced changes made telephone and television companies believe in the second-generations systems. The end of this graded index fiber happened with a test at British Telecom. This company wondered how far 140 million bits per second could go if using the single-mode fibers system. The answer was 49 kilometers. With that distance they could put repeaters in those buildings rendering the system easier and more reliable: “suddenly made people realize that graded index fiber
was a dead duck” recalls Mindwinter [3]. In 1982, Midwinter decided to make a system with the collaboration of STL, between Martlesham and Ipswich. The results were impressive: 565 million bits per second over 62 kilometers without repeaters at 1.3 micrometers. In 1983, after these encouraging tests, British Telecom decided to install single-mode systems. However, Bell Labs had other opinion this type of fibers. They thought their use was condemned to submarine cables. The next step for the American companies was to experiment their second-generation systems from Boston to Washington. This route was very good to make tests because it dealt with much data traffic.

The market of optical fibers

There were several companies that were working with single-mode systems and they could achieve performances of about 4000 million bits per second. This pacific work till the 80’s started to become more difficult from the 80’s on. In order to protect their rights, Corning registered all of their new discoveries and other companies such as ITT faced some penalizations for using some technology that did not belong to them. In the mid-80’s the fiber optics market started to grow. The deregulation of long-distance telephone service created a market for long-distance transmissions in America. This was really the kick-off for the fibers market. The other positive aspect of this deregulation was that the single-mode fiber, as well as the lasers industry, was the right technology to face these new opportunities. All the important companies that were in the telecommunications industry were about to end, so the free market was ready to be implemented and with that implementation a world of opportunities was ready to grow.

Covering the ocean with cables

The last part of this story is the submarine cables linking for instance America with Europe and America with Asia. As Alec Reeves predicted in the 1960’s, one of the major contributions that optical fibers could give to the industry of communication systems was to link and to carry many signals underwater. It is important to notice that other systems, mainly composed by coaxial cables, were working at the depths of the Atlantic Ocean. In fact, the first telephone cable that crossed the Atlantic was installed in 1956. Nowadays, cable’s specifications insist on no more than two failures in the quarter-century lifetime that require hauling the cable to the surface[3]. It is interesting to remember that submarine cables began carrying the electrical signal of the telegraph in the nineteenth century. Since then, a lot of other cables were installed crossing the
In 1953, AT&T and the Post Office decided to implement the first telephone transatlantic cable, the TAT-1.

With this first impetus other companies decided to cover the ocean with some more cables. In the Pacific Ocean, they also installed many cables. The high point of this technology came with the system TAT-6 where it was possible to add more 4000 new voice circuits. But this was the last coax cable crossing the ocean. After that, the cables used would have optical fibers instead. The very first discussion to serve these needs was to choose between graded-index fibers or single-mode fibers. The only demand that was very important in single-mode fibers was to align fibers with the most precision possible. There were other several concerns such as the lifetime of the lasers. After the opening of the third window, it was possible to obtain some interesting results, such as low loss and less pulse dispersion. In fact, at the third window wavelength, 1.3 micrometers, single-mode fibers had near zero dispersion. The next step was to implement the optical fiber inside a big cable and to cover it with any layer that could protect the fiber as well as help it to survive in difficult environments. Standard Telephones and Cables decided to focus its attention in undersea cables. They offered a solution with four-graded index fiber and two single-mode fibers and other materials to endow the fibers. In 1983, after some tests and trainings, it was crystal clear that the next generations of the TAT cables would have inside these four graded-index fibers as well as the two single-mode fibers inside, which could supply 280 million bits per second. In 1982, the first cable and repeater was launched in the Atlantic floor and it worked perfectly! After having some problems with phosphorous, they learned how to protect their fibers by making hydrogen-free cables. An important idea that emerged (and lead to the correspondent industry growth) was related to the amount of smaller

Figure 1.6 - The installation of the first TAT in 1953
cables needed to make a transatlantic one. In Europe the amount of data sent between Belgium, UK, Germany and Netherlands was growing so much that it was needed to create a system that could provide that guarantee. In Japan, NTT designed a system that could transmit 400 million bits per second between islands and the Japanese archipelago. In 1986, many companies decided to work on a new cable, the TAT-9, and its service was to start in 1991. In this cable, the fourth generation of fibers working at 1.5 micrometers, which could guarantee low losses, was used. The evolution of the TAT did not stop with TAT-9. TAT-10 and TAT-11, it followed the same technology of the TAT-9 and developed into the first system that used the new technology, at that time, which used optical amplifiers, that increase the strength of the signal without converting it into electric one. David Payne found that erbium could amplify up to 1.55 micrometers nicely when excited by the infrared lasers. Although this technology was quite good it is easy to understand that is not perfect and would continue breaking.

![Global LIT Submarine Cable Capacity](image)

**Figure 1.7 - The line capacity versus the bandwidth in 2008 (the use of TAT)**

### 1.3 The state of art

As it is obvious, optical fibers shows great advantages in telecommunications. It is possible to carry much more information through just one optical fiber, which improves the amount of data that is possible to send. Nowadays, the systems in use belong to the fourth generation but there are several differences comparing to its beginnings. It is not so incorrect to talk about the new born fifth generation. After facing the problem of the dispersion with the amplifying fibers (EDFA), it is now important to deal with other problems as dispersion. There are several techniques that are being tested to deal with this: dispersion compensation to improve pre-installed systems, dispersion management as a new model to deal with the conventional systems, and the solitons
systems (subject that is referred in this dissertation). Although these techniques are different in their genesis, they have common aspects such as the utilization of EDFA’s, the use of WDM and the dispersion management.

It is important to understand that this technology came and it will be the basis of the next generation of communications. Nowadays, the major contribution of this technology was the submarine cables, but this was the first of many new applications that optical fiber will bring in the future. The great necessity of wideband with the services integration will guide us to the Fiber-To-The-Curb (FTTC) or Fiber-To-The-Home (FTTH). One of the major problems of these systems is that photonics are not as advanced as they should be to follow up the developments occurring with optical fibers. These systems continue using many electronic devices, so the next revolution will be in this area. Words as PDH, SDH and ATM will be the next step of evolution and they will have to coexist in the same telecommunication platform.

1.4 Objective of the dissertation

The work done in this dissertation firstly, has the goal of understanding one of the most interesting mediums of communication: the optical fiber.

In this area, the most important objective is to study the linear and nonlinear regime for different kinds of pulses. To achieve this goal, the attention is focused in some problems that occur inside the optical fibers, such as dispersion. It is easy to explain how dispersion interferes with the efficiency of the optical system. The following example shows the pulse propagating inside an optical fiber.

![Figure 1.8 - A pulse suffering time dispersion when travelling in optical fibers](image)

The major problem of the dispersion is that when it is using a train of pulses the pulse spreading, represented in Figure 1.8, can lead to an undesirable interference called ISI (intersymbolic interference). The problem of the ISI is that it could make the message so difficult for the receiver to detect that it could change the information that was sent
by the emitter. For instance, with ISI, the receiver can detect a 0 instead of a 1 or vice-versa.

Secondly, some examples are proposed on how this interference can be avoided. The technique that it is going to be focused here is the DCF (dispersion compensating fiber).

Lastly, the importance of the nonlinear regime and their advantages comparing with the linear regime will be put in evidence.

All of these aspects are illustrated recurring to some matlab scripts, in order to illustrate all the fundamental aspects of these three topics.

1.5 Organization and structure of the dissertation

The first chapter refers to the historical context, the development of the optical fibers, namely the options that were being studied during the time, the first optical fiber system, the first transatlantic cable and many other aspects. It is also about a perspective of the future, for instance where this technology is going to be used, as well as the other aspects that will be developed in the near future such as photonics.

The second chapter clarifies some basic concepts about optics, by using illustrative problems where it is shown the some basic phenomenon that occurs inside the optical fiber: refraction and reflection. Maxwell's equation as the basis of electromagnetic phenomenon is referred to, and then it flows into the basic aspects of the optical fibers: basic constituents and most important kinds. Many major parameters are shown to simplify the discussion of this theme such as dielectric contrast, wavenumber, normalized frequency and so on. In this chapter it also focused the study about fiber namely LP and fundamental modes.

In the third chapter, it will be deduced the propagation equation in linear regime for single mode fibers. This equation is highly important because it shows the dependence of some coefficient that causes pulse spreading. Then, it follows into a demonstration for a Gaussian pulse of the RMS (root mean square) where it shows the relation between the pulse width at some point of the fiber comparing with the pulse width at the beginning. Then, this chapter becomes more practical with some illustrative examples based on two pulses: the “sech” pulse and the Super Gaussian pulse with Chirp. The influence of the high order dispersion coefficient is also analyzed and illustrated by some figures. Last but not the least, a compensating technique called DCF (dispersion compensating fiber) to avoid pulse spreading is studied. it is done so by presenting a simple case just considering the GVD effect.
Throughout the fourth chapter, it is introduced another way of operating optical fibers: the nonlinear regime. In order to obtain this new regime, it is necessary to explain the interplay between GVD and SPM which will guide to the essential constituent of this regime: the solitons. Then a brief explanation about Kerr’s effect is done and it Nonlinear Schrödinger equation is shown explaining the behavior of the solitons in the optical fiber. After this introduction, an explanation about dark solitons, a type of solitons and why they appear and are not so important in communications systems is given. Then, it is explained that the Split Step Fourier Method was the method used to obtain the mathematical solution of the Nonlinear Schrödinger equation, and that the use of it leads us to show many simulations about bright solitons. The possibility of different input pulses generating solitons, as well as the interaction between two solitons is explored.

In the last chapter, there is a reflection about the most important conclusions in this dissertation as well as some suggestions about a possible future work for others to study.

### 1.6 Major Contributions

In this dissertation the main contribution was my own personal vision of the optical fibers, but there are some important issues to refer:

- Many numerical simulations, both for nonlinear regime and linear regime, in order to understand the problems in optical fiber propagation, namely the group velocity dispersion and the high order effects as well as the soliton interaction.
- Study of the GVD parameter and higher order dispersion parameter in order to understand separately and as a whole their effects in optical fiber propagation.
- The use of DCF in order to compensate GVD in the linear regime.
2. Basic concepts in optical fibers
The main purpose of this chapter is to explain the key concepts about the theory of fiber modes. It is indispensable to study the effect of reflection as the basis of the propagation of the ray when inside the core. In order to do this, it is important to remember the concept of Maxwell’s equations, which are the pillars of the propagation of light inside any medium. To link these basic concepts in optical fibers is essential. It is going to be shown how the constitution and basic equations of optical fibers are vital to its understanding.

2.1 Principle of Fermat: Snell’s law

The subsection introduced here is about one of the simplest processes that happen with light when travelling from a medium to another. Thus, being the basis of optics and optical fibers it is very important to understand it.

The demonstration begins by looking attentively to the following figure

![Ray of light crossing two different mediums](image)

First of all, it is necessary to explain what the refractive index is. In optics, the refractive index or index of refraction of a substance or medium is a measure of the speed of light in that medium. It is expressed as a ratio between the speed of light in vacuum and in the considered medium.

According to Figure 2.1, it is possible to see that the ray of light propagates in two different mediums, which can be represented by two trajectories: the first the one that unites \( A(x_1, y_1) - P(x, 0) \) and the second path will be \( P(x, 0) - B(x_2, y_2) \).
According to Fermat’s Principle the path taken between two points by a ray of light is the path that can be traversed in the least time.

It is possible to write down the following equations

\[ v_1 = \frac{c}{n_1} \]  
(2.1)

and

\[ v_2 = \frac{c}{n_2} \]  
(2.2)

where \( c \) is the velocity of light in vacuum, \( v \) the velocity of light in the considered medium, and \( n \) the refractive index in that medium.

It is possible to write the following equation by using the definition of time, so the total time becomes

\[ T(x) = \frac{1}{v_1} \sqrt{(x-x_1)^2 + y_1^2} + \frac{1}{v_2} \sqrt{(x_2-x)^2 + y_2^2}. \]  
(2.3)

As it is known by using the first derivative of a function, it is possible to obtain a point that minimizes or maximizes that function. In this case by making the first derivative, it is possible to obtain the minimum time needed for the ray to travel this path (Fermat’s Principle) by doing

\[ \frac{dT}{dx} = 0. \]  
(2.4)

Applying this derivative to equation(2.3), it is possible to obtain

\[ \frac{x-x_1}{v_1 \sqrt{(x-x_1)^2 + y_1^2}} = \frac{x_2-x}{v_2 \sqrt{(x_2-x)^2 + y_2^2}} \]  
(2.5)

and, noticing Figure 2.1, and remembering the basic knowledge of trigonometric, it is possible to write

\[ \sin \theta_1 = \frac{x-x_1}{\sqrt{(x-x_1)^2 + y_1^2}} \]  
(2.6)

and

\[ \sin \theta_2 = \frac{x_2-x}{\sqrt{(x_2-x)^2 + y_2^2}}. \]  
(2.7)
Applying equations (2.6) and (2.7) in equation (2.5), it is possible to write

\[ \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}. \]  

(2.8)

The equation (2.8) can be simplified by using equations (2.1) and (2.2), which can be obtained

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2. \]  

(2.9)

This equation is known as Snell’s law and it is the basis of refraction, a phenomenon that occurs in optical fibers. Though it is considered that \( T(x) \) is really a minimum it is indispensable to take its second derivative in order to achieve that conclusion. So,

\[ \frac{d^2T}{dx^2} = \frac{y_1^2}{v_1\sqrt{(x-x_1)^2+y_1^2}} + \frac{y_2^2}{v_2\sqrt{(x_2-x)^2+y_2^2}} > 0 \]  

(2.10)

With equation (2.10), it is possible to conclude that \( T(x) \) is concave up, which means that the point considered, \( x \), is really a minimum.

A similar problem to this one can be solved and has a lot of interest to the discussion about optical fibers. It is important to have a demonstration about reflection, the basic effect that is present inside the optical fibers.

Looking attentively to the next figure, it is possible to see some differences between this figure and Figure 2.1. These major differences are related to mentioned effect: the reflection.

![Figure 2.2 - The reflection effect [4]](image-url)
In this specific case, the ray of light is propagating in the same medium all the time, and when it hits point $P(x,0)$, its path is completely modified propagating in a completely different trajectory. The objective of this problem is to show the relation between the two angles: $\theta$ and $\phi$.

It is possible to write this in the next equations

\[
AP = \sqrt{(x-x_1)^2 + y_1^2}
\]

(2.11)

and

\[
BP = \sqrt{(x_2-x)^2 + y_2^2}.
\]

(2.12)

As it was done previously the total time of propagation will be

\[
T = \frac{AP}{v} + \frac{BP}{v} \iff \frac{c}{n}T = \sqrt{(x-x_1)^2 + y_1^2} + \sqrt{(x_2-x)^2 + y_2^2}.
\]

(2.13)

Knowing that the ray will travel according to the Principle of Least Action, if the first derivative to this equation is used, what will be obtained is the time that minimizes that trajectory of the ray

\[
\frac{c}{n} \frac{dT}{dx} = \frac{x-x_1}{\sqrt{(x-x_1)^2 + y_1^2}} - \frac{x_2-x}{\sqrt{(x_2-x)^2 + y_2^2}} = 0.
\]

(2.14)

Applying the trigonometric knowledge it is easy to obtain the following expressions

\[
\sin \theta = \frac{x-x_1}{\sqrt{(x-x_1)^2 + y_1^2}},
\]

(2.15)

\[
\sin \phi = \frac{x_2-x}{\sqrt{(x_2-x)^2 + y_2^2}}.
\]

(2.16)

When these two equations are applied to equation(2.14), the following conclusion is as follows:

\[
[\theta = \phi].
\]

(2.17)

To finish this process it is indispensable to find the second derivative

\[
\frac{d^2T}{dx^2} = 0
\]

(2.18)

in order to find that this is really a minimum.
2.2 Maxwell's equations

Maxwell’s equations are the most important known equations when the topic of discussion is light. As it is known, they are the basis of classical electrodynamics, classical optics and electric circuits.

For a non-conducting medium without free charges, these equations take the form

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \\
\n\nabla \cdot \mathbf{D} &= 0 \\
\n\nabla \cdot \mathbf{B} &= 0
\end{align*}
\]

Table 2.1 - The four Maxwell’s Equation [5]

where, \( \mathbf{E} \) is the electric field vector and \( \mathbf{H} \) is the magnetic field vector, and \( \mathbf{D} \) and \( \mathbf{B} \) are their correspondent flux densities. These are related with their field vectors by the following constitutive relations

\[
\begin{align*}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{B} &= \mu_0 \mathbf{H} + \mathbf{M}
\end{align*}
\]

Table 2.2 - Constitutive relations [8]

where \( \varepsilon_0 \) is the vacuum permittivity, \( \mu_0 \) is the vacuum permeability, \( \mathbf{P} \) and \( \mathbf{M} \) are the induced electric and magnetic polarizations. For optical fibers, \( \mathbf{M} = 0 \), due to the nonmagnetic nature of silica glass[7].

2.3 The principal constituents of the optical fiber

After retaining some notions about refraction and reflection and understanding how light acts as an electromagnetic radiation, it is now time to go back to the structure of an optical fiber, which is portrayed in Figure 2.3.
Figure 2.3 – The principal constituents of the optical fiber

As it can be seen, the optical fiber is divided into two distinguishable main regions: core and cladding. Those regions are the principal constituents of the optical fiber. The core is surrounded by the cladding and it has a higher refractive index when comparing to the cladding’s one. Both regions are made of dielectric materials, such as silica. Although, the refractive index of the core must be higher than the cladding’s one (this difference will be explained later), it could have two different types of fibers: step-index fibers and graded-index fibers. Their properties and differences can be seen in the following figure.

Figure 2.4 - Two different kinds of optical fiber: Graded-index fiber and Step-index fiber [7].

As it can be seen in the figure above, the step-index fiber has a constant value of the refractive index of the core, although the graded index fiber has different values of refractive index inside the optical fiber core. With this difference and knowing that the
refractive index in the core of a graded-index fiber has a parabolic-index, it leads to a non-dispersive pulse propagation within paraxial approximation (as this dissertation is not based on graded-index fiber this explanation is not going to be given, although that could be seen in [7]. This fact then leads to a reduction of the intermodal dispersion and that was the reason why this type of fiber was used in the first optical systems.

2.4 Important parameters in optical fibers

Since the intention is for the ray-light to propagate inside the core region, the total internal reflection needs to happen.

Therefore, the angle between the light and the boundary (region between core and cladding) needs to be smaller when comparing to the critical angle. In this chapter, the study is going to be focused in the step-index fiber working in single mode.

For now, \( n_1 \) can be considered as the core’s refractive index and \( n_2 \) the cladding’s refractive index. Thus, it can be defined the dielectric contrast

\[
\Delta = \frac{n_1^2 - n_2^2}{2n_2^2}.
\]

The dielectric constant is the parameter that measures this difference between core’s refractive index and cladding’s refractive index. This parameter is responsible for having less dispersion inside the fiber. When \( \Delta \to 0 \), the group velocity dispersion will tend to vanish. The problem is that when this parameter is very low, the fiber will not be able to confine more light inside the core.

The electric field’s axial component can be considered to be

\[
E_z(r,\phi,z,t) = E_0 F(r) \exp(i m \phi) \exp\left[ i (\beta z - \omega t) \right]
\]

where \( E_0 \) is the amplitude’s field, \( F(r) \) is the modal function, \( m \) is a constant that only accepts values for field’s periodicity \( \beta \) is the physical significance of the propagation constant and \( \omega \) stands for frequency.

Since the step-index fiber is being considered, it will have this refractive index function

\[
n(r) = \begin{cases} n_1, & r \leq a \\ n_2, & r > a \end{cases}
\]
where \( a \) is the radius core. Using the ray theory the transverse propagation constant can be obtained

\[
\kappa^2 = n^2(r) k_0^2 - \beta^2
\]  
(2.22)

where \( k_0 = \omega/c = 2\pi/\lambda \) is the vacuum propagation constant. Considering that \( \kappa = h \) (at core) and \( \kappa = q \) (at cladding), the equation (2.22) can be rewritten as

\[
h^2 = n_1^2 k_0^2 - \beta^2 \\
q^2 = n_2^2 k_0^2 - \beta^2.
\]  
(2.23, 2.24)

Since the objective is to have a guided superficial wave, as previously mentioned in the introduction, the internal reflection needs to occur between core and cladding[6], which it means that we need to have \( q = i\alpha \) needs to verify the total internal reflection, i.e.,

\[
\alpha^2 = \beta^2 - n_2^2 k_0^2.
\]  
(2.25)

These equations can be understood by using ray theory, which is demonstrated in Appendix A.

### 2.5 Wavenumber and normalized frequency

Concerning the literature about this subject, it is common to normalize some constants, as it can be seen in [7]. This has proven helpful when the objective is to achieve simpler expressions. The purpose of this section is to explain the methodology of these simplifications.

Considering these variable transformations, known as core and cladding’s wavenumber, accordingly to equations (2.26) and (2.27)

\[
u = h a \]  
(2.26)

\[
w = \alpha a \]  
(2.27)

where \( h \) is the transversal propagation constant and \( \alpha \) is the parameter that measures the attenuation, known as attenuation constant.

As longitudinal propagation constant is defined as \( \beta \), considering this new variable, \( \bar{n} \), the result will be

\[
\beta = \bar{n} k_0
\]  
(2.28)
and based on equation (2.23) and (2.25) the result obtained

\[ u^2 = \left( n_1^2 - \overline{n}^2 \right) (k_0 a)^2 \]  
(2.29)

\[ w^2 = \left( \overline{n}^2 - n_2^2 \right) (k_0 a)^2 . \]  
(2.30)

With the intuition of achieving a new variable defined as normalized frequency, the expressions (2.29) and (2.30) need to be summed up, as in

\[ u^2 + w^2 = v^2 \]  
(2.31)

Normalized frequency can be written as

\[ v = \sqrt{n_1^2 - n_2^2} \ k_0 \ a = n_1 \ k_0 \ a \sqrt{2 \Delta} . \]  
(2.32)

Normalized frequency plays an important role in determining the cutoff condition [7]. It is possible to write longitudinal propagation with respect to other variables such as

\[ \beta = \frac{1}{a} \sqrt{\frac{v^2}{2 \Delta} - u^2} . \]  
(2.33)

It is possible to define normalized modal refractive index as

\[ b = 1 - \frac{u^2}{v^2} = \frac{w^2}{v^2} = \frac{n_1^2 - n_2^2}{n_1^2 - \overline{n}^2} . \]  
(2.34)

There are a few conclusions that can be achieved from equation (2.34):

- As \( n_2 \ k_0 \leq \beta \leq n_1 \ k_0 \) we can notice that \( n_2 \leq \overline{n} \leq n_1 \) and \( 0 \leq b < 1 \).
- The cut-off occurs when total internal reflection starts, i.e., \( \beta = n_2 \ k_0 \)
- At the limit of high frequencies, it is possible to conclude that
  \[ \lim_{v \to \infty} \overline{n} = n_1 \Leftrightarrow \lim_{v \to \infty} b = 1 \]

Physically, it is important to understand these two limits. When it is considered that \( \beta = n_2 \ k_0 \), it is being considered that the total internal reflection will start at this point. Otherwise, when \( \beta \to n_1 k_0 \), it is being considered that the ray of light will see only the core of the fiber propagating without any kind of trouble.

Noticing that

\[ w = v \sqrt{b} \]  
(2.35)
\[ u = v \sqrt{1-b} \]  

(2.36)

and defining \( v_c \) as the cut-off normalized frequency, the result is

\[ v_c = 2 \pi n_i a \frac{c}{\lambda_c} \sqrt{2 \Delta} \]  

(2.37)

and if \( \lambda_i \) is the cutoff wavelength, it can be concluded that

\[ \Delta = \frac{1}{2} \left( \frac{v_c \lambda_i}{2 \pi n_i a} \right)^2. \]  

(2.38)

It is important to notice that a single-mode fiber is being considered, so \( v \leq v_c \), with \( v_c = 2.4084 \). This value (known as the single-mode condition) is obtained by solving two equations with respect to the cutoff frequency of TE\(_{01}\) and TM\(_{01}\) modes[7].

### 2.6 Fiber modes

The notion of fiber mode is a general one when it comes to the comprehension of optical fibers. An optical mode is no more than a specific solution of the wave equation that satisfies the appropriate boundary and has the property of spatial distribution that does not change with propagation [7]. This section will allow the discussion of fundamental aspects, such as the equation that leads to all fiber modes. As shown before, the focus of the argument is in guided modes, despite the fact that other modes of step-index fiber, such as leaky and radiation modes, exist.

A mode is uniquely determined by its propagation constant. According to equation(2.28), each fiber mode propagates with a refractive index whose values lies in between \( n_2 < \pi < n_1 \).

Noticeably, the guided superficial waves are hybrid modes that respect the following equation (this equation is demonstrated in [6])

\[ R_m(u) S_m(u) = m^2 \left( 1 - 2 \Delta u^2 \frac{v}{v^2} \right) \left( \frac{v}{u w} \right)^4 \]  

(2.39)

where

\[ R_m(u) = \frac{J'_m(u)}{u J_m(u)} + \frac{K'_m(w)}{w K_m(w)} \]  

(2.40)
\[ S_m(u) = \frac{J_m^\prime(u)}{u J_m(u)} + (1 - 2\Delta) \frac{K_m^\prime(w)}{w K_m(u)} \] (2.41)

and \( J_m(u) \) is Bessel's function of first kind and \( J_m^\prime(u) \) is its correspondent derivative and \( K_m(u) \) is Bessel's function of second kind and \( K_m^\prime(u) \) is its Bessel's derivative function. These derivatives can be obtained according to

\[ J_m^\prime(u) = \frac{1}{2} [J_{m-1}(u) - J_{m+1}(u)] \] (2.42)

\[ K_m^\prime(u) = -\frac{1}{2} [K_{m-1}(u) - K_{m+1}(u)] \] (2.43)

There are two categories of hybrid modes: \( \text{HE}_{mn} \) and \( \text{EH}_{mn} \). This notation represents whether if \( H_z \) or \( E_z \) dominates, respectively. Moreover, the parameter \( m \) refers to an azimuthal variation and the parameter \( n \) refers to a radial variation. In the special case where \( m = 0 \), these modes change their names to \( \text{TE}_{0n} \) and \( \text{TM}_{0n} \), corresponding to transverse electric \( (E_z = 0) \) and transverse magnetic \( (H_z = 0) \). The equations that describe what is written above are

\[ \text{TE}_{0n} \Rightarrow R_0(u) = 0 \] (2.44)

and

\[ \text{TM}_{0n} \Rightarrow S_0(u) = 0. \] (2.45)

### 2.7 LP modes

As previously mentioned in the subsequent sections, there are other types of fiber modes. In this section the focus is going to be on LP modes. LP stands for linear polarization. This mode is usually a result of weakly guiding fibers, \( i.e., \Delta \ll 1 \). According to equation (2.19) the following approximation is valid:

\[ \Delta \approx \frac{n_2 - n_1}{n_1} \] (2.46)

If Gloge's approximation is going to be used, it obtains \( R_m(u) = S_m(u) \). Applying this knowledge, it is possible to achieve a simpler modal equation, such as
\[ R_m(u) = \pm \frac{mv^2}{u^2 w^2} \] (2.47)

where it can easily be noticed that there are two different signals: \((+)\) stands for \(\text{EH}_{mn}\) mode and \((-)\) stands for \(\text{HE}_{mn}\). As for \(\text{EH}_{mn}\) mode, the following modal equation is obtained

\[ \frac{J_{m+1}(u)}{u J_m(u)} + \frac{K_{m+1}(w)}{w K_m(w)} = 0 \] (2.48)

where

\[ J'_m(u) = -J_{m+1}(u) + \frac{m}{u} J_m(u) \] (2.49)

\[ K'_m(w) = -K_{m+1}(w) + \frac{m}{w} K_m(w). \] (2.50)

Obviously, the same methodology can be applied to \(\text{HE}_{mn}\) mode, which will then obtain

\[ \frac{J_{m-1}(u)}{u J_m(u)} - \frac{K_{m-1}(w)}{w K_m(w)} = 0 \] (2.51)

where, we have considered that

\[ J'_m(u) = J_{m-1}(u) - \frac{m}{u} J_m(u) \] (2.52)

\[ K'_m(w) = -K_{m-1}(w) - \frac{m}{w} K_m(w). \] (2.53)

Taking an attentive look to the equations (2.48) and (2.51), it can easily be concluded that, if \(m = 0\), both equations will result on

\[ \frac{J_1(u)}{u J_0(u)} + \frac{K_1(w)}{w K_0(w)} = 0 \] (2.54)

that corresponds to \(\text{TE}_{0n}\) and \(\text{TM}_{0n}\) modes. These two modes are degenerated (with Gloge’s approximation). According to that, this can be achieved

\[ J_{-m}(u) = (-1)^m J_m(u) \] (2.55)

\[ K_{-m}(w) = K_m(w). \] (2.56)
Thus, the $\text{EH}_{0n}$ and the $\text{HE}_{0n}$ modes are, respectively, $\text{TM}_{0n}$ and $\text{TE}_{0n}$ modes. Considering lower dielectric contrast, all modes are almost linearly polarized and are referred to as $\text{LP}_{pn}$ modes. There are some conclusions that can be achieved with this demonstration:

- $\text{HE}_{nn}$ modes lead us to $\text{LP}_{0n}$ modes.
- $\text{TE}_{0n}$, $\text{TM}_{0n}$ and $\text{HE}_{2n}$ modes are degenerated and lead us to $\text{LP}_{1n}$ modes.
- $\text{HE}_{m+1,n}$ and $\text{EH}_{m-1,n}$ modes, with $m \geq 2$, are degenerated and lead us to $\text{LP}_{mn}$ modes.

It can be concluded that $\text{EH}_{mn} \rightarrow \text{LP}_{mn}$ considering that $p = m + 1$ and $\text{HE}_{mn} \rightarrow \text{LP}_{mn}$ considering that $p = m - 1$.

According to the conditions mentioned above, the $\text{LP}_{pn}$ modal equation can be achieved as

$$u \frac{J_{p-1}(u)}{J_p(u)} + w \frac{K_{p-1}(w)}{K_p(w)} = 0$$

(2.57)

and

$$u \frac{J_{p+1}(u)}{J_p(u)} - w \frac{K_{p+1}(w)}{K_p(w)} = 0$$

(2.58)

According to the equations (2.57) and (2.58), $w$ and $u$ parameters needs to be transformed in terms of $b$ and $v$. The purpose is to achieve a $b(v)$ function. The following figure will show the first six $\text{LP}$ modes of an optical fiber. The results come directly from the equations (2.57) and (2.58)
As can be observed in Figure 2.5, there is no cutoff frequency for the fundamental mode and a higher \( v \) supports a higher variety of modes. There is a calculus that can approximate the number of existing modes using a multimode fiber, which is:

\[
N_{\text{modes}} = \frac{v^2}{2}
\]  

(2.59)

where, obviously, \( N_{\text{modes}} \) is the number of modes.

### 2.8 Fundamental Mode

In this section, the study is going to be focused on the fundamental mode, since it has been focusing on the single-mode fibers. It is easy to understand that SMF (single-mode fibers) only supports one mode and this mode is the fundamental one, i.e., HE_{11} or LP_{01} modes. The equation that corresponds to this fundamental mode is

\[
u J_1(w) K_0(w) = w J_0(u) K_1(w)
\]  

(2.60)

and it is a simplification of the equations (2.57) or (2.58) using \( p = 0 \). An interesting option in order to solve the equation (2.60) is the Rudolph-Neumann’s approximation. This is a numeric solution and can be obtained respecting to the following equation

\[
b(v) = \left(1.1428 - \frac{0.9960}{v}\right)^2
\]  

(2.61)

for the interval between \( 1.5 < v < 2.5 \) [6]. However, if the objective is to simulate other ranges of \( v \) the use of other approximation is required
\[ b = \exp \left( -\frac{4}{v^2} \left( \frac{1.123}{v} \right)^2 \right) \]  

(2.62)

and it is valid at the interval between \( 0 < v < 0.1 \) [6].

According to the equations (2.61) and (2.62), the following graph can be depicted:

![Graph](image)

**Figure 2.6 - Modal equation solution for fundamental mode using Rudolph-Neumann's approximation**

One interesting thing that can be observed is that the LP\(_{01}\) mode curve represented is very similar to the one plotted in Figure 2.5. Therefore, the approximation is valid, and it is accurate to within 0.2% for \( V \) in the range \( 1.5 - 2.5 \) [7].

According to the equation (2.60), a graph that will relates dielectric contrast with the \( b(v) \) function can be depicted:

![Graph](image)

**Figure 2.7 - Dielectric contrast's influence over optical fiber's fundamental mode**
Figure 2.7 shows that with the increase of the dielectric contrast parameter, the required normalized frequency, $\nu$, will increase too. This fact is sustained by the propagation that continues to be possible. As the dielectric contrast parameter becomes lower, the normalized cut off frequency becomes also lower.

Last but not the least, a relationship between core’s radius and dielectric contrast can be proven according to the following graph:

![Figure 2.8 - Influence of the optical fiber's core into dielectric contrast with respect to two different cut off wavelength](image)

Figure 2.8 was obtained with respect to the equation (2.38). As it can be noticed, the dielectric contrast varies indirectly with the radius core, which mean the higher the first one is, the lower the latter gets. It is important to understand this graphic because these curves are related to the second mode, i.e., the cutoff of the mode LP$_{11}$. For both curves if it is considered the region below the curves.

### 2.9 Conclusions

The importance of this chapter was to give many notions about optical fibers. As it was possible to visualize an optical fiber is composed by dielectric materials and has two different regions: the core and the cladding. It is indispensible for the light to propagate inside the optical fiber that total internal reflection occurs. The ray theory cannot explain the effects of the polarization of the field, which means, that to study the effects of dispersion it is necessary to use the modal theory. According to equation (2.39) it is possible to achieve the hybrid modes (solutions of that equation): HE and EH. Though, they are the exact solutions and the modes that really propagate inside the optical fiber, it is valid to use an approximation in order to obtain modes that are mathematically easier to solve, i.e., the LP modes. These modes are valid in a situation...
which the dielectric contrast is very low so the field component $E_z$ and $H_z$ is nearly zero. It was also shown that the cut-off frequency, \textit{i.e.}, the value that limit if it is used a monomodal fiber or a multimodal fiber being $\nu_2 = 2.4048$. It could be seen that the core radius also limits that value, \textit{i.e.}, and considering a monomodal fiber the core radius will be smaller than for the other case.
3. Analytical and numerical analysis in Linear Regime
An optical communication channel is no more than a channel used to transport the optical signal from the transmitter to the receiver without any distortion. However, this is the idealistic goal, since it is impossible to transmit signals without having some distortion. This distortion may appear due to a great variety of reasons but the two most relevant for these studies are the group velocity dispersion (GVD) and third order dispersion. The study of GVD is very important because it leads to pulse broadening which means that undesirable problems could appear, such as inter-symbolic interference (ISI). This issue can drastically influence the quality, speed and bit-rate of the communication. The main goal of this chapter is to prove analytically and numerically the appearance of pulse broadening. Other aspects to avoid the problem of GVD such as DCF (dispersion compensating fibers) are also presented in this chapter.

3.1 Determining propagation equation in linear regime

In order to understand what happens to a pulse during its path along the optical fiber, it is necessary to achieve the propagation equation for a single mode fiber (it is going to be considered this kind of fiber, in order to reduce the dispersive effects). Firstly, the electric field equation for a linear polarization is going to be considered, with respect to the coordinate $x$,

$$\textbf{E}(x, y, 0, t) = \hat{x} E(x, y, 0, t)$$

(3.1)

where

$$E(x, y, 0, t) = E_0 F(x, y) B(0, t)$$

(3.2)

$$B(0, t) = A(0, t) \exp(-i \omega t).$$

(3.3)

$F(x, y)$ is the field distribution of the fundamental fiber mode and $B(0, t)$ represent the amplitude of time variation at the same mode. The $B(0, t)$ term can be written accordingly to the equation (3.3) where $A(0, t)$ represents the pulse envelope at the fiber’s entrance, i.e., at $z = 0$, and $\omega$ represents the carrier frequency.

The first important step is to show the relationship between $B(0, t)$ and $A(0, t)$, using Fourier’s transform. In this dissertation, it was used the following Fourier’s transforms:

$$\tilde{X}(z, w) = \int_{-\infty}^{\infty} X(z, t) \exp(i \omega t) dt$$

(3.4)
According to the equations above, the electric field Fourier transform can be obtained as

\[
X(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(z,\omega) \exp(-i\omega t) dt
\]  

(3.5)

\[
\tilde{E}(x,y,0,\omega) = \int_{-\infty}^{+\infty} E_0 F(x,y) B(0,t) \exp(i\omega t) dt
\]

\[
= E_0 F(x,y) \int_{-\infty}^{+\infty} B(0,t) \exp(i\omega t) dt
\]

\[
= E_0 F(x,y) \int_{-\infty}^{+\infty} A(0,t) \exp(i\omega t) \exp(-i\omega_o t) dt
\]  

(3.6)

Considering the system as linear and time’s invariant, the use of some Fourier’s transform properties, namely the frequency shift property, is allowed. Therefore, the equation (3.6) can be rewritten as

\[
\tilde{E}(x,y,0,\omega) = E_0 F(x,y) \tilde{A}(0,\omega-\omega_o)
\]  

(3.7)

where it can be affirmed that

\[
\tilde{B}(0,\omega) = \tilde{A}(0,\omega-\omega_o).
\]  

(3.8)

The component \(\tilde{B}(z,\omega)\) is the spectral component of the pulse. As it will be demonstrated later, pulse broadening results from the frequency dependence on \(\beta\). As pulses where \(\Delta\omega \ll \omega_o\) are being referred to, it is useful to expand \(\beta\) [7] as

\[
\beta_p(\omega) = \beta_L(\omega) + \beta_{NL}(\omega) + i \frac{\alpha(\omega)}{2}
\]  

(3.9)

where \(\beta_L(\omega)\) and \(\beta_{NL}(\omega)\) are linear and nonlinear parameters of the propagation constant and \(\alpha(\omega)\) is the fiber loss parameter. The focus of this study will be on the linear parameter, so that \(\beta_{NL}(\omega) = 0\) (this term is null because this section is used to study the influence in linear regime) and \(\alpha(\omega) = 0\) (a perfect medium is being considered) so that equation (3.9) will modify to

\[
\beta_p(\omega) = \beta_L(\omega).
\]  

(3.10)

Considering equation (3.10) longitudinal variation will become
\[ \tilde{B}(z, \omega) = \tilde{B}(0, \omega) \exp(i \beta_L(\omega) z). \quad (3.11) \]

According to equation (3.11) and using Fourier’s inverse transform, the \( B(z, t) \) term can be performed as

\[
B(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{B}(z, \omega) \exp(-i \omega t) \, d\omega
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{B}(0, \omega) \exp\left(i \left( \beta_L(\omega) z - \omega t \right) \right) \, d\omega
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0, \omega - \omega_0) \exp\left(i \left( \beta_L(\omega) z - \omega t \right) \right) \, d\omega \quad (3.12)
\]

An easy way to simplify equation (3.12) is to consider a change in the integration variable to

\[
\Omega = \omega - \omega_0, \quad \omega = \Omega + \omega_0, \quad \frac{d\omega}{d\Omega} = 1. \quad (3.13)
\]

Substituting the equation (3.13) for the equation (3.12), it becomes

\[
B(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0, \Omega) \exp\left(i \left( \beta_L(\Omega + \omega_0) z - (\Omega + \omega_0)t \right) \right) \, d\Omega
= \frac{1}{2\pi} \exp(-i \omega_0 t) \int_{-\infty}^{\infty} \tilde{A}(0, \Omega) \exp\left(i \left( \beta_L(\Omega + \omega_0) z - \Omega t \right) \right) \, d\Omega \quad (3.14)
\]

An interesting method to solve equation (3.14) is using Taylor’s series for \( \beta(\omega_0 + \Omega) \) term so that

\[
\beta_L(\omega_0 + \Omega) = \beta_0 + \varphi(\Omega) \quad (3.15)
\]

where,

\[
\beta_0 = \beta(\omega_0)
\]

and

\[
\varphi(\Omega) = \sum_{m=1}^{\infty} \frac{\beta_m}{m!} \Omega^m \quad (3.16)
\]

where
\[ \beta_m = \frac{d^m \beta}{d \omega^m} \bigg|_{\omega=\omega_0} \quad (3.17) \]

Considering this term’s expansion and substituting the equation (3.15) for the equation (3.14), it results in

\[ B(z,t) = \frac{1}{2\pi} \exp \left( -i \omega_0 t \right) \int_{-\infty}^{+\infty} \tilde{A}(0,\Omega) \exp \left( i \left( \beta_0 + \varphi(\Omega) \right) z - \Omega t \right) d\Omega \]

\[ = \exp \left( i \left( \beta_0 - \omega_0 t \right) \right) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(0,\Omega) \exp \left( i \left( \varphi(\Omega) z - \Omega t \right) \right) d\Omega . \quad (3.18) \]

\[ = A(z,t) \exp \left( i \left( \beta_0 z - \omega_0 t \right) \right) \]

To conclude, using the same logic, more terms need to be calculated. The first thing is to look attentively to the equation (3.18) and conclude that \( A(z,t) \) term can be written as

\[ A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(0,\Omega) \exp \left( i \left( \varphi(\Omega) z - \Omega t \right) \right) d\Omega. \quad (3.19) \]

With respect to the equation (3.19) and the equation (3.17) a few terms are missing. The first term that needs to be determined is the \( \beta_1 \) and it is going to be calculated as

\[ \beta_1 = \frac{d \beta}{d \omega} \bigg|_{\omega=\omega_0} = \frac{1}{v_g} \quad (3.20) \]

where \( v_g \) is the group velocity. The \( \beta_2 \) coefficient is determined by the following equation

\[ \beta_2 = \frac{d^2 \beta}{d \omega^2} \bigg|_{\omega=\omega_0} = -\frac{1}{v_g^2} \frac{\partial v_g}{\partial \omega} \quad (3.21) \]

where \( \beta_2 \) is known as the second-order dispersion. There is another coefficient which is known as the third-order dispersion term and it is obtained by

\[ \beta_3 = \frac{\partial \beta_2}{\partial \omega} = \frac{d \beta}{d \omega^2} \bigg|_{\omega=\omega_0} \quad (3.22) \]
These terms are very important for this study since they are the reason for pulse broadening. It is important to notice that the calculus of $A(z,t)$ term with respect to $A(0,t)$ term needs to be performed. Defining the following equation as

$$A_m(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega^m \tilde{A}(0,\omega) \exp\left(i\phi(\Omega)z\right) \exp\left(-i\Omega t\right) d\Omega$$  \hspace{1cm} (3.23)$$

and according to equation (3.23), the general equation can be defined as

$$\frac{\partial A}{\partial z} = i \sum_{m=1}^{\infty} \frac{\beta_m}{m!} A_m(z,t).$$  \hspace{1cm} (3.24)$$

When introducing the optical fiber losses' parameter, the equation (3.24) results in

$$\frac{\partial A}{\partial z} = i \sum_{m=1}^{\infty} \frac{\beta_m}{m!} A_m(z,t) - \frac{\alpha}{2} A(z,t)$$  \hspace{1cm} (3.25)$$

This demonstration is not completed since $A_m(z,t)$ needs to be determined. Focusing on the equation (3.23) and deriving it in order to time these values of $m = 1, 2, 3$ and $4$, the following equations can be obtained

$$\frac{dA}{dt} = -i A_1(z,t)$$  \hspace{1cm} (3.26)$$

$$\frac{d^2A}{dt^2} = i A_2(z,t)$$  \hspace{1cm} (3.27)$$

$$\frac{d^3A}{dt^3} = -i A_3(z,t)$$  \hspace{1cm} (3.28)$$

$$\frac{d^4A}{dt^4} = i A_4(z,t).$$  \hspace{1cm} (3.29)$$

These four equations can be synthesized in the following equation

$$\frac{d^mA}{dt^m} = -i^{2-m} A_m(z,t).$$  \hspace{1cm} (3.30)$$

According to equation (3.30), and specifying the $A_m(z,t)$ term as

$$A_m(z,t) = -\frac{d^mA}{dt^m} t^{m-2}$$  \hspace{1cm} (3.31)$$

and using the equation (3.31) and substituting into equation (3.25) it becomes
\[
\begin{align*}
\frac{\partial A}{\partial z} &= i \sum_{m=1}^{\infty} \frac{\beta_m}{m!} \left( -\frac{d^m A}{dt^m} \right) t^{m-2} - \frac{\alpha}{2} A(z,t) \\
&= \sum_{m=1}^{\infty} \frac{\beta_m}{m!} \left( -\frac{d^m A}{dt^m} \right) t^{m-1} - \frac{\alpha}{2} A(z,t).
\end{align*}
\]

(3.32)

or, in a simpler way,

\[
\frac{\partial A}{\partial z} + \sum_{m=1}^{\infty} \frac{\beta_m}{m!} \frac{d^m A}{dt^m} t^{m-1} + \frac{\alpha}{2} A(z,t) = 0.
\]

(3.33)

In order to simplify this equation and ignoring the attenuation coefficient as well as the superior order terms, equation (3.33) will become

\[
\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{1}{6} \beta_3 \frac{\partial^3 A}{\partial t^3} = 0.
\]

(3.34)

This equation is the basic propagation equation that governs pulse evolution inside a single-mode fiber. So, it is possible to use some tricks such as the normalization of some variables, to obtain a simpler expression. The use of the following normalized variables implies a simpler way of writing equation (3.34), such as,

\[
L_D = \frac{\tau_0}{[\beta_2]}
\]

\[
\begin{align*}
\zeta &= \frac{z}{L_D} \\
\tau &= \frac{t - \beta_1 \zeta}{\tau_0}
\end{align*}
\]

(3.35)

where \(L_D\) is the dispersion length and \(\tau_0\) is the characteristic time of the pulse duration.

Using these new variables, it is necessary to determine \(\partial A/\partial z\) as

\[
\begin{align*}
\frac{\partial A}{\partial z} &= \frac{\partial A}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial A}{\partial \tau} \frac{\partial \tau}{\partial z} \\
\frac{\partial \zeta}{\partial z} &= -\frac{1}{L_D} \\
\frac{\partial \tau}{\partial z} &= \frac{\beta_1}{\tau_0} \\
\frac{\partial A}{\partial z} &= \frac{1}{L_D} \frac{\partial A}{\partial \zeta} - \frac{\beta_1}{\tau_0} \frac{\partial A}{\partial \tau}
\end{align*}
\]

(3.36)

and then, using the same procedure, \(\partial A/\partial t\) term can be obtained as
\[ \begin{align*}
\frac{\partial A}{\partial t} &= \frac{\partial A}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial A}{\partial \tau} \frac{\partial \tau}{\partial t} \\
\frac{\partial \zeta}{\partial t} &= 0 \\
\frac{\partial \tau}{\partial t} &= \frac{1}{\tau_0} \\
\frac{\partial A}{\partial t} &= \frac{1}{\tau_0} \frac{\partial A}{\partial \tau} \\
\end{align*} \tag{3.37} \]

With the introduction of these new variables, the equation (3.34) can be modified to

\[ \frac{1}{L_D} \frac{\partial A}{\partial \tilde{\zeta}} = \frac{\beta_1}{\tau_0} \frac{\partial A}{\partial \tau} + \frac{\beta_2}{\tau_0} \frac{\partial A}{\partial \tau} + i \frac{1}{2} \frac{\beta_2}{\tau_0^2} \frac{d^2 A}{d \tau^2} - \frac{1}{6} \frac{\beta_3}{\tau_0^3} \frac{d^3 A}{d \tau^3} = 0 \tag{3.38} \]

\[
\Leftrightarrow \frac{\partial A}{\partial \zeta} + i \frac{1}{2} L_D \frac{\beta_2}{\tau_0^2} \frac{d^2 A}{d \tau^2} - \frac{1}{6} L_D \frac{\beta_3}{\tau_0^3} \frac{d^3 A}{d \tau^3} = 0
\]

and using the following simplification as

\[ \frac{\beta_2 L_D}{\tau_0^2} = \text{sgn}(\beta_2) \tag{3.39} \]

the equation (3.38) will become

\[ \frac{\partial A}{\partial \zeta} + i \frac{1}{2} \text{sgn}(\beta_2) \frac{d^2 A}{d \tau^2} - \kappa \frac{d^3 A}{d \tau^3} = 0 \tag{3.40} \]

where the parameter \( \kappa \) can be written according to the next equation

\[ \kappa = \frac{1}{6 \tau_0 |\beta_2|} \tag{3.41} \]

which is known as the higher order dispersion coefficient.

Equation (3.40) and (3.34) are called the basic propagation equations. The next part is going to introduce a parameter called chirp. It is necessary to explain why this effect occurs. According to [8] if a Gaussian pulse with the following expression is considered

\[ U(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right) \tag{3.42} \]

Where \( T_0 \) is the half-width. According to the Fourier transforms (equations (3.4) and (3.5)), as well as the property of the integral used in appendix B it is possible to achieve the following equation
\[ U(z,T) = \frac{T_0}{(T_0^2 - i \beta_z z)^{1/2}} \exp \left[ -\frac{T^2}{2(T_0^2 - i \beta_z z)} \right]. \]  

This equation shows that the pulse maintains its shape though its width will increase according to

\[ T_i(z) = T_0 \left[ 1 + \left( \frac{z}{L_0} \right)^2 \right]^{1/2} \]  

This equation shows how GVD broadens the pulse. Now, if equation (3.42) and equation (3.43) are compared, it is possible to visualize some changes, and, these lead to a pulse that is chirped at the end. Writing the following equation

\[ U(z,T) = \left| U(z,T) \right| \exp[i \phi(z,T)] \]  

It is possible to achieve the “new” phase as

\[ \phi(z,T) = -\frac{\text{sgn} (\beta_z) (z/L_0)}{1 + (z/L_0)^2} \frac{T^2}{2T_0^2} + \frac{1}{2} \tan^{-1} \left( \frac{z}{L_0} \right) \]  

and conclude that it is dependent on the normal or anomalous region. This dependence on \( z \) and \( T \) implies that the instantaneous frequency will differ from the central frequency of \( \omega_0 \). This difference is only the first derivative with respect to time, such as

\[ \delta\omega(T) = -\frac{\partial \phi}{\partial T} = \frac{\text{sgn} (\beta_z) (z/L_0)}{1 + (z/L_0)^2} \frac{T}{T_0^2} \]  

and this difference is called Chirp. Though, this effect is called Chirp, in fact, this demonstration occurs in all pulses, i.e., during its path a ray will suffer a deviation in its phase which will cause the broad of the pulse. This effect is referred to as a problem about the fiber. The Chirp that is going to be introduced in the following section is not a fiber problem, but it occurs at the spectral source. It will cause the same deviation at the phase of the pulse, but it is characteristic of the light that it is sent to the fiber.

**3.2 RMS broadening of a Gaussian pulse with Chirp**

The main purpose of this subsection is to obtain an expression that relates the pulse width at any point of the fiber and the pulse width at the entrance of the fiber, \( \sigma/\sigma_0 \).
(expression that is deduced in the appendix B). This equation, as it was previously said, is very important to estimate how the pulse broads along the optical fiber. In this section, a Gaussian pulse was used with the following equation

\[ A(z,t) = A_0 \exp \left( -\frac{1 + i C}{4} \left( \frac{t}{\sigma_0} \right)^2 \right) \]  

(3.48)

where \( A_0 \) stands for the amplitude of the pulse, \( C \) stands for the Chirp's parameter and \( \sigma_0 \) stands for the pulse width at the entrance of the fiber. Using the expression deduced in the appendix B, it was possible to obtain

\[ \frac{\sigma^2}{\sigma_0^2} = \left( 1 + \frac{C \beta_z L}{2 \sigma_0^2} \right)^2 + \left( \frac{\beta_z L}{2 \sigma_0} \right)^2 + \left( 1 + C^2 \right)^2 \left( \frac{\beta_z L}{4 \sqrt{2} \sigma_0^3} \right)^2 \]  

(3.49)

The expression (3.49) shows the ratio between the pulse width at some point of the fiber and the pulse width at the fiber’s entrance. It is important to notice that this ratio depends on the value of the Chirp, the third order dispersion and the second order dispersion. By simplicity, and for the beginning, only the influence of the second order dispersion parameter is going to be considered. Changing some variables as

\[ \zeta = \frac{L}{L_D} \]  

(3.50)

\[ L_D = \frac{2 \sigma_0^2}{|\beta_z|} \]  

(3.51)

\[ \beta_z = |\beta_z| \text{sgn} ( \beta_z) = -|\beta_z| \]  

(3.52)

and substituting these new variables in equation(3.49) the outcome is

\[ \frac{\sigma}{\sigma_0} = \sqrt{(1 + C \zeta)^2 + \zeta^2} . \]  

(3.53)

In order to simulate equation(3.53), three different values of Chirp were used.
The graph shown above allows a better understanding of how the parameter of Chirp influences pulse broadening and how this can be noticed on the curves with non-null Chirp (red and blue curves). According to [7], an unchirped pulse, \( C = 0 \), its width may broaden by a factor of \( \sqrt{2} \) at \( z = L_d \). On the other hand, the chirped pulse may broaden or compress depending on the result of the product between \( \beta_2 C \) being positive or negative. When \( \beta_2 C > 0 \), the pulse will broaden monolithically with a faster rate than an unchirped pulse (red curve). However, when \( \beta_2 C < 0 \), an interesting phenomenon happens: at the beginning the pulse will compress but then the GVD influence will become stronger than the parameter of Chirp which will lead to the broad of the pulse. When comparing this rate of increasing with the unchirped pulse it can be stated that this rate is higher too. The interesting point is that the pulse was compressed at the beginning. The following explanation will justify that behavior. Looking attentively to equation (3.21) and considering that the region considered is anomalous, equation (3.21) needs to be negative, such as

\[
\beta_2 = -\frac{1}{\nu_0^2} \frac{d^2 v_0}{d \omega} \leq 0 \tag{3.54}
\]

which implies that

\[
\frac{d v_0}{d \omega} > 0 \tag{3.55}
\]

so that,
The Figure 3.3.2 implies that higher frequencies travel in a faster speed than lower frequencies. So, due to the DVG, the pulse will broaden according to the figure below.

As mentioned previously, a Gaussian pulse is being considered, with the following equation

\[
\exp \left[ -\frac{1+iC}{2} \left( \frac{t}{\tau_0} \right)^2 \right] = \exp \left( -\frac{t^2}{2\tau_0^2} \right) \exp \left( \frac{iCt^2}{2\tau_0^2} \exp(\phi) \right)
\]

(3.56)

A pulse is said to be chirped if its carrier frequency changes with time, and this change in frequency is related with the phase derivative, so that

\[
\delta \omega(t) = -\frac{d\phi}{dt}
\]

(3.57)

Considering equations (3.56) and (3.57), it can be obtained.
\[ \phi(t) = -\frac{Ct^2}{2\tau_0^2} = -\omega(t). \] (3.58)

According to equation (3.58) it is possible to plot the following graphic:

![Figure 3.4](image)

**Figure 3.4 - Change in the pulse due to positive Chirp**

...and it is possible to conclude that lower frequencies will travel faster than higher frequencies.

This controversial effect will generate a compression at the beginning of the pulse, but then, and according to equation (3.49), DVG parameter will regain its influence and the Chirp’s effect will tend to disappear.

Another thing that can be determined from Figure 3.3.1 is that the pulse broadening is also influenced by the increasing of the fiber’s length. This conclusion was also obtained in equation (3.49).

### 3.3 Pulse propagation

This present section will be the starting point of the simulation analysis. Two different pulses such as the “sech” shape pulse and the super Gaussian pulse are going to be used. All these simulations were obtained recurring to some Matlab scripts and were applied in a linear regime and in a single-mode fiber. It is important to refer that this numerical simulation was based on the Fast Fourier Transform (FFT), about which more information can be found in appendix C.

The propagation equation used in the simulation is

\[ i \frac{\partial u}{\partial \xi} - \frac{1}{2} \text{sgn}(\beta_z) \frac{\partial^2 u}{\partial \tau^2} = 0 \] (3.59)
where \( u \) represents the pulse, and \( \zeta \) and \( \tau \) represent the normalization of some variables according to

\[
\begin{align*}
\zeta &= \frac{z}{L_D} = \frac{\left| \beta_2 \right|}{\tau_0^2}, \\
\tau &= \frac{t - \beta_2 z}{\tau_0}
\end{align*}
\]

Considering the anomalous dispersion zone, i.e., when \( \text{sgn}(\beta_2) = -1 \)

the equation (3.59) results on

\[
\frac{\partial u}{\partial \zeta} = -\frac{1}{2} \frac{\partial^2 u}{\partial \tau^2}
\]

(3.60)

and this equation will be the base for the following simulations.

### 3.4 The “sech” pulse

The first pulse used in this simulation will be a “sech” pulse, which is represented by

\[
u(0, \tau) = \text{sech}(\tau)
\]

(3.61)

where \( \tau \) is the normalized time. The range of this parameter in this simulation was between -15ps and 15ps and the parameter varies between them. With these considerations, it is possible to achieve the following graphs

![Figure 3.5 – The absolute value of the pulse. The relationship between the input pulse and the output pulse for the “sech” pulse](image)
The next two figures show the variation of the pulse along the communication channel.

![Figure 3.3.6 - Pulse broadening along the optical fiber](image)

Despite not having visible attenuation, both figures show that the absolute value of the pulse decreases throughout the fiber, effect that is explained by the dispersion, in this specific case, the DVG. An interesting fact that is observed is that the energy of the pulse stays constant despite the broadening of the pulse. This fact is explained because the energy is constant along the simulation, which means that it is being considered a perfect medium. It is important to understand that when distance=0 and when distance=5 Figure 3.5 is obtained.

### 3.5 Super Gaussian with Chirp’s parameter

In this subsection it is important to remember the Figure 3.3.1. As previously mentioned, Chirp’s parameter influences pulse broadening and normally increases that effect. The idea of this subsection is to obtain several figures of a super Gaussian pulse with Chirp and to analyze what is happening in the different situations. The expression used for these simulations was

\[
u(0,t) = \exp\left(-\frac{1+iC}{2} \tau^{2m}\right)
\]

(3.62)

The next figures will show the pulse at the entrance of the fiber as well as at the fiber’s exit. For this situation the \(m\) parameter will be equal to 3 and, once more, the Chirp’s parameter will change according to the next interval \([-2,0,2]\]. First, considering \(C = -2\) which can provide the following figures.
Figure 3.7 - The absolute value of the pulse. The relationship between the input pulse and the output pulse for the super Gaussian pulse when C=-2.

Figure 3.8 - Pulse broadening along the optical fiber considering a super Gaussian pulse when C=-2.

The following figures correspond to a similar pulse but without the Chirp's effect, which means that $C = 0$, and it becomes
Figure 3.9 - The absolute value of the pulse. The relationship between the input pulse and the output pulse for the super Gaussian pulse if \( C=0 \)

Figure 3.3.10 – Pulse broadening along the optical fiber considering a super Gaussian pulse if \( C=0 \)

Last but not least, it will be presented how a pulse with Chirp’s parameter with \( C = 2 \) behaves
It is important to draw some conclusions related to these figures. The first issue is that the temporal dispersion is presented in figures between Figure 3.5 and Figure 3.3.12. This conclusion is easily taken if the pulse at the entrance and the pulse at the exit are different and the fiber considered has no losses. Besides this, it is important to notice that Chirped pulses suffer the consequences of having the Chirp’s influence, which will cause a big broad of the pulses, especially when the value of the Chirp is negative. As previously mentioned, Chirp’s parameter will cause a broadening in the pulse spectrum, which means that it will have more frequency components. This last conclusion can be proven by the following figures.
There are two more conclusions that can be achieved regarding these figures. One, unchirped pulses generate more fluctuations comparing with Chirped pulses and two; positive values of Chirp will generate a higher broadening in the pulse spectrum, when considering the anomalous region.

### 3.6 Higher order dispersion

Though it has not been defined in this dissertation, one of the most interesting results that is obtained in the optical fibers world

\[ BL |D| \Delta < 1 \]  \hspace{1cm} (3.63)

As it is possible to see, if D=0, an indefinitely higher BL product could be obtained, and that is the intended goal. However, when neglecting the parameter \( \beta_2 \), optical pulses still experience broadening caused by the higher order effects. According to the next equation
\[ D = \frac{d}{d\lambda} \left( \frac{1}{v_g} \right) = -\frac{2\pi c}{\lambda^2} \beta_3 \]  

(3.64)

where \( D \) is the dispersion parameter. In order to achieve the high superior order, it is necessary to derive it one more time to understand what happens to the optical pulse (this expression is more detailed in [6]). For this case another methodology is used, such as the dispersion slope, \( S \), where

\[ S = \frac{\partial D}{\partial \lambda} \]  

(3.65)

and according to equation (3.65) the following dispersion equation can be obtained with respect to \( \beta_3 \) parameter

\[ S_D = \left( \frac{2\pi c^2}{\lambda_D^2} \right) \beta_3 (\lambda_D) \].  

(3.66)

Actually, this value plays an important role in this industry, mainly in the WDM systems. As \( S > 0 \) for most of the fibers, different channels will have slightly different GVD values, so it is very difficult to compensate the dispersion simultaneously for all of the channels. One way of solving this problem is to obtain fibers with reduced \( S \) (reduced-slope fibers) or negative (reverse dispersion fibers). Equation (3.66) is used for two specific cases:

- When the carrier is in the neighboring of the considered wavelength, \( \lambda_D \)
- When the signal has an high spectral width, or, a low temporal width

3.6.1 \( \lambda_0 = \lambda_D \)

The first study will be based on equation (3.66) and for \( \lambda_0 = \lambda_D \) considering a super Gaussian pulse. This demonstration is presented with more detail in appendix C. It is important to notice that the parameters used in this simulation are presented in the following table.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_3 )</td>
<td>( 1 \text{ps}^3/\text{km}^3 )</td>
</tr>
<tr>
<td>( \tau_0 )</td>
<td>( 20 \text{ps} )</td>
</tr>
</tbody>
</table>
Table 3.1 - Used values for the following simulations

\[
\begin{array}{c|c}
L' & 80000km \\
L & 4000km \\
\end{array}
\]

Figure 3.3.14 - Pulse's entrance and exit considering the third order dispersion, when \( \lambda = \lambda_0 \), and \( C = -2 \)

Figure 3.3.15 - Pulse's entrance and exit considering the third order dispersion, when \( \lambda = \lambda_D \), and \( C = 0 \)
These three figures show that without $\beta_2$, there is no time dispersion on the pulse and its patent that the $\beta_1$ parameter creates an oscillating behavior to the pulse. This oscillating behavior is less accentuated in an unchirped pulse than in a chirped one. The other conclusion is that when Chirp’s parameter (Figure 3.3.14 and Figure 3.3.16) is symmetric the pulse will behave the same way for the two cases.

3.6.2 $\ D = L_D$

The other study which is important to make is related to the dispersion length, so the same dispersion length is going to be considered, due to $\beta_1$ and to $\beta_3$. This propagation method is going to be explained in the appendix C. In this subsection, the super Gaussian pulse is going to be used.

The following table details the necessary information used for the following simulations.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$</td>
<td>50 ps</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$-20 \ ps^2/km$</td>
</tr>
<tr>
<td>$L$</td>
<td>100 km</td>
</tr>
<tr>
<td>$L_1$</td>
<td>5 km</td>
</tr>
</tbody>
</table>
Table 3.2 - Values of the following simulations

<table>
<thead>
<tr>
<th>$\beta_3$</th>
<th>$-1 \times 10^{-36} \text{ s/km}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$-0.1667$</td>
</tr>
</tbody>
</table>

The next figure will show the input and the output considering a super Gaussian pulse if $C = -2$.

Figure 3.3.17 - Input and output of a super Gaussian pulse if Chirp = -2

The next figure will show the input and the output considering a super Gaussian pulse when $C = 0$.

Figure 3.3.18 - Input and output of a super Gaussian pulse when Chirp = 0

And for last, the figure below will show the input and the output considering a super gaussian pulse when $C = 2$. 
Figure 3.3.19 - Input and output of a super Gaussian pulse when Chirp = 2

The major conclusion that is possible to achieve with these figures is that the influence of the higher order effect, $\beta_3$, provokes an asymmetrical oscillatory behavior. The reason for this asymmetry is that, as it is possible to verify in the appendix C, this parameter is influenced by an odd behavior. Moreover, when the influence of a non-null Chirp parameter is being considered, the higher order effect is more accentuated than when considering a null value of Chirp.

3.7 DCF

After studying the effects of the pulse broadening, it is important to analyze a technique that can reduce the GVD effect. The answer for this is the dispersion compensating fiber (DCF). The main idea of this technique is to have two different segments of fiber: one that is normal, which means that the pulse will broaden due to GVD and one that will compensate this effect. The next figure will show the main concept of the DCF.

Figure 3.20 - Dispersion Compensating Fibers
This technique appeared because GVD must be compensated for long-haul systems along the transmission line in a periodic fashion [7]. It is important to mention that DCF will totally compensate the GVD effect only if the average optical power is low enough not to consider the nonlinear effects, so the results that are going to be presented are only related to the Pulse propagation section.

To understand the physical concept behind this technique, two different fibers have to be compared: one where the pulse propagates normally and one with the DCF inside. According to equation (3.40) and neglecting the term $\beta_1$ (this term can be neglected when $|\beta_1| > 0.1 \text{ ps}^2/\text{km}$ [7]) it can be obtained

$$A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(0,\omega) \exp \left( \frac{i}{2} \beta_2 z \omega^2 - i \omega t \right) d\omega$$

(3.67)

and considering two different fibers, the sum of equation (3.67) can be obtained, so that

$$A(L,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(0,\omega) \exp \left[ \frac{i}{2} \omega^2 (\beta_1 L_1 + \beta_2 L_2) - i \omega t \right] d\omega$$

(3.68)

where $L = L_1 + L_2$ and $\beta_{i1}$ parameter refers to the GVD parameter in both segments ($i = 1, 2$). According to equation (3.68), to regain the same pulse at the exit of the fiber, the term $\omega^2$ will have to vanish, so that

$$\beta_1 L_1 + \beta_2 L_2 = 0$$

(3.69)

where $D_i$ with ($i = 1, 2$) is referred to the dispersion in that fiber's segment. According to equation (3.69), DCF must have normal GVD at $1.55 \mu m$ ($D_2 < 0$) because $D_1 > 0$ for standard communications [7]. So the second fiber's segment is obtained by

$$L_2 = -(D_1/D_2) L_1$$

(3.70)

Or simply,

$$\beta_{11} L_1 + \beta_{22} L_2 = 0$$

(3.71)

where $\beta_{2j}$ is the second-order dispersion parameter for the fiber length $L_j$.

Despite this solution, there is a big concern about this technique, which is the high loss that DCF exhibits. These losses are caused by the increasing of bending losses
\( \alpha = 0.4 - 0.6 \text{dB/km} \). So, there is a parameter that measures the efficiency of DCF which is the figure of merit, \( M \), and it is calculated by the ratio between \( |D|/\alpha \), where \( \alpha \) represents the losses. A typical value of this parameter is \( M > 250 \text{ ps/nm-dB} \).

The following simulation will be based on the Gaussian pulse, the same used in the previous simulations. The parameters used in this simulation are presented in the next table.

\[
\begin{align*}
\tau_0 &= 50 \text{ps} \\
\beta_{21} &= -20 \text{ ps}^2/\text{km} \\
L_{21} &= 250 \text{km} \\
L_{D1} &= 125 \text{km} \\
\beta_{22} &= 500 \text{ ps}^2/\text{km} \\
L_{22} &= 10 \text{km} \\
L_{D2} &= 5 \text{km}
\end{align*}
\]

Table 3.3.3 - Parameters used for simulation

According to the parameters used, it is possible to show the results for the use of the DCF. The input and the output of the DCF for \( C = -2 \) is shown in the next figure.

![Figure 3.3.21 - Input and output of a pulse with C=-2 using DCF](image)
The next two figures will show the evolution of the compensation, *i.e.*, using DCF, according to time and distance of the fiber.

![Figure 3.3.22 – The use of DCF in an input with C=2](image)

The same pictures are going to be obtained for different values of $C$. The following group of figures will be referred to an input correspondent to $C = 0$.

![Figure 3.3.23 - Input and output of a pulse when C=0 using DCF](image)

Again, the next two figures will show the evolution of the compensation, *i.e.*, using DCF, according to time and distance of the fiber.
Figure 3.3.24 – The use of DCF in an input with C=0

The same procedure is going to be repeated for a pulse correspondent to $C = 2$.

Figure 3.3.25 - Input and output of a pulse if C=2 using DCF

Finally, the following pictures are going to be obtained for a positive Chirp and will show the evolution of DCF compensating the dispersion in time and distance.
Figure 3.3.26 - The use of DCF in an input if C=2

With these amounts of figures, it is easy to understand that the initial pulse is completely recovered with the use of DCF (with the considerations that were used). One important point about this simulation is that the attenuation is negligible, which means that the amplitude of the input and the output of the pulses will be the same, because the energy of the pulse is conserved. Though, it was not done in this dissertation it is possible to compensate the $\beta_i$ parameter by using the DCF technique. The difference to the equation (3.71) is that it is necessary to consider another equation to “kill” the $\beta_i$ effect such as

$$\beta_{31} L_1 + \beta_{32} L_2 = 0.$$  \hspace{1cm} (3.72)

To compensate the two effects it is necessary to respect condition (3.70) as well as adding this new condition

$$\beta_{32} = \left(\frac{\beta_{22}}{\beta_{21}}\right) \beta_{31}.$$  \hspace{1cm} (3.73)

### 3.8 Conclusions

After the realization of this chapter, it is important to make a reflection about it. Firstly, single-mode fibers will have less dispersive effects although they are easy to enter in a region of nonlinearities. This issue is important because the studies done in this chapter are not so valid in the “real” world. In fact, entering in the nonlinear regime is intimately related with the dimensions of this fiber. Single-mode fibers are smaller than the multimode fibers. So, for this chapter makes sense it is important to refer that it is being consider low optical power in order to avoid the nonlinear regime. Other interesting aspect is that the Chirp effect increases the broad of the pulse. It is important to refer, that the Chirp used in this dissertation is a consequence of the
optical source and not a direct problem of the medium. This issue is present because of the use of direct modulation. A way of avoiding this effect is to use an external modulation. It is performed by a light modulator, and so, the light is not modulated by the current injected by the laser. This issue inserts fewer Chirp, which, as it was seen is an undesirable effect. Other important aspect focused in this chapter was the influence of the higher-order effects. As it could be seen, it is important when considering two situations: the first one is when the optical system is being operated at the second-window wavelength. The second window has a great advantage when comparing with the third and the fourth window which is the group-velocity dispersion is nearly zero. In this concrete situation the effect of higher order effect must be considered and as it could be seen has an odd behavior and increases when considering the Chirp effect, the second situation is when the length of the group-velocity dispersion and the higher-order effects are the same. In this situation, both effects contribute to the pulse broadening but the higher order-effects will prevail.
4. Analytical and numerical analysis of the nonlinear regime
The purpose of this chapter is to study the fiber’s nonlinearities. For that, it is very important to understand the concept of soliton. The word soliton appeared due to John Scott Russell’s experiments. In 1834, while trying to determine the most efficient design for canal boats, he discovered a phenomenon that he described as the wave of translation. This was the first soliton ever seen.

![Figure 4.1 - The channel where Scott Russell found the wave transmission (later called soliton)](image)

It is important to notice that solitons are present in many different areas of Science such as fluid dynamics and optics. The optical soliton is a result of the interplay between dispersive and nonlinear effects [8] or, in other words, the result of the balance between GVD and SPM (self-phase modulation) [7]. As previously observed in the Figure 3.3.1, there is a region where a chirped pulse will contract due to $\beta_2$ and $C$ parameters have opposite signs, i.e., $\beta_2 C < 0$. On the other hand, the nonlinear phenomenon of SPM will occur if the Chirp’s parameter is $C > 0$. So, if the anomalous region is considered the condition $\beta_2 C < 0$ is readily satisfied. In this situation, it is easy to understand that under certain conditions GVD and SPM will cooperate in a way that SPM-induced chirp and GVD-induced broadening factor will cancel each other, resulting in an optical pulse propagating undistorted in the shape of a soliton.

### 4.1 Nonlinear Kerr effect

Despite the process to determine the nonlinear regime being similar to what was done in Chapter 3, there are some differences. It is important to remember that the modulus of the pulse’s enveloping at the beginning of the fiber is exactly the same at any point
of the fiber in linear regime. To understand better what was previously said the following diagram should be paid attention to

\[
\begin{align*}
A(z = 0, t) & \rightarrow A(z, t) \\
\mathcal{F} \downarrow & \uparrow \mathcal{F}^{-1} \\
\tilde{A}(0, \Omega) & \rightarrow \tilde{A}(z, \Omega)
\end{align*}
\]

With this diagram and remembering the exponential properties it is possible to conclude that

\[|\tilde{A}(z, \Omega)| = |\tilde{A}(0, \Omega)| \quad (4.1)\]

which makes the work in linear regime easier. The difficulty is with the nonlinear regime, where the same methodology cannot be used due to the appearance of some new frequencies. With this, equation (4.1) is no longer useful for this regime, and the reason why this huge difference is extremely important to understand the nonlinear Kerr effect. It was discovered by John Kerr in 1875, and represents the change in the refractive index of a material in response to an applied electric field. Moreover, it has two different components: the AC and the DC components. As it is being studied in optical fibers, the one that is more important is the AC component (or optical Kerr effect). This effect is one of the most important to obtain solitons and is named as the nonlinearities in the self-phase modulation (SPM). Adding this effect with the GVD it is possible to obtain solitons which have an important characteristic that is to mitigate the dispersion effect. The Kerr effect generates a nonlinear phase and it is given by

\[\phi_{NL}(t) = \gamma P_{in}(t) L \quad (4.2)\]

where \(\gamma\) is the nonlinear parameter given by

\[\gamma = \frac{n_z'}{\lambda w_0^2} \quad (4.3)\]

\(L\) is effective length and it is given by

\[L = \frac{1}{\alpha} \left[ 1 - \exp(-\alpha L) \right] \quad (4.4)\]

and \(P_{in}\) is the input power along the optical fiber and it is given by

\[P(z, t) = P_{in}(t) \exp(-\alpha z) \quad (4.5)\]

The \(\alpha\) parameter is the attenuation coefficient and \(L\) is the length of the fiber.
A pulse will tend to suffer a frequency shift over its carrier due to the SPM such as

\[ \delta \omega(t) = -\frac{d\phi_{NL}}{dt} = -\gamma L \frac{dP_{in}(t)}{dt}. \]  

(4.6)

In the front of the pulse, i.e.,

\[ \frac{dP_{in}}{dt} > 0 \Rightarrow \delta \omega(t) < 0 \]  

(4.7)

and it is possible to conclude that it leads to a deviation to the red zone. In an analogous way it is possible to do the same process for the tail of the pulse. So,

\[ \frac{dP_{in}}{dt} < 0 \Rightarrow \delta \omega(t) > 0 \]  

(4.8)

In which the tail of the pulse will suffer a deviation to the blue zone.

Considering the explanation given in Chapter 3 about the contractions of a pulse, when the product of \( \beta_2 C < 0 \), it is possible to obtain

\[ \beta_2 = -\frac{1}{v_g^2(a_0)} \left. \frac{\partial v_g}{\partial \omega} \right|_{\omega = a_0} \]  

(4.9)

and consider the anomalous dispersion region, where \( \beta_2 < 0 \) so

\[ \left. \frac{\partial v_g}{\partial \omega} \right|_{\omega = a_0} > 0. \]  

(4.10)

Accordingly, the higher frequencies will travel quicker than lower frequencies as it is possible to visualize in the next figure.

Figure 4.2 - Consequence of the DVG in a pulse propagation
As it was said previously AMF do exactly the opposite of DVG, and that is why solitons (bright solitons) appear. If the normal dispersion region is being considered, \( i.e. \), when \( \beta_2 > 0 \), it is possible to obtain the other kind of solitons, named as dark solitons.

### 4.2 Nonlinear Schrödinger Equation

The Nonlinear Schrödinger Equation (NLS) is the mathematical description of the behavior of the solitons with respect to GVD and SPM. The equation is

\[
\frac{\partial A}{\partial z} + \frac{i \beta_2}{2} \frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6} \frac{\partial^3 A}{\partial t^3} = i \gamma |A|^2 A - \frac{\alpha}{2} A
\]

(4.11)

and, as mentioned before, fiber losses are included in the \( \alpha \) parameter and \( \beta_2 \) and \( \beta_3 \) are respectively, the second and the third order dispersion (TOD). In the equation (4.11) the nonlinear parameter is also present with its correspondent parameter being \( \gamma \). This parameter is defined as \( \gamma = 2 \pi n_2 / (\lambda A_{\text{eff}}) \), where \( n_2 \) is the nonlinear-index coefficient and \( A_{\text{eff}} \) is the effective core area. With intuit of simplifying this equation analysis, it is important to set \( \alpha = 0 \) and \( \beta_3 = 0 \). As it was done in the linear regime, it is common normalizing the equation (4.11) with respect to these new variables

\[
\tau = \frac{t}{T_0}, \quad \xi = \frac{z}{L_0}, \quad U = \frac{A}{\sqrt{P_0}}.
\]

The term \( T_0 \) is a measure of the pulse width, \( P_0 \) is the peak power of the pulse, and \( L_D = T_0^2 / |\beta_2| \) is the dispersion length. With some algebraic transformations equation (4.11) can be transformed into

\[
i \frac{\partial U}{\partial \xi} - \frac{\text{sgn}(\beta_2)}{2} \frac{\partial^2 U}{\partial \tau^2} + N^2 |U|^2 U = 0
\]

(4.12)

where, the function \( \text{sgn}(\beta_2) \) could be positive or negative, depending if DVG or anomalous DVG are being considered respectively. The parameter \( N \) presented in equation (4.12) is

\[
N^2 = \frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|}
\]

(4.13)

The equation (4.12) is a well known equation in the area of solitons because it can be solved using a technique called the inverse scattering technique. Historically, this
technique was first used in 1971 by Zakharov and Shabat. Although this is the right method to use, in the simulations that will appear during this chapter another technique called SSFM was used. The NLS equation has solutions for normal GVD or anomalous GVD, but the results are completely different: in the first case result is bright solitons and in the second case is dark solitons.

4.3 Dark solitons

As mentioned previously, there are many solutions to the equation (4.12) which implies that not only dark and bright solitons appear but also bistable solitons. These different solutions take into consideration the different cases of dispersive and nonlinear properties. In this subsection, dark solitons are going to be focused on and to obtain them, normal-GVD need to be considered, which implies \( \text{sgn} (\beta) = 1 \). These solitons were discovered in 1973 and the search in this area has been growing since then. The name of this kind of solitons appeared due to their intensity profile, i.e., they exhibits a dip in a uniform background. This conclusion will become clear with the following figures. The equation that resumes the behavior of dark solitons is

\[
\frac{i}{\xi} \frac{\partial u}{\partial \xi} - \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0. \tag{4.14}
\]

This equation comes directly from equation (4.12) by making \( \text{sgn} (\beta) = 1 \) and \( u = NU \). It can be solved by using the inverse scattering method and imposing a boundary condition; \( |u(\xi, \tau)| \) must tend toward a nonzero constant for large values of \( |\tau| \). Another way of solving this equation is to assume a solution of the form \( u(\xi, \tau) = V(\tau) \exp[i\phi(\xi, \tau)] \) and then solving the ordinary differential equations by \( V \) and \( \phi \). The general solution can be written as

\[
u(\xi, \tau) = \eta B \tanh(\xi - i\sqrt{1-B^2}) \exp(i\eta^2 \xi) \]

where \( \zeta = \eta B (\tau - \tau_s - \eta B \sqrt{1-B^2}) \) [8]. The parameters \( \eta \) and \( \tau_s \) represent the amplitude and the dip location respectively. The parameter \( B \) governs the depth of the dip. When this value is equal to one the dark solitons appear. The next figure will show clearly the distinction between a grey soliton and a dark soliton (as mentioned before it is a special case of the gray soliton).
4.4 Split Step Fourier Method

As in the linear regime, the nonlinear regime faces an identical problem, i.e., the propagation equation does not offer a numerical solution so some approximations need to be done. Accordingly, it is possible to write this equation as

$$\frac{\partial A}{\partial z} = \left( \hat{D} + \hat{N} \right) A \quad (4.16)$$

where, $\hat{D}$ is the differential operator that accounts for dispersion and losses within a linear medium and $\hat{N}$ is a nonlinear operator that governs fiber nonlinearities on pulse propagation.

$$\hat{D} = -i\beta_2 \frac{\partial^2}{\partial T^2} + \frac{\beta_3}{6} \frac{\partial^3}{\partial T^3} - \frac{\alpha}{2} \quad (4.17)$$

$$\hat{N} = i\gamma |A|^2 \quad (4.18)$$

Although, the phenomenon of dispersion and nonlinearity act together along the optical fiber, in this method it is considered that they act independently assuming that the step $h$ is very small. In other words, it is divided into two sub-lengths of optical fiber, where, in the first one, only the nonlinearities are present and, then, in the second, only the dispersion effect is considered.

Mathematically, it is possible to write this equation

$$A(z+h,T) = \exp(h\hat{D})\exp(h\hat{N})A(z,T) \quad (4.19)$$

It is possible
\[
\exp\left(h\hat{D}\right)B(z,T) = F_T^{-1} \exp\left[h\hat{D}(-i\omega)\right]F_T B(z,T)
\]  

where, \(F_T\) denotes the Fourier transform operation, \(\hat{D}(-i\omega)\) is obtained from equation (4.17) and \(\omega\) is the frequency in the Fourier domain. As \(\hat{D}(i\omega)\) is just a number in frequency domain, the evaluation of (4.20) is straightforward. The use of FFT algorithm makes numerical evaluation of equation (4.20) relatively fast [8]. It is important to estimate the accuracy of this method so that it is possible to write the exact solution of equation (4.19), such as

\[
A(z+h,t) = \exp\left(h\left(\hat{D} + \hat{N}\right)\right)A(z,T)
\]  

when considering \(\hat{N}\) independent of \(z\). If the Baker-Hausdorff formula is used for two non-commuting operators \(\hat{a}\) and \(\hat{b}\), then

\[
\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a} + \hat{b} + \frac{1}{2}[\hat{a},\hat{b}]\right)
\]

where \([\hat{a},\hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}\). As it is possible to see in equations (4.20) and (4.22), this method ignores completely the non-commuting nature order of the operators \(\hat{D}\) and \(\hat{N}\). It is possible to find the resultant error of this approximation when considering \(\hat{a} = h\hat{D}\) and \(\hat{b} = h\hat{N}\), the error will be \(\frac{1}{2}h^2 [\hat{D},\hat{N}]\) [8]. This method, used in this dissertation, is called asymmetric. A representation of this method is present in the next figure.

![Figure 4.4 - The SSFM along an optical fiber](image_url)
For those reasons, this method is an approximation given the fact that the non-commuting order of the operators is not considered, as well as the treatment that is made to the integrals by using the FFT (explained in the appendix C). In order to understand the numerical algorithm the next scheme will clearly demonstrate how it works:

![Figure 4.5 - SSFM algorithm scheme](image)

Figure 4.5 shows all the procedures to treat computationally equation (4.16). This algorithm was used for all the simulations that appear in this chapter. As it was mentioned previously, this method is approximated due to two motives:

Although this method was good enough to deal with all pulses that were tested, (given their small value of h), there is another method that is currently used and is more efficient, which is the symmetric one. The difference to the one used is that it considers two zones of dispersion, and between them nonlinear effect. With this method it is possible to obtain an error proportional to $h^3$[8].

### 4.5 Bright solitons

As previously mentioned, in order to obtain this kind of solitons it is necessary to operate in the anomalous GVD, i.e., setting $\text{sgn}(\beta_2) = -1$. With this consideration and using normalized amplitude, i.e., $u = NU$, equation (4.12) will become

$$
\frac{i}{\partial \xi} \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0
$$

for the linear regime. This can be solved using the scattering inverse method. No details of this method are presented in this dissertation; however, there are several books that explain it. The general solution for equation (4.23) is given by

$$
u(\xi, \tau) = \eta \text{sech} \left[ \eta (\tau - \tau_s + \delta \xi) \exp \left[ i \left( \eta^2 - \delta^2 \right) \xi / 2 - i \delta t + i \phi_s \right] \right]
$$

(4.24)
The way to find equation (4.24) is well demonstrated in [8]. It is important to understand that the characterization of the solitons needs four parameters such as $\eta$, representing amplitude, $\delta$, representing frequency, $\tau$, representing position and $\phi$, representing phase. In order to simplify equation (4.24) some considerations need to be done. The factor $\phi$ is an absolute phase constant which has no physical relevance. The factor $\tau$ refers to the position of the soliton peak which can be 0 if the peak is considered at its origin. Looking attentively to the equation (4.24), it is obvious that the parameter $\delta$ represents the frequency shift of the soliton from the carrier frequency $\omega_c$. According to the carrier part, i.e., $\exp(-i \omega_c t)$, the new frequency has to be written like $\omega_c' = \omega_c + \delta/T_0$. Due to this the soliton suffers a speed alteration. Using $\tau = (t - \beta z)/T_0$ and substituting into equation (4.24) what it is going to be obtained is:

$$u(\xi, \tau) = \eta \text{sech} \left[ \eta \left( t - \beta z \right)/T_0 \right]$$

(4.25)

where $\beta' = \beta + \delta \beta'/T_0$. As it has been said the variation in group velocity is a direct consequence of the dispersion.

If the right carrier frequency is chosen, parameter $\delta$ can be left out of discussion, which means it is now possible to rewrite the equation (4.24) as:

$$u(\xi, \tau) = \eta \text{sech} (\eta \tau) \exp \left( i \eta^2 \xi/2 \right).$$

(4.26)

There are interesting conclusions that can be taken from equation (4.26). The first one is that the parameter $\eta$ not only changes the soliton amplitude but also its width. This fact is demonstrated by seeing this happens according to the relation $T_c/\eta$. The soliton amplitude do not need to be equals to the unit.

4.5.1 Fundamental soliton

The usual form for the fundamental soliton is obtained by doing $u(0, 0)$ which implies that $\eta = 1$. With these considerations equation (4.26), becomes

$$u(\xi, \tau) = \text{sech} (\tau) \exp \left( i \xi/2 \right)$$

(4.27)
The beauty of the use of solitons in optical fibers is that, if a hyperbolic-secant pulse is used with $N = 1$ inside an ideal lossless fiber the pulse will propagate undistorted. The figure below will support this information.

![Figure 4.6 – Pulse intensity of the fundamental soliton](image)

As it can been seen, the soliton propagates undistorted, which is very important for the communication systems. This is a result of the perfect compensation between fiber nonlinearity and the GVD.

### 4.5.2 $N^{th}$-order solitons

This subsection names important issues when inputs with $N \neq 1$ are being considered. These are known as the $N^{th}$-order solitons and two special cases about them are going to be developed: the second-order soliton and the third order soliton. The first figure illustrates is the second-order soliton.

![Figure 4.7 - Pulse intensity of the second-order soliton](image)
The next figure represents the third-order soliton. All the important inferences reflections about these two figures to the fundamental soliton one figure are going to be made afterwards.

![Figure 4.8 - Pulse intensity of the third-order soliton](image)

The major conclusion about Figure 4.7 and Figure 4.8 is that there is a periodicity in both. In fact, it can be calculated according to the following expression:

$$z_0 = \frac{\pi}{2} L_n = \frac{\pi}{2} \frac{T_0^2}{|\beta_1|}$$  \hspace{1cm} (4.28)

where $z_0$ is the soliton period and plays a very important role in the theory of the solitons. Working with these solitons is very difficult due to their periodicity, *i.e.*, the initial information that a soliton carries could be different if the length of the fiber was not really well adapted to its periodicity. It is understood that the amplitude of the soliton increases parallel to $N$ (this conclusion is well demonstrated both in the figures and in the equation(4.26)). According to that, it is not valid to say that fundamental solitons only have value 1 of their amplitude.

**4.5.3 “Sech” pulse with different N values**

The other effect can be verified, when considering these nonlinearities with the $N$ value being non-natural numbers.
As it can be seen the new pulse behaves like a $N^{th}$ order solitons. Moreover, there is a periodicity of this soliton for larger values of $\xi$. For lower values of $\xi$, this pulse will be modulated to become a soliton.

### 4.5.4 The Gaussian pulse

The first important issue is that solitons are very stable against perturbations. In this subsection, it is going to be demonstrated that despite being necessary to have a specific shape and a certain peak power, according to equation (4.26), it is possible to obtain a fundamental soliton, through other different pulses. In fact, the shape of a “sech” pulse is not common in nature so this subsection is more representative of what happens in fact. In addition, a Gaussian pulse is going to be used, according to the following equation:

$$ u(0, \tau) = \exp\left(-\frac{\tau^2}{2}\right). \quad (4.29) $$

When simulating the equation (4.29) and considering $N=1$, the following figures are obtained:
As seen here, the pulse adjusts its shape and width in an attempt to become a fundamental soliton and it attains a “sech” profile, though it cannot obtain amplitude of 1 [7]. The amplitude that this pulse is going to stabilize in is 0.7959.

Another interesting approach is to continue using this pulse, but adding the effect of the third-order dispersion. This effect is interesting when, as in linear regime, the pulse wavelength nearly coincides with the zero dispersion wavelength; the same happens for ultrashort pulses (regarding width $T_o < 1 \text{ps}$), because the term that comes from Taylor expansion is no longer small enough for it not to be considered. For this effect the propagation equation used for the next simulations is:

$$i \frac{\partial U}{\partial z} = \beta_2 \frac{\partial^2 U}{\partial T^2} + \beta_3 \frac{\partial^3 U}{\partial T^3}$$

(4.30)
According to equation (4.30), the nonlinear effects are not taken into account. It is now time to solve equation (4.30) by using the same methodology used in to obtain equation (4.24). Using the Fourier transform as

\[
U(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(z, \omega) \exp(-i\omega t) d\omega
\]  

(4.31)

It is possible to obtain

\[
i \frac{\partial \tilde{U}}{\partial z} = -\frac{1}{2} \beta_3 \omega^2 \tilde{U} - \frac{1}{6} \beta_5 \omega^4 \tilde{U}
\]  

(4.32)

where,

\[
\tilde{U}(z, \omega) = \tilde{U}(0, \omega) \exp \left( iz \left( \frac{\beta_3}{2} \omega^2 + \frac{\beta_5}{6} \omega^4 \right) \right)
\]  

(4.33)

so,

\[
U(z, t) = \frac{1}{2\pi} \int \tilde{U}(0, \omega) \exp \left( i \left( \frac{\beta_3}{2} \omega^2 z + \frac{\beta_5}{6} \omega^4 z - i\omega t \right) \right) d\omega.
\]  

(4.34)

In order to simulate this solution, and considering equation (4.17), the following expresses the use in the fragment of the dispersion:

\[
\hat{D} = \frac{i \text{sgn}(\beta_3)}{2} \omega^2 + i\kappa \omega^4
\]  

(4.35)

where

\[
\kappa = \frac{\beta_3}{6\beta_5 |\tau_0|}
\]  

(4.36)

as in Chapter 3. Accordingly the following figures were obtained:
According to Figure 4.12, Figure 4.13 and Figure 4.14 it is possible to achieve interesting conclusions. The first one is that the high order effect has a major influence on ultrashort pulses. It is possible to observe that this influence decreases contrarily to
the width of the pulse. This causes the lower values of \( \eta \) the pulse to broaden, if a perfect medium is considered, so this effect constrains the appearance of solitons.

### 4.6 Soliton Interaction

Despite the good behavior as well as the robustness, solitons have what might be a problem: the interaction between them. This is due to the combined optical field not being a solution for the NLS equation. When thinking about solitons, it is easy to understand that each of them occupies only a fraction of the bit slot (carrying some useful information) and it would be desirable that they could be packed as tight as possible (to send more and more information in these time slots). As the soliton interaction effect is present, it is important to understand how they interact and what these limitations are. The present subsection will show two solitons interacting between them propagating inside the optical fiber.

The first important conclusion is that the parameter that will “control” this effect is the time interval between two consecutive solitons, \( T_b \). This interval influences drastically the bit-rate of the communication as \( B = \frac{1}{T_b} \). So, the written mathematical expression that corresponds to the total field of two solitons interacting between them is:

\[
u_j (\xi, \tau) = \eta_j \text{sech}[\eta_j (\tau - q_j)] \exp (i \phi_j - i \delta_j \tau)
\]

with \( j = 1, 2 \) and noticing that the total field is \( u = u_1 + u_2 \). As mentioned previously, what is important to understand is the sum between the two independent fields of the solitons. If the equation (4.37) is used in equation (4.23), what is obtained the following result

\[
i \frac{\partial u_i}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u_i}{\partial \tau^2} + |u_i|^2 u_i = -2 |u_1|^2 u_1 - u_1^2 u_2^*
\]

Although it is not done in this dissertation the study of the four parameters that influence soliton behavior, it can be concluded that the relative separation between two solitons depends only on their relative phase [8].

According to [9] and after some algebra manipulation and introducing some new variables, the following equation will be obtained:

\[
u_o (\tau) = \text{sech} (\tau - q_o) + r \text{sech} [r (\tau + q_o)] \exp (i \Theta).
\]
This shows how the two solitons interact in the same channel. The following figures will show a simulation of two consecutive bits with value of 1 and spaced by $T_0 = 2q_0\tau_0$.

\[ q_0 = 5.51 \]

**Figure 4.15 - Interaction between two consecutive solitons when using $q_0=5.51$**

In Figure 4.15 it is possible to visualize that both solitons in the same channel do not interact between them, so there will be no interference and the information will arrive clearly to the receiver. It is also understood that the parameter $q_0$ influences the separation between the two solitons and it is called the initial separation. The following figure will show the influence of the same parameter but for a lower value:

\[ q_0 = 3.22 \]

**Figure 4.16 - Interaction between two consecutive solitons when using $q_0=3.22$**

This effect is well demonstrated in [8], but the major idea is that the relative separation of the two solitons is a function of their relative phase. As it is possible to see in Figure 4.16 after some distance the solitons will collide. In the case of two in-phase solitons (as in this case, i.e., $\phi_0 = 0$), the relative separation $q$ changes with propagation periodically as:

\[
q(\xi) = q_0 + \ln \left( \cosh^2 \left( 2\xi e^{-\theta_0} \sin \phi_0 \right) + \cos^2 \left( 2\xi e^{-\theta_0} \cos \phi_0 \right) \right)
\]

(4.40)
An interesting way of seeing this equation is to visualize it in the 2d-plot and changing the value of the initial phase. So:

![Figure 4.17 – $q_0 = 3.22$. Relative spacing between two solitons as a function of fiber length for different values of initial phase.](image)

As it is possible to see in Figure 4.17 the two solitons have different behaviors by changing only their relative phase. One interesting conclusion is that the relative phase has some influence for lower values of $\phi$, as it can be seen for $\phi = 0^\circ$ and $\phi = 20^\circ$. For higher values of $\phi$, the relative spacing does not have influence, in such cases the major influence is related to other parameters such as $q_0$ and others. There is a new parameter that has some interest which is the collision length. It appears due to the periodic nature of $q(\xi)$ where the two solitons have a periodic behavior, and it is given by:

$$L_{col} = \frac{\pi}{2} L_D \exp(q_0) = z_0 \exp(q_0)$$

(4.41)

According to [8] this expression is accurate for $q_0 > 3$. As the two cases where solitons are in phase were already studied, the following figures will show the influence of two important parameters: $\theta$ and $r$. For the following results, the same equation(4.39) was used.
Figure 4.18 – Interaction between two solitons when $\theta = \pi/4$ and $r = 1$

Figure 4.19 – Interaction between two solitons when $\theta = \pi/2$ and $r = 1$

Figure 4.20 – Interaction between two solitons when $\theta = 0$ and $r = 1.1$
At this point, it is important to study these images individually. Looking attentively to Figure 4.18 it is possible to see that the two solitons at the beginning of the input are attracted, but when they continue propagating inside the fiber, they start to separate themselves. In Figure 4.19 it is possible to see that both solitons are repelling themselves in a stronger way. This difference is related to the initial phase difference. In Figure 4.20 another other important parameter: the relative amplitude. As it was considered in the situation of the two in-phase solitons the major difference is that they oscillate periodically although they never collide.

As it is easy to understand, this effect of collapsing consecutive solitons is not desirable. The most direct way to avoid this effect is to increase their initial separation. In [8] it is suggested that $q_0 = 8$ is large enough to avoid this situation. As it is possible to see from Figure 4.18 to Figure 4.20, the relative phase $\theta$ and the relative amplitude $r$ also influence the interaction between two solitons. Even so, for higher values of $q_0$, they are not so important. Other aspects influence the interaction of solitons such as higher-order effects, bandwidth-limited amplification and timing jitter, but they are not going to be developed in this dissertation.

### 4.7 Conclusions

The purpose of this chapter was to explain the origin of the solitons, the differences between the linear regime and the nonlinear regime and the differences between the dark and the bright solitons. Firstly, it is important to mention that to enter into account the fiber’s nonlinearities it is indispensible that the optical power is high enough to consider that effect. The first aspect that is important to mention is that it will origin a nonlinear phase. This issue is responsible for the appearance of new frequencies in the spectrum of the pulse. This is in fact a great change when comparing with the linear regime. This issue is also the reason why the FFT cannot be used to simulate this regime. This new phase will generate a similar behavior has the Chirp effect when considering the anomalous region (the contraction of the pulse). This effect is also known as SPM. This effect, in addition with de GVD will origin the solitons. Solitons are very interesting in communication because as it could be seen they preserve their shape, and are very stable against perturbations. Even when it was introduced the effect of the higher order effects it was possible to visualize that they resist to this effect. Just a little caution in the initial width of the pulse is enough to avoid that effect. Though, it was studied the bright solitons, when it is being considered the normal region, it will lead to the appearance of dark solitons. As it was depicted in this
dissertation, these kind of solitons have a decreasing profile, i.e., the amplitude of the soliton vanishes along the propagation which makes its detection very difficult. That is one of the reasons why this solitons are not used. It was also studied the bright solitons when considering the higher-order effects. It was seen that for ultrashort pulses this effect will regain some importance and could be a reason to destroy the generation of solitons. But it was seen that with some caution, and controlling the initial width of the pulse, this effect will practically not influence the soliton. It was also demonstrated that the periodicity of the $N^{th}$ order soliton could be harmful for the propagation, because the system need to be very coherent with the peak of the pulse. It is not so easy to implement a system using this kind of solitons. The major important parameter is the initial separation between solitons. This cannot be high enough because this will reduce the bit rate, and cannot be low enough because solitons could collide. Though it was not considered in this work, losses are also present in any system. A way of avoiding losses in this regime is to use DDF (dispersion decreasing fibers).
5. - Conclusions
This chapter will enunciate the principal aspects and conclusions achieved with this dissertation.

5.1 Principal Conclusions

The main idea of the second chapter was to give a brief analysis of the optical fibers components as well as to treat it recurring to two different techniques: the ray theory (optical geometry) and the modal theory. It can be concluded that the ray theory can only be used in such conditions as the ratio between the core radius and the wavelength is very high. So, to deepen the study of optical fibers it is indispensable to study modal theory. According to this approach, it is possible to classify optical fibers into single mode and multimode fibers. The number of modes is related to the core radius as it is present in equation (2.59). There are different kinds of modes but the most important are the HE modes and LP modes. These are almost the same, but LP modes are an approximation and they only appear when a low dielectric contrast is considered. This dissertation stated that the limit value between a single mode fiber and a multimode fiber is 2.408.

The third chapter starts with the development of the propagation equation for linear regime. With this equation, it is possible to obtain some interesting results but it also easy to understand that for higher values of GVD and high order effects the pulse will experience a temporal dispersion, which causes ISI. Another interesting conclusion is that the presence of Chirp can cause pulse spreading or pulse contraction when the product of GVD with chirp is positive or negative, respectively. Many simulations recurring to the FFT (fast Fourier transform) were presented such as a “sech” pulse and a Super Gaussian pulse with chirp. For both pulses it was easy to visualize the time dispersion, as well as a reduction of the amplitude of the pulse. Moreover, with the Gaussian pulse it could be seen the effect of the Chirp: it provokes an increasing of the spreading of the pulse, although, in one specific case the effect of GVD is attenuated by Chirp’s effect. These conclusions were in agreement with the analytical results. A study of high order effects and its problems followed. The major visual difference of this parameter is the odd behavior due to the parameter 3rd power. Analyzing the problems, it was now the time had come to show some numerical solutions, and that corresponded to the dispersion compensating fiber. This method worked perfectly when compensating the GVD parameter.

In the fourth chapter studied the nonlinear regime in optical fibers and its model based on the Schrödinger equation. The main difference about this regime is that due to the non linear Kerr effect some new frequencies are generated. This effect is known as
Self Phase Modulation (SPM). Due to DVG and SPM and their contradictory properties, solitons can appear. In order to simulate solitons propagation a mathematical method called SSFM (Split Step Fourier Method) was used. Firstly, it was simulated how the fundamental soliton preserves the initial form and width. Afterwards were the second order soliton and the third order soliton’s turn. In both cases something interesting thing happened, which was their periodicity. Although they achieve a periodic propagation, the conservation of the energy is present because a perfect medium without any perturbations was used. Then, a Gaussian pulse was simulated; also it was observed how the initial Gaussian pulse tends to origin a soliton. Indeed, this is one of the most interesting facts in the nonlinear regime. Then, the effect of the high order effects in this nonlinear regime was studied. The most important conclusion in this part is that this effect is more accentuated when considering lower pulses widths. In fact, when this is increase the effect will tend to vanish according to the GVD effect. It was also simulated, a pulse but considering the high order effects. In this specific case, it was possible to see that with the increment of the width of the pulse this effect is attenuated and it will “try” to become a soliton. When the width of the pulse is smaller, this effect is very intense which makes the “transformation” of the pulse into a soliton much more difficult. The last study done was the interference between two solitons. It is possible to avoid the interaction if their initial separation is higher than a certain value (in this study was used $q_0 = 5.51$. For lower values of $q_0$ this interaction will appear. Other value that permits controlling the interaction of solitons is their initial phase. For certain values of low initial phase, the solitons will attract themselves. The big problem of soliton interaction is when they collide. This situation could lead to a change of the information they carry to the extent of altering the initial message.

5.2 Future work

In order to continue this dissertation, other aspects can be studied:

In nonlinear regime there is an amount of other effects such as the Raman effects and the Brillouin dispersion that leads to difficult propagation of the solitons. A study could be also developed in order to understand what happens to a soliton when using the attenuation coefficient and what techniques can compensate this effect. A mathematical instrument that could also be developed is the inverse scattering method. This method is very complex but it is the right way of solving the NLS.

In the linear regime, it is possible to use DCF to compensate both the GVD and the high order effects.
References


A. Appendix – Geometric interpretation of the guided propagation in a fiber

Equations (2.13) (2.14) and (2.15) needs a brief explanation about their resolution. This explanation will center in the theory of the rays. For that it is important to look attentively to the next figure

![Figure A.5.1 - Core and Cladding scheme in [6].](image)

Considering the region of the core, i.e., \( r \leq a \), the propagation constant is \( k_1 = n_1 k_0 \).

The same knowledge can be applied to the cladding, so the propagation constant to this region is \( k_2 = n_2 k_0 \). It is important that both regions have the same propagation constant with the intuit of verification core and cladding frontier. It is possible to define to traversal propagation constant in the two considered regions.

Visualizing Figure A.5.1 it is possible to write

\[
\beta = n_1 k_0 \sin \theta_1 = n_2 k_0 \sin \theta_2 \quad \text{(A.1)}
\]

known as Snell's law. It is possible to write, using the same methodology this next equations

\[
h = n_1 k_0 \cos \theta_1, \quad \text{(A.2)}
\]

\[
q = n_2 k_0 \sin \theta_2. \quad \text{(A.3)}
\]
Using the last two equations and looking to the figure, it is possible to write the following equations by using Pythagoras theorem

\[ h^2 + \beta^2 = n_1^2 k_0^2 \] (A.4)

\[ q^2 + \beta^2 = n_2^2 k_0^2 \] (A.5)

With these equations it is possible to obtain equations (2.23) and (2.14).

The cutoff corresponds to the start of the total internal reflection. As previously said that is the phenomenon that it is important to happen in optical fibers for the light being confined in the core. Looking to the figure, it is easy to understand that the starting point of total internal reflection is that when \( \theta_i = \frac{\pi}{2} \). According to equations (A.1) and (A.3), in the cutoff \( \beta = n_2 k_0 \) and \( q = 0 \). A guided mode will appear just when total internal reflection occurs

\[ q = i \alpha \] (A.6)

And considering that \( \alpha > 0 \). Solving equation (A.3) with the result of equation (A.6), it is going to lead to

\[ \theta_2 = \frac{\pi}{2} - i \xi. \] (A.7)

Knowing some trigonometric properties such as

\[ \sin(ix) = i \sinh x \] (A.8)

and

\[ \cos(ix) = \cosh(x) \] (A.9)

It is possible to rewrite equations (A.1) and (A.3) as

\[ \beta = n_2 k_0 \cosh \xi \] (A.10)

and

\[ q = i n_2 k_0 \sinh \xi. \] (A.11)

From equations (A.6) and (A.11) it is possible to write

\[ \alpha = n_2 k_0 \sinh \xi. \] (A.12)

Looking attentively to equations (A.10) and (A.12), it is possible to obtain
\[ \beta^2 - \alpha^2 = n_2^2 k_0^2 \]  
(A.13)

which confirms equation (2.25).

it is important to notice that a superficial guided wave only appears when total internal reflection occurs, so that,

\[ \theta_i \geq \theta_c = \sin^{-1} \left( \frac{n_2}{n_1} \right) \]  
(A.14)

according to equation (A.1). At the cutoff, where \( \beta = n_1 k_0 \sin \theta_c = n_2 k_0 \). When the opposite case occurs, i.e., \( \theta_i < \theta_c \), refraction occurs and the ray will pass through the cladding taking with it some amount of energy. Despite guided superficial modes form a discrete spectrum, radiation modes form the radiation continuous spectrum.
B. Appendix – General Formula for Pulse Broadening

The idea of doing this annex is that in the section where it was derived it was assumed that it was used a Gaussian shape pulses and includes dispersive properties only up to the third order.

This annex will show that with the use of root-mean-square (RMS), it could be possible to provide a measure that is ideal for any kind of pulses.

The starting point of this demonstration will be the definition of RMS where the width of the pulse is calculated accordingly to

$$
\sigma^2 = \left[ \langle t^2 \rangle - \langle t \rangle^2 \right]
$$

(B.1)

where $\langle t \rangle$ is the first order moment and $\langle t^2 \rangle$ is the second order moment. There is a general formula for the calculus of the moments, which is

$$
\langle t^m \rangle = \frac{\int_{-\infty}^{+\infty} t^m |A(z,t)|^2 dt}{\int_{-\infty}^{+\infty} |A(z,t)|^2 dt}
$$

(B.2)

The calculus of the moments

For this demonstration, the nonlinear effects have to be negligible. The reason for this simplification lays on the fact that the pulse spectrum does not change in a linear dispersive regime. The purpose of this section is to demonstrate these following equations

$$
\langle t \rangle = -i \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(z,\Omega) \tilde{A}_{\Omega}(z,\Omega) \, d\Omega
$$

(B.3)

and

$$
\langle t^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\tilde{A}_{\Omega}(z,\Omega)|^2 \, d\Omega.
$$

(B.4)

where, $\tilde{A}(z,\Omega)$ is the Fourier transform of $A(z,t)$ and the subscript $\Omega$ is used for a partial derivative with respect to $\Omega$.

Bearing in mind this purpose, the following simplification is going to be used
\[ \int_{-\infty}^{\infty} |A(z,t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{A}(z,\Omega)|^2 \, d\Omega = 1 \quad (B.5) \]

which, it is possible to obtain a simpler equation with respect to the equation (B.2), so that

\[ \langle t^m \rangle = \int_{-\infty}^{\infty} t^m |A(z,t)|^2 \, dt . \quad (B.6) \]

By applying the Fourier’s transforms to \( A(z,t) \) it is obtained

\[ \tilde{A}(z,\Omega) = \int_{-\infty}^{\infty} A(z,t) \exp(i\Omega t) \, dt \quad (B.7) \]

and

\[ A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z,\Omega) \exp(-i\Omega t) \, d\Omega . \quad (B.8) \]

According to these expressions it is now possible to calculate the moment \( \langle t \rangle \). Using the equation (B.6) and considering \( m = 1 \), it comes

\[ \langle t \rangle = \int_{-\infty}^{\infty} t |A(z,t)|^2 \, dt \quad (B.9) \]

and using the following complex property

\[ |A(z,t)|^2 = A(z,t) A^*(z,t) \]

equation (B.9) can be rewritten as

\[ \langle t \rangle = \int_{-\infty}^{\infty} t A(z,t) A^*(z,t) \, dt . \quad (B.10) \]

Substituting the equation (B.8) into the equation (B.10), it comes

\[ \langle t \rangle = \int_{-\infty}^{\infty} t A(z,t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z,\Omega) \exp(-i\Omega t) \, d\Omega \right] \, dt \quad (B.11) \]

\[ = \int_{-\infty}^{\infty} t A(z,t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}^*(z,\Omega) \exp(i\Omega t) \, d\Omega \right] \, dt \]

Switching the integral’s order, it will be accomplished

\[ \langle t \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}^*(z,\Omega) \left( \int_{-\infty}^{\infty} A(z,t) \exp(i\Omega t) \, dt \right) \, d\Omega \quad (B.12) \]
Considering that system is linear and time is invariant, the use of some Fourier properties is allowed. This is useful because the following property will help us solving equation (B.12)

\[ t x(t) \leftrightarrow i \frac{d X(i \omega)}{d \omega} \]  

so, equation (B.12) will become

\[
\langle t \rangle = -i \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}^* (z, \Omega) \frac{d \tilde{A}(z, \Omega)}{d \Omega} d\Omega.
\]  

(B.14)

considering

\[
A_{\Omega} (z, \Omega) = \frac{d \tilde{A}(z, \Omega)}{d \Omega}
\]  

(B.15)

and it was possible to determine equation (B.3). After obtaining the first order moment, it is going to be demonstrated the second order moment. The term \( \langle t \rangle^2 \) is calculated by doing the following algebraic manipulation

\[
\langle t^2 \rangle = \int_{-\infty}^{\infty} \left| A(z,t) \right|^2 dt
= \int_{-\infty}^{\infty} \left[ t A(z,t) \right]^2 dt
= \int_{-\infty}^{\infty} t A(z,t) \left( \int_{-\infty}^{\infty} i \frac{d \tilde{A}^* (z, \Omega)}{d \Omega} \exp(i \Omega t) d\Omega \right)^2 dt
= \frac{1}{2\pi} i \int_{-\infty}^{\infty} \tilde{A}^* (z, \Omega) \left( \int_{-\infty}^{\infty} A(z,t) \exp(i \Omega t) dt \right) \frac{d \tilde{A}(z, \Omega)}{d \Omega} d\Omega
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}^* (z, \Omega) \frac{d \tilde{A}(z, \Omega)}{f} d\Omega
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{A}(z, \Omega)|^2 d\Omega
\]  

(B.16)

With the previous demonstration it was possible to obtain the second order moment, and obtaining the equation (B.4) as it was our desire.
Pulse broadening according to the Chirp’s effect and the group delay

Even though nonlinear effects are not considered in this demonstration, the dispersive effects are included on all orders. In the current subsection the main goal is to achieve an expression that relates the pulse broadening expression with the Chirp’s effect and the group delay.

Defining the pulse at the entrance of the fiber

\[ \tilde{A}(z, \Omega) = \tilde{A}(0, \Omega) \exp\left( i (\beta - \beta_0) z \right) \]  

(B.17)

and considering that

\[ \tilde{A}(0, \Omega) = S(\Omega) \exp\left( i \theta(\Omega) \right) \]  

(B.18)

where parameter \( \theta(\Omega) \) introduces Chirp’s effect at the start of the pulse. Accepting these considerations, it is now time to insert both the parameter \( \theta(\Omega) \) and the group delay \( \tau_g \) in the expression of the pulse broadening. According to the group delay’s definition, it comes

\[ \tau_g(\Omega) = \int_0^L \frac{\partial \beta(z, \Omega)}{\partial \Omega} dz \]  

(B.19)

where the parameter \( \beta \) does not diversify during the propagation, because it is a characteristic of the medium as it was previously seen. Therefore, the group delay is

\[ \tau_g = \frac{d \beta}{d \Omega} L = \beta_1 L = \frac{L}{v_g} \]  

(B.20)

where, \( L \) is the length of the fiber and \( v_g \) the group’s velocity. For this demonstration and for simplicity the derivative of \( \theta(\omega) \)

\[ \theta_\Omega = \frac{d \theta}{d \Omega} \]  

(B.21)

will be used.

Initiating the process, the function \( |A_\Omega(\xi, \Omega)| \) needs to contemplate the new terms that are being considered, \( S(\Omega) \) and \( \theta(\Omega) \), so that, and using the equation (B.15)
\[ \tilde{A}_\omega (z, \Omega) = \frac{\partial \tilde{A}(z, \Omega)}{\partial \Omega} \]

\[ = \tilde{A}_\omega (0, \Omega) \exp (i (\beta - \beta_0) z) \]

\[ + \tilde{A}(0, \Omega) \frac{d}{d\Omega} \exp (i (\beta - \beta_0) z) \]

\[ = \tilde{A}_\omega (0, \Omega) \exp (i (\beta - \beta_0) z) \]

\[ + \tilde{A}(0, \Omega) \frac{d\beta}{d\Omega} i z \exp (i (\beta - \beta_0) z) \]

(B.22)

where, the term \( \tilde{A}_\omega (0, \Omega) \) is calculated by the following way

\[ \tilde{A}_\omega (0, \Omega) = \frac{\partial \tilde{A}(0, \Omega)}{\partial \Omega} \]

\[ = \frac{\partial S(\Omega)}{\partial \Omega} \exp (i \theta(\Omega)) + i S(\Omega) \frac{\partial \theta(\Omega)}{\partial \Omega} \exp (i \theta(\Omega)) \]

(B.23)

\[ = S_\omega \exp (i \theta(\Omega)) + i S(\Omega) \theta_\omega \exp (i \theta(\Omega)) \]

Using some equations, mainly the equations (B.14) and (B.16) term \( \langle \langle r \rangle \rangle \) and term \( \langle r^2 \rangle \), can be written in terms of \( S_\omega \) and \( \theta_\omega \), which results on
\[
\langle t \rangle = -i \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z, \Omega) A_{\alpha}(z, \Omega) \, d\Omega \\
= -i \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0, \Omega) \exp\left(-i(\beta - \beta_0)z\right) \left[\tilde{A}_{\alpha}(0, \Omega) \exp\left(i(\beta - \beta_0)z\right) \right] d\Omega \\
+ \tilde{A}(0, \Omega) \frac{d\beta}{d\Omega} iz \exp\left(i(\beta - \beta_0)z\right) d\Omega \\
= -i \frac{1}{2\pi} \int_{-\infty}^{\infty} S^*(\Omega) \exp\left(-\theta(\Omega)\right) \left[S_{\alpha}(\Omega) \exp\left(i\theta(\Omega)\right)\right] d\Omega \\
+ iS(\Omega) \frac{d\beta}{d\Omega} iz \exp\left(i\theta(\Omega)\right) d\Omega \\
= -i \frac{1}{2\pi} \int_{-\infty}^{\infty} S^*(\Omega) S_{\alpha}(\Omega) d\Omega - i \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\Omega) \exp\left(i\theta(\Omega)\right) d\Omega \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\Omega) \frac{d\beta}{d\Omega} iz d\Omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\Omega) \exp\left(i\theta(\Omega)\right) d\Omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\Omega) \frac{d\beta}{d\Omega} iz d\Omega
\]

and
\[ \langle i^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \tilde{A}(z,\Omega) \right|^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \tilde{A}(0,\Omega) + \tilde{A}(0,\Omega) \frac{\partial \beta}{\partial \Omega} \right|^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| S(\Omega) \exp(i\theta(\Omega)) + iS(\Omega) \theta \exp(i\theta(\Omega)) \right|^2 d\Omega \]

\[ + S(\Omega) \exp(i\theta(\Omega)) \left| \frac{\partial \beta}{\partial \Omega} \right|^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( S(\Omega) + S(\Omega) \left( i\theta + \frac{\partial \beta}{\partial \Omega} \right) \right)^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \sqrt{S^2(\Omega) + S^2(\Omega) \left( \theta + \frac{\partial \beta}{\partial \Omega} \right)^2} \right)^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^2 d\Omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^2(\Omega) \left( \theta + \frac{\partial \beta}{\partial \Omega} \right)^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^2 d\Omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^2(\Omega) \theta^2 d\Omega \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^2(\Omega) 2\theta \frac{\partial \beta}{\partial \Omega} \left( \frac{\beta}{\tau_s} \right) d\Omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^2(\Omega) \left( \frac{\beta}{\tau_s} \right)^2 d\Omega \]  

(B.25)

Introducing the following definition

\[ \langle f \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\Omega) \left| S(\Omega) \right|^2 d\Omega \]  

(B.26)

Equations (B.24) and (B.25) can be simplified, respectively, as

\[ \langle i \rangle = \langle \theta \rangle + \langle \tau_s \rangle \]  

(B.27)

and

\[ \langle i^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| S(\Omega) \right|^2 d\Omega + \langle \theta^2 \rangle + \langle \tau_s^2 \rangle + 2 \langle \theta \tau_s \rangle \]  

(B.28)

Then, replacing the achieved equations (B.27) and (B.28) in equation (B.1), the broadening parameter is obtained as
The equation (B.29) depends on the average of the group delay and the chirp effect. This equation is known as broadening parameter of the pulse.

For the next demonstration it is going to be considered a Gaussian pulse with Chirp in linear regime with the following equation

\[ A(z,t) = A_0 \exp \left[ -\frac{1+iC}{4} \left( \frac{t}{\sigma_0} \right)^2 \right] \]

(B.30)

where \(A_0\) stands for the amplitude of the pulse, \(C\) stands for the Chirp's parameter and \(\sigma_0\) stands for the pulse width at the entrance of the fiber. The group delay can be written as

\[ \tau_g(\Omega) = \left( \beta_1 + \beta_2 \Omega + \frac{1}{2} \beta_3 \Omega^2 \right) L. \]

(B.31)

Firstly, it is important to calculate the \(A(0,t)\) Fourier's transform, for \(z = 0\) so,

\[ \tilde{A}(0,\Omega) = A_0 \int_{-\infty}^{+\infty} \exp \left[ -\frac{1+iC}{4} \left( \frac{t}{\sigma_0} \right)^2 \right] \exp[i\Omega t] dt \]

\[ = A_0 \int_{-\infty}^{+\infty} \exp \left[ -\frac{1+iC}{4\sigma_0^2} t^2 + i\Omega t \right] dt \]

\[ = A_0 \int_{-\infty}^{+\infty} \exp \left[ -\frac{1+iC}{4\sigma_0^2} t^2 + (-i\Omega) t \right] dt \]

(B.32)

where, it can be simplified if using the following integral's property

\[ \int_{-\infty}^{+\infty} \exp \left[ -(ax^2 + bx) \right] dx = \frac{\pi}{\sqrt{a}} \exp \left( \frac{b^2}{4a} \right) \]

(B.33)

So that, and with respect to these new integration variables

\[ a = \frac{1+iC}{4\sigma_0^2} \]

and

\[ b = -i\Omega \]

the equation (B.32) can be altered to
\[ \tilde{A}(0, \Omega) = A_0 \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2}{4a} \right) \]

\[ = A_0 \sqrt{\frac{4\sigma_0^2 \pi}{1+iC}} \exp \left\{ \frac{-\Omega^2 \sigma_0^2}{1+iC} \right\} \]

\[ = A_0 \sqrt{\frac{4\sigma_0^2 \pi}{1+iC}} \left( \frac{1-iC}{1+iC} \right) \exp \left\{ \frac{-\Omega^2 \sigma_0^2}{1+iC} (1-iC) \right\} \]

\[ = A_0 \sqrt{\frac{4\sigma_0^2 \pi}{1+iC}} \left( \frac{1-iC}{1+iC} \right) \exp \left\{ -\frac{\Omega^2 \sigma_0^2}{1+iC} (1-iC) \right\} \]

\[ = A_0 \sqrt{\frac{4\sigma_0^2 \pi}{1+iC}} \left[ \frac{1}{1+C^2} \right] \exp \left\{ -\frac{1}{2} i \tan^{-1}(C) \right\} \exp \left\{ iC \frac{\Omega^2 \sigma_0^2}{1+C^2} - \frac{\Omega^2 \sigma_0^2}{1+C^2} \right\} \]

\[ = A_0 \sqrt{\frac{4\sigma_0^2 \pi}{1+iC}} \left[ \frac{1}{1+C^2} \right] \exp \left\{ -\frac{\Omega^2 \sigma_0^2}{1+C^2} \right\} \times \exp \left\{ i \left( -\frac{1}{2} \tan^{-1}(C) + C \frac{\Omega^2 \sigma_0^2}{1+C^2} \right) \right\}. \] (B.34)

Term \( A_0 \) is still missing. To calculate this parameter, it will be useful to use the following equation

\[ \int_{-\infty}^{\infty} |A(z,t)|^2 \, dt = 1 \iff \int_{-\infty}^{\infty} A_0 \exp \left( -\frac{1+iC \, t^2}{4 \, \sigma_0^2} \right) \, dt = 1 \]

\[ \iff \int_{-\infty}^{\infty} A_0 \exp \left( -\frac{1}{4 \, \sigma_0^2} \, t^2 \right) \, dt = 1 \iff A_0 \exp \left( -\frac{1}{4 \, \sigma_0^2} \, t^2 \right) \, dt = 1 \] (B.35)

\[ \iff A_0^2 \sqrt{2\pi \sigma_0} = 1 \iff A_0 = \frac{1}{\sqrt{2\pi \sigma_0}} \]

and, rewriting this parameter in equation (B.34), \( \tilde{A}(0, \Omega) \) comes
\[
\tilde{A}(0,\Omega) = \frac{1}{i2\pi \sigma_0^2} \sqrt{\frac{4\sigma_0^2 \pi}{1+C^2}} \exp\left\{ -\frac{\Omega^2 \sigma_0^2}{1+C^2} \right\} \times \exp\left\{ i \left( -\frac{1}{2} \tan^{-1}(C) + C \frac{\Omega^2 \sigma_0^2}{1+C^2} \right) \right\}
\]

\[
= \frac{1}{i} \sqrt{\frac{8 \pi \sigma_0^2}{1+C^2}} \exp\left\{ -\frac{\Omega^2 \sigma_0^2}{1+C^2} \right\} \exp\left\{ i \left( -\frac{1}{2} \tan^{-1}(C) + C \frac{\Omega^2 \sigma_0^2}{1+C^2} \right) \right\}
\]

(B.36)

where, as it can be observed in equation (B.36), this parameter can be simplified to

\[
\tilde{A}(0,\Omega) = S(\Omega) \exp\left\{ \theta(\Omega) \right\}.
\]

(B.37)

As the intuit of getting an expression \( \sigma^2 \) in terms of chirp parameter, it is needed the expression of moments so,

\[
\langle \tau_\perp \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau_\perp |S(\Omega)|^2 \, d\Omega
\]

(B.38)

\[
\langle \tau_\parallel \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau_\parallel |S(\Omega)|^2 \, d\Omega
\]

(B.39)

\[
\langle \tau_\perp \theta_\Omega \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau_\perp \theta_\Omega |S(\Omega)|^2 \, d\Omega
\]

(B.40)

where, the parameter \( \theta_\Omega \) is

\[
\theta_\Omega = \frac{\partial \theta(\Omega)}{\partial \Omega} = 2 \frac{C \Omega \sigma_0^2}{1+C^2}
\]

(B.41)

Replacing equation (B.31) into equations (B.38) and (B.39), It is possible the achieve the following equations
\begin{equation}
\langle \tau_s \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau_s |\Omega|^2 d\Omega
\end{equation}

\begin{equation*}
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \beta_1 L + \beta_2 L \Omega + \frac{1}{2} \beta_3 L \Omega^2 \right) |S(\Omega)|^2 d\Omega
\end{equation*}

\begin{equation}
= \frac{1}{2\pi} \beta_1 L \int_{-\infty}^{\infty} |S(\Omega)|^2 d\Omega + \frac{1}{2\pi} \beta_2 L \int_{-\infty}^{\infty} \Omega |S(\Omega)|^2 d\Omega
\end{equation}

\begin{equation}
+ \frac{1}{2\pi} \beta_3 L \int_{-\infty}^{\infty} \Omega^2 |S(\Omega)|^2 d\Omega
\end{equation}

\begin{equation*}
\langle \tau_s^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau_s^2 |\Omega|^2 d\Omega
\end{equation*}

\begin{equation*}
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \beta_1 L + \beta_2 L \Omega + \frac{1}{2} \beta_3 L \Omega^2 \right)^2 |S(\Omega)|^2 d\Omega
\end{equation*}

\begin{equation*}
= \frac{1}{2\pi} \beta_1 L \int_{-\infty}^{\infty} |S(\Omega)|^2 d\Omega + \beta_2 \beta_3 L \int_{-\infty}^{\infty} \Omega |S(\Omega)|^2 d\Omega
\end{equation*}

\begin{equation}
+ \left( \beta_3^2 + \beta_2 \beta_3 \right) \int_{-\infty}^{\infty} \Omega^2 |S(\Omega)|^2 d\Omega + \beta_2 \beta_3 L \int_{-\infty}^{\infty} \Omega^3 |S(\Omega)|^2 d\Omega
\end{equation}

\begin{equation}
+ \frac{1}{4} \beta_3^2 L \int_{-\infty}^{\infty} \Omega^4 |S(\Omega)|^2 d\Omega
\end{equation}

As it was possible to see it is important to achieve some results from the integrals of the equations (B.42) and (B.43). So,

\begin{equation}
\int_{-\infty}^{\infty} |S(\Omega)|^2 d\Omega = 2\pi = \int_{-\infty}^{\infty} |A(0,\Omega)|^2 d\Omega
\end{equation}

\begin{equation}
\int_{-\infty}^{\infty} \Omega |S(\Omega)|^2 d\Omega = 0
\end{equation}

\begin{equation}
\int_{-\infty}^{\infty} \Omega^2 |S(\Omega)|^2 d\Omega = \frac{8\pi \sigma_0^2}{1+C^2} \sqrt{2\pi} \sqrt{1+C^2} \sqrt{\frac{1+C^2}{4\sigma_0^2}}
\end{equation}

\begin{equation}
\int_{-\infty}^{\infty} \Omega^3 |S(\Omega)|^2 d\Omega = 0
\end{equation}

\begin{equation}
\int_{-\infty}^{\infty} \Omega^4 |S(\Omega)|^2 d\Omega = 3*2\pi \left( \frac{1+C^2}{4\sigma_0^2} \right)^2
\end{equation}

With these four equations, it is possible to rewrite equation (B.42) and (B.43) such as
\[
\langle \tau_s \rangle^2 = \left[ \frac{1}{2\pi} \beta_2 L \int_{-\infty}^{\infty} |S(\Omega)|^2 d\Omega + \frac{1}{2\pi} \beta_2 L \int_{-\infty}^{\infty} |\Omega| |S(\Omega)|^2 d\Omega \right]^2
\]
\[
+ \left[ \frac{1}{2\pi} \beta_3 L \int_{-\infty}^{\infty} \Omega^2 |S(\Omega)|^2 d\Omega \right]^2
\]
\[
= L^2 \left( \beta_2 \, 2\pi + \frac{1}{2} \beta_3 \, 2\pi \left( \frac{1+C^2}{4\sigma_0^2} \right) \right)^2
\]
\[
= L^2 \left( \beta_2 \, 2\pi \right)^2 + \left( \frac{1}{2} \beta_3 \, 2\pi \left( \frac{1+C^2}{4\sigma_0^2} \right) \right)^2 + \beta_2 \, \pi \, \beta_3 \left( \frac{1+C^2}{4\sigma_0^2} \right) \]  
(B.49)

\[
\langle \tau_s^2 \rangle = \frac{1}{2\pi} L^2 \left[ 2\pi \beta_2^2 + 2\pi \left( \beta_2^2 + \beta_2 \beta_3 \right) \frac{1+C^2}{4\sigma_0^2} + \frac{3}{4} \beta_3^2 \left( 2\pi \right) \left( \frac{1+C^2}{4\sigma_0^2} \right) \right] \]  
(B.50)

However, there is a term missing and it needs to be calculated and according to the
equation(B.41) and

\[
\langle \theta_\Omega \rangle = \frac{1}{2\pi} \frac{2C\sigma_0^2}{1+C^2} \int_{-\infty}^{\infty} \Omega |S(\Omega)|^2 d\Omega = 0 . \]  
(B.51)

According to equation(B.51), it is now time to calculate the last term needed

\[
\langle \tau_s \theta_\Omega \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} L \left( \beta_2 + \beta_2 \Omega + \frac{1}{2} \beta_3 \Omega^2 \right) \frac{2C\sigma_0^2}{1+C^2} \Omega |S(\Omega)|^2 d\Omega
\]
\[
= \frac{1}{2\pi} \frac{2C\sigma_0^2}{1+C^2} \left[ \beta_2 \int_{-\infty}^{\infty} \Omega |S(\Omega)|^2 d\Omega + \beta_2 \int_{-\infty}^{\infty} \Omega^2 |S(\Omega)|^2 d\Omega \right] + \frac{1}{2} \beta_3 \int_{-\infty}^{\infty} \Omega^3 |S(\Omega)|^2 d\Omega
\]
\[
= \frac{2LC\sigma_0^2}{1+C^2} \beta_2 \frac{1+C^2}{4\sigma_0^2} - L \frac{C\beta_3}{2} \]  
(B.52)

It is possible to write the following equation with the achieved results
\[\sigma^2 = \sigma_0^2 + \left[\langle \tau_g^2 \rangle - \langle \tau_g \rangle^2 \right] + 2\left[\langle \tau_g \beta_3 \rangle \right] \]

\[= \sigma_0^2 + L^2 \left[ \beta_1 \left(1 + \frac{C^2}{4\sigma_0^2}\right) + \frac{1}{2} \beta_3 \left(1 + \frac{C^2}{4\sigma_0^2}\right) \right] + LC \beta_2 \]

\[-L^2 \left[ \beta_1 \left(1 + \frac{1}{4} \beta_3 \left(1 + \frac{C^2}{4\sigma_0^2}\right) \right) + \beta_1 \beta_3 \left(1 + \frac{C^2}{4\sigma_0^2}\right) \right] + LC \beta_2 \]

\[= \sigma_0^2 + L^2 \left(1 + \frac{C^2}{4\sigma_0^2}\right) \beta_2 + L^2 \left(\frac{1}{2} \beta_3 \left(1 + \frac{C^2}{4\sigma_0^2} \right) \right) + LC \beta_2 \]

\[= \sigma_0^2 + LC \beta_2 + \frac{L^2 \beta_2^2}{4\sigma_0^2} + \left(1 + \frac{C^2}{4\sigma_0^2}\right)^2 \]

Or simpler,

\[\frac{\sigma^2}{\sigma_0^2} = \left(1 + \frac{C \beta_2 L}{2\sigma_0^2}\right)^2 + \left(\frac{\beta_1 L}{2\sigma_0^2}\right)^2 + \left(1 + \frac{C^2}{4\sigma_0^2}\right)^2 \]

\[\frac{\beta_1 L}{4\sqrt{2\sigma_0^3}} \]
C. Appendix - Numerical Approach for Pulse Propagation using FFT

In order to solve the propagation equation, it is commonly used the FFT and the IFFT (inverse Fast Fourier Transform). In this annex it is going to be present the mechanism behind the FFT and how computers solve integrals as well as some approximations that needed to be done to obtain a numerical solution.

Let’s start considering the following input pulse

\[ \tilde{A}(z, \omega) = \tilde{A}(0, \omega) \exp(i \beta z) \quad (C.1) \]

and, considering, once more, the propagation equation such as

\[ \frac{\partial A}{\partial \zeta} + i \frac{1}{2} \text{sgn}(\beta_2) \frac{\partial^3 A}{\partial \tau^3} - \kappa \frac{\partial^2 A}{\partial \tau^2} = 0 \quad (C.2) \]

where,

\[ \beta_2 = |\beta_2| \text{sgn}(\beta_2) \quad (C.3) \]

and

\[ \kappa = \frac{\beta_2}{6 |\beta_2| \tau_0}. \quad (C.4) \]

The variable \( A \) is the pulse normalized into variables \( \zeta \) and \( \tau \), \( \kappa \) is the third order dispersion coefficient and \( \tau_0 \) is the width of the pulse.

In order to solve equation (C.2) it is highly recommended to work in the frequency regime so that, using direct and inverse Fourier transforms will help in this procedure and it comes

\[ A(\zeta, \xi) = \int_{-\infty}^{\infty} A(\zeta, \tau) \exp(i \xi \tau) d\tau \quad (C.5) \]

\[ A(\zeta, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(\zeta, \xi) \exp(-i \xi \tau) d\xi \quad (C.6) \]

where, term \( \xi \) is the normalized frequency given by

\[ \xi = (\omega - \omega_0) \tau_0. \quad (C.7) \]
In order to simplify this explanation, high order dispersion parameter will be neglected so that equation (C.2) can be simplified according to

\[
\frac{\partial A}{\partial \bar{\xi}} = -i \frac{1}{2} \frac{\partial^2 A}{\partial \tau^2}
\]  

(C.8)

and the solution for this simplification would be

\[
\tilde{A}(\zeta, \bar{\xi}) = \tilde{A}(0, \bar{\xi}) \exp\left(-i \frac{1}{2} \bar{\xi}^2 \zeta\right)
\]

(C.9)

If it is required the complete solution for equation (C.2), then, its solution comes

\[
\tilde{A}(\zeta, \bar{\xi}) = \tilde{A}(0, \bar{\xi}) \exp\left(-i \frac{1}{2} \text{sgn}(\beta_2) \bar{\xi}^3 \zeta^2 - i \kappa \zeta^3 \zeta\right)
\]

(C.10)

The steps that need to be performed to do a numerical simulation include 3 main steps:

[Diagram showing A(0,τ) → FFT → A(0,ξ) → X → A(ζ,ξ) → IFFT]

\[
\exp(-i*\xi(\text{sgn}(\beta2)*\zeta^2/2 - \kappa*\zeta^3))
\]

This mechanism is used for pulses that we want to obtain a numerical solution. Though, this mechanism is used for the majority of pulses, in some special cases this technique could not be used.

When $\lambda_0 = \lambda_0'$ it needs to have different normalized variables such as

\[
\zeta' = \frac{\zeta}{L_D'}
\]

(C.11)

\[
L_D' = \frac{\tau_0^3}{|\beta|}
\]

(C.12)

where, $L_D'$ is the dispersion length associated with the third order dispersion. In this specific case, the propagation equation used is
\[
\frac{\partial A}{\partial \zeta'} = \frac{1}{6} \text{sgn}(\beta_i) \frac{\partial^3 A}{\partial \tau^3} = 0
\]  

(C.13)

And the pulse is

\[
\tilde{A}(\zeta', \xi) = \tilde{A}(0, \xi) \exp\left(i \frac{1}{6} \text{sgn}(\beta_i) \xi^3 \zeta''\right)
\]  

(C.14)