**VARIATIONAL METHODS IN OPTICAL FIBERS AND DISPERSION COMPENSATION TECHNIQUES**

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**ABSTRACT**

Pulse propagation in optical fibers, operated in the linear or nonlinear regimes, is addressed. The work begins with the study of geometric optics aiming to describe media with a variable profile of the index of refraction. This is done using variational methods based on Fermat’s principle. Namely, the ray trajectory is analyzed using the Euler-Lagrange equations. The main application is the Luneberg lens.

Light propagation is described both through the ray theory and the modal theory, followed by the deduction of the pulse propagation equation in the linear and the nonlinear regimes with the objective of studying pulse evolution along the fiber, namely the effects of dispersion and self-phase modulation.

Numerical simulation of pulses propagating in optical fibers is presented and the use of dispersion compensating techniques such as dispersion compensating fibers for the linear regime and dispersion decreasing fibers for the nonlinear regime is considered.

**Keywords:** variational methods, total internal reflection, optical fiber, propagation, chirp, group velocity dispersion, self-phase modulation, dispersion compensation, solitons

**1. INTRODUCTION**

A communication system is a link between two points through which information is transmitted using a carrier. An optical communication system uses the electromagnetic waves from the optical spectrum, contained between the far infrared (100 μm) and the ultraviolet (0.05 μm) to carry the information.

The basic components of an optical communication system are: an optical transmitter, that converts the electrical signal into optical signals sending them to the optical fiber, an optical fiber and an optical receiver, that receives the optical signal converting it to an electrical signal [1] (Figure 1).

The work developed by the physicists Daniel Collandon and Jacques Babinet was essential to the development of fiber optic based communication systems. In the decade of 1840, they were the first to demonstrate the possibility of redirecting light through refraction, the fundamental principle for the propagation of light in optical fibers [2]. Credit for this discovery was, however, given to physicist John Tyndall in 1854.

In the second half of the twentieth century, optical fibers suffered major breakthroughs. In early 1950s, a partnership between Brian O’Brien and Narinder Kapany [2] resulted in the development of the first communication system to ever use glass fibers, transmitting information through pulses of light.

In the 1960s, optical communication was faced with two obstacles. First, there was the need of a source capable of generating optical pulses. And second, the inexistence of an adequate transmission media.

The first problem was solved with the development of Laser that allowed carrying up to 10000 times more information than the highest radio frequencies used. Still it was not an adequate media for free space propagation due to the sensibility to environmental conditions.

In 1966, Charles Kao (who later received the Nobel Prize of Physics for his work in the field of optical fibers) and Charles Hockham, proposed optical fibers as the ideal light propagation media [2] as long as losses were of the order of 20 dB/km. This goal was reached in 1970 by Robert Maurer, Donald Keck and Peter Schultz making viable the use of optical fibers in communication systems.

In the last decades several generations of optical fiber communication systems were developed. The development of optical amplifiers allowed the amplification of signals without the use of electronics. Erbium doped fiber amplifiers (EDFA’s) [1] where the more important of these allowing the increase of distance between repeaters.

Currently the fourth generation uses optical amplification to increase distance between amplifiers and wavelength-
division multiplexing (WDM) system [1] becoming the first photonic generation. The use of WDM techniques allows bit rates higher than 1 TB/s.

The fifth generation was highly anticipated. Having solved the losses problem by using amplifying fibers, dispersion has become the greater problem to be addressed. To solve this problem, several techniques have been developed [1] such as pre and post compensating dispersion systems, dispersion management systems and soliton based systems [3]. All of these solutions use optical amplification, WDM and dispersion management.

Optical fibers were a revolution in communication systems and, along with photonics, became the backbone of communication systems. The increasing need for high bandwidth digital networks with integrated services is leading to an increase of FTTC (fiber to the curb) and FTTH (fiber to the home) networks.

2. VARIATIONAL METHODS APPLIED TO GEOMETRIC OPTICS

Variational methods are applied to deduct Snell’s law and rays equation trajectory.

2.1. Euler-Lagrange equation

One essential concept of variational calculus [4] is the definition of a functional. A functional establishes a correspondence between a real number and function so that if

\[ I = \int_{x_1}^{x_2} f(x, Y, Y') dx, \]

\( f \rightarrow I \) is a functional.

The objective is to discover, from the infinity of test functions \( Y(x) \), which function \( y(x) \) makes the integral \( I \) stationary [5].

To do this one defines the following relation between \( Y(x) \) and \( y(x) \)

\[ Y(x) = y(x) + \epsilon \eta(x) \]

(2)

where \( \epsilon \) is a real number and \( \eta(x) \) is an arbitrary continuous function with a continuous second derivative.

The solution is obtained by solving

\[ \frac{dI}{d\epsilon}_{\epsilon=0} = 0 \]

(3)

Since the function \( \eta(x) \) is arbitrary the only way to satisfy equation (3) for \( x \in [x_1, x_2] \) is to impose the Euler-Lagrange equation

\[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0. \]

(4)

The function \( y(x) \) that makes \( I \) stationary for \( x \in [x_1, x_2] \) obeys the Euler-Lagrange equation.

2.2. Generalized Snell’s law and trajectory equation

Considering a plane atmosphere with variable index of refraction with the altitude, \( n = n(x) \) (Figure 2) [6]

![Figure 2 - Plane atmosphere with variable index of refraction.](image)

One has, from Figure 2,

\[ ds = dz \sqrt{1 + \left( \frac{dx}{dx} \right)^2} \]

(5)

and considering the stationarity of the optical path, one has

\[ I = \int_{z_1}^{z_2} f(x, x') dz \]

(6)

where \( f(x, x') = n(x) \sqrt{1 + \left( \frac{dx}{dx} \right)^2} \).

Applying the Euler-Lagrange equation one obtains

\[ n(x) \sin[\theta(x)] = D \]

(7)

called the generalized Snell’s law.

From equation (7) one can get the generic equation for the ray trajectory on a plane atmosphere [6]. One has

\[ dz = \frac{dx}{\sqrt{n(u)^2 - n_0^2 \sin^2(\theta_0)}} \]

(8)

Applying equation (8) to the profile of the index of refraction for the corresponding media one gets

\[ z - z_0 = \pm \int_{x_0}^{x} \frac{n_0 \sin(\theta_0)}{\sqrt{n(u)^2 - n_0^2 \sin^2(\theta_0)}} du. \]

(9)

2.3. Generalized Snell’s law and trajectory equation in polar coordinates

Snell’s law is also valid polar coordinates for atmospheres with an index of refraction such as \( n = n(r) \).

Fermat’s principle imposes the stationarity of

\[ I = \int_{r_1}^{r_2} f(r, \theta') dr \]

(10)

where

\[ f(r, \theta') = n(r) \sqrt{1 + (r \theta')^2}. \]

(11)

Applying the Euler-Lagrange equation and simplifying one obtains

\[ n(r) r \sin(\theta) = \kappa. \]

(12)

This is the generalized Snell’s Law for polar coordinates.
Using this result and applying an analogous method as the one in the cartesian coordinates one can reach the trajectory equation for polar coordinates [6]

\[ \theta - \theta_0 = \pm \int_0^r \frac{\kappa}{u \sqrt{u^2 - \kappa^2}} \, du. \]  

(13)

2.4. Luneberg lens

A Luneberg lens is a spherically symmetric structure with a variable index of refraction which will form perfect geometrical images of two concentric spheres onto each other. A Luneberg lens presents two properties: it focuses parallel rays from any direction in a point of the other sphere or it originates parallel rays in the opposite side of their original point [7] [8].

To demonstrate this focusing property one uses the solution proposed by R.K. Luneburg.

Considering a Luneberg lens of unitary ray with an index of refraction given by [11]

\[ n(r) = \sqrt{2 - r^2}, \]  

(14)

from the generalized Snell's law for polar coordinates

\[ \theta - \theta_0 = \pm \int_0^r \frac{\kappa}{u \sqrt{\rho^2(r) + k^2}} \, dr \]  

(15)

with \( \rho(r) = n(r) \) one can obtain the ray turning point coordinates \( (r_\alpha, \theta_\alpha) \)

\[ r_\alpha = \sqrt{2} \sin \left( \frac{\alpha}{2} \right) \]  

(16)

\[ \theta_\alpha = \frac{1}{2} (\pi + \alpha). \]

Having obtained the coordinates of the turning point the trajectory equations for the incident rays can be determined. Solving equation (15) and applying the conditions included in [9] and [10] one gets for the interior of the lens

\[ r(\theta) = \sin \alpha \sqrt{\frac{1 + \cos (\alpha) \cos (2\theta - \alpha)}{\sin^2(2\theta - \alpha) + \sin^2(\alpha) \cos^2(2\theta - \alpha)}} \]  

(17)

and for the exterior

\[ r = \frac{\sin \alpha}{\cos \theta}. \]  

(18)

Figure 3 represents the Luneberg lens focusing property.

3. OPTICAL FIBERS: ANALITICAL TREATMENT

In this section light propagation in an optical fiber is described.

An optical fiber is a cylindrical waveguide with the property of propagating light between two points. It is composed by a core with an index of refraction \( n_1 \) and a cladding with an index of refraction \( n_2 \). The propagation of light inside an optical fiber can be explained by two theories: the ray theory and the modal theory.

3.1. Ray theory

Ray theory describes light as a ray. In an optical fiber two types of rays can exist: meridional rays and skew rays. Meridional rays are ruled by the laws of reflection and refraction propagating inside the optical fiber through total internal reflection (TIR) [1] (Figure 4).

![Figure 4 - TIR and acceptance cone.](image)

Only the rays arriving to the fiber with an incident angle lower than \( \phi_c \), such as

\[ \sin \phi_c = \frac{n_2}{n_1}, \]  

(19)

will propagate through TIR. The others are refracted through the cladding.

Therefore, the necessary condition for total internal reflection is \( n_1 > n_2 \).

So that the ray can enter the fiber (Figure 4) there is also a maximum acceptance angle \( \theta_0 \), given by

\[ n_0 \sin \theta_0 = n_1 \sin \phi_c = \left( n_1^2 - n_2^2 \right)^{1/2}. \]  

(20)

The term \( n_0 \sin \theta_0 \) is called the numerical aperture (NA).

Defining the dielectric contrast by

\[ \Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \]  

(21)

for \( \Delta \ll 1 \), one has

\[ NA \equiv n_1 \sqrt{2\Delta}. \]  

(22)

Skew rays are not confined to planes, and present a helicoidal propagation path. Due to the nature of the fiber there are losses that do not allow these rays to propagate through TIR.

The ray theory is valid when the ratio between the core radius and the wavelength is high enough, which happens more commonly in multimode fibers.
3.2. Modal theory
To obtain a precise model of light propagation within an optical fiber one must recur to the electromagnetic theory, namely, Maxwell’s equations [12]
\[
\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
\n\nabla \cdot \mathbf{D} = \rho \\
\n\nabla \cdot \mathbf{B} = 0
\]
and the following constitutive relations
\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}
\]
The polarization \( \mathbf{P} \) should be given by [1]
\[
\mathbf{P}(r,t) = P_t(r,t) + P_m(r,t).
\]
These equations allow reaching to the wave equation
\[
\nabla^2 \mathbf{E} + \nabla^2 (\omega^2/c^2) \mathbf{E}.
\]
Equation (30) describes wave propagation inside an optical fiber according to the electromagnetic theory and each solution corresponds to a mode.
Both fields \( \mathbf{E} \) and \( \mathbf{H} \) satisfy Maxwell’s equations. Given the cylindrical symmetry of an optical fiber and assuming \( \mathbf{E}_z \) as the supporting component one as
\[
\mathbf{E}_z(r,\omega) = \mathbf{A}(\omega) F(r) \exp(i \phi) \exp(-i \beta z)
\]
and applying it to (30) one gets
\[
\frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} + \left[ k_0^2 n^2 - \beta^2 - \frac{m^2}{r^2} \right] F(r) = 0
\]
The solution to this equation are the Bessel functions
\[
F(r) = \begin{cases} 
BJ_m \left( u \frac{r}{a} \right), & r \leq a \\
CK_m \left( w \frac{r}{a} \right), & r > a
\end{cases}
\]
where \( B \) and \( C \) are arbitrary constants and \( u \) and \( w \) are normalized variables given by
\[
u = a \sqrt{k_0^2 n_1^2 - \beta^2} \quad \text{and} \quad \omega = a \sqrt{\beta^2 - k_0^2 n_2^2}.
\]
An optical fiber supports hybrid modes which are characterized by having both electric and magnetic components in the propagation direction. For guided modes one has
\[
k_0 n_2 < \beta < k_0 n_1.
\]
It is common to define the normalized propagation constant \( b \) and the normalized frequency \( V \) as
\[
b = \frac{\pi^2 - n_2^2}{n_1^2 - n_2^2} \quad \text{and} \quad V = k_0 a \left( n_1^2 - n_2^2 \right)^{1/2} = \left( \frac{2\pi}{\lambda} \right) \alpha n_1 \sqrt{2\Delta}
\]
Since practical optical fibers are characterized by a low dielectric contrast their modes are linearly polarized (LP) [13]. Figure 5 shows the first six modes in an optical fiber, as well as it gives the maximum value for \( V \) in order to have a single mode fiber.

3.3. Fiber optics: Transmission characteristics
Two of principle phenomena that affect pulse propagation in an optical fiber are fiber losses and dispersion. Fiber losses (\( \alpha \)) reduce the available optical power increasing the bit error rate (BER) in the reception and limiting maximum distance between transmitter and receiver. The dispersion on the other hand limits the bandwidth because it causes the broadening of the optical pulse. In long distance communication systems it can cause pulses to interfere with each other (Intersymbol Interference - ISI) [1] causing information loss. There are two kinds of dispersion intramodal and intermodal dispersion. Intramodal dispersion which results from the fact that the group velocity is a function of the wavelength, is the only one relevant for this work since the object of study is single mode fibers.
For a monomodal fiber the group velocity is given by
\[
V_g = \left( \frac{d\beta}{d\omega} \right)^{-1}
\]
and it relates with the group index \( n_g \), by
\[
V_g = \frac{c}{n_g}.
\]
Considering the fact that the frequency dependence causes a time delay one has
\[
D = -\frac{2\pi c}{\lambda^2} \beta^2
\]
where \( D \) is the dispersion coefficient and \( \beta_2 \) is the group velocity dispersion (GVD) coefficient given by
\[
\beta_2 = \frac{d^2 \beta}{d\omega^2}.
\]
The dispersion coefficient includes both the material dispersion (\( D_M \)) and the waveguide dispersion (\( D_w \)). When \( D = 0 \) the fiber is operating near the zero-dispersion wavelength \( \lambda_D \), however the dispersion effects do not disappear. In this case, one has to consider the effects of the high order dispersion which are governed by the dispersion slope \( S = \partial D / \partial \lambda \) which equals
\[
S = \left( \frac{2\pi c}{\lambda_D^2} \right)^2 \beta_3
\]
where \( \beta_3 \) is the high order dispersion coefficient.

### 3.4. Pulse propagation equation in the linear regime

In the linear regime the pulse spectrum shape is invariant along the fiber. As such, one can neglect the term \( P_{NL} \) from equation (29). So, if one considers the field equations [10]
\[
E(x, y, t) = \tilde{X} F(x, y) \exp(\text{i} \omega_0 t)
\]
\[
B(0, t) = A(0, t) \exp(-\text{i} \omega_0 t)
\]
one can obtain the pulse propagation equation for the linear regime given by
\[
\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{1}{6} \beta_3 \frac{\partial^3 A}{\partial t^3} + \frac{\alpha}{2} A = 0.
\]
For the following analysis high order dispersion and fiber losses \( \alpha \) are neglected.

Considering the case of a gaussian pulse
\[
A(0, t) = \exp\left(-\frac{t^2}{2 \tau_0^2}\right)
\]
where \( \tau_0 \) is the initial width of the pulse. Applying equation (46) to equation (45) one gets
\[
A(z, t) = \left(\frac{\tau_0}{\tau_0 - i \beta_2 z}\right)^{1/2} \exp\left[-\frac{t^2}{2 \left(\tau_0^2 - i \beta_2 z\right)}\right].
\]
Attaining to the dispersion length given by \( L_D = \tau_0^2 / |\beta_2| \) one gets the expression for the pulse width along the fiber
\[
\tau(z) = \tau_0 \sqrt{1 + \left(\frac{z}{L_D}\right)^2}.
\]
As Figure 6 shows GVD broadens the pulse’s width.

Also from equation (47) one realizes that the pulse acquires chirp although it did not existed initially. To show it clearly one rewrites the pulse equation in the form
\[
A(z, t) = A(z, t) \exp\left[i \phi(z, t)\right]
\]
where
\[
\phi(z, t) = -\frac{\text{sgn}(\beta_2) (z/L_D)}{1 + (z/L_D)^2} \frac{t^2}{2 \tau_0^2} \tan^{-1}\left(\frac{z}{L_D}\right).
\]
The time dependence of \( \phi(z, t) \) implies that the instant frequency differs from the central frequency \( \omega_0 \) along the pulse. The difference is given by
\[
\delta \omega(t) = -\frac{\partial \phi}{\partial t} = \frac{\text{sgn}(\beta_2) (z/L_D)}{1 + (z/L_D)^2} \frac{t}{\tau_0^2}
\]
and is called chirp.

### 3.5. Bit rate

The bit rate is an important merit figure in an optical communication system as it defines the amount of information transmitted per time unit. It is dependent of the effects of dispersion [13].

The number of pulses to be transmitted per time unit depends on the RMS width of the pulse
\[
\sigma^2 = \left[ \langle t^2 \rangle - \langle t \rangle^2 \right]
\]
where the angle brackets refer to averaging with respect to intensity
\[
\langle t^n \rangle = \frac{\int_{-\infty}^{\infty} t^n |A(z, t)|^2 \, dt}{\int_{-\infty}^{\infty} |A(z, t)|^2 \, dt}.
\]
The broadening factor is given by [1]
\[
\left(\frac{\sigma}{\sigma_0}\right)^2 = \left(1 + \frac{LC \beta_2}{2 \sigma_0^2}\right)^2 + \left(\frac{\beta_2 L}{\sigma_0^2} + \left(1 + C^2\right)^2 \frac{L \beta_1}{4 \sqrt{2} \sigma_0^3}\right)^2
\]
where \( L \) is the fiber length and \( C \) is the chirp parameter. Through some manipulation and considering the normalized variables
\[
\eta = \frac{\sigma}{\sigma_0}^2, \quad \zeta = \frac{z}{L_D}
\]
equation (54) can be written as
\[
\eta = (1 + \text{sgn}(\beta_2) C \zeta^2)^2 + \zeta^2. \tag{56}
\]
Figure 7 shows the effects of the chirp parameter on the broadening of the pulse along the fiber. For the anomalous region (\( \beta_2 < 0 \)) a negative chirp causes a faster broadening of the pulse. However if the chirp is positive one can observe a pulse contraction at least in its initial stage. Later the effect of the chirp is overruled by the GVD effect.

![Figure 6 - Pulse width evolution.](image-url)
\[ n'_{j} = n_{j} + \tilde{n}_{2} \left( P / A_{\text{eff}} \right), \quad j = 1, 2, \] (57)

where \( \tilde{n}_{2} \) nonlinear index coefficient, \( P \) is the optical power and \( A_{\text{eff}} \) is the effective area.

The propagation constant becomes a function of the optical power
\[ \beta' = \beta + \gamma P \] (58)

where \( \gamma = 2\pi \tilde{n}_{2} / A_{\text{eff}} \lambda \).

The nonlinear phase is given by
\[ \phi_{\text{NL}} = \gamma P_{\text{in}}(t) L_{\text{eff}} \] (59)

where \( L_{\text{eff}} \) is the effective length. Self-phase modulation causes chirp.

3.7. Pulse propagation equation in the nonlinear regime

In this situation the term \( P_{\text{NL}} \) from equation (29) is no longer neglected. This term originate the component \( i \nu |U|^{2} U \) that accounts for the nonlinear effects.

The pulse propagation is now given by [14]
\[ i \frac{\partial U}{\partial \zeta} + i \frac{B_{2}}{2} \frac{\partial^{2} U}{\partial t^{2}} = - \frac{\alpha}{2} U + i \nu |U|^{2} U \] (60)

and it is called the nonlinear Schrödinger (NLS) equation.

The equation can be rewritten as
\[ i \frac{\partial U}{\partial \zeta} = \frac{\text{sgn}(B_{2})}{2 L_{\text{D}}} \frac{\partial^{2} U}{\partial t^{2}} - \frac{\alpha/2}{L_{\text{NL}}} |U|^{2} U \] (61)

where \( L_{\text{NL}} = 1/\gamma P_{0} \) is the nonlinear length.

This equation allows to classify the propagation regime in four categories by comparing \( L, L_{\text{D}} \) and \( L_{\text{NL}} \). When \( L \ll L_{\text{NL}} \) and \( L \ll L_{\text{D}} \) the regime is linear and non-dispersive. If \( L \ll L_{\text{NL}} \) and \( L \approx L_{\text{D}} \) the regime is linear and dispersive. For \( L \approx L_{\text{D}} \) and \( L \ll L_{\text{D}} \) the regime is nonlinear and non-dispersive. And if \( L \approx L_{\text{NL}} \) and \( L \approx L_{\text{D}} \) the regime is nonlinear and dispersive. The latter is the most frequent situation in an optical fiber.

For the anomalous region and when the effects of GVD and SPM are balanced it is possible to propagate solitons. In this case and if losses are neglected, equation (60) can be written as
\[ i \frac{\partial U}{\partial \zeta} + \frac{1}{2} \frac{\partial^{2} U}{\partial t^{2}} + N^{2} |U|^{2} U = 0. \] (62)

When \( N = 1 \) the solution of this equation is the fundamental soliton [14] and is given by
\[ U(\zeta, \tau) = \text{sech}(\tau) \exp \left( i \frac{\zeta}{2} \right). \] (63)

The fundamental soliton maintains its shape along the fiber although it suffers a phase shift of \( \zeta / 2 \).

3.8. Variational approach

As referred in a nonlinear dispersive media pulse propagation is governed by NLS equation that can be expressed as
\[ i \frac{\partial U}{\partial \zeta} + \frac{B_{2}}{2} \frac{\partial^{2} U}{\partial t^{2}} + \gamma |U|^{2} U = -i \frac{\alpha}{2} U. \] (64)

Variational methods can give greater insight on the pulse propagation equation. Considering
\[ U(\zeta, t) = B(\zeta, t) \exp \left( \frac{1}{2} \int_{0}^{\zeta} \alpha(z) dz \right) \] (65)

so that the last term of equation (64) can eliminated, one has a Lagrangian density of the form
\[ K = \frac{1}{2} \left[ B^{*} \frac{\partial B}{\partial \zeta} - B \frac{\partial B^{*}}{\partial \zeta} + \frac{1}{2} \left( \frac{\partial B^{2}}{\partial t} + \partial \right) \right]. \] (66)

Using Euler-Lagrange equations one has
\[ \frac{\partial}{\partial \zeta} \left( \frac{\partial K}{\partial q_{t}} \right) + \frac{\partial}{\partial q_{t}} \left( \frac{\partial K}{\partial q_{t}} \right) = 0 \] (67)

where \( q = B^{*}, q_{t} = \frac{\partial B^{*}}{\partial t} \) and \( q_{t} = \frac{\partial B^{*}}{\partial \zeta} \).

The average Lagrangian density is defined by
\[ L = \int_{-\infty}^{\infty} \frac{L(t, q(z))}{dt} \] (68)

and can be written through Euler-Lagrange equations as
\[ \frac{d}{dz} \left( \frac{\partial L}{\partial q_{t}} \right) = 0 \] (69)

with \( q \) being the pulse parameters dependent on \( z \).

Considering a fundamental soliton for the pulse shape, one can described it by
\[ B(\zeta, t) = a \text{sech} \left( \frac{t-T}{\tau} \right) \exp \left( i \phi - i \Omega(t-T) - i C(t-T)^{2}/2 \tau^{2} \right) \] (70)

where the amplitude \( a \), phase \( \phi \), frequency \( \Omega \), time delay \( T \), chirp \( C \) and width \( \tau \) are functions of \( z \).

Applying equation (69) to equation (66) and then to equation (68) one obtains the Lagrangian density
where $E = \int_{-\infty}^{\infty} a^{2} \text{sech}^{2} \left( \frac{t-T}{\tau} \right) dt = 2a^{2}\tau$ is the pulse energy.

One can obtain the equations for the pulse parameters applying equation (71) in equation (69). By making $q = C$ one gets the pulse width

$$\frac{d\tau}{dz} = \frac{\beta^{2} C}{\tau}$$

(72)

and $q = \tau$ obtaining the chirp

$$\frac{dC}{dz} = \frac{4}{\tau^{2}} \left( 1 + C^{2} \right) + \beta_{2} \Omega^{2} + \frac{2\pi E}{\tau^{2}}.$$  

(73)

These equations show that a fundamental soliton does not acquire chirp and that it maintains its shape.

Gaussian pulses can be used to describe dispersion managed solitons [15][16]. The pulse shape is of the form

$$B(z,t) = a \exp \left[ \imath \phi - \imath \Omega (t - T) - \left( \frac{1}{2} C(t - T)^{2} / 2\tau^{2} \right) \right].$$

(74)

Applying equation (74) to equation (66) and then to equation (68) one obtains the Lagrangian density

$$L = E \left[ \frac{d\phi}{dz} - \Omega \frac{dT}{dz} + \left( \frac{1}{2} \frac{dC}{dz} - \frac{C d\tau}{\tau} \right) \right]$$

$$+ \frac{\beta_{2} E}{4\tau^{2}} \left( 1 + C^{2} \right) + \frac{\beta_{2} C^{2}}{2} + \frac{2\pi E}{\sqrt{8\pi} \tau}$$

(75)

$$= \int_{\infty}^{\infty} a^{2} \exp \left[ - \left( \frac{t - T}{\tau} \right)^{2} \right] dt = \sqrt{\pi a^{2}} \tau.$$  

As before one can obtain the equations for the pulse parameters applying equation (75) to equation (69). Using $q = C$ one gets the pulse width

$$\frac{d\tau}{dz} = \frac{\beta_{2} T}{\tau}$$

(76)

and $q = \tau$ obtaining the chirp

$$\frac{dC}{dz} = \frac{\beta_{2}}{\tau^{2}} \left( 1 + C^{2} \right) + \frac{\beta_{2} \Omega^{2}}{2} + \frac{\pi E}{\sqrt{2\pi} \tau}.$$  

(77)

In a linear medium the spectral width is constant and the equations for pulse width and chirp are, respectively,

$$\tau^{2}(z) = \tau(0) + 2 \int_{0}^{z} \beta_{2}(z) dz$$

(78)

$$C(z) = C(0) + \frac{1 + C_{2}^{2}(0)}{\tau^{2}(0)} \int_{0}^{z} \beta_{2}(z) dz.$$  

(79)

Integrating over two sections of the dispersion map, the value of pulse width and chirp at the end of the first map period are given by

$$\tau(L_{m}) = \tau(0) \left[ 1 + C(0) d^{2} + d^{2} \right]^{3/2}.$$  

(80)

$$C(L_{m}) = C(0) + \left( 1 + C^{2}(0) \right) d$$

(81)

where $d = \beta_{m} L_{m} / \tau^{2}(0)$ and $\beta_{m}$ is the average dispersion.

4. OPTICAL FIBER: NUMERICAL SIMULATION

In this section pulse propagation in the linear and the nonlinear regime is addressed as well as dispersion compensating techniques for both cases.

4.1. Linear Regime

Through equation (45) one can simulate the propagation of a pulse along the optical fiber in the linear regime, using the normalized variables

$$\zeta = \frac{z}{L_{D}}$$

(82)

$$\tau = \frac{t - \beta_{2} z}{\tau_{0}}$$

(83)

and applying the algorithm

(i) $A(0, \zeta) = \text{FFT} \left[ A(0, \tau) \right]$  

(ii) $A(\zeta, \xi) = A(0, \zeta) \exp \left[ i \frac{1}{2} \text{sgn}(\beta_{2}) \xi^{2} - \xi^{2} \right]$  

(84)

(iii) $A(\zeta, \xi) = \text{FFT}^{-1} \left[ A(\zeta, \xi) \right]$.

4.1.1. Supergaussian pulse

In this section, the case study is a supergaussian pulse [10] given by

$$A(0, \tau) = \exp \left[ \frac{1 + i C(t/\tau_{0})^{6}}{2} \right]$$

(85)

where $C = 0, \pm 2$.

For $C = 0$ one gets Figure 8.

![Figure 8 - Supergaussian pulse at begin and end of the optical fiber $C = 0$.](image-url)
For $C = -2$ the result is Figure 9.

For $C = 2$ one obtains Figure 10.

The broadening of the pulse due to dispersion is clear in the situation $C = 0$. A decrease of pulse amplitude is caused by the conservation of energy. From the cases $C = \pm 2$ it is possible to observe that chirp increases the broadening effect caused by the GVD.

4.1.2. Dispersion compensating fiber

The broadening of pulses that occurs due to GVD can be compensated using a dispersion compensating fiber (DCF) [1]. This technique allows full dispersion compensation as long as the average optical power of the signal is lower enough so that one can neglect the nonlinear effects.

Considering the propagation of a pulse along two segments of fiber, the corresponding pulse envelope is given by

$$A(L,t) = \int_{-\infty}^{\infty} A(0,\omega) \exp \left[ i \omega (\beta_2 L_1 + \beta_2 L_2) - i \omega t \right] d\omega$$  \hspace{1cm} (86)

where $L = L_1 + L_2$ and $\beta_2$ is the dispersion coefficient of the fiber segment $L_j (j = 1, 2)$.

The DCF is such that the factor in $\omega^2$ is cancelled, meaning that

$$\beta_1 L_1 + \beta_2 L_2 = 0.$$  \hspace{1cm} (87)

Using a DCF to compensate the dispersion in the supergaussian pulse presented previously one obtains Figure 11 and Figure 12.

As expected the pulse recovers its original shape showing full dispersion compensation.

4.2. Nonlinear Regime

To simulate pulse propagation in the nonlinear regime one has to use the Split-Step Fourier Method (SSFM) [14] [17]. This method splits the fiber in segments of length $h$ and considers that the dispersion effects and the nonlinear effects are independent of each other.

Using the equation

$$i \frac{\partial u}{\partial \xi} = \frac{1}{2} \text{sgn}(\beta_2) \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = -i \frac{\Gamma}{2} u$$  \hspace{1cm} (88)

where $\Gamma$ is the normalized losses, the SSFM uses the following algorithm
\[ u(\zeta, \tau) \rightarrow u(\zeta + h, \tau) = w(\zeta, \tau) \]
\[ v(\zeta, \tau) = \exp(hN)u(\zeta, \tau) \]
\[ V'(\zeta, \xi) = \text{FFT}[v(\zeta, \tau)] \]
\[ W'(\zeta, \xi) = \exp(hD')V'(\zeta, \xi) \]
\[ w(\zeta, \tau) = \text{IFFT}[W'(\zeta, \xi)] \]

where \( N \) is the nonlinear operator and \( D \) is the dispersion operator.

4.2.1. Fundamental soliton

In this section, the case study is a fundamental soliton propagating along the optical fiber given by

\[ u_0(\tau) = \text{sech}(\tau) \]

for which one observes Figure 13.

As expected from the theoretical analysis the pulse maintains its shape along the propagation due to the balance between GVD and SPM. Also it stays chirp free.

4.2.2. Gaussian pulse

In this section, the case study is a Gaussian pulse propagating along the optical fiber given by

\[ u_0(\tau) = \exp\left(\frac{-\tau^2}{2}\right). \]

The result is Figure 14.

In this case, there is broadening of the pulse and consequent amplitude is reduction due to the conservation of energy. Also from this simulation one can observe that even if a pulse is not a soliton it will tend to it as it propagates along the fiber. In this case it stabilizes at \( \eta = 0.74235 \).

4.2.3. Dispersion decreasing fiber

The existence of solitons in an optical fiber is dependent on the balance between GVD and SPM. This balance however is broken by the existence of fiber losses. To solve this problem one can use a dispersion decreasing fiber (DDF) [14].

The relation between \( L_D \) and \( L_{NL} \) is given by

\[ N^2 = \frac{L_D}{L_{NL}} \]

and for the fundamental soliton one has \( N = 1 \). Therefore

\[ \beta_2 = \tau_0^2 \gamma P. \]

Assuming the power is \( P = P_0 \exp(-\alpha z) \), one concludes that

\[ \beta_2 = |\beta_2| \exp(-\alpha z) \]

with \( |\beta_2| = \tau_0^2 \gamma P_0 \). This is the ideal profile for \( \beta_2 \) along the fiber in order to solve these problems. In reality this solution is difficult to implement, so it is common to approximate the expression (94) by a step function as shown in Figure 15.

Figure 15 - Step approximation of the dispersion profile.

The evolution of a soliton propagating along a DDF is shown in Figure 16.
The compensation can be seen in each of the steps. It is also more efficient is the number of steps is increased.

5. CONCLUSION

The application of variational methods to study geometric optics allowed to verify Snell’s law and to deduce a ray trajectory equation. Also in the study of the Luneberg lens it confirmed its focusing property.

The analysis on the ray theory leads to the conclusion that it is only valid when the ratio between the optical fiber core radius and the wavelength is very high, as it happens in multimodal fiber. Therefore, to have an accurate description of light propagation in an optical fiber one needs to study the modal theory. This theory shows that for low dielectric contrast optical fibers the modes that propagate are linearly polarized. Also one concludes that to have single mode fibers the normalized frequency needs to be lower than 2.4048.

The study of pulse propagation in the linear regime shows that the pulse broadens due to GVD that leads to the need of reducing the system’s bit rate in order to prevent ISI. The propagation of the pulse also leads to the appearance of chirp.

It is shown that the nonlinear regime is governed by the NLS equation. Also in the nonlinear regime, there is the appearance of new frequency components due to the Kerr effect, leading to SPM effect.

The relation between $L$, $L_D$ and $L_{NL}$ also gives insight on the behavior of the regime the pulse is propagating in.

The numerical analysis of the supergaussian pulse propagating in the linear regime confirmed the broadening of the pulse and the chirp’s effect predicted in the theoretical analysis. It was also shown that the broadening of the pulse leads to a reduction of the pulse amplitude due to the conservation of energy.

To compensate these effects a DCF fiber was introduced and full dispersion compensation was obtained and the pulse’s initial shape was recovered.

The simulation of the propagation of the fundamental soliton showed that, when the GVD and SPM effects are balanced, it does propagate along an optical fiber without suffering changes in its shape. The simulation of a gaussian pulse in this situation shows how robust the system is, as the pulse reshapes into a soliton along the propagation.

When GVD and SPM effects are not balanced due to fiber losses, soliton propagation is not possible. To overcome this problem, the use of a DDF was tested. Using a step approximation for the profile of the dispersion it was shown that the compensation was efficient, being more efficient if the number of steps is increased.

6. REFERENCES


