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On the coefficients of the Hilbert Polynomial of a Cohen-Macaulay local Ring

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Abstract

This work has the intention of being an introduction to the theory of Hilbert Functions and its principal aim the description of the use of superficial sequences, as done with much success by, for example, Valla, Rossi and Sally. It was also my intention to make this work as an introduction to superficial sequences and reductions, which have applications in other fields, such as the study of Rees Algebras. My exposition is done in a sequential way, as if a story is being told, because I think in such a way the objects introduced seem more natural, and a reader can more easily get into the theory.

Resumo

Com este trabalho tive a intenção de fazer uma introdução ao estudo de Funções de Hilbert de anéis locais e Cohen-Macaulay. O principal objectivo deste trabalho é descrever como podemos usar sequências superficiais para estudar estas funções, como foi feito com significativo sucesso por, por exemplo, Valla, Rossi e Sally. Tive a intenção de que este texto pudesse servir como uma introdução ao tema de elementos superficiais e reduções de ideais, que têm outras aplicações para além das contempladas neste texto, tais como Álgebras de Rees. O meu propósito na estruturação deste trabalho foi fazê-lo dum modo sequencial, como quem conta uma história, por ter a ideia de que tal exposição poderia facilitar a compreensão do leitor, parecendo os objectos e teorias introduzidas mais naturais, permitindo, estou em crer, uma mais rápida compreensão do tema.

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Chapter 0

Introduction

In this first chapter we will introduce some of the basic notions that will be used throughout the entire thesis, and some basic results concernig this concepts. Our main goal is to give the definition of Hilbert's Series and Function of a finitely generated graded R-module, with only a little more restrictions to R. We will start with the precise definition of a graded ring and module:

Definition 0.1 - A **graded Ring** is a ring R satisfying the two following conditions:

1. R, as an additive group, can be decomposed in an infinite sum, i.e., $R = \bigoplus_{i \in \mathbb{N}} R_i$
2. $R_i R_j \subset R_{i+j}$, $\forall i, j \in \mathbb{N}$

Similarly, a **graded R-Module** M is a module together with a decomposition $M = \bigoplus_{i \in \mathbb{N}} M_i$ and

$$R_i M_j \subset M_{i+j},$$

$\forall i, j \in \mathbb{N}$.

An example of a ring where the structure of graded ring appears in a natural way is that of a polynomial ring, where the decomposition is made by degrees. More precisely, given a ring R and $\mathbf{R} = R[x_0, ..., x_k]$ the polynomial ring in k-variables, we can rewrite it in the form $\mathbf{R} = \bigoplus_{i \in \mathbb{N}} R_i$, where R_i is the additive subgroup of the homogeneous polynomials of degree i. Evidently, this gives $\mathbf{R}$ a structure of a graded ring. We will return to this example later in this chapter; for now let us introduce some terminology associated to the definition of graded module

Definition 0.2 - Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a graded ring and $M = \bigoplus_{i \in \mathbb{N}} M_i$ a graded R-module. We say that:

1. $x \in M$ is an **homogeneous element** if x belongs to one of the M_i's, and in that case, we say that i is the **degree** of x, and we denote $deg(x) = i$

2. $N \subset M$ is a **homogeneous submodule** if it is generated by homogeneous elements or, equivalently, if $N = \bigoplus_{i\in\mathbb{N}}(N \cap M_i)$
3. we denote $\bigoplus_{i>0} R_i$ as R^+.

The next theorem is not only important because of its statement, but also because its proof will give us an important argument to a sufficient condition for all the pieces of the sum decomposition of a graded R-module to have a finite length.

Theorem 0.3 - A graded ring $R = \bigoplus_{i\in\mathbb{N}} R_i$ is Noetherian iff R_0 is Noetherian and R is a finitely generated R_0-Algebra.

Proof. The 'if' is an immediate consequence of the fact that R is a finitely generated algebra over a Noetherian ring, R_0. To prove the 'only if' we assume that R is Noetherian. It follows immediately that R_0 is Noetherian, because $R_0 \cong R/R^+$. Besides that, we know R^+ is a homogeneous ideal, and it must admit a finite number of generators, because R is Noetherian. Notice that from any set of generators of R we can construct a set of homogeneous generators, picking its homogeneous components as the new basis, so we can pick $\{x_1, ..., x_r\}$ as a set of homogeneous elements of R generating R^+.
We will prove that $R = R_0[x_1, ..., x_r]$ and we will do so proving by induction on i that $R_0, ..., R_i \subset R_0[x_1, ..., x_r], \forall i \in \mathbb{N}$. The case $i = 0$ is obvious, so let's assume that $i > 0$ and $deg(x_j) = d_j$. If we can prove that

$$R_i = R_{i-d_1}x_1 + ... + R_{i-d_r}x_r \tag{1}$$

the result will follow by induction. So let's assume that $y \in R_i \subset R^+$; we must have:

$$y = s_1x_1 + ... + s_rx_r$$

where, by the unicity of representation of y as a sum of homogeneous elements, each s_l belongs to R_{l-d_l} . Then, $y \in R_{i-d_1}x_1 + ... + R_{i-d_r}x_r$, which concludes our result.

□

Assume now that $R = \bigoplus_{i\in\mathbb{N}} R_i$ is a Noetherian graded ring and $M = \bigoplus_{i\in\mathbb{N}} M_i$ is a finitely generated Noetherian graded R-module. As in the prove of 0.1, we can always pick $\{m_1, .., m_r\}$ to be a set of homogeneous elements of M, so we have that:

$$M = Rm_1 + ...Rm_r$$

letting $d_1, ..., d_r$ be, respectively, the degrees of $m_1, .., m_r$ and using an argument similar to the one that lead us to equation (0.1), we have that for each $n \in \mathbb{N}$:

$$M_n = R_{n-d_1}m_1 + ... + R_{n-d_r}m_r$$

and from equality 0.1 each M_n is a finitely generated R_0-module. If we also demand R_0 to be an Artinian ring, we obtain that every M_n is a Noetherian and an Artinian R_0-module, and therefore has finite length as an R_0-module. This allows us to give the following definitions:

Definition 0.4 - If $R = \bigoplus_{i\in\mathbb{N}} R_i$ is a Noetherian graded ring, where R_0 is an Artinian ring, and if $M = \bigoplus_{i\in\mathbb{N}} M_i$ is a finitely generated R-module, we define the **Hilbert Function** of M, that we will denote as $H_M(n) := l_{R_0}(M_n)$, and the **Hilbert Series** of M, that we denote as $P(M,t)$ by:

$$P(M,t) := \sum_{i\in\mathbb{N}} H_M(n)t^i$$

To state this definition was one of the main purposes of this chapter, and in what follows we will study the form of this function more closely, starting to prove that we can express it as a rational function of a special form, given by Theorem 0.7. To prove this, we will need the 'help' of the two following lemmas:

Lemma 0.5 - Given the exact sequence of R-modules:

$$0 \to N \to M \to P \to 0$$

then $l(M) < \infty \iff l(n) < \infty$ and $l(P) < \infty$. In this case we have the following relation:

$$l(M) = l(N) + l(P)$$

Proof. First, let us notice that every short exact sequence as above is equivalent to one of the form:

$$0 \to N \hookrightarrow M \to M/N \to 0$$

To prove the first statement, observe that $l(M)$ being less than infinity is equivalent to M being an Artinian and a Noetherian R-module, and in this case it is obvious that both N, that is a submodule of M, and M/N are Artinian and Noetherian R-modules, and hence have finite length. The other way around comes from the fact that if M/N and N are Artinian, or Noetherian, R-modules, then so is M.
Assume now that M has finite length and let:

$$M_0/N = M/N \supset M_1/N \supset ... \supset M_{l(P)}/N = N/N$$

$$M_{l(P)} = N \supset M_{l(P)+1} \supset ... \supset M_{l(P)+l(N)} = (0)$$

be composition series of M/N and N, respectively. We want to prove that:

$$M = M_0 \supset ... \supset M_{l(P)+l(N)} = 0 \tag{2}$$

is a composition series for M. Obviously, if $i \geqslant l(P)$, M_i/M_{i+1} is simple, but is not immediately evident that it is also the case for $i < l(P)$, but it is a consequence of the fact that:

$$\frac{M_i/N}{M_{i+1}/N} \cong M_i/M_{i+1}$$

And hence (0.1) is a composition series of M and $l(M) = l(N) + l(P)$, as we intended.

□

Lemma 0.6 - Given the exact sequence of R-modules of finite length:

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\varphi} P \longrightarrow L \longrightarrow 0$$

we have following relation:

$$l(M) - l(P) = l(N) - l(L)$$

Proof. As in the previous Lemma, we have that every exact sequence as above is equivalent to one of the form:

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\varphi} P \longrightarrow L \longrightarrow 0$$

$$0 \to Ker\varphi \hookrightarrow M \to P \to P/Im\varphi \to 0$$

From this we get the two following sequences:

$$0 \to Ker\varphi \hookrightarrow M \to M/Ker\varphi \to 0$$

$$0 \to Im\varphi \hookrightarrow P \to P/Im\varphi \to 0$$

Applying Lemma 0.5 to these sequences we get the following relations:

$$l(M) = l(N) + l(M/N) \tag{3}$$

$$l(P) = l(Im\varphi) + l(P/Im\varphi)$$

And subtracting the second one from the first we get:

$$l(M) - l(P) = l(N) + l(M/N) - l(Im\varphi) - l(P/Im\varphi)$$

Now, applying the First Fundamental Isomorphism Theorem for R-Modules to the function φ, we conclude that $M/N \cong Im\varphi$, which implicates that $l(M/N) = l(Im\varphi)$, and so:

$$l(M) - l(P) = l(N) - l(P/Im\varphi)$$

wich concludes our result.

□

This two results admit a generalization by induction for an arbitrary long exact sequence with extremes 0, where Lemma 0.5 is the first step and the argument to the induction step is similar to the one we used in Lemma 0.6, but we will only need this two cases for our theorem, that we now state:

Theorem 0.7 - Given $R = \bigoplus_{i\in\mathbb{N}} R_i$ a graded ring with R_0 Artinian, M a finitely generated graded R-Module and supposing that $R = R_0[x_1, ..., x_r]$, with each x_i of degree d_i, we have that:

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{r}(1-t^{d_i})} \tag{4}$$

where $f(t) \in \mathbb{Z}[t]$.

Proof. We will deduct this result by induction on the number of generators of R as an R_0-Algebra. In the case $r = 0$, we have that $R = R_0$, and so $\bigoplus_{i=0}^{n} M_i$ are all R-submodules of M, and thus we cannot have $M_n \neq 0$ for n arbitrarily large, otherwise the chain:

$$M_0 \subset M_0 \oplus M_1 \subset ... \bigoplus_{i=0}^{n} M_i \subset ...$$

will contain an infinitely strictly crescent chain, thus contradicting the fact that M is a Noetherian R-Module. By this, we know $l(M_n)$ must be 0 for n sufficiently large, and so $P(M,t) = \sum_{i\in\mathbb{N}} l(M_i)t^i$ is a polynomial in $\mathbb{Z}[t]$.
Assume now that $r > 0$, and consider the following R homomorphism:

$$M_n \to M_{n+d_r}$$

$$m \mapsto x_r m$$

which we will denote by x_r. Denoting the kernel of this map by K_n and the co-kernel by C_n we get the following exact sequence:

$$0 \to K_n \to M_n \to M_{n+d_r} \to C_{n+d_r} \to 0$$

and by Lemma 0.6 we know that:

$$l(K_n) - l(M_n) + l(M_{n+d_r}) - l(C_{n+d_r}) = 0. \tag{5}$$

Besides this, we know $\bigoplus K_n$ is a submodule of M, $\bigoplus C_n = M/x_rM$, $x_r \bigoplus K_n = x_r \bigoplus C_n = 0$. The last statement is equivalent to say that $x_r \in Ann_R(\bigoplus K_n) \cap Ann_R(\bigoplus C_n)$, hence $\bigoplus K_n$ and $\bigoplus C_n$

are also $R/(x_r) = R[x_1, ..., x_{r-1}]$-modules, and seen as such have the same structure as R-modules, and so we can apply our inductive hypothesis to get:

$$P(\bigoplus K_n, t) = \frac{g(t)}{\prod_{i=1}^{r-1}(1 - t^{d_i})}$$

$$P(\bigoplus C_n, t) = \frac{h(t)}{\prod_{i=1}^{r-1}(1 - t^{d_i})}$$

where $g(t), h(t) \in \mathbb{Z}[t]$. Multiplying equality (0.5) by t^{n+d_r} and summing over n we get:

$$t^{d_r}P(\bigoplus K_n, t) - t^{d_r}P(M, t) + P(M, t) - P(\bigoplus C_n, t) = l(t)$$

where $l(t) \in \mathbb{Z}[x]$. This equality is equivalent to:

$$P(M, t) = \frac{P(\bigoplus C_n, t) + l(t) + t^{d_r}P(\bigoplus K_n, t)}{1 - t^{d_r}}$$

And our results follows immediately.

□

A big part of the study of the Hilbert series, and function, is concerned with the restrictions in the type of functions we can obtain, and obviously this result is crucial in that study, besides the fact that is the first of that nature. As we have observed in the abstract, this theory came from a geometrical motivation, and of particular interest in geometry is the case where our modules are generated by elements of degree one, being the ring of polynomials, regarded as a module over its base ring, one such example. Also in this case our equality (0.4) will take the simpler form:

$$P(M, t) = \frac{h(t)}{(1 - t)^r}.$$

Notice that we can always assume that $(1 - t) \nmid h(t)$, canceling the terms $(1 - t)$ of $f(t)$, rewriting the equality above as:

$$P(M, t) = \frac{f(t)}{(1 - t)^d}. \tag{6}$$

A very good consequence of this simpler equality is that we can use it to get a polynomial that will be equal to the Hilbert function of M for large enough n, but in order to do so we will state and prove another lemma.

Lemma 0.8 - The following equality is valid for every $d \in \mathbb{N} \setminus 0$:

$$\frac{1}{(1 - t)^d} = \sum_{n=0}^{\infty} \binom{d + n - 1}{d - 1} t^n.$$

Proof. We will prove this result by induction on d, the case $d = 1$ being the well known formula:

$$\frac{1}{(1-t)} = \sum_{n=0}^{\infty} t^n$$

Assuming that $d > 1$ and that this result is true for $d - 1$, we have that:

$$\frac{1}{(1-t)^{d-1}} = \sum_{n=0}^{\infty} \binom{d+n-2}{d-2} t^n$$

Differentiating each members of this equality with respect to t we obtain:

$$\frac{d-1}{(1-t)^d} = \sum_{n=0}^{\infty} n \binom{d+n-2}{d-2} t^{n-1} \Leftrightarrow$$

$$\frac{d-1}{(1-t)^d} = \sum_{n=0}^{\infty} (n+1) \binom{d+n-1}{d-2} t^n \Leftrightarrow$$

$$\frac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \frac{n+1}{d-1} \frac{(d+n-1)!}{(d-2)!(n+1)!} t^n \Leftrightarrow$$

$$\frac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \frac{(d+n-1)!}{(d-1)!n!} t^n \Leftrightarrow$$

$$\frac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \binom{d+n-1}{d-1} t^n$$

as we intended.

□

Using this lemma we obtain:

Lemma 0.9 - Given $R = \bigoplus_{i \in \mathbb{N}} R_i$ a graded ring with R_0 Artinian, M a finitely generated graded R-Module, supposing that R is generated as an R_0-algebra by elements of degree one and writing:

$$P(M,t) = \frac{f(t)}{(1-t)^d}$$

where $(1-t) \nmid f(t) = a_0 + a_1 t + ... + a_s t^s$ the following equality holds:

$$l(M_n) = a_s \binom{d+n-s-1}{d-1} + ... + a_1 \binom{d+n-2}{d-1} + a_0 \binom{d+n-1}{d-1} \tag{7}$$

where in here we set $\binom{m}{d-1} = 0$, for $m < d-1$

Proof. We will star to notice that:

$$l(M_n) = \frac{P^{(n)}(M,0)}{n!}.$$

Now, we observe:

$$P^{(n)}(M,t) = f^{(n)}(t)\frac{1}{(1-t)^d} + ... + \left(\begin{array}{c} n \\ i \end{array}\right) f^{(n-i)}(t)\frac{(d+i-1)!}{(d-1)!}\frac{1}{(1-t)^{d+i}} +$$

$$+ ... + f(t)\frac{(d+n-1)!}{(d-1)!}\frac{1}{(1-t)^{d+n}}$$

and so:

$$\begin{cases} P^{(n)}(M,0) = \left(\begin{array}{c} n \\ s \end{array}\right) a_s s! \frac{(d+n-s-1)!}{(d-1)!} + ... + a_1 n \frac{(d+n-2)!}{(d-1)!} + a_0 \frac{(d+n-1)!}{(d-1)!} & \text{for} \quad n \geqslant s \\ P^{(n)}(M,0) = a_n n! + ... + a_1 n \frac{(d+n-2)!}{(d-1)!} + a_0 \frac{(d+n-1)!}{(d-1)!} & \text{for} \quad n < s \end{cases}$$

Dividing each member by $n!$ we get:

$$\begin{cases} l(M_n) = a_s \frac{(d+n-s-1)!}{(n-s)!(d-1)!} + ... + a_1 \frac{(d+n-2)!}{(n-1)!(d-1)!} + a_0 \frac{(d+n-1)!}{n!(d-1)!} & \text{for} \quad n \geqslant s \\ l(M_n) = a_n + ... + a_1 \frac{(d+n-2)!}{n!(d-1)!} + a_0 \frac{(d+n-1)!}{n!(d-1)!} & \text{for} \quad n < s \end{cases}$$

Or, as in the statement of the lemma:

$$l(M_n) = a_s \left(\begin{array}{c} d+n-s-1 \\ d-1 \end{array}\right) + ... + a_1 \left(\begin{array}{c} d+n-2 \\ d-1 \end{array}\right) + a_0 \left(\begin{array}{c} d+n-1 \\ d-1 \end{array}\right)$$

where in here we set $\left(\begin{array}{c} m \\ d-1 \end{array}\right) = 0$, for $m < d-1$, which concludes our result.

□

The notation used in here, that $\left(\begin{array}{c} m \\ d \end{array}\right) = 0$ whenever $m < d$, will be used in the rest of the thesis. Notice that when $m \geqslant 0$, whether it is less or greater than d, this combination is equal to the polynomial in m given by:

$$\frac{m(m-1)...(m-(d+1))}{d!}$$

and so, from a certain point on, H_M coincides with the polynomial function with rational coefficients:

$$h_M(n) = a_s \left(\begin{array}{c} d+n-s-1 \\ d-1 \end{array}\right) + ... + a_1 \left(\begin{array}{c} d+n-2 \\ d-1 \end{array}\right) + a_0 \left(\begin{array}{c} d+n-1 \\ d-1 \end{array}\right)$$

$$= a_0 \frac{(d+n-1)(d+n-2)...(n+1)}{(d-1)!} + a_1 \frac{(d+n-2)...(n+1)n}{(d-1)!} + \tag{8}$$

$$+ ... + a_s \frac{(d+n-s-1)...(n-s+1)}{(d-1)!} =$$
$$= \frac{a_0 + ... + a_s}{(d-1)!} n^{d-1} + terms\, of\, lower\, degree$$

From the comment above and equality (0.7) we notice that h_M and $l(M_n) = H_M$ coincide for $d+n-s-1 \geqslant 0 \iff n \geqslant s+1-d$. We also may notice that this polynomial has degree $d-1$. We will state this important result as a corollary of Theorem 0.7, but first let us assign names to this functions:

Definition 0.10 - Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ a graded ring generated as an R_0-algebra by elements of degree one with R_0 Artinian, M a finitely generated graded R-module and:

$$P(M,t) = \frac{a_0 + ... + a_s t^s}{(1-t)^d}$$

the Hilbert Series of M, where $1 - t \nmid a_0 + ... + a_s t^s$. Then we define

1. h**-vector** of M to the vector $(a_0, ..., a_s)$ and h**-polynomial** to $a_0 + ... + a_s t^s$
2. we call **Hilbert Polynomial** of the graded module M to the polynomial $h_M(n)$

And so we can reformulate the previous comment as follows:

Corollary 0.11 - The Hilbert function and the Hilbert polynomial coincide for $n \geqslant s+1-d$.

This will be the last result of this chapter, but before ending it, let us an illustrative example: Let R be an Artinian ring, $\mathbf{R} = R[x_1, .., x_k]$ the ring of polynomials in k variables over R and $\bigoplus_{i \in \mathbb{N}} R_i$ the decomposition into the aditive subgroups of homogeneous polynomials referenced in the beginning of the chapter. Start to observe that $R[x_1, .., x_k]$ satisfies every condition required in Corollary 0.1, and, if I is the set of the homogeneous monomials of degree i:

$$R_i = \bigoplus_{x \in I} Rx$$

where the direct decomposition above is as a R-module. Hence:

$$H_{\mathbf{R}}(n) = l(R)|I|.$$

Since the number of monomials of degree n is given by:

$$\left(\begin{array}{c} n+k-1 \\ k-1 \end{array} \right)$$

we obtain:

$$H_{\mathbf{R}}(n) = l(R) \left(\begin{array}{c} n+k-1 \\ k-1 \end{array} \right)$$

Chapter 1

Samuel Function and the Dimension Theorem

In this chapter we will continue to give the introductory concepts. We will focus now on rings and in our main object of study: the Hilbert Function of the graded ring associated to an ideal I of a Noetherian local ring R. One of the main interests in our theory is the connections between the properties of this Hilbert Function and the properties of R itself. We will also meet the first relevant result of this nature, the dimension theorem, which relates the dimension of our ring with the degree of the Hilbert Function.
To start with, let us give the definition of graded ring:

Definition 1.1 Let (R, m) be a Noetherian local ring and I an ideal in R. We call **Graded Ring of R associated to** I to:

$$gr_I(R) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$$

If $I = m$ we will just call it the graded ring of R and denote it by $gr(R)$.
Let now M be a finitely generated R-module, we call **G**raded Module associated to I to the graded $gr_I(R)$-module given by:

$$gr_I(M) = \bigoplus_{n \geqslant 0} I^n M/I^{n+1}M$$

In order to apply to the new objects the theory developed in the first chapter, we will first see that $gr_I(R)$ is a Noetherian ring. Obviously, it is generated, over R/I, by elements of I/I^2, and as R is itself Noetherian, we know I is finitely generated. Fix $\{x_1, ..., x_s\}$ to be a set of generators of I, then:

$$gr_I(R) = (R/I)[x_1 + I^2, ..., x_s + I^2]$$

and as R is Noetherian, R/I is Noetherian, and so $gr_I(R)$ is a finitely generated algebra over a

Noetherian ring, and so is itself Noetherian.
Notice that this also proves that $gr_I(R)$ is finitely generated by elements of degree one and, as M is finitely generated, being $\{m_1, ..., m_r\}$ such a set of generators, it is easy to see that $gr_I(M)$ is generated over $gr_I(R)$ by $\{m_1+IM, ..., m_r+IM\}$, so in order to apply in full generality the results obtained in the last chapter we are just missing the Artinian hypothesis over R/I. As we have just demanded R to be Noetherian, it is not always true that R/I is Artinian, so we could add this to our set of hypothesis, but it would be too restrictive. Instead, we will look for ideals of R for which R/I is Artinian. One such example is the maximal ideal m, because we know then that R/m is a field, and so trivially it is Artinian. Imagine now that I is an ideal for which there is an n such that:

$$m^n \subseteq I \subseteq m$$

As R/I is Noetherian we know it is Artinian iff it is of dimension 0, so assume P/I is any prime ideal, then:

$$m^n \subseteq I \subseteq P.$$

But as P is a prime ideal in R, we would have $P = m$, and so m/I is the only prime ideal in R/I. Therefore R/I has dimension 0 and so it is Artinian.
From the previous comments we can apply Theorem 0.7, that assures us:

$$P(gr_I(M), t) = \frac{f(t)}{(1-t)^d}$$

for some $f(t) = a_0 + ... + a_s t^s \in \mathbb{Z}[t]$ such that $(1-t) \nmid f(t)$ and $d \in \mathbb{N}$. To simplify our notation, we will write $P_{(I,M)}(t) := P(gr_I(M), t)$.
We can also apply Lemma 0.9, that tells us:

$$H_{gr_I(M)}(n) = a_s \left(\begin{array}{c} d+n-s-1 \\ d-1 \end{array} \right) + ... + a_1 \left(\begin{array}{c} d+n-2 \\ d-1 \end{array} \right) + a_0 \left(\begin{array}{c} d+n-1 \\ d-1 \end{array} \right)$$

in a similar fashion, we will simplify our notation to $H_{(I,M)}(n) := H_{gr_I(M)}(n)$. By Corollary 0.11, from which we know that $H_{(I,M)}(n)$ and $h_{gr_I(M)}(n)$, which we will simplify to $h_{(I,M)}(n)$, coincide for n greater than $s+1-d$. Let us now fix some more notation and introduce some new concepts:

Definition 1.2 Let (R, m) be a Noetherian local ring and I an ideal of R. We say that I is an **Ideal of Definition** of R if there exists an $n > 0$ such that:

$$m^n \subseteq I$$

In the case that I is an ideal of definition, as we said above, we will denote the Hilbert Function, Hilbert Polynomial and Hilbert Series of $gr_I(M)$ as $H_{(I,M)}(n)$, $h_{(I,M)}(n)$ and $P_{(I,M)}(t)$, respectively. We will also use the notation $s(I, M)$ and $d(I, M)$ to denote the degrees of the h-polynomial of $gr_I(M)$ and $h_{(I,M)}(n)$, respectively. We will simplify this notations to $H_I, h_I, P_I, s(I), d(I)$ in the case $M = R$, and we will refer to them as Hilbert functions of R. Furthermore, in the case $I = m$

we will use the notations $H_R, h_R, P_R, s(R)$ and $d(R)$.

Definition 1.3 Let (R, m) be a Noetherian local ring, M a finitely generated R-module, I an ideal of definition of R and $(a_0, ..., a_s)$ the h-vector of $gr_I(M)$. We call **Multiplicity of** (I, M) to the sum $a_0 + ... + a_s$ and denote it by $e(I, M)$.

Let us now introduce a new object, the Samuel Function and Samuel polynomial of M. In the following chapters, it will be as relevant as the Hilbert Function, but in here we will just use it as an auxiliary tool to prove the dimension theorem, which will be the purpose of the rest of this chapter.

Definition 1.4 Let (R, m) be a Noetherian local ring, M a finitely generated R-module and I an ideal of definition of R. We call **Samuel Function** of (I, M) to:

$$H^1_{(I,M)}(n) = \sum_{i=0}^{n} H_{(I,M)}(i)$$

There is an equivalent way of defining the Samuel Function, as we can see from the following equality:

$$H^1_{(I,M)}(n) = \sum_{i=0}^{n} H_{(I,M)}(i) = l(M/IM) + l(IM/I^2M) + ... + l(I^nM/I^{n+1}M) =$$

$$= l(M) - l(IM) + l(IM) - l(I^2M) + ... + l(I^nM) - l(I^{n+1}M) \iff$$

$$H^1_{(I,M)}(n) = l\left(\frac{M}{I^{n+1}M}\right) \tag{1.1}$$

Both definitions are useful, and is best to keep in mind the two of them, as we will be using both. Being defined as a sum of the previous terms of the Hilbert Function, it is not surprising that we can also get a polynomial that is equal to the Samuel Function from a certain point on. In order to pursue this result, we will have to prove the following lemma:

Lemma 1.5 For every m and n the following equality holds:

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m-1 \\ n-1 \end{pmatrix} + \begin{pmatrix} m-1 \\ n \end{pmatrix}$$

Proof. First, regard that if m is less or equal than 0 or n is greater than m, by the assumption we have made on combinations in chapter 0, this equality just becomes $0 = 0 + 0$. Assuming now that

$m \geqslant n \geqslant 0$, we have:

$$\binom{m-1}{n-1} + \binom{m-1}{n} =$$

$$\frac{(m-1)!}{(n-1)!(m-1-(n-1))!} + \frac{(m-1)!}{n!(m-1-n)!} =$$

$$(m-1)!\left(\frac{1}{(n-1)!(m-n)!} + \frac{1}{n!(m-1-n)!}\right) =$$

$$(m-1)!\frac{n+m-n}{n!(m-n)!} =$$

$$= \binom{m}{n}$$

as we wanted. □

Lemma 1.6 Let (R, m) be a Noetherian local ring, M a finitely generated R-module, I an ideal of definition of R, $(a_o, ..., a_s)$ the h-vector of (I, M) and $d(I, M) = d$ the degree of the Hilbert polynomial of $gr_I(M)$. Then the following equality holds:

$$H^1_{(I,M)}(n) = \sum_{j=0}^{s} a_j \binom{d+n-j}{d}$$

Proof. We will proceed by induction on n. If n is equal to zero, then:

$$\sum_{j=0}^{s} a_j \binom{d-j}{d} = a_0 \binom{d}{d} = a_0 \binom{d-1}{d-1} = \sum_{j=0}^{s} a_j \binom{d-1-j}{d-1}$$

By Lemma 0.9:

$$= H_{(I,M)}(0) = H^1_{(I,M)}(0).$$

Assume now that $n > 0$:

$$H^1_{(I,M)}(n) = \sum_{j=0}^{n} H_{(I,M)}(j) = H^1_{(I,M)}(n-1) + H_{(I,M)}(n).$$

By the inductive hypothesis, it follows that:

$$H^1_{(I,M)}(n) = \sum_{j=0}^{s} a_j \binom{d+n-1-j}{d} + H_{(I,M)}(n).$$

By Lemma 0.9 we obtain:

$$H^1_{(I,M)}(n) = \sum_{j=0}^{s} a_j \binom{d+n-1-j}{d} + \sum_{j=0}^{s} a_j \binom{d+n-j-1}{d-1} =$$

$$= \sum_{j=0}^{s} a_j \left(\binom{d+n-1-j}{d} + \binom{d+n-j-1}{d-1} \right)$$

and so lemma 1.5 assures us that:

$$H^1_{(I,M)}(n) = \sum_{j=0}^{s} a_j \binom{d+n-j}{d}$$

as we wanted.

□

Exactly as in the case of the Hilbert Function, we conclude there is a polynomial, in this case of degree $d(I,M)$, such that whenever $n \geqslant s(I,M) - d(I,M)$, it coincides with the Samuel Function. We will call it Samuel Polynomial and denote it by $h^1_{(I,M)}(n)$. As before, we will simplify our notation to H^1_I, h^1_I in the case $M = R$ and H^1_R, h^1_R in the case $I = m$.
We are now ready to make our first step towards the Dimension Theorem: the degree $d(I,M)$ actually does not depend on I and is an invariant of M. The remarkable result referenced in the beginning of this chapter is that this coefficient is actually equal to the dimension of M and the rest of this chapter will be entirely dedicated to the proof of this result.

Lemma 1.7 Let (R,m) be a Noetherian local ring, M a finitely generated R-module and I an ideal of definition of R. Then, $d(I,M) = d(m,M)$, and so this number is an invariant of M, which we will denote by $d(M)$.

Proof. We know that there exists an $i \geqslant 0$ for which:

$$m^i \subseteq I \subseteq m \Rightarrow m^{in} \subseteq I^n \subseteq m^n.$$

But then, for $n \geqslant max\{s(I,M) - d(I,M), s(m,M) - d(m,M)\}$:

$$h^1_{(m,M)}(ni-1) = l(M/m^{ni}M) = l(M) - l(m^{ni}M) \geqslant l(M) - l(I^nM) = h^1_{(I,M)}(n-1).$$

Similarly, we get that:

$$h^1_{(I,M)}(n-1) = l(M) - l(I^nM) \geqslant l(M) - l(m^nM) = h^1_{(m,M)}(n-1)$$

and so we conclude that $h^1_{(m,M)}$ and $h^1_{(I,M)}$ have the same degree, i.e., $d(I,M) = d(m,M)$, as we wanted.

□

Consider the following result:

Theorem 1.8 - Let (R, m) be a Noetherian local ring, and:

$$0 \to M' \to M \to M'' \to 0$$

a exact sequence of finitely generated R-modules, then the following equality holds:

$$d(M) = max\{d(M'), d(M'')\}$$

Also if I is any ideal of definition of R, then $h^1_{(I,M)} - h^1_{(I,M'')}$ and $h^1_{(I,M')}$ have the same leading coefficient.

Proof. As usual we will start to assume that $M'' = M/M'$. With this assumption, we get that:

$$M''/I^nM'' = \frac{M/M'}{I^n(M/M')} = \frac{M/M'}{(I^nM + M')/M'} \cong$$

$$\cong \frac{M}{I^nM + M'}.$$

From this we obtain:

$$l(M/I^nM) = l(M) - l(I^nM + M') + l(I^nM + M') - l(I^nM).$$

By the second isomorphism theorem and the equality above:

$$l(M/I^nM) = l(M''/I^nM'') + l(M'/(M' \cap I^nM).$$

Then, we have that, for large enough n:

$$h^1_{(I,M)}(n) = h^1_{(I,M'')}(n) + l(M'/(M' \cap I^{n+1}M))$$

As $h^1_{(I,M)}$ and $h^1_{(I,M'')}$ are both polynomial functions, we conclude that also $l(M'/(M' \cap I^nM)$ is a polynomial function in n. Evenmore, since any of the three polynomial in this last equality only take positive values, we know that $d(M) = max\{d(M''), deg\, l(M'/(M' \cap I^nM)\}$.
By the Artin-Rees Lemma ([Mats], page 59, Theorem 8.5), we know that there exists a $c > 0$ for which:

$$\forall n \geqslant c, I^{n+1}M' \subseteq M' \cap I^{n+1}M \subseteq I^{n-c+1}M'$$

and hence, for large enough n:

$$h^1_{(I,M')}(n) = l(M') - l(I^{n+1}M') \geqslant l(M') - l(M' \cap I^{n+1}M) =$$

$$= l(M'/(M' \cap I^{n+1}M)) \geqslant l(M') - l(I^{n-c+1}M') = h^1_{(I,M')}(n - c).$$

But then, both $h^1_{(I,M')}$ and $l(M'/(M' \cap I^{n+1}M))$ have the same degree and the same leading coefficient, which allows us to conclude both of our statements. □

Besides proving that $d(M)$ is equal to the dimension of M, the Dimension Theorem also establishes an equality with another invariant for M:

Definition 1.9 Let (R, m) be a Noetherian local ring and M a finitely generated R-module. We denote by $\delta(M)$ the minimum natural number such that there exists a sequence $x_1, ..., x_{\delta(M)}$ for which $l(M/(x_1M+...+x_{\delta(M)}M)$ is finite. If M happens to have finite length, we interpret $\delta(M)$ as 0.

Notice that as (R, m) is a Noetherian local ring and M is a finitely generated R-module this number always exist. For instance, if I is an ideal of definition of R then M/IM has finite length, and hence a generating set of I is as in our definition.
We will finally be able to state our theorem:

Theorem 1.10 (Dimension Theorem) Let (R, m) be a Noetherian local ring and M a finitely generated R-module. Then the following equalities hold:

$$dim(M) = d(M) = \delta(M)$$

Surprisingly enough, this theorem doesn't require much to prove, but it is rather long. Because of this, we will break it into smaller lemmas, each of which gives us a relation between the numbers above:

Lemma A - Let (R, m) be a Noetherian local ring. Then $d(R) \geqslant dimR$

Proof. We will proceed by induction on $d(R)$. Start to assume that $d(R) = 0$, then we know that its Hilbert Function starts to be zero at a certain point, but then there exists an n s.t.:

$$m^n = m^{n+1}$$

and so, by Nakayama Lemma, $m^n = 0$. But then, if P is a prime ideal of R, $m^n \subset P$, which implies $P = m$, but hence P is maximal, and hence every prime ideal of R is maximal, i.e., $dimR = 0$.
Assuming that $d(R) > 0$, if $dimR = 0$ we are obviously done, so suppose that $dimR > 0$ and let:

$$P_0 \subset P_1 \subset ... \subset P_l$$

be any strictly increasing chain of primes of R, pick up any x in $P_1 \setminus P_0$ and consider the following exact sequence:

$$0 \longrightarrow R/P_0 \xrightarrow{x} R/P_0 \longrightarrow R/(P_0 + xR) \longrightarrow 0$$

Where the endomorphism of R/P_0 is, as the notation indicates, given by multiplication by x and the other one is just the canonical projection. By Theorem 1.1 we know that $d(R/P_0) > d(R/(P_0+xR))$, but then, by induction, we get:

$$l - 1 \leqslant dimR/(P_0 + xR) \leqslant dR/(P_0 + xR) \leqslant d(R) - 1$$

where the first inequality is a consequence of the fact that the strictly chain of prime ideals above, without P_1, is a chain of primes in $R/(P_0+xR)$. But, because this chain was arbitrary, we conclude that the length of any chain of prime ideals of R is less or equal than $d(R)$, i.e., $dimR \leqslant d(R)$, as we intended. □

Lemma B - Let (R, m) be a Noetherian local ring and M a finitely generated R-module, we have that $d(M) \geqslant dimM$

Proof. We know ([Mats], Page 39, Theorem 6.4) that we can have a R-submodule M' of M such that:

$$M/M' \simeq R/p$$

For some $p \in Spec(R)$. Now from the exact sequence:

$$0 \to M' \to M \to R/p \to 0$$

And using Theorem 1.8 and Lemma A we get that:

$$d(M) \geqslant d(/p) \geqslant dim(R/p) \geqslant dim(R/Ann(M)) = dim(M)$$

□

Lemma C Let (R, m) be a Noetherian local ring and M a finitely generated R-module, then $d(M) \leqslant \delta(M)$.

Proof. If $\delta(M) = 0$, then $l(M)$ is finite, and as

$$l(M/M') = l(M) - l(M')$$

We conclude that the Samuel Function of M is bounded, and then it has degree $d(M) = 0$.
Assume now that $\delta(M) > 0$, let $\{x_1, ..., x_{\delta(M)}\}$ be such that $l(M/(x_1, ..., x_{\delta(M)})M)$ is finite. Let $M_i = M/(x_1, ..., x_i)M$, the assumption of minimality on $\{x_1, ..., x_{\delta(M)}\}$ assures us that $\delta(M/(x_1, ..., x_i)) = \delta(M) - i$, for all $1 \leqslant i \leqslant s$, and consequently $\delta(M_s) = 0$, which by the above paragraph shows that $d(M_s) = 0$. Also:

$$l(M_1/m^n M_1) = l\left(\frac{M/x_1M}{m^n(M/x_1M)}\right) = l(M/(x_1M + m^nM)) =$$

$$l(M) - l(m^n M) + l(m^n M) - l(x_1 M + m^n M) = l(M/m^n M) - l((x_1 M + m^n M)/m^n M) =$$

By the second isomorphism theorem:

$$= l(M/m^n M) - l(x_1 M/(x_1 M \cap m^n M)) =$$

Now notice that $x_1 M \cap m^n M = x_1(m^n M : x_1)$, and so trivially the map from $M/(m^n M : x_1)$ to $x_1 M/(x_1 M \cap m^n M)$ given by x_1 multiplication is an isomorphism:

$$= l(M/m^n M) - l(M/(m^n M : x_1)) \geqslant$$

as $m^{n-1} M \subseteq (m^n M : x_1)$:

$$\subseteq l(M/m^n M) - l(M/m^{n-1} M)$$

And so we conclude that $h^1_{(m,M_1)}(n-1) \geqslant h^1_{(m,M)}(n-1) - h^1_{(m,M)}(n-2)$, and hence $d(M_1) \geqslant d(M)-1$. Repeating this procedure we get that $d(M_s) = 0 \geqslant d(M) - s$, and so $d(M) \leqslant s = \delta(M)$, as we intended. □

Let us just pause the proof to make a remark: this will be the first time where the comparison between $(m^n : x)$ and m^{n-1}, for x an element in m, will be crucial in the study of the properties of the Hilbert Function. As we will see, they will prove to have a major role in it. We are now ready for the forth, and last, step of our proof:

Lemma D Let (R, m) be a Noetherian local ring and M a finitely generated R-module, then $dim(M) \geqslant \delta(M)$.

Proof. We will proceed by induction on $dim(M)$. If $dim(M) = dim(R/Ann(M)) = 0$, then $(R/Ann(M), m/Ann(M))$ is a Noetherian local ring of dimension zero, and hence also an Artinian ring. Setting $\tilde{m} = m/Ann(M)$, it follows that the chain:

$$\tilde{m} \supseteq \tilde{m}^2 \supseteq ... \supseteq \tilde{m}^n \supseteq ...$$

has to stop, and so there is an n for which $\tilde{m}^n = \tilde{m}^{n+1}$ and we conclude, by Nakayama, that $\tilde{m} = 0$, and so $Ann(M) = m$ and so M is a finitely generated vector space, and so its length is equal to its dimension, and in particular it is finite, hence $\delta(M) = 0$.
Assume now that $dim(M) > 0$ and let $P_1, ..., P_r$ be the minimal primes over $Ann(M)$. As $dim(M)$ is not zero, we conclude that none of this primes can be m, and so by Prime Avoidance ([Swan, Hun], Appendix A) we can choose an $x_1 \in m$ not in any P_i. But then $dim(M_1) < dim(M)$ and by the inductive hypothesis we conclude that $\delta(M_1) \leqslant dim(M_1)$. We then conclude our result from the trivial fact that $\delta(M) \leqslant \delta(M_1) + 1$. □

The most important consequence of this theorem for our purposes is that $d(I) = d(R)$ is actually equal to $dim(R)$, which gives one more restriction to the form the Hilbert Series, and the Hilbert polynomial, of $gr_I(R)$.

Chapter 2

Higher Iterated Hilbert Functions

In the first two chapters we just introduced the preliminary concepts, which from now on we will restrict and study in a particular case: the graded ring of a Noetherian local ring R associated to an ideal of definition I, $gr_I(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. Recall that applying Theorem 0.7 to this graded R-module, which is generated by elements of I/I^2, and hence is generated by elements of degree one, we obtain that its Hilbert Series has the special form:

$$P_I(t) = \bigoplus_{n=0}^{\infty} l(I^n/I^{n+1})t^n = \frac{f(t)}{(1-t)^d}$$

where $f(t) = a_0 + a_1 t + ... + a_s t^s \in \mathbb{Z}[t]$ is not divisible by $1-t$. We called h-vector of R associated to I to $(a_o, ..., a_s)$, and h-polynomial of R associated to I to $f(t)$. The dimension theorem, proved in the last chapter, stated that the d that appears above is in fact equal to the dimension of R. We kept on going, and from Lemma 0.9 were able to prove that the Hilbert Function of $gr_I(R)$ can be written as:

$$H_I(n) = l(I^n/I^{n+1}) = a_s \begin{pmatrix} d+n-s-1 \\ d-1 \end{pmatrix} + ... + a_1 \begin{pmatrix} d+n-2 \\ d-1 \end{pmatrix} + a_0 \begin{pmatrix} d+n-1 \\ d-1 \end{pmatrix}.$$

In chapter one we introduced an iteration of the Hilbert Function, which we called Samuel Function, H_I^1, and proved, in Lemma 1.6, it can be written in a similar fashion of the Hilbert Function:

$$H_I^1(n) = \sum_{i=0}^{n} H_I(i) = l\left(\frac{R}{I^{n+1}}\right) = a_s \begin{pmatrix} d+n-s \\ d \end{pmatrix} + ... + a_1 \begin{pmatrix} d+n-1 \\ d \end{pmatrix} +$$

$$+a_0 \begin{pmatrix} d+n \\ d \end{pmatrix}.$$

Making the simple observation that $\begin{pmatrix} k+n \\ k \end{pmatrix}$ can be identified with a polynomial in n of degree m for $n+k \geqslant 0$, this two formulas allowed us to state that the Hilbert Function and Samuel Function are equal to a polynomial for $n \geqslant s-d-1$, in the case of the Hilbert Function, and

$n \geqslant s-d$, in the case of the Samuel Function, with degrees $d-1$ and d, respectively, to which we called Hilbert Polynomial and Samuel Polynomial, denoted by h_I and h_I^1, respectively.
The Samuel Function is just the first iterated Hilbert Function, and we will start this chapter to define one for each natural number, as is showed below:

Definition 2.1: Let (R, m) be a Noetherian local ring and I an ideal of definition of R. The i^{th} **iterated Hilbert Function**, denoted H_I^i, is defined as follows:

$$\begin{cases} H_I^i(n) = l(\frac{m^n}{m^{n+1}}) & \text{for} \quad i=0 \\ H_I^{i+1}(n) = \sum_{j=0}^{n} H_I^i(j) & \text{for} \quad i>0 \end{cases}$$

As before, we will denote this function by H_R^i in the case $I = m$

One considerable simple equality that follows directly from this definition is the following:

$$H_I^{i+1}(n) - H_I^{i+1}(n-1) = \sum_{j=0}^{n} H_I^i(j) - \sum_{j=0}^{n-1} H_I^i(j) = H_I^i(n) \tag{2.1}$$

Regardless its simpleness, we have stated it because of its usefulness, with one first aplication being the following lemma:

Lemma 2.2 Let $f(t) = a_0 + ... + a_s t^s$ be the $h-polynomial$ of $gr_I(R)$, for each $i, n \in \mathbb{N}$ the following equality holds:

$$H_I^i(n) = \sum_{j=0}^{s} a_j \left(\begin{array}{c} d+i+n-j-1 \\ d+i-1 \end{array} \right)$$

Where in here $\left(\begin{array}{c} m \\ d-1 \end{array} \right) = 0$, for $m < d-1$. In particular, we conclude, as in the case of the Hilbert Function, that there exists a is a polynomial function in n of degree $d+i-1$, such that it coincides with $H_R^i(n)$ since $n \geqslant s+1-d-i$.

Proof. We know that this equality is true for $i=0$ and $i=1$ from Lemma 0.9 and Lemma 1.6, respectively. Besides that, for n=0 we have that:

$$H_I^i(0) = H_I(0) = l(R/I) = \sum_{j=0}^{s} a_j \left(\begin{array}{c} d-j-1 \\ d-1 \end{array} \right) =$$

$$= \sum_{j=0}^{s} a_j \begin{pmatrix} d-j-1 \\ -j \end{pmatrix}.$$

And because $\begin{pmatrix} d+i+-j-1 \\ -j \end{pmatrix}$ is 0 unless j is 0, we have that:

$$H_R^i(0) = a_0 = \sum_{j=0}^{s} a_j \begin{pmatrix} d+i-j-1 \\ -j \end{pmatrix}$$

so the result follows from induction on i and n, because equality (2.1) implies that:

$$H_R^i(n) = H_R^{i-1}(n) + H_R^i(n-1)$$

and using the inductive hypothesis this becomes:

$$= \sum_{j=0}^{s} a_j \left(\begin{pmatrix} d+i-1+n-j-1 \\ n-j \end{pmatrix} + \begin{pmatrix} d+i+n-1-j-1 \\ n-1-j \end{pmatrix} \right)$$

and using Lemma 1.5 we obtain:

$$= \sum_{j=0}^{s} a_j \begin{pmatrix} d+i+n-j-1 \\ n-j \end{pmatrix}$$

as we wanted. □

From this result we conclude that, as it happened with the Hilbert and Samuel Functions, there exists a polynomial that from a certain point on is equal to the $i^t h$ iterated Hilbert Function:

Definition 2.3 : We call the function:

$$h_R^i(X) = \sum_{j=0}^{s} a_j \begin{pmatrix} d+i+X-j-1 \\ X-j \end{pmatrix}$$

the **higher i^{th} iterated Hilbert Polynomial** of R.

Notice that in order to determine any of the iterated Hilbert Functions and Polynomials the only thing we have to know is the h-vector of R associated to I. It will prove to be useful to rewrite our Hilbert Polynomial in a different fashion, using a different basis for $\mathbb{Q}[X]$. To do this consider the following notation:

$$\begin{pmatrix} X+p \\ p \end{pmatrix} = \frac{(X+p)...(X+1)}{p!}$$

Thus, we can write:

$$h_R^i(X) = \sum_{j=0}^{d+i-1} (-1)^j e_j^{(i)} \begin{pmatrix} X+d+i-j-1 \\ d+i-j-1 \end{pmatrix}$$

For some rational coefficientes $e_j^{(i)} \in \mathbb{Q}$. Considering that the Hilbert higher iterated functions are obtained from summing the previous ones, it is not surprising that we have the following result:

Proposition 2.4: The coefficientes $e_j^{(i)}$ introduced above satisfy the following properties:

1. The coefficients $e_j^{(i)}$ do not depend on i;
2. $e_k = \sum_{j=k}^{s} \begin{pmatrix} j \\ k \end{pmatrix} a_j$.

and in particular we conclude that $e_k \in \mathbb{Z}$, $e_k = 0$, for $k > s$ and $e_0 = e = a_0 + ... + a_s$

Proof. Once again, the equality (2.1) and lemma 1.5 will play their role:

(1):

Using lemma 2.2 and 1.5 we have that, for $n >> 0$:

$$H_R^{i+1}(n) - H_R^{i+1}(n-1) =$$

$$= \sum_{j=0}^{d+i} (-1)^j e_j^{(i+1)} \left(\begin{pmatrix} n+d+i-j \\ d+i-j \end{pmatrix} - \begin{pmatrix} n-1+d+i-j \\ d+i-j \end{pmatrix} \right) =$$

$$= \sum_{j=0}^{d+i} (-1)^j e_j^{(i+1)} \begin{pmatrix} n+d+i-1-j \\ d+i-1-j \end{pmatrix} =$$

$$= \sum_{j=0}^{d+i-1} (-1)^j e_j^{(i+1)} \begin{pmatrix} n+d+i-1-j \\ d+i-1-j \end{pmatrix}.$$

And from equality (2.1) we get:

$$H_R^i(n) = \sum_{j=0}^{d+i-1} (-1)^j e_j^{(i)} \begin{pmatrix} n+d+i-1-j \\ d+i-1-j \end{pmatrix}$$

$$= \sum_{j=0}^{d+i-1} (-1)^j e_j^{(i+1)} \begin{pmatrix} n+d+i-1-j \\ d+i-1-j \end{pmatrix}$$

From which (1) follows.

(2):

In order to prove (2) we will have to use the following equality:

$$\binom{m-j}{p} = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{m-k}{p-k}$$

From this we get:

$$h_R^i(X) = \sum_{j=0}^{s} a_j \binom{X+d+i-1-j}{d+i-1}$$

$$= \sum_{j=0}^{s} a_j \left(\sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{X+d+i-1-k}{d+i-1-k} \right)$$

$$= \sum_{j=0}^{s} \left(\sum_{k=0}^{j} (-1)^k a_j \binom{j}{k} \binom{X+d+i-1-k}{d+i-1-k} \right) =$$

Now, putting $\binom{X+d+i-1-k}{d+i-1-k}$ in evidence we get:

$$= \sum_{k=0}^{s} (-1)^k (\sum_{j=k}^{s} a_j \binom{j}{k}) \binom{X+d+i-1-k}{d+i-1-k}$$

And (2) also follows. □

From this proposition we know that if we can determine this new coefficients we can determine the h-vector, and hence the Hilbert Series, Hilbert Function and Hilbert Polynomial of R. They will prove to be much better to work with than the h-vector, so we will give them a proper name:

Definition 2.5: Given (R, m) a Noetherian local ring and I an ideal of definition of R, we call **Hilbert coefficients** of R associated to I to the $\{e_i(I)\}_{i \in \mathbb{N}}$ defined above. As we have done in the previous calculation, we will drop the I in the notation when no confusion arises. In the case $I = m$, we will use the notation $e_i(R)$.

As we will see in the last chapter of this thesis, a lot can be said about our ring from its Hilbert Coefficients. In many cases, we are not interested in calculating them explicitly, what can be rather hard, but in some relations they satisfy and how can we translate them into algebraic properties. In the next chapter we will introduce techniques to simplify their calculation, being able to replace, in some cases, the calculation of the Hilbert coefficients of a Noetherian Cohen-Macaulay local ring R by a ring of smaller dimension. A first step towards this end is the following very useful equality:

Lemma 2.6 (Singh's equality) - Let (R, m) be a Noetherian local ring, I an ideal of definition of R, then the following equality holds for every $x \in I$:

$$l\left(\frac{I^n}{I^{n+1}}\right) = H^1_{I/(x)}(n) - l\left(\frac{I^{n+1} : x}{I^n}\right)$$

Proof. We have the following:

$$l\left(\frac{I^n}{I^{n+1}}\right) = H^1_{(R/(x),I/(x))}(n) - l\left(\frac{I^{n+1}:x}{I^n}\right)$$
$$-l(I^{n+1}) = l(R/(x)) - l((I/(x))^{n+1}) - l(I^{n+1}:x)$$
$$-l(I^{n+1}) = l(R) - l((x)) - l\left(\frac{I^{n+1}+(x)}{(x)}\right) - l(I^{n+1}:x)$$
$$-l(I^{n+1}) = l(R) - l(I^{n+1}+(x)) - l(I^{n+1}:x)$$
$$l\left(\frac{I^{n+1}+(x)}{I^{n+1}}\right) = l\left(\frac{R}{I^{n+1}:x}\right)$$

Now, from the second isomorphism theorem we know that $\frac{I^{n+1}+(x)}{I^{n+1}} \cong \frac{(x)}{I^{n+1}\cap(x)}$, so if we can prove that $\frac{R}{I^{n+1}:x} \cong \frac{(x)}{I^{n+1}\cap(x)}$ our result will follow. A possible isomorphism is given by the following function, as we show below

$$\frac{R}{I^{n+1}:x} \to \frac{(x)}{I^{n+1}\cap(x)}$$
$$r + (I^{n+1}:x) \mapsto xr + (I^{n+1}\cap(x))$$

We have that:

1. the function is well defined:
   if $r \in I^{n+1}:x$, then $xr \in I^{n+1}$ and obviously xr is in (x), and we conclude that the function is well defined;

2. the function is obviously surjective and a R-modules morphism;

3. the function is injective:
   if $xr + (I^{n+1}\cap(x)) = I^{n+1}\cap(x)$ then, by definition, r is in $I^{n+1}:x$

□

As mentioned previously, in the following chapter our purpose will be to introduce techniques that allow us to replace the calculation of the Hilbert coefficients of a Cohen-Macaulay Noetherian local ring (R,m) for one of dimension strictly lower, being this a good motivation to pursue an understanding of the Hilbert Function of a ring of dimension one. The next Lemma will prove to be perfect to this purpose:

Lemma 2.7: Let (R,m) be a 1-dimensional Noetherian Cohen-Macaulay local ring, I an ideal of definition of R, $(a_0,...,a_{s(I)})$ the h-vector of R and x a regular element that satisfies:

$$\frac{(I^{n+1}:x)}{I^n}=0 \tag{2.2}$$

for large enough n. Then the following equality holds for every $n\in\mathbb{N}$:

$$H_I(n)=l\left(\frac{I^n}{I^{n+1}}\right)=e_0(I)-\rho_n$$

where $\rho_j>0$, if $j<s(I)$, and $\rho_j=0$ if $j\geqslant s(I)$

Proof. We will prove this result in four steps, being the first one that the Hilbert Function of R is constant equal to $e_0(I)$ from $s(I)$ on. This is in fact an easy statement, since Corollary 0.11 assures us that the Hilbert polynomial is of degree $d-1$, and since the Dimension theorem tells us that this d is in fact equal to the dimension of R, we obtain that it is constant and it coincides with the Hilbert Function for $n\geqslant s(I)+d-1=s(I)$. Then, for $n\geqslant s(I)$:

$$l\left(\frac{I^n}{I^{n+1}}\right)=H_I(n)=\sum_{j=0}^{s}a_j(I)=e_0(I)$$

and so H_I is in fact equal to $e_0(I)$ from $s(I)$ on, and so we also conclude that this ρ_j's are zero from $s(I)$ on.
The second step will be to prove that $H_I(n)\leqslant e_0(I)$. To do so, notice that our assumption (2.2) on x combined with Singh's equality gives us that for large enough n, $e_0(I)=H_I(n)=H_{I/(x)}(n)$. The result now follows from the fact that by definition the Samuel Function is always an increasing function, wich proves that $H_{I/(x)}(n)\leqslant e_0(I)$, for all n, and so:

$$H_I(n)=H_{I/(x)}(n)-l\left(\frac{(I^{n+1}:x)}{I^n}\right)\leqslant H_{I/(x)}(n)\leqslant e_0(I)$$

as we wanted. Notice that statement assures us the existence of this nonnegative ρ_j's, and so to complete our proof we only have to show that they are strictly greater than 0 for $j<s(I)$.
The third step is again an easy consequence of Corolarry 0.11, this tells us that:

$$\forall n<s(I):H_I(n)=\sum_{i=0}^{n}a_i$$

and in particular we conclude that $H_I(s(I)-1)=a_0+...+a_{s(I)-1}$ is different from $e_0(I)=a_0+...+a_{s(I)}$, since $a_{s(I)}$ is by definition different from zero.
Finally, we will prove that :

$$H_I(n)=e_0(I)\Rightarrow H_I(n+1)=e_0(I)$$

which will end our proof. In order to do it, we will start to prove that $H^1_{(I/(x)}$ is strictly increasing until it starts to be constant. Recall that:

$$H^1_{(I/(x)}(n) = \sum_{i=0}^{n} l\left(\frac{(I/(x))^i}{(I/(x))^{i+1}}\right)$$

. and Nakayama tells us that:

$$\frac{(I/(x))^k}{(I/(x))^{k+1}} = 0 \iff (I/(x))^k = (I/(x))^{k+1} \iff (I/(x))^k = 0 \iff$$

$$\forall n \geqslant k, (I/(x))^n = 0$$

so our claim is indeed true. Fix now some n for which $H_I(n) = e_0(I)$, using Singh's equality this tells us that:

$$H_I(n) = e_0(I) \Rightarrow e_0(I) = H^1_{(I/(x)}(n) - l\left(\frac{(I^{n+1} : x)}{I^n}\right)$$

and since $H^1_{(I/(x)}$ is strictly increasing until it starts to be constant equal to $e_0(I)$, this tells us that:

$$H^1_{(I/(x)}(n) = e_0(I); H^1_{(I/(x)}(n+1) = e_0(I)$$

$$l\left(\frac{(I^{n+1} : x)}{I^n}\right) = 0 \iff I^{n+1} : x = I^n.$$

Now observe that:

$$I^{n+1} : x = I^n \tag{2.3}$$

$$(I/(x))^n = 0 \Rightarrow I^n + (x) = (x) \Rightarrow I^n \subseteq (x) \Rightarrow$$

$$\forall m \geqslant n, I^m \subseteq (x) \tag{2.4}$$

From this we conclude:

$$xI^{n+1} = I^{n+2} = (x) \cap I^{n+2} = x(I^{n+2} : x)$$

And as x is regular, this implies $I^{n+1} = (I^{n+2} : x)$, but then $H_I(n+1) = H^1_{I/(x)}(n+1) = e_0(I)$, as we have claimed. □

So we have asserted three constants that give alternative descriptions of the Hilbert Function of a Cohen-Macaulay: first, the h-vector $(a_0, ..., as)$, given by the numerator of the Hilbert Series, secondly the Hilbert coefficients $\{e_j\}_{j\in\mathbb{N}}$, obtained by rewriting our Hilbert Polynomial using a different basis for $\mathbb{Q}[t]$, and finally this new $\rho_{j_{j\in\mathbb{N}}}$. Proposition 2.4 already asserts the way the Hilbert Coefficients and the h-vector are related, but to make our most recent description, given by the $\rho_{j_{j\in\mathbb{N}}}$, useful we have to relate it to the previous ones, which is done in this proposition, that ends this chapter:

Proposition 2.8 Let (R, m) be a 1-dimensional Cohen-Macaulay local ring, $(a_0, ..., a_s)$ the h-vector of R and $(e_0, ..., e_s)$ its Hilbert coefficients. Then, defining the set of $\{\rho_j\}_{j\in\mathbb{N}}$ as in lemma 2.3 , we have the following properties:

1. $\rho_0 = e - 1$, $\rho_1 = e - h - 1$
2. $l(\frac{m^j}{m^{j+1}}) = \sum_{i=0}^{j} a_i$
3. $\forall k \geqslant 1, e_k = \sum_{j=k-1}^{s-1} \left(\begin{array}{c} j \\ k-1 \end{array} \right) \rho_j$
4. $\sum_{j=0}^{k} (-1)^j e_j - 1 = (-1)^k \sum_{t=k}^{s-1} \left(\begin{array}{c} t-1 \\ k-1 \end{array} \right) \rho_t$

Proof. The proof of (1) is obvious from the fact that $l(R/m) = 1$ and $l(m/m^2) = h$.
To prove (2) let us notice that:

$$P(R,t) = \sum_{n=0}^{\infty} l\left(\frac{m^n}{m^{n+1}}\right) t^n = l(R/m) + ... + l\left(\frac{m^{s-1}}{m^s}\right) + \sum_{n=s}^{\infty} l\left(\frac{m^n}{m^{n+1}}\right) t^n =$$

From proposition 2.3 we know that $l(\frac{m^n}{m^{n+1}}) = e$, $\forall n \geqslant s$, so:

$$= l(R/m) + ... + l(\frac{m^{s-1}}{m^s}) + et^s \sum_{n=0}^{\infty} t^n =$$

$$= l(R/m) + ... + l(\frac{m^{s-1}}{m^s}) + \frac{et^s}{1-t} =$$

$$= \frac{(l(R/m) + ... + l(\frac{m^{s-1}}{m^s}))(1-t) + et^s}{1-t} =$$

$$= \frac{l(R/m) + (l(m/m^2) - l(R/m))t + ... + (l(m^l/m^{l+1}) - l(m^{l-1}/m^l))t^l + ... + et^s}{1-t}$$

But because the R has dimension 1 we can conclude that its h-polynomial is given by:

$$l(R/m) + (l(m/m^2) - l(R/m))t + ... + (l(m^l/m^{l+1}) - l(m^{l-1}/m^l))t^l + ... + et^s$$

And so:

$$a_0 = l(R/m)$$

$$a_s = e$$

$$a_l = l(m^l/m^{l+1}) - l(m^{l-1}/m^l), \forall 1 < l < s$$

And from this the result follows. In order to prove (3) we will start to recall proposition 2.1 part (2):

$$e_k = \sum_{j=k}^{s} \left(\begin{array}{c} j \\ k \end{array} \right) a_j$$

Now, from (2) we get that:

$$a_k = e - \rho_k - \sum_{i=0}^{k-1} a_i =_{(3)} e - \rho_k - (e - \rho_{k-1}) = \rho_{k-1} - \rho_k$$

Applying tis to the first equality we get that:

$$e_k = \sum_{j=k}^{s} \begin{pmatrix} j \\ k \end{pmatrix} (\rho_{j-1} - \rho_j) =$$

Recalling that $\rho_s = 0$:

$$= \rho_{k-1} + \sum_{j=k}^{s} (\begin{pmatrix} j+1 \\ k \end{pmatrix} - \begin{pmatrix} j \\ k \end{pmatrix})\rho_j$$

And applying lemma 0. we get that:

$$= \sum_{j=k-1}^{s-1} \begin{pmatrix} j \\ k-1 \end{pmatrix} \rho_j$$

As wanted. To prove the final statement let us state the following equality:

$$\sum_{j=1}^{k}(-1)^j \begin{pmatrix} t \\ j-1 \end{pmatrix} = \begin{cases} 0 & \text{if} \quad t \leqslant k-1 \\ (-1)^k \begin{pmatrix} t-1 \\ k-1 \end{pmatrix} & \text{if} \quad t \geqslant k \end{cases} \tag{2.5}$$

Now we have that:

$$(\sum_{j=0}^{k}(-1)^j e_j) - 1 = e_0 - 1 + \sum_{j=1}^{k}(-1)^j e_j =$$

Using (2):

$$= e_0 - 1 + \sum_{j=1}^{k}(-1^j) \left(\sum_{t=j-1}^{s-1} \begin{pmatrix} t \\ j-1 \end{pmatrix} \rho_t \right) =$$

From (1) we know that $e_0 - 1 = e - 1 = \rho_0$:

$$= -\sum_{t=1}^{s-1} \begin{pmatrix} t \\ j-1 \end{pmatrix} \rho_t + \sum_{j=2}^{k}(-1^j) \left(\sum_{t=j-1}^{s-1} \begin{pmatrix} t \\ j-1 \end{pmatrix} \rho_t \right)$$

Now putting the ρ_t in evidence:

$$= \sum_{t=1}^{s-1} \left(\sum_{j=1}^{k}(-1)^j \left(\begin{array}{c} t \\ j-1 \end{array} \right) \right) \rho_t$$

And applying equality (2.2):

$$= (-1)^k \sum_{t=k}^{s-1} \begin{pmatrix} t-1 \\ k-1 \end{pmatrix} \rho_t$$

Wich ends our proof. □

Chapter 3

Quotient through superficial elements

3.1 Superficial elements

Let us start to recall the Singh's equality, which we have met in the last chapter:

$$H_R(n) = H^1_{R/(x)}(n) - l(\frac{m^{n+1} : x}{m^n})$$

From this we immediately see the importance of elements that satisfy:

$$\frac{m^{n+1} : x}{m^n} = 0$$

Because from them we can reduce the calculation of the Hilbert Polynomial of R to the Samuel Polynomial of $R/(x)$, which is a ring of lower dimension. A first question that immediately comes to mind is in what conditions we can obtain such elements. The main purpose of this first section will be to pursue the conditions under which every ideal has such an element, but first let us give it a proper name:

Definition 3.1 Let (R, m) be a Noetherian local ring, I a proper ideal in R and $x \in I \setminus I^2$. We say that x is a **superficial element** of I if there exists a $c \geqslant 0$ such that:

$$\forall n \geqslant c, (I^{n+1} : x) \cap I^c = I^n \tag{3.1}$$

We call (3.1) the condition of superficiality of x.

Notice that equality (3.1) is only important by the inequality:

$$(I^{n+1} : x) \cap I^c \subseteq I^n$$

since the other direction always hold.
Another simple question is why we demand that x is not in I^2. In order to explain it, assume that x satisfies the conditions above but that x is also in I^2. We know that there exists a c such that:

$$\forall n \geqslant c, (I^{n+1} : x) \cap I^c = I^n$$

But, because x is also in I^2, if $n \geqslant c+1$, we conclude:

$$I^{n-1} \subseteq (I^{n+1} : x) \cap I^c = I^n$$

But then:

$$I^c = I^{c+1} = ... = I^n = ...$$

And because R is local and Noetherian, we know that:

$$\cap_{l \geqslant 0} I^l = 0$$

And so $I^c = 0$, but then the study of the Hilbert polynomial of $gr_I(R)$, our main interest in the study of superficial elements, is trivial.

Let K be a field and denote by $K[[X]]$ the ring of formal power series in one variable over K. Since the invertible elements are the ones that have constant part different from zero, this ring is local with maximal ideal X. Fix n to be any natural number and f to be an element in $((X)^{n+1} : X)$. Then $fX \in (X)^{n+1}$, therefore we know there exists a $g \in K[[X]]$ for which $fX = gX^{n+1}$, but since X is regular:

$$X(f - gX^n) = 0 \Rightarrow f = gX^n.$$

This gives us that $f \in (X)^n$, and hence X is a first example of a superficial element. Actually this example can be generalized: in any Noetherian local ring R, if a is a regular element, a is a superficial element of (a).
Let us start with a simple proposition about superficial elements:

Proposition 3.2: Let (R, m) be a Noetherian local ring, I a proper ideal of R and x an element of $I \setminus I^2$, then:

1. if x is a superficial element of I, then for all n x^n is a superficial element of I^n;
2. if x is a regular element of R, then x is a superficial element of I iff for all n sufficiently large $I^n : x = I^{n-1}$.

Proof. **(1)**:

Let us start to fix c such that:

$$\forall n \geqslant c, (I^{n+1} : x) \cap I^c = I^n.$$

We claim that:

$$(I^{k(n+1)} : x^k) \cap I^{ck} = I^{nk}.$$

To prove the claim observe that:

$$r \in (I^{k(n+1)} : x^k) \cap I^{ck} \Rightarrow$$

$$rx^k \in I^{k(n+1)} \Rightarrow$$

$$rx^{k-1}x \in I^{k(n+1)}.$$

Because x is superficial for I we obtain:

$$rx^{k-1} \in (I^{k(n+1)} : x^k) \cap I^c = I^{kn+k-1}$$

Repeating the process a finite number of times, we obtain:

$$r \in I^{kn}.$$

(2):

Obviously if x satisfies $(I^n : x) = I^{n-1}$ for all n sufficiently large, then x is superficial. To see the converse, assume that x is superficial and regular, then, by the Artin-Rees Lemma (which can be found in [Mats], page 59, theorem 8.5):

$$\exists k : \forall n \geqslant k, I^n \cap < x >= I^{n-k}(I^k \cap < x >).$$

Therefore,

$$x(I^n : x) = I^n \cap < x > \subseteq xI^{n-k}$$

and, because x is regular, this implies:

$$I^n : x \subseteq I^{n-k}. \tag{3.2}$$

If we fix a c such that for n greater than c x satisfies the condition of superficiality, and we pick up an $n \geqslant c + k$,

$$I^n : x = (I^n : x) \cap I^c = I^{n-1}$$

where the first equality is valid because of (3.2) and the second one because of the assumption of superficiality. This proves our result. □

From the second part of this result one may wonder that in order to find an element that is suitable for Singh's equality, one has to find an element that is both regular and superficial. In matter of fact, we just have to ask for the existence of a regular element in I in order to have all superficial elements in I regular:

Proposition 3.3 Let (R, m) be a Noetherian local ring and I a proper ideal in R with positive grade. Then all superficial elements in I are regular.

Proof. Let x be a superficial element for I and c such that:

$$\forall n \geqslant c, (I^{n+1} : x) \cap I^c = I^n \Rightarrow$$

$$\forall n \geqslant c, (I^{n+1} : x) I^c \subseteq I^n \Rightarrow$$

$$\forall n \geqslant c, (0 : x) I^c \subseteq I^n.$$

Therefore,

$$(0 : x) I^c \subseteq \bigcap_{n \geqslant 0} I^n.$$

Because R is Noetherian and local we know the right-hand side of the last equality is 0, so:

$$(0 : x) I^c = 0$$

And because $grade(I)$ is positive, we conclude that $(0 : x) = 0$, as we wanted. □

Unfortunately, the requisition of R to be Noetherian and local is not sufficient to assure the existence of a superficial element in all ideals, as the next example shows:

Example 3.4 Fix $\mathbb{Z}_2[[x, y]]$ to be the ring of power series with coefficients in $\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$. We know that in this ring the only elements that are not units are the ones that have 0 as the constant part. One easy consequence from this fact is that in any proper ideal all of its elements have the form:

$$p(x, y)x + q(x, y)y$$

for some $p(x, y), q(x, y) \in \mathbb{Z}_2[[x, y]]$. Therefore, any proper ideal is contained in $m = (x, y)$, and so this ring is local. From a special form of the Hilbert basis theorem, we also know this ring is Noetherian. Actually, this fact is not only valid for the ring of formal power series over $\mathbb{Z}_2$, one can show that any ring of power series over any field is Noetherian and local (the local part is actually proved above, since we don't explicitly use the fact that we are working with the field $\mathbb{Z}_2$).
From the paragraph above we also know that the ring $\mathbf{R} = \mathbb{Z}_2[[x, y]]/(xy(x+y))$ is Noetherian and local. Let $I = (x, y)\mathbf{R} = (x, y)/(xy(x+y))$, we claim that in this ideal we don't have any superficial

element. To prove it, assume $r \in I \setminus I^2$ is a superficial element. Therefore, there exists a $c \geqslant 0$ for which:

$$\forall n \geqslant c, (I^{n+1} : r) \cap I^c = I^n.$$

As r is not in $I^2 = [(x,y)/(xy(x+y))]^2 = (x^2, y^2, xy)/(xy(x+y))$, we know that the degree 1 part of r must be nonzero. We may assume without loss of generality that its degree 1 part is X, therefore:

$$r = a(x,y)x + b(x,y)y^2 + (xy(x+y))$$

for some $a(x,y), b(x,y) \in \mathbb{Z}_2[[x,y]]$. Therefore, if $n > max\{2, c\}$:

$$y^{n-2}(x+y)r + (xy(x+y)) = y^{n-2}x(x+y)a(x,y) + y^n(x+y)b(x,y) + (xy(x+y)).$$

As $n > 2$, we know $y^{n-2}x(x+y)a(x,y) \in (xy(x+y))$, therefore:

$$y^{n-2}(x+y)r + (xy(x+y)) = y^n(x+y)b(x,y) + (xy(x+y)) \in I^{n+1}.$$

From this we conclude that $y^{n-2}(x+y) + (xy(x+y)) \in (I^{n+1} : r)$. On the other hand, as $n > c$, $y^{n-2}(x+y) + (xy(x+y)) \in I^c \ I^n$, thus contradicting the superficial assumption on r.

The problem with this example comes from the fact that we have not enough space to maneuver. Actually, it can be shown that for every ideal I in a Noetherian local ring there is a power of it for which it has a superficial element, but in order to obtain such an element in I itself, our ring has to be big enough:

Theorem 3.5 Let (R, m) be a Noetherian local ring and I a proper ideal in R. There exists $n \geqslant 1$ and an element x in I^n such that x is a superficial element for I^n. Furthermore, if we require R to have an infinite residue field, then n can be taken to be 1.

Proof. Let us start to fix a primary decomposition of the 0 ideal in $gr_I(R)$:

$$0 = N_1 \cap ... \cap N_r$$

and let P_i be the radical of each of the N_i's, i.e., $\sqrt{N_i} = P_i$. Because 0 is a homogeneous ideal, and each of the P_i's is a prime divisor of zero, we know that they are also homogeneous (**Mats**, page 101, theorem 13.7 part 1).
Assume that the first s primes all contain $gr_I(R)^+$ and the last $r - s$ do not contain it. We know for each i between 0 and s there exists a $c_i \geqslant 0$ for which $P_i^{c_i} \subseteq N_i$. Since each of these P_i's contain $gr_I(R)^+ = \bigoplus_{k \geqslant 1} I^k/I^{k+1}$, we know there exists a $c > max\{c_1, ..., c_s\}$ that satisfies:

$$I^c/I^{c+1} \subseteq P_1^{c_1} \cap ... \cap P_s^{c_s} \subseteq N_1 \cap ... \cap N_s.$$

By prime avoidance we can get a homogeneous element $h = x + I^{n+1}$ that is not in $P_{s+1} \cup ... \cup P_r$, and that this n may be taken to be 1 if R/m is infinite (**HunSwa** , Appendix A). We will prove that x is a superficial element of I^n. In order to do so, suppose by contradiction that we can get:

$$y \in (I^q : x) \cap I^c \backslash I^{q-n}. \tag{3.3}$$

Let k be the greatest integer such that $y \in I^k$. We have that:

$$q - n \geqslant k + 1 \iff q \geqslant n + k + 1$$

because $k + 1$ is the first power of I that does not contain y, and by assumption I^{q-n} does not contain y. But then we conclude that:

$$yx \in I^q \subseteq I^{n+k+1} \Rightarrow$$

$$(y + I^{k+1})(x + I^{n+1}) = yx + I^{n+k+1} = 0 \tag{3.4}$$

We have $(y + I^{k+1})(x + I^{n+1}) = 0 \in N_{s+1} \cap ... \cap N_r$ and

$$h = x + I^{n+1} \notin P_{s+1} \cup ... \cup P_r$$

since $N_{s+1}, ..., N_r$ are primary ideals, we must have $y + I^{k+1} \in N_{s+1} \cap ... \cap N_r$.
Now, since $y \in I^c$, we must have $k \geqslant c$. As $I^c/I^{c+1} \subseteq N_1 \cap ... \cap N_s$, we have:

$$y + I^{k+1} \in N_1 \cap ... \cap N_s.$$

But then we conclude that $y + I^{k+1} \in N_1 \cap ... \cap N_r = 0$, which is a contradiction. So, we conclude that for all $q \geqslant c, n$, we have $(I^q : x) \cap I^c = I^{q-n}$. Taking nq instead of q, we get:

$$\forall q \geqslant c : ((I^n))^q : x) \cap I^c = (I^n)^{q-1}$$

As we intended. □

So we now have an answer to the question made in the beginning of the chapter: in order to have an element x in I that satisfies:

$$(I^{n+1} : x) = I^n$$

for n sufficiently large we have to demand that R has to be local, Noetherian, have an infinite residue field and I has to have positive grade.
The fact that the residue field of R is isomorphic to the residue field of $R/(x)$ gives us hope that we can iterate this process. The next two sections will deal with techniques to obtain such a process. A very simple first question related with it is the following: we already know that, if R/m is infinite, every ideal I has a superficial element, but to pursue an iterative process we have to ask ourselves how big is the set of superficial elements in I. Fortunately, the answer to this question is not hard and can be obtained as a very simple corollary from Theorem 3.1.
First, let us formulate it properly: observe that I/mI is a R/m finite dimensional vector space, with dimension, say, t, so we can identify it with $(R/m)^t$. In $(R/m)^t$ we can consider the topological

structure given by the usual Zariski topology, i.e., the topology for which the closed sets are the common zeros of ideals of polynomials in t variables and coefficients in R/m. We know that in the Zarisky topology the open sets are dense, so we can state our question as: can we obtain an open set in I/mI for which every element "in" it is a superficial element?

Corollary 3.6 Let (R, m) be a Noetherian local ring with infinite residue field and I a proper ideal in R. Consider in I/mI the Zarisky topology specified in the previous paragraph and let $P_1, ..., P_r$ be prime ideals that do not contain I. Then, there exists an open subset U of I/mI for which whenever $x \in I$ is such that $x + mI$ is in U, x is a superficial element of I.

Proof. The proof of this result can be made as a set of reformulations of the following observation: the superficial element of the last proof was obtained by picking up an element that was not in any of the prime divisors of 0 that do not contain $gr_I(R)^+$. Fixing $P_1, ..., P_r$ to be such primes, we can reformulate this observation as: if $x + I^2$ is not in $P_i \cap I/I^2$, for all $i = 1, ..., r$, then x is a superficial element for I. We know that, for each i, we can write:

$$P_i \cap I/I^2 = P_i'/I^2$$

For some ideal P_i' strictly contained in I. Then an element that is not in any of this $P_i's$ is necessarily superficial. Notice also that the sets:

$$P_i'' = \frac{P_i' + mI}{mI}$$

Are proper vector subspaces of I/mI, because if this was not the case, then:

$$P_i' + mI = I$$

But then, by Nakayama, we would have $P_i' = I$, which is absurd. Each of the P_i'' is a closed set of I/mI, and hence a closed set for the Zarisky topology (it is given by the common zeros of a set of linear, and hence polynomial, equations), and so it's complement is an open set U'. We know that all the pre-images in I of the elements of U' are superficial, and besides that we know that U' is not empty because the Zarisky topology of a vector space over an infinite field is irreducible, and the union of proper closed sets in an irreducible topological space is also proper.
Now fix $P_1, ..., P_s$ to be primes in R that avoid I. By the same set of reasons as above, $(P_i + mI)/mI$ are proper vector subspaces of I/mI which the union cannot be the entire set, but then, the complement of this union, call it U'', intersected with U' cannot be empty, once again because I/mI with the Zariski topology is irreducible, and this intersection, U, fulfill the requirements of our result. □

We have showed that for every proper ideal I in a Noetherian, local ring (R, m) with infinite residue field has a superficial element, x. If I happens to have positive grade, we can apply proposition 3.3, to get that for n sufficiently large, we have:

$$\frac{I^{n+1} : x}{I^n} = 0. \tag{3.5}$$

Hence, through Singh's equality, we have for large enough n:

$$H_I(n) = H^1_{I/(x)}(n)$$

and so, the Hilbert Polynomial of R related to I and the Samuel polynomial of $R/(x)$ related to $I/(x)$ must coincide.
It would be useful if we can get such nice conditions that assure us the existence of an element that, also through Singh's equality, gives us an equality between the Hilbert and Samuel Functions. Obviously, such an element would have to satisfy the superficial condition for all n. The next result gives us a set of reformulations, in the case $I = m$, of this condition:

Proposition 3.7 Let (R, m) be a Noetherian local ring and $x \in m \setminus m^2$. The following conditions are equivalent:

1. $\forall n, m^{n+1} : x = m^n$
2. $(1 - t)P(R, t) = P(R/(x), t)$
3. $x + m^2$ is regular in $gr(R)$
4. $\forall j, e_j(R) = e_j(R/(x))$

Proof. Recall that, by definition:

$$P(R, t) = \sum_{n\geqslant 0} H_R(n)t^n =$$

By the Singh's equality we get:

$$= \sum_{n\geqslant 0} \left(H^1_{R/(x)}(n) - l\left(\frac{m^{n+1} : x}{m^n}\right)\right) t^n =$$

$$= \sum_{n\geqslant 0} H^1_{R/(x)}(n)t^n - \sum_{n\geqslant 0} l\left(\frac{m^{n+1} : x}{m^n}\right) t^n =$$

$$= \sum_{n\geqslant 0}\sum_{j=0}^{n} H_{R/(x)}(j)t^n - \sum_{n\geqslant 0} l\left(\frac{m^{n+1} : x}{m^n}\right) t^n =$$

$$= \sum_{n\geqslant 0} H_{R/(x)}(n)\sum_{j\geqslant n} t^j - \sum_{n\geqslant 0} l\left(\frac{m^{n+1} : x}{m^n}\right) t^n =$$

$$= \sum_{n\geqslant 0} H_{R/(x)}(n)t^n\sum_{j\geqslant 0} t^j - \sum_{n\geqslant 0} l\left(\frac{m^{n+1} : x}{m^n}\right) t^n =$$

$$= \frac{1}{1-t}\sum_{n\geqslant 0} H_{R/(x)}(n)t^n - \sum_{n\geqslant 0} l\left(\frac{m^{n+1} : x}{m^n}\right) t^n.$$

So we immediately get that (1) is equivalent to (2).
We know that $x + m^2$ is regular in $gr(R)$ if and only if:

$$\forall n \geqslant 0, \forall y \in m^n, (y + m^{n+1})(x + m^2) = 0 \Rightarrow y + m^{n+1} = 0$$

$$\forall n \geqslant 0, \forall y \in m^n, yx + m^{n+2} = 0 \Rightarrow y + m^{n+1} = 0$$

$$\forall n \geqslant 0, \forall y \in m^n, yx \in I^{n+2} \Rightarrow y \in m^{n+1}$$

$$\forall n, m^{n+1} : x = m^n$$

and so (3) is also equivalent to (1).
Finally, if (2) holds, then, by the relation between the Hilbert coefficients and the h-vector given in Proposition 2.5 of the last chapter, R and $R/(x)$ share the same h-vector, and then (4) also holds. Conversely, if (4) holds, then they have the same h-vector, hence the same h-polynomial, say $f(t)$. Recalling Theorem 0.7 and the dimension theorem, we know that:

$$P(R, t) = \frac{f(t)}{(1-t)^{dim(R)}}$$

$$P(R/(x), t) = \frac{f(t)}{(1-t)^{dim(R/(x))}}$$

so we just have to prove that:

$$d = dim(R) = dim(R/(x)) + 1 = d' + 1.$$

As $x \notin m^2$ we know that:

$$H_R(1) = l(m/m^2) > l(m/((x) + m^2)) =$$

$$= l\left(\frac{m/(x)}{((x) + m^2)/(x)}\right) + 1 = H_{R/(x)}(1) + 1$$

But then the Hilbert functions of R and $R/(x)$ cannot coincide, and we conclude that their Hilbert series cannot coincide also. Since they have the same h-polynomial, this proves that $d \neq d'$. Also,

$$H_R(n) = l(m^n/m^{n+1}) = dim_{R/m}(m^n/m^{n+1}) \geqslant dim_{R/m}\left(\frac{m^n + (x)}{m^{n+1} + (x)}\right) =$$

$$= H_{R/(x)}(n)$$

and, therefore, $d \geqslant d'$. But since $d \neq d'$, $d > d'$, i.e., $d \geqslant d' + 1$. The other inequality, $d' + 1 \geqslant d$, is not difficult to see, but since it involves some theory of minimal primes over elements, that is not developed in this thesis, I can refer [7], Theorem 154. The combination of this two gives us that $d = d' + 1$, and so (4) is also equivalent to (2). □

From the equivalence between (1) and (3) we know that our request is satisfied by the existence of a regular element of degree one in $gr(R)$. The next result simplify this demand for the existence of any regular element in $gr(R)$:

Proposition 3.8 Let (R, m) be a Noetherian local ring with infinite residue field such that $depth(gr(R)) \geqslant 1$. Then there is an element x in m such that $x + m^2$ is a regular element.

Proof. Start to fix $P_1, ..., P_s$ to be the associated primes of $gr(R)$. As in the proof of Corollary 3.6, we know that this primes are homogeneous, and so they must be contained in an homogeneous maximal ideal ([4], Theorem 13.4, part 1 and 2). But, it is easily seen that $gr(R)^+$ is the only such ideal in $gr(R)$, so all the minimal prime ideals are contained in it. As $gr(R)^+$ is the only homogeneous maximal ideal, we know there must be a regular element in it. By Prime Avoidance, this allow us to pick up an homogeneous element h that is regular and is not in $P_1 \cup ... \cup P_s$, and because R/m is infinite we can pick this element to be of degree 1 ([6], Appendix A), which concludes the proof.

□

Notice that from proposition 3.7 we know that if we quotient through a regular element in gr_R, we can save all the information regarding Hilbert coefficients.
If, instead, we have just a superficial and regular element of R, the information is not all preserved, but we can still save most part of it:

Proposition 3.9 Let (R, m) be a Noetherian local ring with positive depth and dimension d and $x \in m \setminus m^2$ a superficial element for m. The following conditions hold:

1. $e_j(R/(x)) = e_j(R)$, for all $j = 0, ..., d-1$
2. for n sufficiently large, $(-1)^d e_d(R) = (-1)^d e_d(R/(x)) - \sum_{j=0}^{n} l\left(\frac{m^{j+1}:x}{m^j}\right)$
3. $e_d(R) = e_d(R/(x)) \iff x + m^2$ is a regular element for $gr(R)$.

Proof. From proposition 3.3 we know that x is also a regular element for R. As we have mentioned before, if x is a regular and superficial element for m, we obtain an equality between the Hilbert polynomial of R and the Samuel polynomial of $R/(x)$:

$$h_R(X) = h^1_{R/(x)}(X).$$

Because x is regular we also know that $dimR/(x) = d-1$, but then, using the description of the iterated Hilbert polynomials given in Chapter 2:

$$\sum_{j=0}^{d-1}(-1)^j e_j(R) \begin{pmatrix} X+d-1-j \\ d-1-j \end{pmatrix} =$$

$$= \sum_{j=0}^{d-1}(-1)^j e_j(R/(x)) \begin{pmatrix} X+d-1+1-1-j \\ d-1+1-1-j \end{pmatrix}$$

and so (1) follows.
In order to prove (2), Singh's equality gives us that:

$$H_R^1(n) = \sum_{j=0}^{n} H_R(j) = \sum_{j=0}^{n} \left(H_{R/(x)}^1(j) - l\left(\frac{m^{j+1} : x}{m^j}\right)\right).$$

And this can be simplified to:

$$H_R^1(n) = H_{R/(x)}^2(n) - \sum_{j=0}^{n} l\left(\frac{m^{j+1} : x}{m^j}\right).$$

For n sufficiently large this equality becomes:

$$\sum_{j=0}^{d}(-1)^j e_j(R) \begin{pmatrix} n+d-j \\ d-j \end{pmatrix} = \sum_{j=0}^{d}(-1)^j e_j(R/(x)) \begin{pmatrix} n+d-j \\ d-j \end{pmatrix} -$$

$$-\sum_{j=0}^{n} l\left(\frac{m^{j+1} : x}{m^j}\right)$$

From (1) this equality simplifies to:

$$(-1)^d e_d(R) = (-1)^d e_d(R/(x)) - \sum_{j=0}^{n} l\left(\frac{m^{j+1} : x}{m^j}\right)$$

as we wanted to prove.
The claim made in (3) is just an easy consequence of (2), because from it we can deduce that $e_d(R) = e_d(R/(x))$ is equivalent to:

$$\forall j \geqslant 0, l\left(\frac{(m^{j+1} : x)}{m^j}\right) = 0.$$

And so the equivalence between (1) and (3) of Proposition 3.7 gives us the result. □

3.2 A Miracle and a Machine

In the last section we were able to prove that in any proper ideal I of any Noetherian local ring (R, m) of dimension d with infinite residue field, there exists a superficial element x. We have also proved that if I has positive grade, then this element is also regular, and this two properties combined gives us that for n sufficiently large:

$$l\left(\frac{(I^{n+1} : x)}{I^n}\right) = 0$$

And hence Singh's equality assures us an equality between the Hilbert polynomial of R and the Samuel polynomial of $R/(x)$. In the case $I = m$, we even proved that we have an equality between the first $d-1$ Hilbert coefficients of R and $R/(x)$.
Since $R/(x)$ is also Noetherian, local and its residue field is isomorphic to the one of R, one can only

hope that this process can be iterated. By this, we mean that we can possibly get an element $y+(x)$ that is a superficial element for $I/(x)$ and a regular element of $R/(x)$. This would allow us to get an equality between the Hilbert polynomial of $R/(x)$ and the Samuel polynomial of $R/(x,y)$, but more important than that, in the case $I = m$ an equality between the first $d-2$ Hilbert coefficients of R and of $R/(x,y)$. This motivates the following definition:

Definition 3.10 Let (R, m) be a Noetherian local ring and I a proper ideal of R. The ordered set $\{x_1, ..., x_s\} \subseteq I$ is called a **superficial sequence** for I if for all $1 \leqslant i \leqslant s, x_i + (x_1, ..., x_{i-1})$ is a superficial element for $I/(x_1, ..., x_{i-1})$.

Superficial sequences satisfy a very good property, that is given by the following lemma, which will be very useful in what comes ahead:

Lemma 3.11 Let (R, m) be a Noetherian local ring, I a proper ideal of R and $\{x_1, ..., x_s\}$ a superficial sequence for I, then, for all n sufficiently large, the following equality holds:

$$I^n \cap (x_1, ..., x_s) = (x_1, ..., x_s)I^{n-1}.$$

Proof. We will proceed by induction on s. Let s be equal to 1 and fix c to be the constant of superficiality of x_1. We know, from the Artin-Rees Lemma, that there exists a k for which:

$$\forall n \geqslant l, I^n \cap (x_1) \subseteq x_1 I^{n-k} \tag{3.6}$$

Now, fix $n \geqslant max\{k, c\}$ and let y be an element in $I^n \cap (x_1)$. We can write y as:

$$y = ax_1 = bx_1$$

For b an element in $I^{n-k} \subseteq I^c$ ((2.1) and a in $(I^n : x_1)$). From this we conclude:

$$a - b \in (0 : x_1) \subseteq (I^n : x_1) \Rightarrow$$

$$b \in I^c \cap (I^n : x_1)$$

But then:

$$I^n \cap (x_1) \subseteq x1(I^c \cap (I^n : x_1))$$

And from the fact that x_1 is superficial:

$$I^n \cap (x_1) \subseteq x_1 I^{n-1}$$

Which concludes the case $s = 1$.
Assume now that $s > 1$, we know by the inductive hypothesis that there exists a $c \geqslant 0$ for which:

$$\forall n \geqslant c, I^n \cap (x_1, ..., x_{s-1}) = (x_1, ..., x_{s-1})I^{n-1} \tag{3.7}$$

Besides that, from the case $s = 1$ we know that there exists a $l \geqslant 0$ for which:

$$\forall n \geqslant l, \frac{I^n + (x_1, ..., x_{s-1})}{(x_1, ..., x_{s-1})} \bigcap \frac{(x_1, ..., x_s)}{(x_1, ..., x_{s-1})} = \frac{x_s I^{n-1} + (x_1, ..., x_{s-1})}{(x_1, ..., x_{s-1})} \iff$$

$$(I^n + (x_1, ..., x_{s-1})) \cap (x_1, ..., x_s) = x_s I^{n-1} + (x_1, ..., x_{s-1})$$

So, if we pick up n to be greater than both c and l we would obtain:

$$I^n \cap (x_1, ..., x_s) \subseteq I^n \cap (x_1, ..., x_{s-1}) + x_s I^{n-1}) =$$

From the fact that $x_s I^{n-1} \subseteq I^n$ we can rewrite it as:

$$= I^n \cap (x_1, ..., x_{s-1}) + xsI^{n-1} =$$

By (2.2) we get:

$$= (x_1, ..., x_{s-1})I^{n-1} + x_s I^{n-1} =$$

$$= (x_1, ..., x_s)I^{n-1}$$

And our result is done. □

This result as a remarkable corollary, such that it was called "A miracle", regarding the graded ring of I:

Theorem 3.12 (The Miracle) Let (R, m) be a Noetherian local ring, I a proper ideal of R and x a superficial element of I. Then for n sufficiently large the following equality holds:

$$\forall k \geqslant n, [gr_I(R)/(x + I^2)]_k \cong [gr_{I/(x)}(R/(x))]_k.$$

In particular, if $x + I^2$ is a regular element in $gr_R(I)$, we conclude that $gr_R(I)/(x + I^2)$ and $gr_{R/(x)}(I/(x))$ are isomorphic.

Proof. Let us start to give a description of the grading of both rings:

$$(x + I^2) = \frac{(x) + I^2}{I^2} \oplus \frac{xI^2 + I^3}{I^3} \oplus ... \oplus \frac{xI^{k-1} + I^{k+1}}{I^{k+1}} \oplus ...$$

As $(x + I^2)$ is an homogeneous ideal, we get:

$$gr_I(R)/(x + I^2) = \bigoplus_{k \geqslant 0} \frac{I^k / I^{k+1}}{(xI^{k-1} + I^{k+1})/I^{k+1}} =$$

$$= \bigoplus_{k \geqslant 0} \frac{I^k}{xI^{k-1} + I^{k+1}}$$

By definition, setting $\overline{I} = I/(x)$ and $\overline{R} = R/(x)$:

$$gr_{\overline{I}}(\overline{R}) = \bigoplus_{k \geqslant 0} \frac{\overline{I}^k}{\overline{I}^{k+1}} =$$

$$= \bigoplus_{k \geqslant 0} \frac{(I/(x))^k}{(I/(x))^{k+1}} = \bigoplus_{k \geqslant 0} \frac{(I^k + (x))/(x)}{(I^{k+1} + (x))/(x)} =$$

$$= \bigoplus_{k \geqslant 0} \frac{I^k + (x)}{I^{k+1} + (x)} =$$

As $I^{k+1} \subseteq I^k$:

$$= \bigoplus_{k \geqslant 0} \frac{I^k + (I^{k+1}(x))}{I^{k+1} + (x)} =$$

And using the second isomorphism theorem:

$$\bigoplus_{k \geqslant 0} \frac{I^k}{(I^{k+1} + (x)) \cap I^k} = \bigoplus_{k \geqslant 0} \frac{I^k}{I^{k+1} + ((x) \cap I^k)}$$

So our statement is equivalent to the existence of an n for which:

$$\forall k \geqslant n, \frac{I^k}{xI^{k-1} + I^{k+1}} \cong \frac{I^k}{I^{k+1} + ((x) \cap I^k)}$$

Lemma 3.11 assure us that there exists an n for which:

$$\forall k \geqslant n, I^k \cap (x) = xI^{k-1}$$

And from this we get that $I^{k+1} + ((x) \cap I^k) = I^{k+1} + xI^{k-1}$, and so our result follows.
In order to prove the last statement, fix x to be such $x + I^2$ is a regular element in $gr_I(R)$, we will prove that:

$$\forall k \geqslant 0, I^k \cap (x) = xI^{k-1}$$

From the fact that $x + I^2$ is regular the equivalence (1) $\iff$ (3) gives us:

$$\forall k \geqslant 0, (I^{k+1} : x) = I^k \Rightarrow x(I^{k+1} : x) = xI^k$$

Further, notice that any element in $x(I^{k+1} : x)$ is an element in (x) that is also in I^{k+1}, and so $xI^k = x(I^{k+1} : x) = (x) \cap I^{k+1}$, as we wanted. □

As we have observed in the last section, the residue field of $R/(x)$ is isomorphic to the field of R, so in principle every superficial sequence can be extended to a longer one. Obviously, we have some limitations. For instance, because R is Noetherian, at some point we will reach a superficial sequence that will generate our ideal, and so we will have to stop. Other limit is concerned with the grade of our ideal, even if we have reached a sequence of the length of the grade of I, we know we can get a longer superficial sequence, but we are not interested in it because our elements stop to be regular, and so are not suitable for our purposes.

At least if we are talking about superficial elements of m, other limit is the dimension of R, because every step of the way we are making a quotient through a regular element, and so our dimension decreases in one. From this, it is obvious that Cohen-Macaulay rings will be more suitable for our intentions.
Imagine now that we have a superficial sequence $\{x_1, ..., x_s\} \subseteq I$ that satisfies the following:

$$\exists n \geqslant 0 : \forall m \geqslant n, I^m(x_1, ..., x_s) = I^{m+1}$$

First, notice that in this case I and $(x_1, ..., x_s)$ have the same dimension, because if P was a prime ideal strictly between them, then we would have:

$$(x_1, ..., x_s) \subset P \subseteq I \Rightarrow P \subseteq I^{n+1} = I^n(x_1, ..., x_s) \Rightarrow P \subseteq (x_1, ..., x_s)$$

which is absurd. This makes it a suitable candidate to stop our process, but evenmore, from our property we know that:

$$(I/(x_1, ..., x_s))^{n+1} = \frac{I^{n+1} + (x_1, ..., x_s)}{(x_1, ..., x_s)} = I^n(x_1, ..., x_s)/(x_1, ..., x_s) = 0$$

and so our ideal is nilpotent, and so it has depth 0. This two facts may point out that this kind of superficial sequence are a good point to stop, and this motivates the following definition:

Definition 3.13 Let R be a ring and and $J \subseteq I$ proper ideals of R, then we say that J is a **reduction** of I if there exists an $n \geqslant 0$ such that:

$$JI^n = I^{n+1}.$$

We denote this relation by $J \preceq I$. If J is minimal with this property we call it a **minimal reduction**.

Notice that reductions always exist, because I is trivially a reduction of itself, but unfortunately the existence of minimal reductions is not so trivial. Nonetheless later on we will be able to prove that they exist for all Noetherian local rings. With this intention in mind, the next lemma will prove to be rather useful:

Lemma 3.14 Let R be a Noetherian ring, m its Jacobson radical, I an ideal in R, L any ideal contained in mI and $J, J' \subseteq I$ ideals such that $J + L = J' + L$. Then $J \preceq I$ iff $J' \preceq I$.

Proof. Assume $J \preceq I$ and fix an n such that:

$$I^{n+1} = JI^n.$$

By our hypothesis:

$$JI^n \subseteq (J + L)I^n = (J' + L)I^n \subseteq .$$

As $L \subseteq mI$:

$$\subseteq (J' + mI)I^n = J'I^n + mII^n = J'I^n + mI^{n+1}$$

and by Nakayama:

$$I^{n+1} \subseteq J'I^n$$

as we wanted. □

Remember that we are not interested in general reductions: our aim is reductions that are generated by a superficial sequence. From the previous section, we know that in order for those reductions exist for all ideals, we at least have to require for our ring to be Noetherian, local and with infinite residue field. A good thing is that in the next section we will prove that this conditions are sufficient.
Before pursuing our study of reductions we will finish this chapter with a remarkable result, which will be, together with Theorem 3.5, the most important in this chapter. It will prove that, besides the properties we already mentioned, minimal reductions generated by superficial sequences have an extra good property: they allow us to control not just the depth of R, but also the depth of $gr_I(R)$:

Theorem 3.15 (Sally's Machine) Let (R, m) be a Noetherian local ring and I an ideal of R. If $(x_1, ..., x_n) \subseteq I$ is a minimal reduction of I generated by a superficial sequence of I, then, fixing $r \leqslant n$ and $J = (x_1, ..., x_r)$ we obtain:

$$depth(gr_I(R)) \geqslant r + 1 \iff depth(gr_{I/J}(R/J)) \geqslant 1$$

Proof. See [5] Lemma 2.2.

□

3.3 Reductions and how to avoid infinity

In order for us to use Sally's Machine we have to deal with two problems, the first of them being the already mentioned existence of minimal reductions generated by a superficial sequence. Other problem is concerned with our discussion made before Definition 3.13: if (R, m) is a Noetherian local ring with $depth(R) \geqslant k$ and $\{x_1, ..., x_k\}$ is a superficial sequence for m of length k we obtain, through an iterative use of Proposition 3.5 (1), an equality between the first $d - k$ Hilbert coefficients of R and of $R/(x_1, ..., x_k)$. This points out that it would be good to get some control over the general length of superficial sequences. Both of this problems are dealt with in this section. Finally, we will end this section by giving a simple trick to avoid the infinite condition on the residue field of R. First of all, we will have to star with the following definition:

Definition 3.16 Let (R, m) be a Noetherian local ring and I an ideal of R. We call **Rees Algebra of I** to the subring of $R[t]$ given by:

$$R[It] = \bigoplus_{n \geqslant 0} I^n t^n$$

and **Fiber cone of** I to:

$$F_I = \frac{R[It]}{mR[It]} = \bigoplus_{n \geqslant 0} \frac{I^n}{mI^n}.$$

The dimension of F_I is called the **analytic spread of** I and denote it by $\alpha(I)$.

A first connection between Rees Algebras and reductions is given by the following lemma:

Lemma 3.17 Let R be a Noetherian ring and $J \subseteq I$ ideals of R. Then J is a reduction of I, $J \preceq I$, iff $R[It]$ is module finite over $R[Jt]$. If either of this condition holds, the greatest degree of the elements of a homogeneous minimal generating set of $R[It]$ over $R[Jt]$ is the minimal number n such that:

$$JI^n = I^{n+1}.$$

.

Proof. Start to assume that $J \preceq I$; then there exists an n for which:

$$\forall m \geqslant n, JI^m = I^{m+1} \Rightarrow J^2 I^m = JJI^m = JI^{n+1} \Rightarrow ... \Rightarrow J^k I^m = I^{m+k}.$$

But then:

$$\forall k \geqslant 0, (R[It])_{k+n} = I^n t^n (R[Jt])_k.$$

Let now, for each $0 \leqslant i \leqslant n$, $s_{i_1}, ..., s_{i_l}$ be the generators of I^i, we have that:

$$\forall 0 \leqslant i \leqslant n, R[It]_i = (s_{i_1}R + ... + s_{i_l}R)t^i \subseteq s_{i_1}R[Jt] + ... + s_{i_l}R[Jt]$$

and:

$$(R[It])_{k+n} = I^n t^n (R[Jt])_k = (s_{n_1}R + ... + s_{n_l}R)t^n (R[Jt])_k.$$

So we conclude that $R[It]$ is generated over $R[Jt]$ by the $s_{i_m}t^i$'s, which are finitely many.
Now assume that $R[It]$ is finitely generated over $R[Jt]$ and let $\{a_{0_0}, ..., a_{0_{n_0}}, a_{1_0}t, ..., a_{n_{n_n}}t^n\}$ be a minimal homogeneous generating set, we have that:

$$I^{n+1}t^{n+1} = \sum_{j=0}^{n} \sum_{i=0}^{n_j} a_{j_i} t^j J^{n+1-j} t^{n+1-j} \subseteq$$

$$RJ^{n+1}t^{n+1} + ItJ^nt^n + ... + I^nt^nJt \subseteq JI^nt^{n+1}.$$

And so we conclude that $J \preceq I$. Evenmore, the minimal number m such that:

$$JI^m = I^{m+1}$$

has to be equal to n. This comes from a observation derived from the proof of the first implication, because if this number were to be less than n then we would get a homogeneous generating set for $R[It]$ over $R[Jt]$ for which the element of maximal degree would have degree less than n, thus contradicting the minimality of our basis. □

This result motivates the following definition:

Definition 3.18 To the minimal number such that:

$$JI^n = I^{n+1}$$

we call the **Reduction Number of I with respect to J**.

A connection between reductions and the fiber cone is given by:

Proposition 3.19 Let (R, m) be a Noetherian local ring, n a positive integer, $J \subseteq I^n$ ideals of R and B the subalgebra of $F_{I^n}(R)$ generated over R/m by $(J + mI^n)/mI^n$. Then $J \preceq I^n$ iff $B \subseteq F_I(R)$ is a finitely generated B-module.
If either condition holds, the reduction number of I^n with respect to J is the largest degree of an element in a homogeneous minimal generating set of $F_{I^n}(R)$ over B.

Proof. We will start by assuming that the result holds and prove the last statement. First, notice that:

$$B = \frac{R[Jt] + mR[I^nt]}{mR[I^nt]}$$

and let $A = \{i_0t^{k_0}, ..., i_ht^{k_h}\}$ be a minimal basis of $R[I^nt]$ over $R[Jt]$. Then the reduction number of I^n with respect to J is h and:

$$\sum_{l=0}^{h} i_lt^{k_l}B = \sum_{l=0}^{h} i_lt^{k_l}\frac{R[Jt] + mR[I^nt]}{mR[I^nt]} =$$

$$\frac{R[I^nt]}{mR[I^nt]} = F_{I^n}(R).$$

So A is a basis of F_{I^n} over B. This is minimal, because if $\tilde{A}$ is a subset of A then:

$$\tilde{A}B = F_{I^n} \Rightarrow$$

$$\tilde{A}R[Jt] + mR[I^n t] = R[I^n t]$$

and by Nakayama we conclude:

$$\tilde{A}R[Jt] = R[I^n t]$$

which contradicts the minimality of A. So A is a minimal basis of $F_{I^n}(R)$ over B and the last statement holds.
Assume now that we have proved the result for the first case and that J is a reduction of I^n, then $B \subseteq F_{I^n}$ is module-finite. Fixing, for every $0 \leqslant k \leqslant m-1$, $\{i_{k,l}\}_{l=0}^{n_k}$ to be a basis of I^{k+1} over R we get:

$$F_I = \sum_{k=0}^{n-1} \sum_{l=0}^{n_k} i_{l,k} F_{I^n}.$$

We conclude that F_I is finitely generated over F_{I^n} and so $B \subseteq F_I$ is module-finite.
Conversely, if $B \subseteq F_I$ is module-finite, then $B \subseteq F_{I^n}$ is module-finite and because:

$$B = \frac{R[Jt] + mR[I^m t]}{mR[I^m t]}$$

$$F_{I^m} = \frac{R[I^m t]}{mR[I^m t]}$$

We conclude, by Nakayama, that $R[Jt] \subseteq R[I^m t]$ is module-finite and finally, by the last lemma 3.17, this implies $J \preceq I^n$.
To conclude our result we are only missing the proof of the case $n = 1$:

Assume that J is a reduction of I. From the second isomorphism theorem we get:

$$B = \frac{R[Jt] + mR[It]}{mR[It]} \cong \frac{R[Jt]}{mR[It] \cap R[Jt]} \subseteq F_I(R).$$

But then, as $J \preceq I$ implies that $R[Jt] \subseteq R[It]$ is a module-finite extension, we conclude that $B \subseteq F_I(R)$ is also a module-finite extension.
Assume that $B \subseteq F_I(R)$ is a module-finite extension and that all the generators of $F_I(R)$ over B have degree less or equal than d, then:

$$I^{d+1}/mI^{d+1} = ((J + mI)/mI)(I^d/mI^d) \Rightarrow$$

$$\Rightarrow I^{d+1} = JI^d + mI^{d+1}.$$

By Nakayama, $I^{d+1} = JI^d$, and so J is a reduction of I, as we wanted. □

This result allow us to compare the analytic spread of I with the minimal number of generators of one of its reductions, but in order to fulfill it we will need to make the following easy observation: if $A \subseteq B$ is a module-finite extension of R-modules, then $Ann(B) \subseteq Ann(A)$, but the converse also holds because we have for some $\{a_1, ..., a_n\} \subset B$:

$$B = \sum_{i=1}^{n} a_i A \Rightarrow Ann(A) \subseteq Ann(B).$$

So we conclude that $Ann(A) = Ann(B)$ and consequently $dim(A) = dim(B)$. From this we can get the following corollary:

Corollary 3.20 Let (R, m) be a Noetherian local ring and J and I ideals of R such that $J \preceq I$. Then, if $\mu(J)$ denotes the minimal number of generators of J, $\mu(J) \geqslant \alpha(I)$.

Proof. Let B be as in the last proposition 3.19. As $J \preceq I$ we know that $B \subseteq F_I$ is a module-finite extension, and then:

$$\alpha(I) = dimF_I = dimB.$$

It is a well know consequence of Nakayama that the minimal number of generators of J is equal to the dimension of J/mJ as a vector-space over R/m, i.e., $\mu(J) = dim_{R/m}(J/mJ)$. As in the case of a vector-space its length and dimension over its base field coincide, we get that $dim_{R/m}(J/mJ) = l(J/mJ)$. As $mJ \subseteq (J \cap mI)$, because $J \subseteq I$, we conclude that:

$$\mu(J) = l(J/mJ) \geqslant l\left(\frac{J}{J \cap mI}\right).$$

From the second isomorphism theorem we know that $J/(J \cap mI) \cong (J + mI)/mI$, thus concluding:

$$\mu(J) \geqslant l\left(\frac{J + mI}{mI}\right).$$

Now set a to be equal to $l\left(\frac{J+mI}{mI}\right)$ and $\{v_1, ..., v_a\}$ a basis of this module as a vector-space over R/m, we have that:

$$B = (R/m)[\frac{mI + J}{mI}] = (R/m)[v_1, ..., v_a].$$

But then B is an algebra over a field generated by at minimum a elements, thus having dimension less or equal than a. Uniting those facts we conclude that:

$$\alpha(I) = dim(B) = a \leqslant \mu(J)$$

as we wanted ◻

Now that we have asserted the connections between reductions, Rees Algebras and the Fiber cone we are ready to pursue our theorem concerning the existence of minimal reductions. Let us start with some of its properties:

Proposition 3.21 Let (R, m) be a Noetherian local ring and $J \preceq I$ a minimal reduction. Then the following conditions hold:

1. $J \cap mI = mJ$
2. for any ideal K such that $J \subseteq K \subseteq I$, every minimal generating set of J can be extended to a minimal generating set of K

Proof. **1**:

We know that $J/(J \cap mI)$ is an R/m vector space and as R is Noetherian it has to be of finite dimension, so we can write:

$$J/(J \cap mI) \cong (R/m)^t$$

for some t. Fixing $\{x_1 + (J \cap mI), ..., x_t + (J \cap mI)\}$ to be a basis of $J/(J \cap mI)$, we obtain:

$$J = (x_1, ..., x_t) + (J \cap mI).$$

As $J \cap mI \subseteq J$:

$$J + (J \cap mI) = (x_1, ..., x_t) + (J \cap mI).$$

Then by Lemma 3.14 we get that $(x_1, ..., x_t) \preceq I$, which implies that $(x_1, ..., x_t) = J$, because J is a minimal reduction of I. We conclude that J is minimally generated by t elements, because if J could be generated by $q < t$ then:

$$\frac{J}{J \cap mI} \cong (R/m)^q$$

a contradiction. Then:

$$dim(J/mJ) = t = dim(J/(J \cap mI))$$

And as they are both R/m vector spaces we conclude that:

$$J/mJ \cong J/(J \cap mI). \tag{3.8}$$

Considering the function:

$$\frac{J}{mJ} \longrightarrow \frac{J}{J \cap mI}$$
$$j + mJ \mapsto j + J \cap mI$$

We know that $mJ \subseteq J \cap mI$, so it is well defined, and combining the facts that it is surjective and (3.8) we conclude that it is injective, but then:

$$j + mJ = mJ \iff j + J \cap mI = j \cap mI \iff$$

$$j \in mJ \iff j \in J \cap mI \Rightarrow$$

$$mJ = J \cap mI.$$

And we conclude our result.

2:

Consider I, J and K as in the conditions of the statement and $\{x_1, ..., x_t\}$ as in the proof of **1**. Then:

$$mJ \subseteq J \cap mK \subseteq J \cap mI = mJ \Rightarrow$$

$$mJ = J \cap mK.$$

If we set f to be the canonical surjection from K to $K/(mK)$, then we have:

$$f(J) = \frac{J + mK}{mK} \cong \frac{J}{J \cap mK} \cong$$

$$\cong \frac{J}{mJ} \cong (R/m)^t.$$

Therefore,

$$f(J) = \frac{R}{m}(x_1 + mK) + ... + \frac{R}{m}(x_t + mK).$$

This basis of $f(J)$, $\{x_1 + mK, ..., x_t + mK\}$, over R/m can be extended to a basis of K/mK over R/m, which in turn can be lifted to a minimal generating set of K, thus ending our proof. □

Using this result we are ready to establish the existence of minimal reductions:

Theorem 3.22 Let (R, m) be a Noetherian local ring and I an ideal in R. Then, there exists J that is a minimal reduction of I. More strongly, for every $J \preceq I$, there exists an ideal $K \subseteq J$ that is a minimal reduction of I.

Proof. Let us start to fix J as any reduction of I and $\sum$ the set of reductions of I that are contained in J. As J is in $\sum$ we know $\sum$ is not empty, and so we can get a K in $\sum$ such that:

$$\tilde{K} = \frac{K + mI}{mI}$$

is a minimal vector subspace of I/mI over R/m in $\sum$. Considering $\{k_0 + mI, ..., k_n + mI\}$ to be a basis of $\tilde{K}$ we know that:

$$(k_0, ..., k_n) + mI = K + mI$$

and so by Lemma 3.14 $(k_0, ..., k_n)$ is also a reduction of I. Without loss of generality we will call it K and prove that it is minimal. As in the last proof we have that:

$$\frac{K}{mK} \longrightarrow \frac{K}{K \cap mI}$$

$$k + mK \mapsto k + (K \cap mI)$$

is an isomorphism, and therefore we have that :

$$K \cap mI = mK. \tag{3.9}$$

Then, if $L \subseteq K$ is another reduction of I, then $L \in \sum$ and by the minimal assumption on K we obtain that:

$$L + mI = K + mI.$$

But then,

$$K \subseteq (L + mI) \cap K$$

and as $L \subseteq K$:

$$(L + mI) \cap K = L + (mI \cap K).$$

Therefore, $K \subseteq L + (mI \cap K) = L + mK$, and so, by Nakayama, we conclude that $K = L$, as wanted. □

As we have said in the beginning of this section, the existence of minimal reductions is not sufficient for our purposes, we need to establish the existence of minimal reductions that are generated by superficial sequences. In order to do it, we need a better study a little bit further the properties of minimal reductions. Of particular importance will be Corollary 3.20, where we were able to prove that the analytic spread of an ideal I is always less or equal than the minimal number of generators of any reduction of I. This fact point that reductions generated by $\alpha(I)$ are good candidates to be minimal reductions, and this is in fact true, as we show in the following corollary of the last Theorem:

Corollary 3.23 Let (R, m) be a Noetherian local ring and $J \preceq I$ a reduction such that the number of generators of a minimal generating set of J, $\mu(J)$, is equal to the analytic spread of I, $\alpha(I)$, then:

1. J is a minimal reduction of I
2. F_J is isomorphic to the subalgebra of F_I generated over R/m by $\frac{J+mI}{mI}$, the same subalgebra B of Proposition 3.19, and is also isomorphic to a polynomial ring in $\alpha(I)$ variables over R/m
3. $\forall k > 0, J^k \cap mI^k = mJ^k$.

Proof. Let $\{k_1, ..., k_{\alpha(I)}\}$ be a generating set of J, then:

$$B = R/m[k_1 + mI, ..., k_{\alpha(I)} + mI]$$

As $J \preceq I$, we can apply Proposition 3.19 to get that F_I is finitely generated over B, therefore:

$$\alpha(I) = dim F_I = dim B.$$

By the Noether Normalization Theorem, we know there exists a subalgebra $\tilde{A}$ of B that is a ring of polynomials and over which B is finitely generated. Therefore:

$$\alpha(I) = dim B = dim \tilde{A}.$$

But the dimension of $\tilde{A}$ is also equal to the number of variables that generate $\tilde{A}$, and so we conclude that $\tilde{A} = B$ and B is a polynomial ring in $\alpha(I)$ variables.
As trivially, $J \preceq J$, we can apply Corollary 3.20 to conclude that $\alpha(J) \leqslant \mu(J) = \alpha(I)$. Then the natural surjection $\prod$ from $F_J(R)$ to B is a bijection and $F_J(R)$ is a polynomial ring in $\alpha(I)$ variables. We also get that $\prod |_{J^k/mJ^k}$:

$$\frac{J^k}{mJ^k} \longrightarrow \frac{J^k + mI^k}{mI^k} \cong \frac{J^k}{J^k \cap mI^k}$$

has kernel 0, or equivalently, $mJ^k = J^k \cap mI^k$, which finishes the proof of both **2** and **3**. In order to prove **1**, start to fix $K \subseteq J$ a minimal reduction of I, which exists by last theorem. Then, by Corollary 3.20:

$$\mu(K) \geqslant \alpha(I) = \mu(J).$$

But we also know that any minimal generating set of K can be completed in a minimal generating set of J, by Proposition 3.21 (2). Then $\mu(J) = \mu(K)$ and consequently $J = K$. So J is a minimal reduction of I, as we wanted. □

A simple question would be if the first part of this Corollary as a reverse, that is, if any minimal reduction of I can be generated by $\alpha(I)$ elements. This is not true in general, but at least in the case that our ring has an infinite residue field we can prove it:

Proposition 3.24 Let (R, m) be a Noetherian local ring with infinite residue field and I an ideal of R. Then every minimal reduction of I is minimally generated by $\alpha(I)$ elements.

Proof. Fix J to be any reduction of I. As R/m is infinite, we know there exists $\{\tilde{a_1}, ..., \tilde{a_{\alpha(I)}}\}$ in $(J + mI)/mI$ ([Hun,Swa], Theorem 4.2.3) such that:

$$A = R/m[\tilde{a_1}, ..., \tilde{a_{\alpha(I)}}]$$

is a polynomial ring and B is finitely generated over it. By proposition 3.19, F_I is finitely generated over B, and hence over A.
Let $\{a_1, ..., a_{\alpha(I)}\} \subseteq J$ be such that $a_i + mI = \tilde{a}_i$ and set $K = (a_1, ..., a_{\alpha(I)})$, then:

$$K + mI = J + mI$$

and F_I is finitely generated over $A = (R/m)[\frac{K+mI}{mI}]$. By Proposition 3.19, K is also a reduction of I, and by Corollary 3.20:

$$\mu(k) \geqslant \alpha(I).$$

But K is generated by $a_1, ..., a_{\alpha(I)}$, i.e., by $\alpha(I)$ elements, and therefore $\mu(J) = \alpha(I)$. By Corollary 3.23 part (1) we get that K is a minimal reduction of I.
We have proved that for any reduction of I we can obtain a minimal reduction contained in it that is minimally generated by $\alpha(I)$ elements. Applying this argument to any minimal reduction J, we get a $K \subseteq J$ that is a minimal reduction of I and is minimally generated by $\alpha(I)$ elements. Since J is minimal, we obtain that $J = K$, and therefore J is generated by $\alpha(I)$ elements, as we wanted. □

Now we are almost done. We have proved that minimal reductions exist for any ideal I, and in the case that our ring has infinite residue field, we have even proved that they can be minimally generated by $\alpha(I)$ elements. Obviously, if J is a minimal reduction, itself can be generated by a superficial sequence for J, but we need it to be generated for a superficial sequence for I. What is good is that we can combine the two results obtained above, and obtain a minimal reduction for I that is generated by a superficial sequence:

Theorem 3.25 Let (R, m) be a Noetherian local ring with infinite residue field and I an ideal of R. Then, there exists a superficial sequence $\{x_1, ..., x_{\alpha(I)}\}$ of I which which generates a minimal reduction.

Proof. We will prove this result by induction on $\alpha(I)$. We know that F_I is a finitely generated R/I algebra, because R is Noetherian. As F_I is generated over R/I by the elements of I/I^2, and because R/I is Noetherian, we conclude that F_I is Noetherian. Assuming now that $dim(F_I(R)) = \alpha(I) = 0$ we conclude that $\alpha F_I(R)$ is also Artinian, and so the chain:

$$F_I \supseteq \bigoplus_{n \geqslant 1} \frac{I^n}{mI^n} \supseteq ... \supseteq \bigoplus_{n \geqslant i} \frac{I^n}{mI^n} \supseteq ...$$

must stop. Therefore there exists a k such that $I^k/mI^k = 0$, and by Nakayama we conclude that $I^k = 0$. So, 0 is a minimal reduction of I and our result follows.
Assume now that $\alpha(I) \geqslant 1$. As R/m is infinite, we know that we can get an element $x_1 + mI$ in I/mI such that x_1 is superficial, but even more, because F_I has positive dimension, we know that no minimal prime ideal can contain all of I/mI (otherwise it would contain $gr_I(R)^+$, and so it would be maximal). This allow us to obtain a $x_1 + mI$ that is not in any of those primes, and so it is regular. From this we obtain that:

$$dimF_I/(x_1 + mI) = dimF_I - 1.$$

Consider now $J = I/(x_1)$ and the fiber cone of J. We know that we have a surjection from $F_I/(x_1 + mI)$ to F_J (see the proof of the "Miracle"), and so:

$$\alpha(J) = dimF_J \leqslant dim(F_I/(x_1 + mI)) = \alpha(I) - 1.$$

By applying our inductive hypothesis, we can obtain a superficial sequence $\{x_2 + (x_1), ..., x_{\alpha(J)+1} + (x_1)\}$ that generates a minimal reduction of J. From this we conclude that there exists an n such that:

$$(x_2 + (x_1), ..., x_{\alpha(J)+1} + (x_1))J^n = J^{n+1} \iff$$

$$\frac{(x_2, ..., x_{\alpha(J)+1}) + (x_1)}{(x_1)} \frac{I^n + (x_1)}{(x_1)} = \frac{I^{n+1} + (x_1)}{(x_1)}.$$

Therefore:

$$I^{n+1} \subseteq (x_2, ..., x_{alpha(J)+1})I^n + (x_1) \Rightarrow$$

$$I^{n+1} \subseteq (x_1) \cap I^{n+1} + (x_2, ..., x_{\alpha(J)+1})I^n.$$

But by Lemma 3.11 we know that for n sufficiently large $I^{n+1} \cap (x_1) = x_1 I^n$, and so:

$$I^{n+1} \subseteq (x_1, x_2, ..., x_{l(J)+1}) I^n.$$

Then $(x_1, x_2, ..., x_{l(J)+1})$ is a reduction of I, and because:

$$\alpha(J) \leqslant \alpha(I) - 1 \Rightarrow \alpha(J) + 1 \leqslant \alpha(I)$$

We conclude, by corollary 3.23 part (1), that it is a minimal reduction, as we wanted. □

Apparently we are done: given a proper ideal I in a Noetherian local ring (R, m) with infinite residue field, we have stated the existence of minimal reductions minimally generated by a superficial sequence, and we are even able to control their length, which is equal to the analytic spread of our ideal. The problem is that this control is only apparent, since in principle we have no way of, in general, of knowing the dimension of the fiber cone of I.
For a general ideal I we have no way of dealing with this problem, but in the case $I = m$ we can resolve it. Start to notice that:

$$F_m = \bigoplus_{n \geqslant 0} m^n \ mm^n = gr(R).$$

So in this case $\alpha(R) = dimgr(R)$, and the good thing is that this number is indeed equal to the dimension of R. Despite having all the techniques needed to prove this result, it is only a consequence of the dimension theorem, it would be too long to prove, so we will only state it and give a proper reference to the interested reader:

Theorem 3.26 Let (R, m) be a Noetherian local ring. Then $dim(R) = dim(gr(R))$.

Proof. See [4], Theorem 13.9, page 102. □

In fact the following more general theorem holds: for I a proper ideal in a Noetherian local ring R, $dim(R) = dim(gr_I(R))$, but this is far more complicated to prove, and we only be needing this version.
So we can now recapitulate the technique we have developed in this chapter. Fix (R, m) to be a Noetherian local ring, Theorem 3.26 assures us the existence of a superficial sequence for m of the form $\{x_1, ..., x_{\alpha(m)}$ that generates a minimal reduction. We know that $\alpha(m) = dim(gr(R))$, and from the last theorem this number is indeed equal to the dimension of R. An iterative use of Proposition 3.9 part (1) assures us an equality between the first $d-k$ Hilbert coefficients of R and $R/(x_1, ..., x_k)$, since $grade(m) = depth(R) \geqslant k$. If in addition R turns out to be Cohen-Macaulay, we can express this equality for all k between 0 and d.
Suppose that we have established an equality between the first d Hilbert coefficients of R and $R/(x_1)$. Part (3) of Proposition 3.9 then assures us that $x_1 + m^2$ is indeed a regular element

for $gr(R)$. If we can also establish an equality between the first $d-1$ coefficients of $R/(x_1)$ and $R/(x_1, x_2)$, then $x_2 + (m/(x_1))^2$ is a regular element for $gr(R/(x_1))$, and so its depth is greater or equal than one. Sally's Machine then assures us that the depth of $gr(R)$ is indeed greater or equal than two, and hence we can establish an equality between all the Hilbert coefficients of R and of a ring of dimension $dim(R) - 2$. In the following chapter, the final one, we will give plenty of applications of this technique.

Before ending this chapter, we will deal with one problem we have come across in it: the imposition that R has to have an infinite residue field, but as example 3.2 shows, this is indeed a necessary condition in order to obtain superficial elements in any ideal. Fortunately, we can apply a clever and usual technique to avoid this problem: we construct a ring that has the same Hilbert Function of any ideal of definition of R but has an infinite residue field. Fixing (R, m) to be a Noetherian local ring, I an ideal in R and x a variable over R, we denote by $I[x]$ the following ideal of R:

$$I[x] = \bigoplus_{n \geqslant 0} I x^n$$

Notice that this ideals are always homogeneous, as they are generated by I. In the case of $m[x]$, the following sequence of isomorphisms:

$$R[x]/m[x] \cong \bigoplus_{n \geqslant 0} (Rx^n)/(mx^n) \cong (R/m)[x]$$

Shows that $m[x]$ is a prime ideal, since $(R/m)[x]$ is a domain. This allow us to construct the Noetherian local ideal $R(X) = R[x]_{m[x]}$, which, as we prove on the following result, is the construction promised in the previous paragraph.

Theorem 3.27 Let (R, m) be a Noetherian local ring, then the following properties hold:

1. $R(X)$ is a Noetherian local ring with an infinite residue field
2. I is an ideal of definition of R iff $I[x]_{m[x]}$ is an ideal of definition of $R(X)$
3. if I is an ideal of definition of R, I and $I[x]_{m[x]}$ have the same Hilbert Function
4. $dim(R) = dim(R(X))$
5. if R is Cohen-Macaulay then $R(X)$ is Cohen-Macaulay

Proof. **(1)**
The fact that it is local comes from the fact proved before that $m[x]$ is a prime ideal. It is also prime because, R being prime the Hilbert basis theorem assure us that $R[x]$ is prime, and hence $R(X)$ is also prime. To see that it has an infinite residue field notice that:

$$R[x]_{m[x]}/m[x]_{m[x]} \cong (R[x]/m[x])_{m[x]} \cong (R/m)[x]_0$$

And this is the field of fractions of $(R/m)[x]$, hence an infinite field, and our proof is done.

(2)

Let us first fix an n such that:

$$m^n \subseteq I$$

It is also quite clear that $(m[x])^n = m^n[x]$, and so we get that $(m[x])^n \subseteq I[x]$, and hence:

$$m^n[x]_{m[x]} = (m[x]_{m[x]})^n \subseteq I[x]_{m[x]}$$

As we wanted.

(3) and (4)

Fixing I to be an ideal of definition, our intention is to prove that:

$$l(I^n/I^{n+1}) = l\left(\frac{I^n[x]_{m[x]}}{I^{n+1}[x]_{m[x]}}\right) = l\left(\left(\frac{I^n[x]}{I^{n+1}[x]}\right)_{m[x]}\right)$$

In both cases, this lengths are given by the maximal number of ideals we can put between them, so we will start to built a suitable saturated chain of ideals between I^n and I^{n+1}:
First, if $I^n = mI^n + I^{n+1}$, then we know by Nakayama that $I^n = I^{n+1}$, and consequently:

$$I^n[x] = I^{n+1}[x] \Rightarrow I^n[x]_{m[x]} = I^{n+1}[x]_{m[x]}$$

And our equality is valid. If this is not the case, then we can get an I_1 such that:

$$I^n \supsetneq I_1 \supseteq mI^n + I^{n+1} \supseteq I^{n+1}$$

and for which there are no ideals between it and I^n. If $I^{n+1} = I_1$ we stop here, otherwise we repeat this procedure with I_1. As R is Noetherian, this process has to stop, and when it does we obtain a chain:

$$I_0 = I^n \supsetneq I_1 \supsetneq ... \supsetneq I_l = I^{n+1}$$

for which there are no ideals between two consecutive terms, hence $l = l(I^n/I^{n+1})$, and for every i between 0 and $l-1$:

$$I_i \supsetneq I_{i+1} \supseteq I_i m + I^{n+1} \tag{3.10}$$

Our aim will be to prove that:

$$I_0[x]_{m[x]} \supseteq I_1[x]_{m[x]} \supsetneq ... \supseteq I_l[x]_{m[x]}$$

Is a saturated chain.
Firstly, we will prove the inequalities above are indeed strict: in order to do it fix an i between 0 and $l-1$ and observe that if a is an element in $I_i \setminus I_{i+1}$ then, as there are no ideals between them, we have that $I_i = (a) + I_{i+1}$. Assuming we have an equality between $(I_i[x])_{m[x]}$ and $I_{i+1}[x]_{m[x]}$, we would have:

$$\exists w(x) \notin m[x], \exists p(x) \in I_{i+1}[x] : a/1 = \frac{p(x)}{w(x)} \iff$$

$$\exists w'(x) \notin m[x] : aw(x)w'(x) = p(x)w'(x)$$

And so we would conclude there exists a polynomial $h(x) \notin m[x]$ for wich $ah(x) \in I_{i+1}[x]$. But this would imply that all the coefficients of this polynomial are in I_{i+1}, and as all the terms of h either are in m or invertibles, and we know h itself is not in m, we would conclude that a is in I_{i+1}, wich is absurd, and so the inequalities are indeed strict.

To prove that there are no ideals between two of them, fix i, a as in the previous paragraph and J to be an ideal that strictly contains $I_{i+1}[x]_{m[x]}$ and is contained in $I_i[x]_{m[x]}$. From equality (.), we conclude that if $p(x) \in I_i[x] \setminus I_{i+1}[x]$, then $p(x) = ap'(x) + p''(x)$, where $p''(x) \in I_{i+1}[x]$ and $p'(x) \notin m[x]$. Any element in $J \backslash I_{i+1}[x]_{m[x]}$ can be written as $p(x)/w(x)$ for some $p(x) \in I_i[x] \backslash I_{i+1}[x]$ and $w(x) \notin m[x]$, we can fix p' and p" as before and conclude that:

$$\frac{p(x)}{w(x)} \in J \Rightarrow \frac{ap'(x) + p''(x)}{1} \in J, \Rightarrow$$

$$p'(x)\frac{ap'(x)}{1} \in J \Rightarrow a/1 \in J$$

And so our result follows. The statement (4) is just a consequence of (3) and the Dimension Theorem.

(5)

Under the stronger assumption that R is also Cohen Macaulay, and being $R(X)$ local and Noetherian, we know $depth(R(X) \leqslant dim(R(X)) = dim(R) = depth(R) = d$, so in order to prove that $R(X)$ is also Cohen Macaulay we just have to prove that we can get a regular sequence as large as we can for R. With this in mind, fix $\{x_1, ..., x_d\}$ to be a regular sequence in R, our aim will be to prove that $\{x_1/1, ..., x_d/1\} \subseteq m[x]_{m[x]}$ is a regular sequence in $R(X)$:

$$\frac{x_1}{1}\frac{p(x)}{w(x)} = \frac{0}{1} \iff$$

$$\exists w'(x) \notin m[x] : x_1 p(x)w'(x) = 0$$

But as x_1 is a regular element in R this would imply that $p(x)w(x) = 0$, but this clearly implies that $p(x)/w(x)$ is zero, and hence $x_1/1$ is regular.

Fix now i to be a natural number between 2 and d, we have that:

$$\frac{x_i}{1}\frac{p(x)}{w(x)} + (x_1, ..., x_{i-1})_{m[x]} = (x_1, ..., x_{i-1})_{m[x]} \iff$$

$$\exists p'(x) \in (x_1, ..., x_{i-1}) \exists w'(x) \notin m[x] : \frac{x_i p(x)}{w(x)} = \frac{p'(x)}{w'(x)} \iff$$

$$\exists w''(x) \notin m[x] : x_i p(x)w'(x)w''(x) = p'(x)w(x)w''(x)$$

As x_i is a regular element in $R/(x_1, ..., x_{i-1})$ all the coefficients of $p(x)w'(x)w''(x)$ are in $(x_1, ..., x_{i-1})$, and hence:

$$\frac{p(x)}{w(x)} = \frac{p(x)w'(x)w''(x)}{w(x)w'(x)w''(x)} \in (x_1, ..., x_{i-1})_{m[x]}$$

And our result follows. □

Chapter 4

Applications

We are now ready to fulfill our final results: we will give some applications, and refer some others, of the techniques developed in the last chapter to the calculus of the Hilbert Function. A first point we have to make is a reference to the last result obtained in the previous chapter: it allows us to assume without loss of generality that R has an infinite residue field, and we will assume as such in this entire chapter. The major part of our results will concern the two first Hilbert Coefficients, e_0 and e_1, being able to prove some classical results with little effort. In this work of major importance will be the description we have obtained of the Hilbert Function of a 1-dimensional Cohen-Macaulay local ring given in Lemma 2.7:

$$\begin{cases} H_R(n) = e - \rho_n & \text{for} \quad n < s \\ H_R(n) = e & \text{for} \quad n \geqslant s \end{cases}$$

where $\{\rho_0, ..., \rho_{s-1}\}$ is a set of positive natural numbers. Recall that we gave this description under the assumption of the existence of a superficial and regular element, but as we have proved in the last section such an element always exist this description is always valid.
The importance of this result shouldn't came as a surprise, as Proposition 3.9 showed, quotients through superficia elements allows us to save the information regarding the first $d(R) - 1$ Hilbert Coefficients, so a good description of the 1-dimensional case is almost enough to any result concerning the two first Hilbert coefficients. Much more difficult seems to be any information regarding the Hilbert coefficients of higher order, but we will still be able to prove same results about e_2, and in what came to me as a surprise, our description of the 1-dimensional case plays an important role.
The first result is mainly important by its 1-dimensional case, where the requisition in $gr(R)$ is vacuous:

Proposition 4.1 Let (R, m) be a Cohen-Macaulay Noetherian local ring with dimension $d = d(R) \geqslant 1$, $depth(gr(R)) \geqslant d - 1$, then the following properties hold:

1. $\forall s > k \geqslant 0, e_k > 0$, and they are zero thereafter

2. $\forall s \geqslant k \geqslant 0, H_R(k) \geqslant \left(\begin{array}{c} n+d \\ d \end{array} \right)$

Proof. **(1)**
We will prove the result by induction on $d(R)$. First of all as R is Cohen-Macaulay $depth(R) = d(R) = 1 > 0$, so we can get an x in m that is both regular and superficial. From this, we know that Lemma 2.3 holds, and so by Proposition 2.2 (4) we get:

$$e_k = \sum_{j=k-1}^{s-1} \left(\begin{array}{c} j \\ k-1 \end{array} \right) \rho_j$$

And as $\rho_j > 0$ between 0 and $s-1$, our result follows. Assume now that $d(R) > 1$, because $depth(gr_m(R)) \geqslant d(R) - 1 \geqslant 1$, we know we can get an x in m such that $x + m^2$ is regular in $gr_m(R)$. From this Proposition 3.3 assures us that R and $R/(x)$ have the same Hilbert coefficients. Besides that, by the miracle we get that:

$$gr_{m/(x)}(R/(x)) \cong gr_m(R)/(x+m^2)$$

and by the regularity of $x + I^2$ we obtain:

$$depth(gr_{m/(x)}(R/(x))) = depth(gr_m(R)/(x+m^2)) = depth(gr_m(R)) - 1 \geqslant d - 2$$

And as x is regular $dim(R/(x)) = d - 1$, so our result follows by induction.

(2) Once again we will proceed by induction on the dimension of R. We will prove the case $d = 1$ by induction on k. If $k = 0$, then:

$$H_R(0) = 1 \geqslant \left(\begin{array}{c} 0+1 \\ 1 \end{array} \right).$$

Assume now that $k \leqslant s$ and $k \geqslant 0$. Then the inductive hypothesis and Singh's equality assure us that, for any $x \in m$:

$$H_R(k-1) = H^1_{R/(x)}(k-1) - l\left(\frac{m^k : x}{m^{k-1}}\right) \geqslant k - 1 + 1 = k \iff$$

$$H^1_{R/(x)}(k-1) \geqslant k + l\left(\frac{m^k : x}{m^{k-1}}\right)$$

and so:

$$H_R(k) = H^1_{R/(x)}(k) - l\left(\frac{m^{k+1} : x}{m^k}\right) = H^1_{R/(x)}(k-1) + H_{R/(x)}(k) - l\left(\frac{m^{k+1} : x}{m^k}\right) \geqslant$$

$$\geqslant k + l\left(\frac{m^k : x}{m^{k-1}}\right) - l\left(\frac{m^{k+1} : x}{m^k}\right) = k + l(m^k : x) - l(m^{k-1}) - l(m^{k+1} : x) + l(m^k) =$$

$$k + l\left(\frac{m^k : x}{m^{k+1} : x}\right) + l\left(\frac{m^k}{m^{k-1}}\right) \geqslant$$

$$\geqslant k + H_R(k-1) \geqslant k + k \geqslant k+1$$

as we wanted.
Assume now that $d > 1$. Then as in (1) we can get an element x that is regular and satisfies the superficial condition for all n. For that element we get that $H_R(n) = H^1_{R/(x)}(n)$, and so:

$$\forall k \leqslant s, H_R(k) = H^1_{R/(x)}(k) = \sum_{j=0}^{k} H_{R/(x)}(j).$$

Also as in (1), we can apply the inductive hypothesis to $R/(x)$, and use the formula of Lemma 1.5 to obtain:

$$H_R(k) \geqslant \sum_{j=0}^{k} \begin{pmatrix} j+d-1 \\ d-1 \end{pmatrix} = \begin{pmatrix} k+d \\ d \end{pmatrix}$$

as we wanted to prove. □

As we have said, this result is important mainly because of the 1-dimensional case, where the condition over $gr(R)$ is trivially satisfied. In bigger dimensions this result does not have to hold. In [9], Marley showed that this does not have to be true, giving an example, Example 2.3, where he showed that if K is a field, $K[[x,y,z]]/I$, where $I = (x^3, y^3, z^3, x^2y, xy^2, yz^2, xyz)$, has $e_3 = -1$, with the help of a computer program called "Macaulay".
We will start our study with a classical inequality that relates the multiplicity of R and its embedding co-dimension, which we now define:

Definition 4.2 Let (R, m) be a Noetherian local ring of dimension d. We define its **embedding co-dimension** to be equal to $l(m/m^2) - d = H_R(1) - d$ and denote it by $h(R)$, or h in case no ambiguity arises. If $h = 0$ we say that R is a **Regular Local Ring**.
Notice that, through Lemma 0.9, we obtain that:

$$1 = l(R/m) = H_R(0) = a_0$$

$$H_R(1) = a_0 + da_1 = 1 + da_1$$

So we can rewrite the embedding co-dimension as:

$$h = H_R(1) - d = 1 + da_1 - d$$

which will be useful to our purposes, since it enables us to relate it to our Hilbert coefficients.
Let now x be a regular element in R. As x is regular we know it is an element different from zero of m, therefore the dimension as a vector space over R/m of:

$$\frac{m/m^2}{(x+m^2)}$$

is one less that the dimension of m/m^2. As this dimensions equal their lengths over R/m and the dimension of $R/(x)$ equals the dimension of R minus one, we conclude that their embedding codimensions coincide.

Proposition 4.3 Let (R, m) be a Cohen-Macaulay Noetherian local ring of dimension $d \geqslant 1$ and embedding co-dimension h. Then $e_0 \geqslant h + 1$.

Proof. As usual we will prove the result by induction, starting in the case $d = 1$. By Lemma 2.7 and Proposition 2.8 (1), we know that $0 \leqslant \rho_1 = e_0 - h - 1 \iff h + 1 \leqslant e_0$, as we wanted.
Assume now that $d > 1$; as usual we can get an element that is both a regular element of R and a superficial element for m, then Proposition 3.9 (1) assure us that R and $R/(x)$ share the Hilbert coefficients up to $d - 1$, and in particular they have the same e_0. Now, if R has embedding co-dimension h, then m can be generated by $d + h$ elements, and as these elements came from lifting a basis of m/m^2 over R/m, we conclude that we can have a basis of the form $\{x, x_1, ..., x_{d+h-1}\}$. But then $\frac{(x)+m^2}{m^2}$ is a subspace of m/m^2 of dimension 1. This allows us to conclude that:

$$m/(x)/(m/(x))^2 = \frac{m}{m^2 + (x)}$$

has dimension $d+h-1$, and so $R/(x)$ has embedding co-dimension equal to $d+h-1-dim(R/(x)) = d + h - 1 - (d - 1) = h$, and our result follows. □

The next result, first proved by Nagata, in [12], in a more general case that the one we cover here, shows us that this inequality is as sharp as we can get. It has important geometrical consequences, since this theorem says that a point on an analytically irreducible variety is nonsingular if and only if the multiplicity of its associated local ring is one :

Theorem 4.4 Let (R, m) be a Noetherian, Cohen-Macaulay, local ring of dimension $d \geqslant 1$ and embedding co-dimension h. Then $e_0 = 1$ iff R is regular.

Proof. Suppose that R is regular. Notice that $gr(R) \cong F_m$, so in this case the minimal number of generators of m, $\mu(m)$, equals $dim(R) = dim(gr(R)) = dim(F_m) = \alpha(m)$. So, by corollary 3.23 (2), as $m \preceq m$, we get that $F_m \cong gr(R)$ is a polynomial ring over R/m in $\alpha(m) = d(R)$ variables. From this we conclude that:

$$H_R(n) = \binom{n + d - 1}{d - 1}$$

Therefore,

$$P_R(t) = \sum_{n \geqslant 0} H_R(n) t^n =$$

$$= \sum_{n \geqslant 0} \binom{n + d - 1}{d - 1} t^n$$

which, by Lemma 0.8:

$$P_R(t) = \frac{1}{(1-t)^d}.$$

But then the h-vector of R is just $a_0 = 1$, and hence, by Proposition 2.4 (2), $e_0 = a_0 + ... + a_s = 1$, as wanted.
Assume now that $e_0 = 1$ Proposition 4.3 tells us that $h + 1 \leqslant e_0 = 1$, but then $h = 0$ and R is a regular ring. □

This result is an extreme case of the inequality proved in Proposition 4.3, and is a special case of the following theorem, firstly proved by Sally, in [10]:

Theorem 4.5 Let (R, m) be a Cohen-Macaulay Noetherian local ring of dimension $d \geqslant 1$ and embedding co-dimension h. If $e_0 = h + 1$, then $gr(R)$ is Cohen-Macaulay and:

$$P_R(t) = \frac{1 + (e-1)t}{(1-t)^d}$$

This would be our last result concerning the relations between the multiplicity and embedding co-dimension of R. We will now pursue some results concerning the relation between the multiplicity of R and its first Hilbert Coefficient, e_1, and will be able to deduce the previous theorem as a consequence of this study. We will start to prove one classical inequality that relates e_1 and $e_0 = e$, $e_1 \geqslant e - 1$, that was first proved by Northcott in [3]. Applying these techniques this inequality cames almost for free:

Proposition 4.6 Let (R, m) be a Cohen-Macaulay Noetherian local ring of dimension $d \geqslant 1$; then $e_1 \geqslant e_0 - 1$.

Proof. As usual we will prove this result by induction on d. If $d = 1$, fix $\{\rho_j\}_{j=0}^{s-1}$ as in Lemma 2.7. Then from Proposition 2.8 (1) $e_0 = \rho_0 + 1$, from part (3) we get that $e_1 = \rho_0 + \rho_1 + ... + \rho_{s-1}$, and as $\rho_i > 0$, for all i between 0 and $s - 1$, our result follows.
Assume now that $d > 1$; because R is Cohen-Macaulay we can get a superficial element x that is also a regular element. From the fact that x is regular we know that $dim(R/(x)) = d - 1$ and $R/(x)$ is Cohen-Macaulay, so the inductive hypothesis apply, $e_1(R/(x)) \geqslant e_0(R/(x)) - 1$. Because x is regular and superficial Proposition 3.9 part (1) tells us that $e_0(R) = e_0(R/(x))$ and $e_1(R) = e_1(R/(x))$, so our result follows. □

Assume that (R, m) is a Cohen-Macaulay local ring with dimension equal to $d(R) = 1$. A simple observation from the proof of the case $d = 1$ of Proposition 4.6 tell us that $e_1 = e_0 - 1$ iff

$\rho_j = 0$, for all j different from 0. So, in this case $s(R) = 1$ and the Hilbert Series of R has the form:

$$P_R(t) = \frac{1 + a_1 t}{1 - t}$$

As R has dimension 1, we know $h = da_1 + 1 - d = da_1 = a_1$, hence:

$$P_R(t) = \frac{1 + ht}{1 - t}.$$

Therefore the h-polynomial of R is given by $1+ht$. Proposition 2.4 (1) tell us then that $e_0 = 1+h$, $e_1 = h$ and all the other Hilbert coefficients are nule. Obviously, if we require that $gr(R)$ has depth greater or equal than $d-1$ we could extend, in the same fashion as the proof of Proposition 4.1, this result for rings of bigger dimension. Surprisingly enough, through Sally's Machine we not only don't need this assumption as it is as a consequence of our equality:

Corollary 4.7 Let (R, m) be a Cohen-Macaulay, Noetherian local ring of dimension $d \geqslant 1$ and embedding dimension h. If $e_1 = e_0 - 1$ then the Hilbert Series of R is given by:

$$P_R(t) = \frac{1 + ht}{(1 - t)^d}$$

And even more, we conclude that $gr(R)$ is Cohen-Macaulay.

Proof. We will prove the result by induction, the first part of the case $d = 1$ being covered by the previous paragraph. As R is Cohen-Macaulay, we know it has positive depth and so we can pick up an x that is both a regular element of R and a superficial element for m. Singh's equality tells us that:

$$H_R(n) = H^1_{R/(x)}(n) - l\left(\frac{m^{n+1} : x}{m^n}\right).$$

As we have showed in the proof of Lemma 2.7, $H^1_{R/(x)}(n)$ is a strictly increasing function till it reaches e_0, where it starts to be constant, but then:

$$\forall n \geqslant 1, H^1_{R/(x)}(n) \geqslant H_R(n) = e_0$$

so we conclude that:

$$H^1_{R/(x)}(n) = e_0$$

for $n > 0$. So H_R and $H^1_{R/(x)}$ coincide for all n bigger than 0. As in $n = 0$ they are both equal to 1, we conclude they are the same, and so:

$$\forall n \geqslant 0, l\left(\frac{m^{n+1} : x}{m^n}\right) = 0$$

and we conclude by Proposition 3.7 (3), that $x + m^2$ is a regular element in $gr(R)$. From this we get:

$$dim(R) = dim(gr(R)) = 1 \geqslant depth(gr(R)) \geqslant 1 \Rightarrow dim(gr(R)) = depth(gr(R))$$

and so $gr(R)$ is Cohen-Macaulay.
Assume now that $d > 1$, because $gr(R) \cong F_m$, we conclude that $\alpha(m) = dim(gr(R)) = d$, and so Theorem 3.25 tells us that we can get a superficial sequence $\{x_1, ..., x_d\}$ for m that generates a minimal reduction of it. Setting $J = (x_1, ..., x_{d-1})$, we know R and $R/(x_1)$ share the Hilbert coefficients up to $d - 1$ and have the same embedding co-dimension. By the same reasoning, R and $R/(x_1, x_2)$ share the Hilbert coefficients up to $d - 2$, and continuing in this way we conclude R and R/J share the Hilbert Coefficients up to $d - (d - 1)) = 1$ and have the same embedding co-dimension. So R/J is a Cohen-Macaulay local ring of dimension $d-(d-1) = 1$, and by the case above we conclude that $e_1(R/J) = e_0(R/J) - 1$ and $gr(R/J)$ is Cohen-Macaulay, but by Sally's Machine we get:

$$depth(gr(R/J)) = 1 \Rightarrow depth(gr(R)) = 1 + d - 1 = d$$

and so $gr(R)$ is Cohen-Macaulay and:

$$P_R(t) = \frac{1 + ht}{(1 - t)^d}$$

as we wanted. □

We will now go back to Theorem 4.5. If $d = 1$, then assuming that $e_0 = h + 1$ implies that $\rho_1 = e_0 - h - 1 = 0$, and so we conclude that $\rho_j = 0$ for j bigger than zero, and consequently $e_1 = \rho_0 = e_0 - 1$, so in this case our theorem comes as a consequence of this corollary. The proof for bigger dimensions is done by induction in a similar fashion as we have done in Corollary 4.7. Similarly as in the case of Corollary 4.7, if $e_1 = e_0$ we conclude that $\rho_j = 0$, for all j bigger than 1 and $\rho_1 = 1$. So we conclude that $s(R) = 2$, and:

$$P_R(t) = \frac{1 + a_1 t + a_2 t^2}{1 - t}.$$

As in the previous case we know that $a_1 = h$. Besides this, from Proposition 2.4 (2) we get:

$$e_0 = a_0 + a_1 + a_2 = 1 + a_1 + a_2 = e_1 = a_1 + 2a_2 \Rightarrow a_2 = 1$$

and so:

$$P_R^I(t) = \frac{1 + ht + t^2}{1 - t}.$$

and as before, we can generalize this result to bigger dimensions:

Corollary 4.8 Let (R, m) be a Cohen-Macaulay, Noetherian local ring of dimension $d \geqslant 1$ and embedding co-dimension h. If $e_1 = e_0$ then the Hilbert Series of R is given by:

$$P_R(t) = \frac{1 + ht + t^2}{(1 - t)^d}$$

And even more, we conclude $gr(R)$ is Cohen-Macaulay.

Proof. The proof is almost equal to the one of Corollary 4.7. In the case $d = 1$ we will start to pick up an x that is both a regular element and a superficial element for m. As before, we conclude that H_R and $H^1_{R/(x)}$ coincide for $n \geqslant 2$ and for $n = 0$. We know that $(m^2 : x)$ is a proper ideal of R, x being superficial assures us that x is not in m^2, and so it is contained in m. As trivially m is contained in $(m^2 : x)$, we conclude that $(m^2 : x) = m$. Therefore, through Singh's equality, we know that H^R and $H^1_{R/(x)}$ also coincide in $n = 1$, and therefore are equal. The rest of the proof is equal to Corollary 4.7. □

Both of this results can be found in [1], and similar ones were obtained in the case that $e_1 = e_0 + 1$ in [2].
All of this results were proved, or at least weaker versions of them, before the Sally's Machine was proved. The use we've made of this result up to this point only allowed us to give much simpler proofs of this results. On the contrary, the result we will now pursue, concerning the Hilbert coefficient e_2, was only stated after its proof.
Before pursuing it, we will state a famous inequality given by Elias in [8]:

$$e_1 \leqslant \binom{e_0}{2} - \binom{h}{2}.$$

Elias was also able to prove that this inequality is sharp. In [11], Kirby and Mehran found a similar bound:

$$e_2 \leqslant \binom{e_1}{2}$$

and by its similarity with the previous inequality, we would fill compelled to think that it is as sharp as we can get. This is not the case. Our final result will be to prove that this inequality can be improved, as it was showed in [2]. As mentioned before, this will be much harder than the results proved before, but with an ingenious use of our description of the 1-dimensional Hilbert function we will be able to do it. Start to define:

$$\pi = e_1 - 2e_0 + h + 2$$

in the 1 dimensional case. Following Proposition 2.8 (3), we get:

$$\pi = \sum_{j=0}^{s-1} \rho_j - 2e_0 + h + 2 = \rho_0 + \rho_1 - 2e_0 + h + 2 + \sum_{j=2}^{s-1} \rho_j =$$

By the part (1) of the same proposition this becomes:

$$= e_0 - 1 + e_0 - h - 1 - 2e_0 + h + 2 + \sum_{j=2}^{s-1} \rho_j = \sum_{j=2}^{s-1} \rho_j \tag{4.1}$$

And in particular we obtain that $\pi \geqslant 0 \iff e_1 \geqslant 2e_0 - h - 1$, and in a similar fashion as before we can extend this inequality to rings of a bigger dimension. We now define:

$$t = min\{\pi + 2, e - 1\}$$

And n the only integer such that:

$$\binom{n}{2} + t \leqslant \pi + 2 < \binom{n+1}{2} + t$$

Obviously, such an integer always exist, since if $t = \pi + 2$ we can take it to be 1 (remark the way we extended the notation for combinations in chapter 0). We needed this just to establish the inequality we pretend to prove:

$$e_2 \leqslant e_0 - h - 2 + (t-n-1)(\pi+2) - \binom{t-n}{2} + \binom{n+2}{3}$$

We will start, once again, to deal with the case $d(R) = 1$. In this case Proposition 2.7 tells us that:

$$e_2 = \sum_{j=2}^{s-1} j\rho_j$$

Our aim will be to compare the sequence of the ρ_j's with one we will obtain from the left hand side of this inequality. In order to built it we will have to break it into pieces:

$$e_0 - h - 2 + (t-n-1)(\pi+2) - \binom{t-n}{2} + \binom{n+2}{3} =$$

$$= e_0 - h - 2 + (t-n-1)\left(\pi + 2 - \frac{t-n}{2}\right) + \binom{n+2}{3} =$$

$$= e_0 - h - 2 + (t-n-1)\left(\pi + 3 + \frac{t-n-2}{2} - 2\frac{t-n}{2}\right) + \binom{n+2}{3} =$$

$$= e_0 - h - 2 + (t-n-1)\left(\pi + 3 + \frac{t-n-2}{2} - t + n\right) + \binom{n+2}{3} =$$

as $n = \binom{n+1}{2} - \binom{n}{2}$:

$$= e_0 - h - 2 + (t-n-1)\left[\frac{t-n-2}{2} + \pi + 3 - t + \binom{n+1}{2} - \binom{n}{2}\right] +$$

$$+ \binom{n+2}{3} =$$

$$= e_0 - h - 2 + \binom{t-n-1}{2} + (t-n-1)\left(\pi - \binom{n}{2} - t + 3\right) +$$

$$+(t-n-1)\binom{n+1}{2} + \binom{n+2}{3} =$$

$$= e_0 - h - 2 + \binom{t-n-1}{2} + (t-n-1)\left(\pi - \binom{n}{2} - t + 3\right) +$$

$$+(t-n)\binom{n+1}{2}+\binom{n+1}{3}=$$

$$=e_0-h-1+(2+...+t-n-2)+(t-n-1)\left(\pi-\binom{n}{2}-t+3\right)+$$

$$+(t-n)\binom{n+1}{2}+\binom{n+1}{3}$$

Now let us prove the following lemma:

Lemma 4.9 For all t,n positive natural numbers the following equality holds:

$$(t-n)\binom{n+1}{2}+\binom{n+1}{3}=\sum_{j=t-n}^{t-1} j(t-j)$$

Proof. We will prove this equality by induction in n, but before notice that:

$$\sum_{j=t-n}^{t-1} j(t-j)=(t-n)(t-(t-n))+(t-n+1)(t-(t-n+1))+...+(t-1)(t-(t-1))=$$

$$=(t-n)n+(t-(n-1))(n-1)+...+1(t-1)=\sum_{j=1}^{n} j(t-j)$$

Assume that $n=1$, then:

$$(t-n)\binom{1+1}{2}+\binom{1+1}{3}=$$

$$=(t-1)=\sum_{j=t-1}^{t-1} j(t-j)$$

So our equality holds. Assume now that $n>1$, then:

$$\sum_{j=1}^{n} j(t-j)=\sum_{j=1}^{n-1} j(t-j)+n(t-n)=$$

Applying our inductive hypothesis:

$$=(t-(n-1))\binom{n}{2}+\binom{n}{3}+n(t-n)=$$

$$=(t-n)\left(\binom{n}{2}+n\right)+\binom{n}{2}+\binom{n}{3}=$$

$$=(t-n)\binom{n+1}{2}+\binom{n+1}{3}$$

And so our result follows. □

Using this lemma, our equality becomes:

$$e_0 - h - 2 + (t-n-1)(\pi+2) - \binom{t-n}{2} + \binom{n+2}{3} =$$

$$= e_0 - h - 1 + (2 + ... + t - n - 2) + (t-n-1)\left(\pi - \binom{n}{2} - t + 3\right) + \sum_{j=t-n}^{t-1} j(t-j)$$

And so, if:

$$\sigma_j = \begin{cases} \rho_1 = e_0 - h - 1 & j = 1 \\ 1 & 2 \leqslant j \leqslant t-n-2 \\ \pi - \binom{n}{2} - t + 3 & j = t-n-1 \\ t-j & t-n \leqslant j \leqslant t-1 \\ 0 & j \geqslant t \end{cases}$$

Our result can be restated as:

$$e_2 \leqslant \sum_{j=1}^{t-1} j\sigma_j$$

From Proposition 4.1 (2), we know that $H_R(j) \geqslant j+1$, and so $H_R(e_0-1) \geqslant e_0$, and by our description of the Hilbert function we get that $e_0 - 1 \geqslant s$. Further, we know that:

$$\pi = \sum_{j=2}^{s-1} \rho_j \geqslant \sum_{j=2}^{s-1} 1 = s-2 \iff s \leqslant \pi + 2$$

so we conclude that $t = min\{\pi+2, e_0-1\} \geqslant s$, and as $\rho_j = 0$ for $j > s-1$, we get that our inequality can be translated as:

$$e_2 = \sum_{j=1}^{s-1} j\rho_j = \sum_{j=1}^{t-1} j\rho_j \leqslant \sum_{j=1}^{t-1} j\sigma_j$$

As promised we were able to translate our inequality as a comparison between the sequence of the ρ_j's and a new sequence, the σ_j's. We will start to observe that:

$$\sum_{j=2}^{t-1} \sigma_j = \sum_{j=2}^{t-n-2} 1 + \pi - \binom{n}{2} - t + 3 + \sum_{j=t-n}^{t-1} t - j =$$

$$t - n - 3 + \pi - \binom{n}{2} - t + 3 + t - (t-n) + ... + t - (t-1) =$$

$$= -n + \pi - \binom{n}{2} + n + (n-1) + ... + 1 =$$

$$= \pi - \binom{n+1}{2} + \binom{n+1}{2} = \pi =$$

and by equality (4.1) we get:

$$= \sum_{j=2}^{t-1} \rho_j.$$

We will now define an operation in sequences that, as our two, have the same sum. If $\alpha = (\alpha_2, ..., \alpha_{t-1})$ is a sequence of nonnegative integers with sum equal to π and k is a positive number such that $2 < k \leqslant s-1$, we define a new sequence $\tau_j(\alpha) = (\alpha'_2, ..., \alpha'_{t-1}) := (\alpha_2, ..., \alpha_{k-1}+1, \alpha_k - 1, ..., \alpha_{t-1})$. Obviously the sum of the enters of α and $\tau_k(\alpha)$ is equal, but it is also true that:

$$\sum_{j=2}^{t-1} j\alpha_j = \sum_{j=2}^{t-1} j\alpha'_j - (j-1) + j = \sum_{j=2}^{t-1} j\alpha'_j + 1$$

Our aim will be to prove that our sequence of $\rho = (\rho_0, ..., \rho_{t-1})$ can be obtained from σ by this kind of operations, and so our result will follow. In order to do it, let us prove another lemma:

Lemma 4.10 Let $\alpha = (\alpha_2, ..., \alpha_{t-1})$ and $\zeta = (\zeta_2, ..., \zeta_{t-1})$ be two sequences of nonnegative integers with equal sum and k a number between 2 and $t-2$ for which:

$$\forall j \leqslant k, \alpha_j \leqslant \zeta_j$$

$$\forall j \geqslant k+1, \alpha_j \geqslant \zeta_j$$

Then:

$$\sum_{j=2}^{t-1} j\alpha_j \geqslant \sum_{j=2}^{t-1} j\zeta_j$$

And equality holds if and only if they are equal.

Proof. First let us introduce a notation to simplify our proof: if $\gamma = (\gamma_2, ..., \gamma_{t-1})$ is a sequence as above, we denote by $f(\gamma)$ the following integer:

$$f(\gamma) = \sum_{j=2}^{t-1} j\gamma_j$$

To prove this result, we will prove it is possible to transform α into ζ by transforms of the form τ_k. Start to apply τ_{t-1}, $\alpha_{t-1} - \zeta_{t-1}$ times, to α, we obtain the sequence:

$$\alpha^1 = (\alpha_2, ..., \alpha_{t-2} + \alpha_{t-1} - \zeta_{t-1}, \zeta_{t-1})$$

And notice that $f(\alpha) = f(\alpha^1)$ if and only if $\alpha_{t-1} = \zeta_{t-1}$, otherwise $f(\alpha) > f(\alpha^1)$. We will now apply τ_{t-2}, $\alpha_{t-1} - \zeta_{t-1} + \alpha_{t-2} - \zeta_{t-2}$ times, to α^2, we obtain:

$$\alpha^2 = (\alpha_2, ..., \alpha_{t-3} + \alpha_{t-1} - \zeta_{t-1} + \alpha_{t-2} - \zeta_{t-2}, \zeta_{t-2}, \zeta_{t-1})$$

And $f(\alpha^1) = f(\alpha^2)$ if and only if $\alpha_{t-2} = \zeta_{t-2}$. Iterating this procedure, at some point we will obtain:

$$\alpha' = (\alpha_2, ..., \alpha_{k-1}, \alpha_k + \sum_{j=k+1}^{t-1} (\alpha_j - \zeta_j), \zeta_{k+1}, ..., \zeta_{t-1})$$

And $f(\alpha) = f(\alpha')$ if and only if $\alpha_j = \zeta_j$, for all j between $k+1$ and $t-1$. We will now apply τ_3, $\zeta_2 - \alpha_2$ times, to α', we obtain:

$$\alpha'^1 = (\zeta_2, \alpha_3 - (\zeta_2 - \alpha_2), ..., \alpha_k + \sum_{j=k+1}^{t-1} (\alpha_j - \zeta_j), \zeta_{k+1}, ..., \zeta_{t-1})$$

And, once again, $f(\alpha') = f(\alpha'^1)$ if and only if $\zeta_2 = \alpha_2$. Iterating this procedure we will obtain:

$$\alpha'' = (\zeta_2, ..., zeta_{k-1}, \alpha_k - \sum_{l=2}^{k-1}(\zeta_l - \alpha_l) + \sum_{j=k+1}^{t-1} (\alpha_j - \zeta_j), \zeta_{k+1}, ..., \zeta_{t-1})$$

And $f(\alpha'') = f(\alpha)$ if and only if $\alpha_j = \zeta_j$ for all j different from k. Because all this sequences have the same sum of their entries we conclude that $\alpha'' = \zeta$, and consequently $f(\alpha) \geqslant f(\zeta)$ and this inequality is an equality if and only if they are equal, as we intended. □

To finish the proof of our inequality we will apply this lemma to ρ and σ. In this case, k will be equal to $t-n-1$, and it is obvious that:

$$\forall j \leqslant t-n-1-1 = t-n-2, 1 = \sigma_j \leqslant \rho_j$$

Since the ρ_j's are positive integers. To prove the second condition, start to assume that $t = min\{\pi+2, e_0-1\} = e_0 - 1$. Then, as we have proved in Proposition 4.1 (2):

$$\forall j < s, H_R(j) = e_0 - \rho_j \geqslant j+1 \iff \rho_j \leqslant e_0 - j - 1 \Rightarrow$$

$$\rho_j \leqslant e_0 - 1 - j = t - j = \sigma_j$$

And we are done. If $t = \pi + 2$, then, recalling the definition of n, we obtain that $n = 1$, so the only j that is bigger than $k = t-n-1 = t-2$ is $t-1$. Besides that:

$$t = \pi + 2 = \sum_{j=2}^{t-1} \rho_j + 2 = \sum_{j=2}^{\pi+1} \rho_j + 2$$

Now, if $t-1 > s-1$, then $\rho_{t-1} = 0 \leqslant 1 = t-(t-1) = \sigma_{t-1}$, if $t-1 = s-1$, then by the equality above we obtain that all the ρ_j's have to be 1, and so we also conclude that $\rho_{t-1} \leqslant \sigma_{t-1}$. Then our Lemma applies, and we conclude that:

$$e_2 = \sum_{j=1}^{t-1} j\rho_j = \rho_1 + \sum_{j=2}^{t-1} j\rho_j =$$

$$= \sigma_1 + \sum_{j=2}^{t-1} j\rho_j \leqslant \sigma_1 + \sum_{j=2}^{t-1} j\sigma_j$$

From this it is easy to generalize this result for bigger dimensions:

Theorem 4.11 Let (R, m) be a Cohen-Macaulay Noetherian local ring of dimension $d \geqslant 1$ and embedding co-dimension h, then the following inequality holds:

$$e_2 \leqslant e_0 - h - 2 + (t - n - 1)(\pi + 2) - \binom{t-n}{2} + \binom{n+2}{3}$$

Proof. We will prove this result by induction, the case $d = 1$ being covered by the argument sketched above. If $d > 1$ then we can get an x that is a regular element for R and a superficial element for m. As usual, we know that R and $R/(x)$ have the same embedding co-dimension, $e_0 = e$ and e_1. From this we see that $\pi = e_1 - 2e_0 + h + 2$ is also equal for the both of them, and consequently t and n are also equal, and so by induction we get that:

$$e_2(R/(x)) \leqslant e_0 - h - 2 + (t - n - 1)(\pi + 2) - \binom{t-n}{2} + \binom{n+2}{3}$$

As Proposition 3.9 (2) assures us that $e_2(R) \leqslant e_2(R/(x)$, our result follows. □

As before, we will now study the behavior of the Hilbert Function of R in the extreme case of this inequality:

Theorem 4.12 Let (R, m) be a Cohen-Macaulay Noetherian local ring of dimension $d \geqslant 1$ and embedding co-dimension h, and assume that:

$$e_2 = e_0 - h - 2 + (t - n - 1)(\pi + 2) - \binom{t-n}{2} + \binom{n+2}{3}$$

Then:

$$P_R(z)(1 - z)^d = 1 + hz + (e_0 - h - 2)z^2 + \left[\binom{n}{2} + t - 2 - \pi\right] z^{t-n-1}$$
$$+ \left[\pi - \binom{n+1}{2} - t + 3\right] z^{t-n} + \sum_{j=t-n+1}^{t} z^j$$

and $gr(R) \geqslant d - 1$

Proof. We will prove this result by induction on d. If $d = 1$, the part concerning $gr(R)$ is trivially true. To prove the first part, fix $\rho = (\rho_1, ..., \rho_{t-1})$ and $\sigma = (\sigma_1, ..., \sigma_{t-1})$ as in the proof of Theorem

4.11, and recall that Lemma 4.2 tells us that equality is valid if and only if $\rho = \sigma$, and in particular $s = t$. From Proposition 2.8 (2) we know that:

$$\forall 1 \leqslant n \leqslant s, H_R(n) = e_0 - \rho_n = e_0 - \sigma_n = \sum_{j=0}^{n} a_j$$

But then:

$$a_0 = 1$$

$$a_1 = e_0 - \sigma_1 - a_0 = e_0 - 1 - \sigma_1 = \rho_0 - \sigma_1$$

$$a_2 = e_0 - \sigma_2 - a_1 - a_0 = e - 1 - \sigma_2 - (\rho_0 - \sigma_1) = \sigma_1 - \sigma_2$$

$$\ldots$$

$$a_s = \sigma_s - \sigma_{s-1}$$

So, we conclude that, for $j \geqslant 1$:

$$a_j = \sigma_j - \sigma_{j-1} = \begin{cases} h & j = 1 \\ e_0 - h - 2 & j = 2 \\ 0 & 3 \leqslant j \leqslant t - n - 2 \\ \binom{n}{2} + t - 2 - \pi & j = t - n - 1 \\ \pi - \binom{n+1}{2} - t + 3 & j = t - n \\ 1 & t - n + 1 \leqslant j \leqslant t \\ 0 & j \geqslant t + 1 \end{cases}$$

And our result follows.
Assume now that $d = 2$, as usual we pick up an x that is superficial for m and a regular element of R. As in Theorem 4.11, we know that $R/(x)$ and R share the same e_0, e_1, h, n and t, and evenmore:

$$e_2(R) = e_0 - h - 2 + (t - n - 1)(\pi + 2) - \binom{t-n}{2} + \binom{n+2}{3} \leqslant e_2(R/(x) \leqslant$$

By Theorem 4.11:

$$\leqslant e_0 - h - 2 + (t - n - 1)(\pi + 2) - \binom{t-n}{2} + \binom{n+2}{3}$$

So equality holds throughout, and $e_2(R) = e_2(R/(x))$. But by Proposition 3.9 (3), this implies that $x + m^2$ is a regular element for $gr(r)$, and in particular R and $R/(x)$ have the same h-polynomial, and so our result follows by the previous case.
Finally, assume that $d > 2$ and, as in the proof of Corollary 4.7, pick up $(x_1, ..., x_d)$ to be a minimal reduction of m generated by a superficial sequence. In a similar fashion as the proof of that corollary, we will fix $J = (x_1, ..., x_{d-2})$. We know that R and R/J share the same $e_0, e_1, e_{d-(d-2)} = e_2, h, t$ and n, but then, by our inductive hypothesis, we obtain that the h-polynomial of R/J is as we

pretend and $depth(gr(R/J)) \geqslant d(R/J) - 1 = d - (d-2) - 1 = 1$. Applying Sally's Machine we get that $depth(gr(R)) \geqslant 1 + d - 2 = d - 1$, and the second part of our statement follows.
Because $d-1 \geqslant 1$, we know that we can get an x such that $x+m^2$ is a regular element of $gr(R)$, but then R and $R/(x)$, which has dimension $d-1$, have the same Hilbert Coefficients and h-polynomial, and so the first part of our statement also follows by induction. □

Before ending our thesis, we will deal with one last promise: we will prove the inequality find in Theorem 4.11 is indeed an improvement of the one founded by Kirby and Mehran. Recall that this inequality is given by:

$$e_2 \leqslant \binom{e_1}{2}$$

Proposition 4.13 Let (R, m) be a Cohen-Macaulay Noetherian local ring of dimension d and embedding co-dimension h. Fixing n, π and t as in the proof of Theorem 4.3, the following inequality holds:

$$\binom{e_1}{2} \leqslant e_0 - h - 2 + (t-n-1)(\pi+2) - \binom{t-n}{2} + \binom{n+2}{3}$$

Proof. Let us remind that t was defined to be the minimum between $\pi + 2$ and $e_0 - 1$, so we know that $t - 1 \leqslant \pi + 1$. Let us now define the sequence $\alpha = (\alpha_2, ..., \alpha_{\rho+1}) = (1, ..., 1)$. We claim that:

$$\sum_{j=2}^{\pi+1} j\alpha_j \geqslant \sum_{j=2}^{\pi+1} j\sigma_j$$

Now, if $\pi + 1 = t - 1$, then we conclude, as in the proof of Theorem 4.11, that $\sigma_j \leqslant 1$, and so the inequality above holds. On the other hand, if $\pi+1 > t-1$ we know that $\sigma_j \geqslant \alpha_j$, for all j between 2 and $t-1$, and because $\sigma_j = 0$ for j bigger than $t-1$, we also know that in this case $\sigma_j \leqslant \alpha_j$. Observing that:

$$\sum_{j=2}^{\pi+1} \sigma_j = \sum_{j=2}^{t-1} \sigma_j = \pi = \sum_{j=2}^{\pi+1} 1 = \sum_{j=2}^{\pi+1} \alpha_j$$

we know that Lemma 4.10 applies, and so our claim follows.
From this we get:

$$e_0 - h - 2 + (t-n-1)(\pi+2) - \binom{t-n}{2} + \binom{n+2}{3} =$$

As was observed in the proof of Theorem 4.11:

$$= e_0 - h - 1 + \sum_{j=2}^{t-1} j\sigma_j \leqslant e_0 - h - 1 + \sum_{j=2}^{\pi+1} j\alpha_j =$$

$$= e_0 - h - 1 + \binom{\pi+2}{2} - 1 = e_0 - h - 2 + \binom{\pi+2}{2}.$$

Now we claim that:

$$e_0 - h - 2 + \binom{\pi+2}{2} \leqslant \binom{e_1}{2} - \binom{h}{2}$$

In order to do it, recall that $\pi = e_1 - 2e_0 + h + 2$, and so we want to prove that:

$$2\left[e_0 - h - 2 + \binom{\pi+2}{2}\right] = 2e_0 - 2h - 4+$$

$$+(e_1 - 2e_0 + h + 3)(e_1 - 2e_0 + h + 4) \leqslant$$

$$\leqslant e_1(e_1 - 1) - h(h-1) \iff$$

$$2e_0 - 2h - 4 + e_1^2 + e_1(-2e_0 + h + 4) - 2e_0e_1 + 4e_0^2 - 2e_0h - 8e_0 + he_1 - 2e_0h + h^2 + 4h+$$

$$+3e_1 - 6e_0 + 3h + 12 \leqslant$$

$$\leqslant e_1^2 - e_1 - h^2 + h \iff$$

Putting all the terms with e_1 on the right hand side:

$$4e_0^2 - 12e_0 - 4e_0h + 2h^2 + 4h + 8 \leqslant -e_1(e_1 - 2e_0 + h + 4 - 2e_0 + h + 3 - e_1 + 1) \iff$$

$$4e_0^2 - 12e_0 - 4e_0h + 2h^2 + 4h + 8 \leqslant e_1(4e_0 - 2h - 8) \iff$$

$$2e_0^2 - 6e - 2e_0h + h^2 + 2h + 4 \leqslant e_1(2e_0 - h - 4)$$

Since from Proposition 4.3 we know that $e_1 \geqslant e_0 - 1$, it suffices to prove:

$$2e_0^2 - 6e_0 - 2e_0h + h^2 + 2h + 4 \leqslant (e_0 - 1)(2e_0 - h - 4) \iff$$

$$2e_0^2 - 6e_0 - 2e_0h + h^2 + 2h + 4 \leqslant 2e_0^2 - e_0h - 4e_0 - 2e_0 + h + 4 \iff$$

$$-e_0h + h^2 + h \leqslant 0 \iff h(e_0 - (h+1)) \geqslant 0$$

As we have proved in Proposition 4.3 that $e_0 \geqslant h + 1$ we know this inequality is true, and so our result follows. □

Notice that in particular we have proved:

$$e_2 \leqslant \binom{e_1}{2} - \binom{h}{2}$$

In a very similar result to the one obtained for e_1. Nonetheless, as we have mentioned in the case of e_1 the inequality was sharp, while in here we proved, if you pay attention to our calculations, that our bound only equals this one in the case R regular or in the case $e = h + 1$. Both cases are extreme ones, where our Hilbert Function is perfectly determined by this equalities, as we have showed in Theorem 4.4 and 4.5.

All this results point out how rigid the Hilbert Functions can be, intuiting out that no much freedom is allowed in their behave. Notice that every time we gave an inequality concerned with Hilbert coefficients, we were able to specify its Hilbert Function in the extreme case, thus confirming the general philosophy that near to the border there is not much choice to the Hilbert function.

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