Electromagnetic Waves in Metamaterials: Waveguides and Lenses

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Abstract — This work is focused on the study of the propagation of electromagnetic waves in double negative (DNG) metamaterials. The work begins with the study of the properties of DNG metamaterials, including the negative refraction since, in these media, the electric permittivity and magnetic permeability are both negative, leading to the appearance of backward waves, because the Poynting vector and wave vector have opposite directions. For the study of guided electromagnetic wave propagation in structures containing DNG metamaterials, we have considered the DPS-DNG interface and the DNG dielectric slab. In the analysis of DPS-DNG interface it is necessary to use the Lorentz dispersive model with losses, so that the results have physical meaning. In the analysis of a DNG dielectric slab we have shown the existence of slow modes and super slow modes with double solutions. Finally we study metamaterial lenses, where the lens design is analyzed, verifying that the lens contour is dependent on its refractive index. The Veselago's flat lens and the Pendry's perfect lens are both studied, observing that in these lenses the image resolution is no longer limited to the wavelength, as happens in conventional lenses.

Keywords — Metamaterials, Double Negative Media, Negative Refraction, Backward Waves, Planar Waveguides, Perfect Lens

I. INTRODUCTION

In recent years, there has been a growing interest in artificial materials due to their electromagnetic properties that cannot be found in nature. However, the first attempt to explore the concept of artificial materials occurred in 1898, when Jagadis Chunder Bose conducted the first microwave experiment on twisted structures [1]. In 1914, Karl Ferdinand Lindman worked on artificial chiral media by embedding many randomly oriented small wire helices in a host medium [2]. In 1948, Winston Edward Kock made microwave lenses by arranging conducting spheres, disks and strips periodically, to adjust the effective refractive index of an artificial medium [3]. Since then, the artificial complex materials have been studied for many researchers around the world.

In 1967, Victor Georgievich Veselago theoretically investigated the plane wave propagation in a medium with electric permittivity and magnetic permeability simultaneously negative [4]. Veselago’s ideas remained ignored for three decades, due to the absence of known materials with negative refractive index. In the 90s, John Brian Pendry et al. attempted to achieve negative permittivity in the frequency range of GHz using metal wires [5]. They also showed that periodic arrays of split ring resonators (SRRs) allowed to obtain negative permeability in the range of GHz [6]. In 2000, David R. Smith et al., produced for the first time a material with negative refractive index in the microwave regime, combining the structures proposed by Pendry in a single structure [7].

II. DOUBLE NEGATIVE METAMATERIALS

Considering a double negative metamaterial (DNG) characterized by two complex constants, the electric permittivity \( \varepsilon = \varepsilon(\omega) \in \mathbb{C} \) and the magnetic permeability \( \mu = \mu(\omega) \in \mathbb{C} \). In general

\[
\varepsilon = \varepsilon' + i\varepsilon''
\]

\[
\mu = \mu' + i\mu''
\]

with \( \varepsilon', \varepsilon'', \mu', \mu'' \in \mathbb{R} \).

Figure 1 graphically illustrates the classification of each medium according to their permittivity and permeability values.

The constitutive relations of the medium, in frequency domain, are given by

\[
\begin{align*}
\mathbf{D} &= \varepsilon_0 \varepsilon(\omega) \mathbf{E} \\
\mathbf{B} &= \mu_0 \mu(\omega) \mathbf{H}
\end{align*}
\]

Assuming that the electric field is polarized according to x-axis and that the electromagnetic waves propagate according to z-axis, the electric and magnetic fields can be written as

\[
\begin{align*}
\mathbf{E} &= \hat{\mathbf{E}}_0 \exp[i(kz - \omega t)] \\
\mathbf{H} &= \hat{\mathbf{H}}_0 \exp[i(kz - \omega t)]
\end{align*}
\]

where the propagation constant \( k \) is related with the propagation constant in vacuum \( k_0 \), given by

\[
k_0 = \omega \sqrt{\varepsilon_0 \mu_0} = \frac{\omega}{c},
\]
through the refractive index of the medium $n = n(\omega)$ such that

$$k = \mathbf{z}k \rightarrow k = k(\omega) = n(\omega)k_0 = n(\omega)\frac{\omega}{c}.$$  \hspace{1cm} (6)

The wave impedance of the medium is given by

$$\eta = \frac{E_0}{H_0} = \xi \eta_0 = \frac{\mu}{n} \eta_0.$$  \hspace{1cm} (7)

The refractive index of the medium is given by

$$n = \sqrt{\varepsilon \mu}.$$  \hspace{1cm} (8)

In general

$$n = n' + in''$$  \hspace{1cm} (9)

then, from (4), (6), (7) and (9), one obtain the complex amplitudes of the electric and magnetic fields in the form

$$\begin{align*}
\mathbf{E} &= \mathbf{z}E_0 \exp(-n''k_0x) \exp(i n'k_0z) \\
\mathbf{H} &= \frac{\mathbf{y}}{\xi \eta_0} E_0 \exp(-n''k_0x) \exp(i n'k_0z).
\end{align*}$$  \hspace{1cm} (10)

The average value of the Poynting vector for the frequency $\omega$ is given by

$$\mathbf{S}_\omega = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{H}^*) = \mathbf{z} \frac{1}{2n_0} \Re(E_0 H_0^*) = \mathbf{z} \frac{|E_0|^2}{2n_0} \Re \left( \frac{1}{\xi} \right) \exp(-2\Im(k)z).$$  \hspace{1cm} (11)

For a DNG medium

$$\Re \left( \frac{1}{\xi} \right) = \frac{n''\mu' + n''\varepsilon'}{(\mu')^2 + (\varepsilon')^2} > 0 \Rightarrow \mathbf{z} \cdot \mathbf{S}_\omega > 0$$  \hspace{1cm} (12)

and

$$\Re(k) = \frac{n'\omega}{c} < 0 \rightarrow \mathbf{S}_\omega \cdot \Re(\mathbf{k}) < 0.$$  \hspace{1cm} (13)

The triplet vectors $[\mathbf{E}_0, \mathbf{H}_0, \Re(\mathbf{k})]$ and $[\mathbf{E}_0, \mathbf{H}_0, \mathbf{S}_\omega]$ for DPS and DNG media are shown in the Figure 2.

As shown in Figure 2, in a DPS medium the triplet vector $[\mathbf{E}_0, \mathbf{H}_0, \Re(\mathbf{k})]$ is right, while in a DNG medium the same triplet vector is left. On the other hand, the triplet vector $[\mathbf{E}_0, \mathbf{H}_0, \mathbf{S}_\omega]$ is right in both media. In a DPS medium the Poynting vector $\mathbf{S}$ and the wave vector $\mathbf{k}$ have the same direction, providing forward waves, while in a DNG medium the vectors $\mathbf{S}$ and $\mathbf{k}$ have opposite directions, which create a backward wave.

In order to study the expression of the refractive index of a DNG medium, one must first analyze the behavior of electromagnetic characteristics of the medium in the complex plane [8]. The refractive index described by (8) can be decomposed into two components

$$\begin{align*}
n_x &= \sqrt{\varepsilon} \\
n_\mu &= \sqrt{\mu} \rightarrow n = n_x n_\mu,
\end{align*}$$  \hspace{1cm} (14)

where $n_x$ depends exclusively on the electric permittivity and $n_\mu$ on the magnetic permeability.

As depicted in Figure 3,

$$n_x = n_x' + in_x'' = \sqrt{\varepsilon} \exp \left( \frac{i \theta_x}{2} \right).$$  \hspace{1cm} (15)

Fig. 3. $n_x$ component of the refractive index in the complex plane.

Variables of (15) can be described by

$$\begin{align*}
\rho_x &= \sqrt{(\varepsilon')^2 + (\varepsilon'')^2} = |\varepsilon'| \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} \\
\cos \left( \frac{\theta_x}{2} \right) &= \frac{1}{\sqrt{2}} \left[ 1 + \frac{\varepsilon'}{(\varepsilon')^2 + (\varepsilon'')^2} \right]^{\frac{1}{2}} \\
\sen \left( \frac{\theta_x}{2} \right) &= \frac{1}{\sqrt{2}} \left[ 1 - \frac{\varepsilon'}{(\varepsilon')^2 + (\varepsilon'')^2} \right]^{\frac{1}{2}}
\end{align*}$$  \hspace{1cm} (16)

and the real and imaginary part of the $n_x$ component are given by

$$\begin{align*}
n_x' &= \frac{|\varepsilon'|}{\sqrt{2}} \left[ 1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right]^{\frac{1}{2}} + \frac{\text{sgn}(\varepsilon')}{\sqrt{2}} \left[ 1 - \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right]^{\frac{1}{2}} \\
n_x'' &= \frac{|\varepsilon'|}{\sqrt{2}} \left[ 1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right]^{\frac{1}{2}} - \frac{\text{sgn}(\varepsilon')}{\sqrt{2}} \left[ 1 - \left( \frac{\varepsilon''}{\varepsilon'} \right)^2 \right]^{\frac{1}{2}}
\end{align*}$$  \hspace{1cm} (17)

The analysis of the $n_\mu$ component of the refractive index leads to
analogous results. From (17), it is clear that \( n''_e > n''_h \) then, \( n \) can be written as

\[
n = n_e n_h = (n'_e n'_h - n''_e n''_h) + i(n'_e n''_h + n''_e n'_h).
\]

(18)

As a result, in a DNG medium one has

\[
\begin{align*}
    n' &= n'_e n_h - n''_e n''_h < 0 \\
    n'' &= n'_e n''_h + n''_e n'_h > 0.
\end{align*}
\]

(19)

A. Dispersion

For an isotropic, nondispersive medium, characterized by permittivity \( \varepsilon \) and permeability \( \mu \), the time-averaged electric and magnetic energy densities, \( \langle W_e \rangle \) and \( \langle W_m \rangle \) are, respectively, given by

\[
\langle W_e \rangle = \frac{1}{4} \varepsilon_e \varepsilon_0 |\textbf{E}|^2
\]

(20)

\[
\langle W_m \rangle = \frac{1}{4} \mu_0 \mu |\textbf{H}|^2
\]

(21)

In the case of DNG media, the equations (20) and (21) cannot be used because it would lead to negative values for \( \langle W_e \rangle \) and \( \langle W_m \rangle \). Thus, the equations that allow the calculation of these quantities for dispersive media [9] are

\[
\langle W_e \rangle = \frac{1}{4} \varepsilon_0 \varepsilon_0 \frac{\partial [\text{e}(\omega)\varepsilon]}{\partial \omega} |\textbf{E}|^2
\]

(22)

\[
\langle W_m \rangle = \frac{1}{4} \mu_0 \mu_0 \frac{\partial [\text{m}(\omega)\mu]}{\partial \omega} |\textbf{H}|^2
\]

(23)

that in the particular case in which the permittivity and permeability do not depend on frequency, reduced to equations (20) and (21).

B. Phase Velocity and Group Velocity

The phase and group velocities are given by

\[
\begin{align*}
    v_f &= \frac{\omega}{k} = \frac{c}{n} \\
    v_g &= \frac{\partial \omega}{\partial k}.
\end{align*}
\]

(24)

From (6) we have

\[
k = nk_0 = \frac{\omega}{c}.
\]

(25)

Deriving (25) we obtain

\[
\frac{\partial k}{\partial \omega} = \frac{1}{c} (n + \omega \frac{\partial n}{\partial \omega}).
\]

(26)

Replacing (24) in (26)

\[
\frac{1}{v_g} = \frac{1}{v_f} + \frac{\omega}{c} \frac{\partial n}{\partial \omega}.
\]

(27)

That shows us that \( v_g = v_f \) is only possible when there is no frequency dependence of the refractive index.

For an isotropic medium we have

\[
k^2 = \omega^2 \varepsilon_0 \mu_0 \varepsilon \mu.
\]

(28)

Deriving it in order of the frequency we obtain

\[
\frac{\partial (k^2)}{\partial \omega} = \varepsilon_0 \frac{\partial (\omega \varepsilon_\omega)}{\partial \omega} (\omega \varepsilon_\omega \mu) + (\omega \varepsilon_\omega \mu) \frac{\partial (\omega \varepsilon_\omega)}{\partial \omega}.
\]

(29)

From (22) and (23) we know that

\[
\begin{align*}
    \frac{\partial (\omega \varepsilon)}{\partial \omega} &> 0 \\
    \frac{\partial (\omega \mu)}{\partial \omega} &> 0,
\end{align*}
\]

(30)

since the stored energy is positive. As the medium is DNG we have \( \varepsilon < 0 \) and \( \mu < 0 \), then from (29) we can conclude that \( \partial (k^2)/\partial \omega < 0 \).

We can also write (29) as

\[
\frac{\partial (k^2)}{\partial \omega} = 2k \frac{\partial k}{\partial \omega} = 2n \frac{\omega}{c} \frac{\partial k}{\partial \omega} = 2\omega \frac{1}{v_g} \frac{1}{v_f} < 0
\]

(31)

Thus, we conclude that in a dispersive DNG medium the group and phase velocities have opposite directions, i.e., \( v_g v_f < 0 \).

C. Negative Refraction

The phenomenon of negative refraction [8], [10] is studied by considering the scattering of a wave that is obliquely incident on a DPS-DNG interface as shown in Figure 4.

![Fig. 4. Oblique incidence of a wave in a DPS-DNG interface.](image)

Applying Snell’s law

\[
n_1 \sin(\theta_{\text{inc}}) = n_2 \sin(\theta_{\text{trans}}),
\]

(32)

as the refractive index of the DNG medium is negative, \( n_2 < 0 \), one has that

\[
\theta_{\text{trans}} = \text{sgn}(n_2) \sin^{-1} \left( \frac{n_1}{|n_2|} \sin(\theta_{\text{inc}}) \right).
\]

(33)
III. PROPAGATION OF ELECTROMAGNETIC WAVES IN DNG GUIDES

A. DPS-DNG Interface

In this section we analyze a DPS-DNG interface, illustrated in Figure 5. The $z$-axis represents the direction of the propagation, $x$-axis the transverse direction and $y$-axis the direction in which the fields do not vary, i.e., $\partial / \partial y \equiv 0$.

In a DPS-DNG interface, the field $E_y$, for the TE modes, can be written as

$$E_y = \begin{cases} A \exp(-\alpha_1 x) \exp(ik_z z), & x > 0 \\ B \exp(\alpha_2 x) \exp(ik_z z), & x < 0 \end{cases}$$

where $\alpha_1$ and $\alpha_2$ are the transverse attenuation constants in the DPS and DNG media, respectively, given by

$$\alpha_1^2 = n_{eff}^2 - \varepsilon_1 \mu_1$$

$$\alpha_2^2 = n_{eff}^2 - \varepsilon_2 \mu_2.$$  

Applying the boundary conditions, the modal equation of TE modes is

$$\frac{\alpha_2}{\alpha_1} = -\frac{\mu_2}{\mu_1}.$$  

Similarly for the TM modes, we have

$$\frac{\alpha_2}{\alpha_1} = -\frac{\varepsilon_2}{\varepsilon_1}.$$  

In a DPS-DNG interface modal equations (37) and (38) lead to valid solutions, which does not happen in a DPS-DPS interface.

In order to study the DPS-DNG interface we will use the Lorentz dispersive model

$$\varepsilon(\omega) = 1 + \frac{\omega_{pe}^2 - \omega^2}{\omega_{pe}^2 - \Gamma_e \omega - \omega^2}$$  

$$\mu(\omega) = 1 + \frac{\omega_{pm}^2 - \omega^2}{\omega_{pm}^2 - \Gamma_m \omega - \omega^2}.$$  

We choose the following values $\omega_{pe} = 2\pi \times 6 \times 10^9$ rad/s, $\omega_{pm} = 2\pi \times 5 \times 10^9$ rad/s, $\omega_{qe} = 2\pi \times 2.7 \times 10^9$ rad/s, $\omega_{qm} = 2\pi \times 2.1 \times 10^9$ rad/s and $\Gamma_e = \Gamma_m = 0$, i.e., we consider that there are no losses.

Figure 6 illustrates the variations of electric permittivity, $\varepsilon_2$, and magnetic permeability, $\mu_2$, with the frequency. The variation of refractive index, $n$, with the frequency is shown in Figure 7.

![Fig. 6. Lossless Lorentz dispersive model for $\varepsilon$ and $\mu$.](image)

![Fig. 7. Lossless Lorentz dispersive model for $n$.](image)

From Figures 6 and 7 it becomes clear that $n$ is only real when $\varepsilon_2$ and $\mu_2$ have both positive or negative values. Note that for a DNG medium, where $\varepsilon_2$ and $\mu_2$ are both negative, there is negative refraction, while for a DPS medium, where $\varepsilon_2$ and $\mu_2$ are both positive, there is positive refraction. In a ENG medium, where $\varepsilon_2$ is negative and $\mu_2$ is positive, the refractive index is purely imaginary.

From (35), (36) and (37), we obtain the following equation for the effective refractive index of TE modes

$$n_{eff}(\omega) = \sqrt{\frac{\varepsilon_2(\omega) \mu_2(\omega) - \left(\frac{\mu_2(\omega)}{\varepsilon_2(\omega)}\right)^2 \varepsilon_1(\omega) \mu_1(\omega)}{1 - \left(\frac{\mu_2(\omega)}{\varepsilon_2(\omega)}\right)^2}},$$

and its variation with frequency is shown in Figure 8.

From Figure 8, one can easily see that the effective refractive index is purely real. We can also see that when $\mu_1(\omega) = 1$ and $\mu_2(\omega) = -1$, the effective refractive index goes to infinity, since it cancels the denominator of (41), which represents an unphysical result.

Figure 9 illustrates the variation of transverse attenuation constants $\alpha_1$ and $\alpha_2$ with the frequency, for the TE modes. From (34) one can see that $\alpha_1$ and $\alpha_2$ must always be positive, to the field $E_y$ do not tend to infinity. From (35), $\alpha_1$ is real when $n_{eff}$ is greater than one and purely imaginary when $n_{eff}$ is lower than one.
Fig. 8. Dispersion diagram for TE modes in the lossless case.

Fig. 9. Variation of $\alpha_1$ and $\alpha_2$ with the frequency for TE modes.

Now uses the Lorentz dispersive model with the same values for the plasma and resonance frequencies, but considering the existence of losses, i.e., $\Gamma_p = \Gamma_m = 0.05 \times \omega_0$.

Figure 10 illustrates the variations of electric permittivity, $\varepsilon_2$, and magnetic permeability, $\mu_2$, with the frequency. The variation of refractive index, $n$, with the frequency is shown in Figure 11.

Fig. 10. Lossy Lorentz dispersive model for $\varepsilon$ and $\mu$.

From Figure 11, one can see that for a DNG medium the refractive index is negative, for a DPS medium the refractive index is positive and for an ENG medium there is an interval in which the refractive index is also negative. Thus, we conclude that for a medium to have a negative refractive index is sufficient, but not a necessary condition, that this medium is DNG.

The effective refractive index of TE modes for the lossy Lorentz dispersive model does not tend to infinity, as shown in Figure 12, which means that there are physical results. One can also see that the imaginary part of $n_{eff}$ grows considerably in the frequency range where previously there was an unphysical result, while in the frequency range where there was physical results has relatively low values.

Fig. 11. Lossy Lorentz dispersive model for $n$.

Fig. 12. Dispersion diagram for TE modes in the lossy case.

The transverse attenuation constants $\alpha_1$ and $\alpha_2$ for the TE modes are shown in Figure 13.

Fig. 13. Variation of $\alpha_1$ and $\alpha_2$ with the frequency for TE modes.

The real part of the transverse attenuation constants has to be positive to have propagation along the interface and exponential attenuation as we move away from it.

For TM modes, one uses the same procedure to construct the dispersion diagrams and the graphics of the variation of the transverse attenuation constants with the frequency.
B. DNG Dielectric Slab

In this section we study a DNG dielectric slab, shown in Figure 14.

![Diagram of DNG dielectric slab](image)

Fig. 14. DNG dielectric slab immersed on a DPS medium.

The modal equation of the medium can be separated into TE and TM modes. Given the geometric symmetry of the structures, the modes can be even or odd.

Thus, the electric field of the even TE modes is given by

$$E_y = \begin{cases} A \exp[-\alpha_1(x-d)], & x \geq d \\ B \cos(h_2x), & 0 \leq x \leq d \end{cases} \quad (42)$$

and for odd TE modes is

$$E_y = \begin{cases} A \exp[-\alpha_1(x-d)], & x \geq d \\ B \sin(h_2x), & 0 \leq x \leq d \end{cases} \quad (43)$$

where $\alpha_1$ is the attenuation constant in the DPS medium, expressed by

$$\alpha_1^2 = k_x^2 - \omega^2 \varepsilon_2 \mu_1$$

and $h_2$ is the transverse propagation constant in the DNG medium, given by

$$h_2^2 = \omega^2 \varepsilon_2 \mu_2 - k_x^2. \quad (45)$$

Applying the boundary conditions and through algebraic manipulation, we obtain for the even TE modes

$$\alpha_1 d = \frac{\mu_1}{\mu_2} h_2 \text{dctg}(h_2 d) \quad (46)$$

and for odd TE modes

$$\alpha_1 d = -\frac{\mu_1}{\mu_2} h_2 \text{dctg}(h_2 d). \quad (47)$$

To simplify (46) and (47) it is considered that $w = \alpha_1 d$ and $u = h_2 d$.

Thus, the modal equation of even TE modes is of the form

$$w = \frac{\mu_1}{\mu_2} u \text{ctg}(u) \quad (48)$$

while for the odd TE modes is

$$w = -\frac{\mu_1}{\mu_2} u \text{ctg}(u). \quad (49)$$

The relation between the normalized propagation constants is expressed by

$$u^2 + w^2 = \upsilon^2, \quad (50)$$

where the normalized frequency is given by

$$\upsilon = \frac{k_0 d}{\sqrt{\varepsilon_2 \mu_2 - \varepsilon_1 \mu_1}}. \quad (51)$$

We will now study the surface modes of the DNG dielectric slab, where $k_x = \beta$. From (45) we can see that the transverse propagation constant $h_2$ take real values if $\beta < \omega \sqrt{\varepsilon_2 \mu_2}$ and imaginary values if $\beta > \omega \sqrt{\varepsilon_2 \mu_2}$. Assuming that $U = -i\upsilon$ and considering that $\text{tg}(ix) = i\text{tgh}(x)$ and $\cotg(ix) = -i\text{cotgh}(x)$ is possible to rewrite (48), (49) and (50)

$$w = -\frac{\mu_1}{\mu_2} u \text{tgh}(U) \quad (52)$$

$$w = -\frac{\mu_1}{\mu_2} u i \text{cotgh}(U) \quad (53)$$

$$w^2 = U^2 + \upsilon^2. \quad (54)$$

The modal solutions can be represented graphically by the intersection of the modal equations with equation (50) or (54).

In Figures 15 and 16 we analyze the DPS and DNG dielectric slabs, respectively, where the positive semi-axis of abscissa represents the transverse propagation constant $u$ and the negative semi-axis represents its imaginary part, that we have previously called $U$.

![Graph of modal solutions](image)

Fig. 15. Representation of the modal solutions of a DPS dielectric slab with $\varepsilon_1 = \mu_1 = 1$, $\varepsilon_2 = \mu_2 = 1.5$ and $\upsilon = 2$.

From Figures 15 and 16 one can see that, unlike the DPS dielectric slab that only allows modal solutions with real $u$, the DNG dielectric slab enables modal solutions with imaginary $u$. These modes are known as super slow modes, since the phase velocity, given by

$$v_f = \frac{\omega}{\beta}, \quad (55)$$

is lower than the speed of light in the medium

$$v_f < \frac{c}{\sqrt{\varepsilon_2 \mu_2}}. \quad (56)$$
Figure 17 graphically illustrates the modal solutions of the DNG dielectric slab to a different value of \( v \).

In Figures 15 and 17 one verifies the existence of modal solutions when \( u \) is real. This implies that the surface modes are called slow modes, since the phase velocity takes values in the range

\[
\frac{c}{\sqrt{\epsilon_2\mu_2}} < v_f < \frac{c}{\sqrt{\epsilon_2\mu_2}}.
\]

(57)

In the DNG medium the modal equations suffer a sign change due to \( \mu_2 < 0 \), which causes that for some slow modes, for a given range of frequencies, one has two solutions to the same value of \( v \), as illustrated in Figure 17.

The dispersion diagram of the DNG dielectric slab is shown in Figure 18.

The two dashed lines represent transition limits. The first one, given by \( k_{0d} = \beta d / \sqrt{\epsilon_1\mu_1} \), corresponds to the cutoff condition of the surface modes, where \( w = 0 \). The second one, of the form \( k_{0d} = \beta d / \sqrt{\epsilon_2\mu_2} \), represents the transition border between slow modes and super slow modes. The fundamental mode is a super slow odd mode, that becomes a slow mode when \( u = 0 \), i.e., \( v = \mu_1/|\mu_2| \) and propagates until \( v = \pi/2 \).

From the results in Figure 18, we conclude that there is a direct relation between the constitutive parameters of the medium and the dispersion diagrams obtained, as the point in which takes place the transition from super slow mode to slow mode, which depends on the value of \( \mu_1 \) and \( \mu_2 \).

To get the results of Figure 18 was assumed that \( \epsilon_1 \mu_1 < \epsilon_2 \mu_2 \) and \( \mu_1 < |\mu_2| \), however, if we consider the case where the inner medium is less dense than the outer medium, \( \epsilon_1 \mu_1 > \epsilon_2 \mu_2 \), is obtained from (51) that

\[
v^2 < 0.
\]

(58)

From (50) and (58), one has that

\[
u^2 + w^2 < 0.
\]

(59)

As \( w \geq 0 \) it follows from (59), that is only possible to have propagation if the following condition was verified

\[
u^2 < 0.
\]

(60)

The condition (60) is satisfied if we are in the presence of super slow modes, this means that the propagation in a less dense medium is only possible when using a DNG dielectric slab.

We will now analyze the dispersion diagrams for the case where the inner medium is less dense than the outer medium, but considering the influence of the constitutive parameters. First we consider the case where \( \mu_1 < |\mu_2| \), shown in Figure 19.

\[
\text{Fig. 16. Representation of the modal solutions of a DNG dielectric slab with } \\
\epsilon_1 = \mu_1 = 1, \epsilon_2 = \mu_2 = -1.5 \text{ and } v = 0.5.
\]

\[
\text{Fig. 17. Representation of the modal solutions of a DNG dielectric slab with } \\
\epsilon_1 = \mu_1 = 1, \epsilon_2 = \mu_2 = -1.5 \text{ and } v = 3.
\]

\[
\text{Fig. 18. Dispersion diagram for TE modes of a DNG dielectric slab with } \\
\epsilon_1 = \mu_1 = 1 \text{ and } \epsilon_2 = \mu_2 = -2.
\]

\[
\text{Fig. 19. Dispersion diagram for TE modes of a DNG dielectric slab with } \\
\epsilon_1 = 2, \mu_1 = 1, \epsilon_2 = -0.8 \text{ and } \mu_2 = -1.3.
\]
both modes tend to the same value.

Figure 20 illustrates the case where $\mu_1 > |\mu_2|$.

![Fig. 20. Dispersion diagram for TE modes of a DNG dielectric slab with $\varepsilon_1 = \mu_1 = 2$ and $\varepsilon_2 = \mu_2 = -1.5$.](image)

In this case, propagates only one super slow even mode, which is limited in frequency. For all the frequency band in which the mode propagates, there are always two solutions, that tend to the same value with the increase of frequency. When the two solutions intersect the surface mode ceases to propagate.

IV. METAMATERIAL LENSES

A. Optical Path and Lens Design

In general, the total optical path is equal to the geometrical path in each medium multiplied by the corresponding refractive index.

To convert the light rays emanating from a source located at point O in a plane wave is necessary to use a lens. Thus, we ensure that the optical paths of different trajectories are equal when they reach the plane wavefront $P_1P_2$ [11], Figure 21.

![Fig. 21. Optical path representation.](image)

In Figure 21 there are two optical paths that must be equal, then

$$OP_1 = OC + nCP_2.$$  \hspace{1cm} (61)

Equation (61) can be written in polar coordinates as

$$r = d + n[r\cos(\theta) - d],$$  \hspace{1cm} (62)

where

$$r = d(1 - n) \frac{1}{1 - n\cos(\theta)}.$$  \hspace{1cm} (63)

Equation (63) describes a hyperbola in polar coordinates, with an asymptote at $\cos(\theta) = 1/n$.

Transforming into Cartesian coordinates, we obtain

$$\begin{cases} x = \frac{r}{d} \cos(\theta) \\ y = \frac{r}{d} \sin(\theta) \end{cases}.$$  \hspace{1cm} (64)

After some algebraic manipulation, we obtain the equation of a hyperbola in Cartesian coordinates

$$\left(\frac{x - n}{n+1}\right)^2 - \frac{y^2}{n^2-1} = \frac{1}{(n+1)^2}. $$  \hspace{1cm} (65)

From (65) we can obtain the lenses contours. We can also see that for $n = 0$ the equation of the hyperbola is approximately the same as that of a circle. Figure 22 illustrates the lenses contours for different refractive indexes.

![Fig. 22. Lenses contours for different values of n.](image)

B. Veselago’s Flat Lens

In 1968, Veselago showed that for a particular geometry a flat lens with $n = -1$ will bring a diverging beam to a converging beam [11], as shown in Figure 23.

![Fig. 23. Passage of light rays through a Veselago flat lens.](image)

The new feature was the presence of an internal focus and the fact that the optical path from the external focus to the internal focus is zero.

Based on Figure 24 is possible to study the lenses contours with internal focus.

Due to symmetry, it is sufficient to look at the problem between the points $O_1$ and $O_2$ representing the two foci. The optical paths observed in Figure 24 should be equal, then
9

From Figure 24 it is possible to establish a second relationship between \( r_1 \) and \( r_2 \), such that

\[
\begin{align*}
 r_2^2 &= r_1^2 + (d_1 + d_2)^2 - 2r_1(d_1 + d_2)\cos(\theta). 
\end{align*}
\]  
(67)

Knowing that the optical path from the external focus to the internal focus is zero, we have

\[
\begin{align*}
 d_1 + nd_2 &= 0. 
\end{align*}
\]  
(68)

The equation (68) can only be satisfied if \( n < 0 \), then the interval of interest for the study of the lenses contours with internal focus is \(-\infty < n < 0\).

After some algebraic manipulation, we obtain the equation in polar coordinates

\[
\begin{align*}
 (n + 1)\left(\frac{r_1}{d_1}\right)^2 - 2n\cos(\theta)\left(\frac{r_1}{d_1}\right) + (n - 1) &= 0. 
\end{align*}
\]  
(69)

Converting into Cartesian coordinates we obtain the equation for the lenses contours with internal focus

\[
\begin{align*}
 (x - \frac{n}{n + 1})^2 + y^2 &= \frac{1}{(n + 1)^2}. 
\end{align*}
\]  
(70)

The equation (70) is the equation of a circle centered at \((n/(n + 1), 0)\), which in the limit of \( n \to -1 \) becomes a straight line, obtaining the flat lens. Figure 25 illustrates the lenses contours with internal focus for different refractive indexes.

C. Pendry’s Perfect Lens

Pendry has proposed that a slab of thickness \( d \), with \( \varepsilon = -1 \) and \( \mu = 1 \), will be able to image an object with infinite resolution. The positions of the object, lens and image correspond to those of Veselago shown in Figure 23. The key to this remarkable behavior is that the refractive index of the slab is \( n = \sqrt{\varepsilon\mu} = -1 \). However, the impedance of the medium, given by \( \eta = \sqrt{(\mu_0\mu)/(\varepsilon\varepsilon_0)} \), is positive so that, when \( \varepsilon = \mu = -1 \), the impedance of the medium is a perfect match to free space and the interfaces show no reflection. At the far boundary of the lens there is again an impedance match and the light is perfectly transmitted into vacuum.

If the energy transport is done in the \( z \) direction, so that all the energy is perfectly transmitted into the medium, it is necessary that

\[
k_z^2 = -\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2. 
\]  
(71)

In general, the transmission coefficient of the medium is

\[
T = \cos(k'd) = \exp\left(-i\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2}d\right), 
\]  
(72)

the negative phase results from the choice of the propagation constant imposed by causality and it is this phase reversal that allows the medium to refocus light by cancelling the phase acquired by light as it moves away from the source [12].

Let us consider a TE wave propagating in the vacuum. The electric field is given by

\[
E_0 = \exp(ik_xz + ik_xx - i\omega t), 
\]  
(73)

where the propagation constant is of the form

\[
k_z = i\sqrt{k_x^2 + k_y^2 - \left(\frac{\omega}{c}\right)^2}, 
\]  
(74)

with \((\omega/c)^2 < k_x^2 + k_y^2\).

At the interface with the medium some of the light is reflected and some is transmitted into the medium. In order to maintain causality the fields must decay as they move away from the interface. Thus, the expression of the electric field for the reflected wave is

\[
E_{0r} = r\exp(-ik_xz + ik_xx - i\omega t), 
\]  
(75)

while for the transmitted wave is

\[
E_{1t} = t\exp(ik_x'z + ik_xx - i\omega t), 
\]  
(76)

where

\[
k_x' = i\sqrt{k_x^2 + k_y^2 - \varepsilon\mu\left(\frac{\omega}{c}\right)^2}, 
\]  
(77)

with \(\varepsilon\mu(\omega/c)^2 < k_x^2 + k_y^2\).

A wave in the vacuum incident on the interface with the medium...
presents a transmission and a reflection as follows
\[ t = \frac{2\mu_k}{\mu_k + k_z'z} , \quad r = \frac{\mu_k - k_z'}{\mu_k + k_z'}. \]

(78)

On the other hand, for a wave inside the medium incident on the interface with the vacuum we obtain the transmission and reflection
\[ t' = \frac{2k_z'}{k_z' + \mu_k}, \quad r' = \frac{k_z' - \mu_k}{k_z' + \mu_k}. \]

(79)

To calculate transmission through both surfaces of the slab we must sum the multiple scattering events [12],
\[ T = tt'\exp(ik_z'd) + tt'r'\exp(3ik_z'd) + \cdots = \frac{tt'\exp(ik_z'd)}{1 - r'^2\exp(2ik_z'd)}. \]

(80)

Substituting (78) and (79) in (80) and applying the limit, we obtain the transmission coefficient
\[ \lim_{\mu \to -1} T = \exp(-ik_z'd) = \exp(-ik_zd). \]

(81)

The reflection coefficient is given by
\[ \lim_{\mu \to -1} R = \lim_{\epsilon \to -1} \frac{tt'r'\exp(2ik_z'd)}{1 - r'^2\exp(2ik_z'd)} = 0. \]

(82)

As they move away from the object plane the evanescent waves decay in amplitude, not in phase, therefore to focus the image the lens amplifies the evanescent waves rather than correcting their phase. From Figure 26 we can see that evanescent waves emerge from the far side of the medium enhanced in amplitude by the transmission process.

Fig. 26. Variation of the evanescent waves in the presence of a perfect lens.

V. CONCLUSION

We have shown that in this type of media the real part of electric permittivity and magnetic permeability are both negative, which leads to a negative refractive index, and that in a DNG medium the Poynting vector and the wave vector have opposite directions, creating backward waves. The dispersion in a DNG medium was also analyzed. Based on electromagnetic theory we obtained the equations of the time-averaged electric and magnetic energy densities. These equations have been modified to DNG media since it would lead to negative values, because these media are necessarily dispersive. Then we studied the phase velocity and group velocity, concluding that in a non-dispersive medium these velocities are equal, in a dispersive medium have different values and in a dispersive DNG medium have opposite directions.

We have also study the propagation of electromagnetic waves in two structures, the DPS-DNG interface and the DNG dielectric slab. We began to analyze the DPS-DNG interface, where we deduced the modal equations of TE and TM modes. We also found that for a medium to have a negative refractive index is sufficient, but not a necessary condition, that this medium is DNG. Then we analyzed the surface modes of the DNG dielectric slab, concluding that it allows the propagation of super slow modes, which are characterized by a phase velocity lower than the speed of light in the medium. We also concluded that the propagation in a less dense medium is only possible when using a DNG dielectric slab.

Finally we have studied the metamaterial lenses. First, we analyzed the lens design using the concept of optical path. From this concept we obtained an equation, dependent on the refractive index of the lens, which allowed us to determine the geometric shape of the lens contour. After making some simulations, we found that the lens is convex when \( n > 1 \) and concave when \( n < 1 \). Next, we studied the Veselago’s flat lens, where \( n = -1 \), which focuses the rays from a source at a point inside the lens. When analyzing the Pendry’s perfect lens, in that \( \epsilon = \mu = -1 \), we found that the impedance of the medium is a perfect match to free space and the interfaces show no reflection. The evanescent waves, that decay as they move away from the object, are amplified inside of the lens. When these waves emerge on the other side of the lens they decay until its amplitude reaches the original level in the image plane. Thus, we conclude that with this new lens both evanescent and propagating waves contribute to the resolution of the image. This result is very important, since the image resolution is no longer limited to the wavelength of light, as is the case of conventional lenses.

REFERENCES