On the Hilbert coefficients of a Cohen-Macaulay local ring

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The theory of Hilbert Functions started, hence its name, with David Hilbert. In his remarkable paper "Uber die Theorie der algebraischen Formen" (where, among others, he proved the Hilbert Basis Theorem), Hilbert proved that if I is an homogeneous ideal in $\mathbb{C}[x_1, ..., x_n]$, then the length of its homogeneous components follows a polynomial for n sufficiently large.
Afterwards, Samuel was able to generalize this result, proving that it still holds if we substitute $\mathbb{C}$ for an Artinian ring. Moreover, he proved that in the case that (R, m) is a local ring the function $H_I^1(n) = l_R(R/I^n)$ also becomes a polynomial for n sufficiently large. Due to this, this function was named after him. Samuel used this to study the intersection multiplicities of algebraic varieties, specially through the connections between the Hilbert coefficients of the graded ring of R related to I, $gr_I(R)$, and the multiplicity of the ring. He was not alone in this study, as this direction was also pursued by Serre, Nagata or Northcott. Of remarkable importance in the study of the Hilbert Function were the introduction of superficial elements and sequences. Through them, there was obtained a technique to substitute the calculation of the Hilbert coefficients of R to one of lower dimension, at least in the case that $depthgr_I(R) \geq d-1$. Later on, in the mid-90s, Huckaba and Marley proved a result that gave a new life to this aproach, Sally's Machine. Being able to relate superficial sequences and the depth of the graded ring of R, it was a powerful technique widely used to obtain new results, simplify and improve old ones. Such work was carried out mainly by the well known Genova school, namely by Sally, Elias, Rossi and others. This would be the aproach followed in this work, having the intention of both serving as an introduction to the study of Hilbert Functions, but also to detail, and give several applications, to this technique.

In the Introduction, we introduce the main definitions, following Matsumura in his book Commutative ring theory. We start from the beginning, with the definition of graded ring, $\mathbf{R} = \bigoplus_{n \geq 0} R_n$, and module, $\mathbf{M} = \bigoplus_{n \geq 0} M_n$. Under the assumption that $\mathbf{R}$ is Noetherian, with R_0 Artinian, and $\mathbf{M}$ is a finitely generated $\mathbf{R}$-module we were able to define the natural valued function

$H_M(n) := l(M_n)$, and:

$$P(M,t) = \sum_{n\geq 0} H_M(n)t^n.$$

If we make the extra condition that R is finitely generated by elements of degree one over R_0, then we obtain that:

$$P(M,t) = \frac{a_0 + ... + a_s t^s}{(1-t)^d}$$

where $a_0 + ... + a_s t^s \in \mathbb{Z}[t]$. Through this equality we were able to prove that:

$$H_M(n) = a_s \left(\begin{array}{c} d+n-s-1 \\ d-1 \end{array} \right) + ... + a_1 \left(\begin{array}{c} d+n-2 \\ d-1 \end{array} \right) + a_0 \left(\begin{array}{c} d+n-1 \\ d-1 \end{array} \right) \tag{1}$$

thus establishing an equality between the Hilbert Function and a polynomial function, for large enough n. We have called to this polynomial Hilbert polynomial.

In the second chapter, we introduced the main focus of our study: the Hilbert Function and Polynomial of the graded ring of an ideal of definition I:

$$gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

We denote its Hilbert Series by $P_R(t)$ and Hilbert Function by $H_R(n)$. When the graded ring was first introduced, it was done so by Krull, with the intention of relating it to the study of theory of ideals in a polynomial ring over a field. We end this chapter with one such result, being able to prove that if:

$$P_R = \frac{a_0 + ... + a_s t^s}{(1-t)^d}$$

where $1 - t \nmid a_0 + ... + a_s t^s$, then d equals the dimension of R. This result is known as Dimension Theorem, and is widely used throughout the rest of this work, mainly because it gives us a better insight in the behaviour of the Hilbert Function (recall (0.1)). In its proof we have come across other important function: the already mentioned Samuel Function. As we have said this equals:

$$H_R^1(n) = l(R/I^{n+1})$$

but in matter of fact it can be equivalently defined by:

$$H_R^1(n) = \sum_{j=0}^{n} H_R(n)$$

thus being also known as the first iterated Hilbert Function.

In the second chapter, in a similar fashion to way we defined the Samuel Function, we defined the higher iterated Hilbert Functions:

$$H_R^{i+1}(n) = \sum_{j=0}^{n} H_R^i(j)$$

where $H_R^0(n)$ is the Hilbert Function of the graded ring of R. As we have done with the Hilbert Function, we were able to obtain that:

$$H_R^i(n) = a_s \left(\begin{array}{c} d+i+n-s-1 \\ d+i-1 \end{array} \right) + ... + a_1 \left(\begin{array}{c} d+n+i-2 \\ d+i-1 \end{array} \right) + a_0 \left(\begin{array}{c} d+i+n-1 \\ d-1 \end{array} \right).$$

So we can establish the existence of polynomials that coincide with the iterated Hilbert Functions from a certain point on. Rewriting these polynomials in a different basis of $\mathbb{Q}[t]$, we were able to define the Hilbert coefficients:

$$h_I^i(X) = \sum_{j=0}^{d+i-1} (-1)^j e_j \left(\begin{array}{c} X+d+i-j-1 \\ d+i-j-1 \end{array} \right)$$

From this coefficients, one can argue that e_0 is the most important one. This is know as the multiplicity of R, and as we have said before it played an important role in Algebraic Geometry through Samuel's research. Until the 90s, since the first $d-1$ coefficients are in one-to-one correspondence with the Hilbert Function, only those were studied. What came as a surprise was that in fact there was much more that could be said about the Hilbert Function if we consider the entire set of Hilbert coefficients. In the rest of the chapter we were able to relate them with the h-vector:

$$e_k = \sum_{j=k}^{s} \left(\begin{array}{c} j \\ k \end{array} \right) a_j$$

hence proving they are integers. We ended this chapter with a very important result, Singh's equality, that establishes that for any $x \in I$ and for any n:

$$H_I(n) = H^1_{I/(x)}(n) - l\left(\frac{I^{n+1} : x}{I^n}\right).$$

In chapter 3, motivated by the Singh's equality, we introduced the notion of superficial elements for an ideal I. Those are elements $x \in I \setminus I^2$ that for a fixed $c \geq 0$ and $n \geq c$ satisfy:

$$(I^{n+1} : x) \cap I^c = I^n.$$

We were able to prove that under the stronger hypothesis that I has positive depth this can be translated as:

$$(I^{n+1} : x) = I^n \tag{2}$$

for large enough n.
In the first section we proved they existed for any ideal I in a Noetherian local ring (R, m) with infinite residue field, R/m. We were also able to establish that if $I = m$, R has positive grade and x is a superficial element for R, then the first $d-1$ Hilbert coefficients of R and $R/(x)$ coincide.

In the second section, we tried to study how we could iterate the process described in the last section, and what problems could arise in it. We ended up introducing the notion of superficial sequences, that are in nature similar to regular sequences. If R is also Cohen-Macaulay such a sequence $\{x_1, ..., x_k\}$ establishes an equality between the first $d-k$ coefficients of R and of $R/(x_1, ..., x_k)$. We ended up stating Sally's machine, for which we needed to establish the notion of reductions. If $J \subseteq I$ are ideals in R, then we say J is a reduction of I if there is an n for which:

$$JI^n = I^{n+1}.$$

In the third section we proved that minimal reductions, i.e., reductions which are minimal among all reductions, exist. Even more, we needed to establish the existence of minimal reductions generated by a superficial sequence. We were able to do so, and also proved that this sequences always have minimal length equal the analytic spread of I, i.e., the dimension of:

$$F_I(R) = \bigoplus_{n \geq 0} I^n/mI^n.$$

This result has major consequences when $I = m$, since in that case $F_m \cong gr_m(R) = gr(R)$, which dimension is equal to the dimension of R itself.
We ended by giving a simple trick to avoid the infinite assumption on R/m, being it fully detailed in the last result.

Finally, in the last chapter we gave several applications of this techniques. We started by giving several improvements and simplified proofs of classical results in the theory of Hilbert coefficients. Among those, one of the more famous one is the equivalence between R being regular and its multiplicity, e_0, being equal to 1. We ended up the thesis by giving a sharp upper bound for e_2, an improvement of a previous result only obtained through the use of Sally's Machine.