Abstract In this thesis, the design of quasi-satellites orbits in the elliptic restricted three-body problem is addressed from a preliminary space mission design perspective. The stability of these orbits is studied by an analytical and a numerical approach and findings are applied in the study of the Mars-Phobos system. In the analytical approach, perturbation theories are applied to the solution of the unperturbed Hill’s equations, obtaining the first-order approximate averaged differential equations on the osculating elements. The stability of quasi-satellite orbits is analyzed from these equations and withdrawn conclusions are confirmed numerically. We also use the fast Lyapunov indicator, a chaos detection technique, to analyze the stability of the system. The study of fast Lyapunov indicator maps for scenarios of particular interest provides a better understanding on the characteristics of quasi-satellite orbits and their stability. Both approaches are proved to be powerful tools for space mission design.

Keywords Quasi-Satellite Orbits · Elliptic Restricted Three-Body Problem · Stability · Perturbation Theory · Fast Lyapunov Indicator · Mars-Phobos System

1 Introduction

The Martian moon Phobos is one of the most prominent candidates for future space exploration missions. The interests of a mission to this moon range from scientific to engineering purposes that make it an appealing research subject. One of the major challenges in the design of such a mission is the search for sufficiently stable orbits around the moon.

The selection of a sufficiently stable orbit to circumnavigate Phobos is very important. Depending on the mission, the spacecraft must orbit the moon enough time to update ephemerides, perform scientific experiments, select a landing site, or prepare for a landing approach (Gil and Schwartz, 2010). The stability of such orbit will increase the probability of mission success.

The small mass of Phobos, when compared to Mars’ (about seven orders of magnitude larger), relinquishes any possibility of a Keplerian orbit as the region of influence of Phobos is below its own surface, and its Hill sphere, with the Lagrange points \( L_{1,2} \) on its surface, is just a few kilometers above its surface (see fig. 1.1) with radius (Hamilton and Burns, 1992)

\[
R_H \approx a \left(1 - e^2\right) \left(\frac{m_{Ph}}{3(m_M + m_{Ph})}\right)^{1/3}
\]  

(1.1)

where \( a \) is the semi-major axis, \( e \) the orbital eccentricity of Phobos, and \( m_{Ph} \) and \( m_M \) the mass of Phobos and Mars, respectively (values are presented in table 1.1. Consequently, the problem should be treated as a three-body problem (hereafter 3BP) with Mars, Phobos, and the spacecraft (Gil and Schwartz, 2010).

Although Keplerian orbits can not be found around the moon, it is possible to find sufficiently stable orbits around Phobos in the setup of the 3BP. In this case a type of orbits assumes particular interest — the so-called quasi-satellite orbits (QSOs). They are a special type of orbits as they are not closed periodic trajectories although they tend to occupy the same region in space. In some literature, QSOs are called distant retrograde orbits (DROs).

This work concerns with the stability of QSOs, but first we must define the concept of stability. In the literature there are many definitions of stability and we adopt the one that
is best fit to help us achieve our objective, i.e., find suitable orbits around Phobos for the purposes of mission design.

One of the most used definitions states that an orbit is stable if the distance to an initially nearby orbit increases linearly with time, whereas it is chaotic if the distance to an initially nearby orbit increases exponentially with time (Meyer et al, 2009). This concept is known as exponential divergence. In this work we use chaos detection techniques to analyze the stability of QSOs that are based in this definition. However, apart from the use of the chaos indicator, this definition does not suit our purposes as there are orbits that escape from Phobos and enter in a orbit around Mars that present a stable nature.

The stability definition best fit to our purpose is one that can fulfill the orbit’s objective, i.e., orbit Phobos during enough time to perform any reconnaissance, or scientific activities required for the mission without colliding with the moon or escaping from its vicinity. This way we define a sufficiently stable orbit as follows.

A sufficiently stable orbit about Phobos is one that will orbit the moon for, at least, a period of 100 revolutions of the moon around Mars without colliding against it or get more than 1,000 km away from it.

The number of 100 revolutions of Phobos about Mars, about a month, is chosen. Other studies in the literature use similar timespans (25 days in (Wiesel, 1993) and 30 days in (Gil and Schwartz, 2010)). The upper limit of 1,000 km is chosen following (Gil and Schwartz, 2010).

### 2 Dynamics

In our work we analyze the motion of a third body in the so-called elliptic restricted three-body problem (ER3BP). The mass of the two primaries is assumed to be much larger than the mass of the third body, \( m_1, m_2 \gg m_3 \). The two primaries, which are not influenced by the third body, move in ellipses around their barycenter with the same eccentricity, \( e \), semi-major axis, \( a \), and true anomaly, \( f \) (Szebehely, 1967).

The influence of the primaries’ irregular shape and rotation is neglected and their masses are assumed to be evenly distributed — the problem is described with the point mass approximation. Smaller perturbations as the solar radiation pressure or the influence of other celestial bodies are neglected.

#### 2.1 Equations of Motion

The problem is described in a synodic reference frame centered in the second primary and rotating with the same angular velocity that the primaries orbit around their barycenter. When the reference frame is centered in the primaries’ barycenter, the pulsating coordinates defined in (Szebehely, 1967) are

\[
x = \frac{y_d(1 + e \cos f)}{a(1 - e^2)}
\]

\[
y = \frac{y_d(1 + e \cos f)}{a(1 - e^2)}
\]

\[
z = \frac{z_d(1 + e \cos f)}{a(1 - e^2)}
\]

where the index \( d \) denotes the dimensional coordinates. The shift to the second primary is performed with \( x \to x + \mu - 1 \) and, this way, the first and second primaries are at the constant positions \((1, 0, 0)\) and \((0, 0, 0)\), respectively.

The true anomaly \( f \) is introduced as the independent variable by the equation

\[
\frac{d}{dt^*} = \frac{df}{dt^*} \frac{df}{d f}
\]
where \( t^* \) is the dimensional time and, from the conservation of the angular momentum of the two primaries,

\[
\frac{df}{dt^*} = \mathcal{G}^{1/2}(m_1 + m_2)^{1/2} a^{1/2} (1 - \epsilon \cos f)^2
\]

with \( \mathcal{G} \) as the gravitational constant.

The Hamiltonian with respect to the independent variable \( f \) is found from the Lagrangian function \( L = T - U \) by

\[
H = \sum_i p_i \dot{q}_i - L
\]

resulting in

\[
H = \frac{1}{2} \left[ (p_x + y)^2 + (p_y - x - \mu + 1)^2 + p_z^2 \right] + \frac{1}{2} \Omega^2 - \Omega
\]

with the amended potential

\[
\Omega = \frac{1}{1 + \epsilon \cos f} \left[ \frac{1}{2} ((x + \mu - 1)^2 + y^2 + z^2) \right. \\
+ \frac{1}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu (1 - \mu) \right]
\]

where \( r_1^2 = (x - 1)^2 + y^2 + z^2 \) and \( r_2^2 = x^2 + y^2 + z^2 \) are the distances to the first and second primaries, respectively, \( p_i \) are the conjugate momenta of the position coordinates \( q_i \), and \( \mu \) is the mass parameter of the 3BP

\[
\mu = \frac{m_2}{m_1 + m_2}
\]

The Hamilton's equations of motion

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}
\]

can be combined into a set of three second-order differential equations

\[
\ddot{x} - 2\dot{y} = \Omega_x \\
\ddot{y} + 2\dot{x} = \Omega_y \\
\ddot{z} + \dot{z} = \Omega_z
\]

with \( \Omega_i \) being the derivative of \( \Omega \) with respect to the coordinate \( i \).

An Invariant Relation The ER3BP does not possess the Jacobi integral characteristic of the 3BP. However, it is possible to derive an approximate invariant relation (Szebehely, 1967)

\[
C = 2 \Omega' - V^2
\]

where \( C \) is the modified Jacobi integral, \( \Omega' = \Omega - 1/2 \dot{z}^2 \), and \( V \) is the third body velocity in the synodic reference frame.

2.2 Variational Equations

The initial deviation vector contains the displacements of an initial state of the system. The stability of a system can be analyzed by the evolution of these initial displacements through the so-called chaos indicators, CIs. The time evolution of orbits and deviation vectors is addressed in (Skokos, 2010) and reviewed here.

The Hamilton’s equations of a system with \( N \) degrees of freedom can be written in matrix notation as

\[
x = f(x) = \begin{bmatrix} \frac{\partial H}{\partial p} - \frac{\partial H}{\partial q} \end{bmatrix}^T = \nabla_x H(t, x)
\]

where \( x(t) = (q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t)) \), and \( \nabla_x \) is the vector differential operator in respect to the state of the system \( x \), where

\[
J = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix}
\]

with \( I_N \) being the \( N \times N \) identity matrix and \( 0_N \) the \( N \times N \) matrix with all its elements equal to zero. The solution of (2.11) is formally written as

\[
x(t) = \Phi'(x(0)).
\]

where \( \Phi' \) is the fundamental matrix of solutions of (2.11) and maps the evolution of \( x(0) \) to \( x(t) \).

Now, let us see how can we determine the evolution of the deviation vector. We denote by \( \nabla_x \Phi' \) the linear mapping of the evolution of the deviation vector \( w(t) \)

\[
w(t) = \nabla_x \Phi' w(0)
\]

where \( w(0) \) and \( w(t) \) are deviation vectors with respect to the given orbit at times \( t = 0 \) and \( t > 0 \), respectively.

An initial deviation vector

\[
w(0) = (\delta q_1(0), \ldots, \delta q_n(0), \delta p_1(0), \ldots, \delta p_n(0))
\]

from an orbit \( x(t) \) evolves according to the so-called variational equations

\[
w = \frac{\partial f}{\partial x}(x(t)) \cdot w = \nabla^2_x H(t, x) \cdot w = A(t) \cdot w
\]

where \( \nabla^2_x H(t, z) \) is the Hessian matrix of the Hamiltonian computed on the reference orbit \( x(t) \), i.e.,

\[
\nabla^2_x H(t, x)_{i,j} = \frac{\partial^2 H}{\partial x_i \partial x_j} \bigg|_{\Phi'(x(0))} \quad i, j = 1, 2, \ldots, 2N
\]

Note that equation (2.16) represents a set of linear differential equations with respect to \( w \), having time-dependent coefficients, since matrix \( A(t) \) depends on the particular reference orbit, which is a function of time \( t \). The solution of (2.16) can be written as

\[
w(t) = Y(t) \cdot w(0)
\]
where $\mathbf{Y}(t)$ is denominated the fundamental matrix of solutions of (2.16), satisfying the following equation
\[
\frac{\partial \mathbf{Y}}{\partial x} \cdot \mathbf{Y}(t) = \mathbf{A}(t) \cdot \mathbf{Y}(t), \quad \mathbf{Y}(0) = \mathbf{I}_2N. \tag{2.19}
\]
The analysis of the deviation vector is the basis for most CIs taking advantage of the concept — introduced by Lyapunov — of exponential divergence, i.e., that of initially nearby chaotic orbits separate exponentially with time, whereas nearby regular orbits separate linearly with time.

2.2.1 Fast Lyapunov Indicator

Chaos indicators are numerical techniques developed to accomplish one of the most important aspects in the study of the behavior of dynamical systems: the differentiation of trajectories of regular and chaotic nature. Such task is of difficult realization as the difference between two trajectories of different nature can be very subtle (Maffione et al, 2011). This was the motivation behind the development of CIs.

In (Maffione et al, 2011), a comparison of some of the most used methods is performed. In this paper, the fast Lyapunov indicator (FLI) presented the best relation between speed of convergence and capability to identify chaotic motion and, thus, it was selected as the CI to be used in our work.

The FLI was introduced in (Froeschlè et al, 1997) motivated by the need to have a faster method to distinguish between chaotic and regular orbits. In some of the FLI definitions, it can also distinguish resonant from non-resonant motion (Skokos, 2010).

In our work we use an adapted FLI definition from (Villac and Aiello, 2005)
\[
FLI = \sup_{t \leq T} \sup_{\tau \leq |t|} \ln \left| \frac{\left\| \mathbf{w}_i(f + \tau) \right\|}{\left\| \mathbf{w}_i(f) \right\|} \right| \tag{2.20}
\]
where $\tau$ is the step-size and $\mathbf{w}_i(f)$ is a basis of $n$ deviation vectors with initial conditions
\[
\mathbf{w}_i(0) = (\mathbf{w}_1(0), \mathbf{w}_2(0), \ldots, \mathbf{w}_n(0)) = \mathbf{I}_n \tag{2.21}
\]
with $n = 6$. This definition provides only one value per set of initial conditions, making it possible to construct FLI maps where chaotic and regular regions are easily distinguishable.

There is still one issue in the definition of the FLI: the normalization. It is stated and proved in (Skokos, 2010) that, although for different chosen norms the value of the FLI changes, its capacity to distinguish regular from chaotic motion remains intact, i.e., the norm choice only affects the FLI value quantitatively but not qualitatively. In the computation of the FLI we use the norm
\[
\left\| \mathbf{w} \right\| = \sqrt{\frac{1}{r} (w_x^2 + w_y^2 + w_z^2) + \frac{1}{p} \left( w_{x,1}^2 + w_{x,2}^2 + w_{x,3}^2 \right)} \tag{2.22}
\]
where $r$ and $p$ represent the Euclidean norms of the position and momenta of the third body state at the current time $t$, respectively.

3 QSO Solutions and Stability

The equations of motion of the ER3BP (2.9) can be simplified if we consider $\mu \ll 1$ and $x, y, z \ll 1$ to obtain
\[
\begin{align*}
\dot{x} - 2\dot{y} - \frac{3x}{1 + e \cos f} &= f_x = - \frac{1}{1 + e \cos f} \left( \frac{\mu x}{r^2} \right) \\
\dot{y} + 2\dot{x} &= f_y = 0 \\
\dot{z} + z &= f_z = - \frac{1}{1 + e \cos f} \left( \frac{\mu z}{r^2} \right) \tag{3.1}
\end{align*}
\]
where $f_i$ is the perturbing function caused by the second primary in respect to the coordinate $i$.

3.1 Unperturbed Hill’s Solutions

We start by considering the simplified Hill’s case, $\mu \to 0$, where the equations of motion (3.1) become
\[
\begin{align*}
\dot{x}_{np} - 2\dot{y}_{np} - \frac{3x_{np}}{1 + e \cos f} &= 0 \\
\dot{y}_{np} + 2\dot{x}_{np} &= 0 \\
\dot{z}_{np} + z_{np} &= 0 \tag{3.2}
\end{align*}
\]

The system of equations (3.2) is independent in $z$ and its solution is easily obtained
\[
\begin{align*}
z_{np}(f) &= z_{0np} \cos f + \dot{z}_{0np} \sin f \\
\dot{z}_{np}(f) &= - z_{0np} \sin f + \dot{z}_{0np} \cos f \tag{3.3}
\end{align*}
\]
where $z_{0np}$ and $\dot{z}_{0np}$ are the initial conditions for the non-perturbed Hill case.

The other two coordinates form a system of differential equations with time-dependent periodic coefficients. There is no explicit method of finding the solution of this type of systems. However, it is possible to find the solution in $x$ and $y$ using the program Mathematica (Wolfram Research, Inc., 2011) to obtain one independent solution and reference (Zwillinger, 1997) provides a method to obtain the second independent solution from the first.

The obtained solutions present discontinuities due to their trigonometric functions. The practical meaning is that the solutions are unstable due to secular terms that appear after correcting the discontinuities caused by the trigonometric terms. Nevertheless, there is a family of stable solutions when a condition that depends on the initial true anomaly $f_0$
and on the initial coordinates is fulfilled. This condition for \( f_0 = 0 \) is, in dimensional coordinates,
\[
\dot{y}_{\text{unpert}} = -\frac{2 + e}{1 + e} n x_{\text{unpert}} \tag{3.4}
\]
where \( n \) is the mean motion of the primaries, and leads to the solutions
\[
x_{\text{unpert}}(f) = (1 + e \cos f)(C_1 \sin f - C_2 \cos f)
\]
\[
y_{\text{unpert}}(f) = \frac{1}{2} [(e + 4 \cos f + e \cos (2f))C_1
+ 2C_2(2 + e \cos f) \sin f + 2C_4] \tag{3.5}
\]
where \( C_i \) are integration constants that can be computed from the ‘unperturbed initial conditions’. In fig. 3.1 a sample orbit of this family of stable orbits is presented.

![Fig. 3.1 Parametric plot of the solutions of the unperturbed equations under the stability condition, \( C_3 = -eC_2 \), with the following parameters: \( e = 0.25, C_1 = -0.3, C_2 = 0.2, C_3 = -0.05, C_4 = -0.2 \).](Image)

3.2 Osculating Elements

Experimentation shows that the constants \( C_1 \) and \( C_2 \) influence mainly the amplitude of the orbit whereas \( C_4 \) influences the position of the origin of the orbit. This suggests that a change to some variation of ‘polar’ constants would be advantageous.

The constants \( C_1, C_2, \) and \( C_4 \) can be transformed to a set of three alternative equivalent constants \( \alpha, \phi, \text{ and } \delta_i \) by
\[
\begin{align*}
C_1 &= -\alpha \sin \phi \\
C_2 &= -\alpha \cos \phi \\
C_4 &= \delta_i \\
\dot{z}_{\text{unpert}} &= \gamma \cos \psi \\
\dot{\theta}_{\text{unpert}} &= -\gamma \sin \psi
\end{align*}
\]
\[
\begin{align*}
\phi &= \arctan \left( \frac{C_1}{C_2} \right) \\
\psi &= \arctan \left( -\frac{\dot{\theta}_{\text{unpert}}}{\dot{z}_{\text{unpert}}} \right) \\
\alpha &= (C_1^2 + C_2^2)^{1/2} \\
\gamma &= \left( \dot{z}_{\text{unpert}} + \dot{\theta}_{\text{unpert}} \right)^{1/2} \\
\delta_i &= C_4 \tag{3.6}
\end{align*}
\]
The solutions for Hill’s problem (equations (3.2)) can now be rewritten as
\[
\begin{align*}
x_{\text{pert}}(f) &= \alpha(1 + e \cos f) \cos (f + \phi) + \delta_i \\
y_{\text{pert}}(f) &= -\alpha(2 + e \cos f) \sin (f + \phi) + \delta_j \\
z_{\text{pert}}(f) &= \gamma \cos (f + \psi) \tag{3.7}
\end{align*}
\]
with \( \delta_i = 0 \) in the unperturbed case.

These constants are known as osculating elements (Kogan, 1989) and are represented in fig. 3.2. The orbit’s projection on the \( x-y \) plane resembles an ellipse (distorted by the eccentric terms) with semi-axes \( \alpha \) and \( 2\alpha \) along the \( x \) and \( y \) direction, respectively. The point oscillates in the \( z \)-direction with amplitude \( \gamma \). The angles \( \phi \) and \( \psi \) define, respectively, the motion phase of the point in the projection of the orbit in the \( x-y \) plane and the motion phase in the oscillations along the \( z \) direction. The point travels in the orbit in a retrograde direction with period \( 2\pi \) (in the unperturbed case). The center of the epicycle is shifted along the \( x \) and \( y \) directions by \( \delta_i \) (zero for the unperturbed case) and \( \delta_j \), respectively. Finally, the contour of the orbit lies in a plane which intersects the \( x-y \) plane along a line. This line has an inclination of \( \beta = \psi - \phi \) to the \( x \)-axis.

![Fig. 3.2 Geometric representation of the osculating elements. \( O \) is the origin of the Cartesian reference frame and \( C \) is the orbit’s center. Figure adapted from (Gil and Schwartz, 2010).](Image)

3.3 Influence of the Second Primary

The study of the unperturbed Hill problem serves as the basis for the perturbation theories applied when considering the influence of the second primary.

The equations of motion for the perturbed case (3.1) cannot be solved analytically but, under convenient assumptions, an analysis of the motion of QSOs is possible and approximate solutions can be obtained. For this purpose we
assume that the displacements of the orbit and its amplitude in the z direction are much smaller than the amplitude of the orbit in the x-y plane, i.e., $\delta_t/\alpha, \delta/\alpha, \gamma/\alpha \ll 1$.

### 3.3.1 Region of Stability

The solutions of the perturbed case can be considered as the sum of the solutions to Hill’s case plus a small enough perturbation

\[
\begin{align*}
    x(f) &= \alpha(1 + e \cos f) \cos(f + \phi) + u \\
    y(f) &= -\alpha(2 + e \cos f) \sin(f + \phi) + \delta_t + v \\
    z(f) &= \gamma \cos(f + \psi) + w
\end{align*}
\]  

(3.8)

where $u, v, w$ are the perturbation variables and $\alpha$ and $\phi$ are assumed to be constant which limits this analysis to quasi-synchronous QSOs. After substitution on (3.1), the equations of motion in the perturbation variables become

\[
\begin{align*}
    \ddot{u} - 2\dot{v} - \frac{3u}{1 + e \cos f} &= f_x(u, v, w) \\
    \ddot{v} + 2\dot{u} &= f_y(u, v, w) \\
    \ddot{w} + w &= f_z(u, v, w)
\end{align*}
\]  

(3.9)

The perturbation functions $f_x, f_y, f_z$ are linearized about the perturbation variables, the eccentricity and the small quantities $\delta_t/\alpha, \delta/\alpha, \gamma/\alpha$. Although simpler, the resulting equations of motion cannot be solved analytically yet. Nevertheless, the equations of motion are averaged over a period of $2\pi$ for the angle $\theta = f + \phi$ to provide enough insight into the stability properties of the system. This technique neglects the periodic effects on the system but allow the analysis of the secular effects. This way, the averaged equations of motion on the perturbation variables are

\[
\begin{align*}
    \ddot{u} - 2\dot{v} - \frac{3u}{\sqrt{1 - e^2}} &= \bar{f}_x(u, v, w) \\
    \ddot{v} + 2\dot{u} &= \bar{f}_y(u, v, w) \\
    \ddot{w} + w &= \bar{f}_z(u, v, w)
\end{align*}
\]  

(3.10)

The equation on the $w$ variable is independent and stable. The analysis of the stability of the system is reduced to the other two equations that form the fourth-order system

\[
\begin{pmatrix}
    \dot{u} \\
    \dot{v} \\
    \dot{w}
\end{pmatrix} =
\begin{pmatrix}
    0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & \mu \alpha^3 A_1 & 0 \\
    0 & 0 & \mu \alpha^3 A_2 & -2
\end{pmatrix}
\begin{pmatrix}
    u \\
    v \\
    w
\end{pmatrix}
\]  

(3.11)

with

\[
\begin{align*}
    A_1 &= \frac{4E - K}{3\pi} > 0 \\
    A_2 &= \frac{K - E}{3\pi} > 0
\end{align*}
\]  

(3.12)

where $K$ and $E$ are the complete integrals of the first and second kind, respectively, with module $k = \sqrt{3}/2$.

The system is stable when its eigenvalues have a non-positive real part. This is accomplished under the following conditions

\[
\begin{align*}
    \alpha > \left(\frac{\mu(A_1 + A_2)}{4 - \frac{3}{\sqrt{1 - e^2}} \alpha^3 (A_1 + A_2)}\right)^{1/3} \\
    4 - \frac{3}{\sqrt{1 - e^2}} \alpha^3 (A_1 + A_2)^2 > 0 \\
    -4 \left(\frac{\mu^2}{\alpha^3} A_1 + \frac{3}{\sqrt{1 - e^2}} \frac{\mu}{\alpha^3} A_2\right) > 0
\end{align*}
\]  

(3.13)

### 3.3.2 Approximate Solutions in the Osculating Elements

We now obtain the approximate solutions in the osculating elements. We start from the solutions to the unperturbed problem (3.7) and assume that under the perturbation of the second primary the previously constant osculating elements have a variation. The differential equations on the osculating elements are computed with the method of variation of the arbitrary constants (Danby, 1962). These equations are too complex to be useful but, again, the equations are linearized in the small quantities and averaged to obtain

\[
\begin{align*}
    \dot{\alpha} &= 0 \\
    \dot{\phi} &= \frac{\mu}{\pi \alpha^3 E} \\
    \dot{\delta}_x &= \frac{\mu}{6 \pi \alpha^3}(K - E)\delta_x - \frac{\mu}{9 \pi \alpha^3} e(5K - 8E)\sin \phi \\
    \dot{\delta}_y &= \frac{2\mu}{3 \pi \alpha^3}(4E - K)\delta_y - \frac{\mu}{6 \pi \alpha^3} e(70E - 19K)\cos \phi \\
    \dot{\gamma} &= \frac{\mu}{6 \pi \alpha^3} \gamma(5E - 2K)\sin(2\beta) \\
    \dot{\psi} &= \frac{\mu}{6 \pi \alpha^3}(3E + (5E - 2K)\cos(2\beta)) \\
    \dot{\beta} &= \dot{\psi} - \dot{\phi} = \frac{\mu}{6 \pi \alpha^3}(-3E + (5E - 2K)\cos(2\beta))
\end{align*}
\]  

(3.14)

The differential equation on $\dot{\phi}$ is independent from the other osculating elements and its solution is easily derived

\[
\dot{\phi} = \frac{\mu}{\pi \alpha^3 E} f + \phi_0
\]  

(3.15)

After substitution of $\dot{\phi}$, the system composed by $\delta_x$ and $\delta_y$ becomes separable from the other variables and can be represented as a second-order system

\[
\begin{align*}
    \frac{d}{df} \begin{pmatrix}
        \delta_x \\
        \delta_y
\end{pmatrix} =
\begin{pmatrix}
    0 & a \\
    -b & 0
\end{pmatrix}
\begin{pmatrix}
    \delta_x \\
    \delta_y
\end{pmatrix} +
\begin{pmatrix}
    -c \sin \phi \\
    -d \cos \phi
\end{pmatrix}
\end{align*}
\]  

(3.16)

with $a, b, c, d$ as positive constants. The motion of both $\delta_x$ and $\delta_y$ is composed by two periodic functions with different
periods and amplitudes. The first has a larger amplitude, is not influenced by the primaries eccentricity, and has period

$$P_1 = \frac{2\pi}{\sqrt{ab}} = \frac{6\pi^2}{(5EK - 4E^2 - K^2)^{1/2}} \frac{\tilde{\alpha}^3}{\mu} \approx 37.1483 \frac{\tilde{\alpha}^3}{\mu}$$

(3.17)

whereas the second periodic function, much smaller in amplitude, has period equal to $\phi$

$$P_2 = \frac{2\pi}{\phi} = \frac{2\pi^2 \tilde{\alpha}^3}{E \mu} \approx 16.2992 \frac{\tilde{\alpha}^3}{\mu}$$

(3.18)

The system (3.14) is also independent on the variable $\gamma$, and its solution is

$$\tilde{\beta} = -\arctan \left( \frac{m - n}{m + n} \tan \left( \sqrt{m^2 - n^2 (f + C_\tilde{\beta})} \right) \right)$$

(3.19)

with $m = 3\mu E/(6\pi \tilde{\alpha}^3)$ and $n = 3\mu (5E - 2K)/(6\pi \tilde{\alpha}^3)$. The frequency and period of this oscillation are obtained in the same fashion, $\omega_{\tilde{\beta}} = \sqrt{m^2 - n^2}$ and

$$P_{\tilde{\beta}} = \frac{2\pi}{\sqrt{m^2 - n^2}} = \frac{6\pi^2}{(5EK - 4E^2 - K^2)^{1/2}} \frac{\tilde{\alpha}^3}{\mu} \approx 37.1483 \frac{\tilde{\alpha}^3}{\mu}$$

(3.20)

After substitution of $\tilde{\beta}$ the solution on $\gamma$ is obtained

$$\gamma = C_\gamma (m - n \cos(2\tilde{\beta}))^{1/2}$$

(3.21)

which oscillates two times faster than $\tilde{\beta}$ and, thus, $P_{\gamma} = P_{\tilde{\beta}}/2$.

The relation between the orbital mean motions of the QSO, $n_{QSO}$, and of the second primary, $n$, in their orbits around the first primary can also be obtained. The QSO has orbital mean motion $n_{QSO} = 1 + \tilde{\phi}$ and $n$ is 1.

$$\frac{n_{QSO}}{n} = 1 + \tilde{\phi} = 1 + \frac{\mu E}{\pi \tilde{\alpha}^3} \approx 1 + 0.385491 \frac{\mu}{\tilde{\alpha}^3}$$

(3.22)

The second primary, located on the origin of the reference frame, acts as a restoring force on the third-body. This force varies inversely with the distance between these two bodies. If the third-body gets too far from the origin, there is a chance that this restoring force will not be strong enough to maintain the third-body in orbit around the second primary.

The inclination of the QSO influences the distance of the third body to the second primary as the motions in the $z$ direction and on the $x$-$y$ plane are independent. Consequently, the inclination of the QSO also influences the restoring capability of the second primary. A critical value for the ratio $\gamma/\alpha$, for which this restoring capability vanishes, can be derived.

The motion of the third-body in the $x$-$y$ plane is nearly elliptic and the distance to the second primary is maximum when it crosses the $y$-axis in $y = \pm 2\alpha \pm \delta_y$. The motion in the $z$-direction has also to be considered. The maximum distance between the second and third bodies is achieved when the third-body achieves the maximum height (in absolute value) $z = \pm \gamma$ in the same point that achieves the maximum distance to the second primary in the $x$-$y$ plane, $y = \pm 2\alpha \pm \delta_y$. This is the worst-case scenario for the analysis of the restoring capability of the second primary with the inclination of the QSO and it is defined by an angle $\beta = \pi/2$ (angle between the intersection line of the QSO plane with the primaries’ orbital plane and the positive $x$ semi-axis).

If we give up on the assumption that the quantity $q = \gamma/\alpha$ is small, the system composed by $\tilde{\delta}_x$ and $\tilde{\delta}_y$ in (3.16) maintains the same form but has different values for the constants $a$ and $b$. The complete elliptic integrals now have module $k = \sqrt{(q^2 + 3)/(q^2 + 4)}$. The period of the main motion in this case is

$$P = \frac{2\pi}{\sqrt{ab}} = \frac{2\pi^2 (q^2 + 3)(q^2 + 4)}{[(4K - (q^2 + 3q^2 + 4)E)(4(q^2 + 4)E - 4K)]^{1/2}} \frac{\tilde{\alpha}^3}{\mu}$$

(3.23)

We now want to compute the critical ratio $q_c$ for which separation occurs. Analytically, the period is infinite for ejected orbits, hence, the quantity $q_c$ can be computed numerically by finding the root of the denominator in (3.23)

$$[\{(4K - (q_c^2 + 3q_c^2 + 4)E)(4(q_c^2 + 4)E - 4K)\}]^{1/2} = 0$$

$\rightarrow q_c = 0.961073$

(3.24)

This conclusion is based on the averaged equations of motion where the peaks of periodic effects are neglected. Thus, it is expected that separation occurs before the value found for $q_c$. The computed value is merely a statement that QSOs with $q > q_c$ will suffer orbit separation.

3.4 Application to the Mars-Phobos System

We now apply the results to the case of a QSO around Phobos. All the parameters computed here are for the case of initial true anomaly $f_0 = 0$. The values in table 1.1 are used to compute the different parameters in dimensional values.

This way, the stability conditions for quasi-synchronous orbits are

$$\begin{cases} 
\alpha > 17.3774 \text{ km} \\
\alpha > 29.4262 \text{ km}
\end{cases}$$

(3.25)

with the second condition prevailing over the first.
The periods of the osculating elements are, as a function of the dimensional amplitude (in kilometers)

\[
\begin{align*}
P_1 &= P_0 = 39.1483 \left( \frac{\alpha^3}{\mu a^3 (1-e)^3 n} \right) = 13.1961 \alpha^3 \\
P_2 &= 16.2992 \left( \frac{\alpha^3}{\mu a^3 (1-e)^3 n} \right) = 5.49413 \alpha^3 \\
P_3 &= 39.1483 \left( \frac{\alpha^3}{2 \mu a^3 (1-e)^3 n} \right) = 6.59806 \alpha^3
\end{align*}
\]

(3.26)

The relation between the orbital mean motions of the QSO and Phobos’ can also be studied through equation (3.22). In table 3.1 the amplitudes of QSOs with ratios represented by two consecutive small integers are presented. From (Wiesel, 1993), it is known that sufficiently stable resonant orbits exist for these values of the ratio \( n_{QSO}/n \). Note that these non-synchronous orbits are not restricted by the conditions (3.25).

<table>
<thead>
<tr>
<th>( n_{QSO}/n )</th>
<th>Amplitude [km]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:1</td>
<td>17.117</td>
</tr>
<tr>
<td>3:2</td>
<td>21.5661</td>
</tr>
<tr>
<td>4:3</td>
<td>24.687</td>
</tr>
</tbody>
</table>

Table 3.1 Amplitudes for values of the ratio \( n_{QSO}/n \) represented by a fraction of two small consecutive integers.

4 Numerical Exploration of QSOs

We now integrate numerically the equations of motion and the variational equations to obtain the evolution of both the orbit and the deviation vector. The solutions obtained numerically are the basis for the study of the stability of the Mars-Phobos system with the FLI maps.

Here we only present some of the conclusions of the performed study. The complete study is found in the thesis dissertation.

4.1 FLI Maps

The representation of the values of the FLI over a set of initial conditions generates a FLI map (or stability map) where regions filled with sufficiently stable orbits can be identified.

We study the problem of the ER3BP which is defined by 6 parameters plus the origin of the normalized time \( f_0 \). The FLI maps represent the FLI value as a function of two varying parameters, hence, the set of initial conditions needs to have 5 fixed initial parameters. A complete study of the stability of the system is not possible but it is possible to study the stability in a plane of two varying parameters. This approach provides enough insight into the stability properties of the system to help in the choice of a sufficiently stable QSO.

The invariant relation defined in equation (2.10) can be used to compute an initial parameter from the initial modified Jacobi integral \( C_0 \) and the two varying parameters. The characterization of FLI maps as a function of the modified Jacobi integral provides insight into the sensibility of the stability regions to the initial conditions.

Orbits are integrated over 100 periods of Phobos around Mars and the FLI value is computed in respect to this time-span. The dark regions of the FLI maps correspond to regions of mostly regular motion. The lighter areas are not true values of the FLI but values assigned to orbits that escaped Phobos’ vicinity or crashed against the moon. Most of these orbits are stable in nature, hence, a FLI value is assigned to them so they can be distinguished from sufficiently stable QSOs with

\[
FLI_{esc} = 20 - 10 \frac{f_{esc}}{f_f}
\]

(4.1)

where \( f_{esc} \) is the normalized time when the third body escaped from Phobos and \( f_f \) is the final time of integration.

4.1.1 Planar QSOs

Let us begin with the stability analysis of planar QSOs with the variation of \( x_0 \) and \( y_0 \). This analysis is performed for the case of initial true anomaly \( f_0 = 0 \), i.e., Phobos’ perigee passage. The other initial parameters \( y_0, z_0, \dot{x}_0, \dot{z}_0 \) are set to zero.

The orbital amplitude in the \( x-y \) plane, \( \alpha \), is best measure in the passage of the orbit by \( y_0 = 0 \) because experimentation suggests that the displacement of the QSO in the \( x \) direction is much smaller than in the \( y \) direction, i.e., \( \delta_x << \delta_y \).

![FLI Maps for 2D QSOs in the Mars-Phobos system with initial conditions negative \( x_0 \) and positive \( y_0 \). The remainder of the initial parameters are set to zero.](image)
The results presented in figure 4.1 suggest an almost linear relation between $x_0$ and $\dot{y}_0$ for sufficiently stable QSOs. This FLI map also suggests that amplitudes as large as 400 km are possible although the stability in $y_0$ seems to decrease with the increase of the amplitude $\alpha \approx x_0$. The stability map also suggests that the minimum amplitude for sufficiently stable quasi-synchronous orbits is in the interval 35–40 km, a value not far from the value of 29.43 km that we estimated in the analytical approach of our problem.

Let us now address the stability of QSOs with the variation of $y_0$ and $\dot{x}_0$. The initial true anomaly and the other initial parameters are set to zero. We are interested in the comparison with the previous case to assess which of the two cases presents as a better candidate for the transfer to a QSO or for the escape from a QSO. The FLI map on the analyzed

parameters $y_0$ and $\dot{x}_0$ (fig. 4.2) suggests that the stability region in this case is larger (regarding the analyzed parameters) than the previous case where $x_0$ and $y_0$ were varied and, thus, the QSO stability is less sensitive to the variation of the initial velocity when maintaining the original direction. For now, no conclusions can be made regarding the sensitivity of the QSO stability to changes in the direction of the velocity which analysis is addressed in the thesis dissertation. A sample QSO is presented in figure 4.3. Note that the maximum distance for which sufficiently stable QSOs are found is, roughly, 270 km, a value that is not in accordance with the value obtained in the previous analysis since, from our analytical approach, it is predicted that a QSO reaches larger distances in the $y$-axis than in the $x$-axis. In the previous case we found sufficiently stable QSOs at larger distances in the $x$-axis. A possible explanation is that large amplitude QSOs require a velocity in the $y$-axis to remain stable. A hypothesis that is analyzed in the thesis dissertation. It is found that with an initial velocity in the $y$ direction sufficiently stable QSOs can be found 900 km from the moon for an initial modified Jacobi integral $C_0 = 2.9547$.

4.1.2 Three-Dimensional QSOs

Let us now analyze the stability of three-dimensional QSOs varying the initial vertical velocity and the initial position in the negative $y$ semi-axis. The velocity on the $x$ axis is computed with the modified Jacobi integral and the remainder of the initial conditions are set to zero, including the initial true anomaly.

The amplitude of a QSO in the $z$ direction, $\gamma$, is in dimensional coordinates

$$\gamma = \sqrt{\frac{x_0^2}{n^2} + \frac{\dot{z}_0^2}{n^2}}$$  \hspace{1cm} (4.2)

where $n$ is the mean orbital motion. Consequently, the larger the initial vertical velocity, the larger $\gamma$ will be.
The FLI map in figure 4.4 suggests that the maximum $\gamma$ (or maximum initial vertical velocity) increases with the increase of the amplitude up to the point where the maximum distance of the orbit reaches values for which the restoring force exercised by Phobos is not strong enough to keep the probe in orbit. After this point, the maximum $\gamma$ starts to decrease with the amplitude of the orbit to not surpass this maximum distance for which Phobos can still keep the probe in orbit.

In figure 4.5, we present an example of a QSO for the limiting case where a value of $q = \gamma/\alpha \approx 65/70 = 0.93$ is achieved, a value close to our prediction of $q^* = 0.96$. Notice that the angle between the line of intersection of the plane of the orbit with the $x$-$y$ plane and the positive $x$ semi-axis, $\beta$, varies over its whole domain, from 0 to $2\pi$, hence, the worst-case scenario, $\beta = \pi/2$, is met.

5 Conclusions

This work focused on the use of analytical and numerical techniques to obtain and assess the stability of the so-called quasi-satellite orbits in the elliptic restricted three-body problem.

The non-autonomous equations of motion for Hill’s problem, derived from the Hamiltonian formulation, were solved to obtain the unperturbed solutions. These serve as the basis for the perturbation theories applied in two different formulations, each providing its own motion and stability considerations from the analysis of the approximate averaged equations of motion for $\mu << 1$, $e << 1$, and $x, y, z << 1$.

The fast Lyapunov indicator, a chaos indicator based on the analysis of the evolution of the deviation vector, was used to compute the FLI maps to study stability regions in which sufficiently stable QSOs are found. Chaotic regions were not found in the studied regions. The study of the FLI maps over sets of initial conditions provided insight about stability regions in the Mars-Phobos system and their relation with the osculating elements was addressed.

The present work comprend some restrictions regarding both the analytical and numerical approaches which pose interesting subjects for future work. The validity of the analytical approach is limited by the assumptions $\mu << 1$, and $e << 1$. The estimated stability considerations seem to hold only for a range of values of the amplitude. The overcome of these limitations is an interesting challenge.

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