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Probabilistic Distributed Temporal Logic

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Abstract

We present a logic for reasoning about temporal properties of distributed systems and finitely additive probability. The language for this logic allows us to make statements such as "The probability of 'sometime in the future φ' is at least s".

We start by presenting a linear temporal logic LTL with a sound and complete axiomatic system. We next present a distributed temporal logic DTL and show that this logic is reducible to LTL.

To introduce probabilities in the linear temporal logic we define pLTL. We then make a first attempt for probabilities in a distributed temporal logic with $\mathsf{PDTL}_{\mathsf{L}}$ by having probabilities in the temporal states. Finally we adopt for an exogenous approach, $\mathsf{PDTL}_{\mathsf{G}}$, with probabilities over the models of a distributed temporal logic.

Contents

Chapter 1

Introduction

The term temporal logic is used to describe any system of rules and symbolism for representing, and reasoning about, propositions qualified in terms of time. It was first introduced as tense logic, by Artur Prior in the 1960s , as a particular modal logic-based system [33].

Temporal logic is a branch of logic focusing on propositions whose truth values depends on time. It has been studied and developed by practitioners mainly in the areas of computer science and logic. One main concern is with the study of *distributed systems*, which are collections of connected autonomous components. The first advances in the use of temporal logics to study distributed systems were made by Pnueli, followed by Sernadas [35, 31]. This logic was also used to study *reactive systems*, which are system that maintain ongoing processes, and to study any form of *concurrency* which is the property of systems running several processes at the same time.

Temporal logic provides a formalism for describing the occurrence of events in time that is suitable for reasoning about concurrent programs [23]. When a program runs two processes in parallel we can't infer the input-output relation based solely on the input-output relations computed by each of the individual processes. The reason for this is that, running in parallel, the processes may interfere with one another, resulting in a different behavior than if they run alone. Temporal logic was then used to describe the desired behavior or operation of a system, while avoiding references to the method or details of it's implementation. In order to analyze systems, Manna and Pnueli used a simplified model in which the executions were restricted to be interleaved [23]. An interleaved execution is one in which at any instant only one process is executing an instruction. Once the instruction was completed, another process would initiate an instruction. Under this model, the execution of the program proceeds as a sequence of discrete steps. The selection of the next process to be executed was made by a *scheduler*. This however raised a problem

of *fairness*. An infinite process could be constantly selected by the scheduler, which could be neglecting another process that was ready to execute an instruction. A fair scheduler would ensure that no process would be neglected forever.

Hence the behavior of a concurrent program is characterized by the set of its fair execution sequences. Temporal logic is then utilized to state properties of the execution sequences of a given program, thus describing properties of the dynamic behavior of the program.

Another task of great importance is the verification of a program, proving or disproving the correctness of the algorithms. The old approach was to construct a proof by hand using axioms and inference rules in a deductive system such as temporal logic [23]. These proofs took some effort and brilliance to organize and, due to the complexity of testing validity, mechanical theorem provers were of little help. Clarke and Emmerson presented a model-theoretic approach for the case of finite-state concurrent systems that would mechanically determine if the system met a specification expressed in propositional temporal logic [9]. This technique is known as *model checking*.

They used a specification language they called computation tree logic (CTL) that is a proposition, branching-time temporal logic. The global state graph of the concurrent system was viewed as a finite Kripke structure, and an efficient algorithm could be given to determine whether the structure was a model of a particular formula (therefore determining if the program met its specification).

However, they encountered the problem that with CTL it was not possible to assert that correctness only holds on fair execution sequences [3, 10]. They overcame this problem by moving the fairness requirements into the semantics of CTL.

When defining temporal logic, there are two possible views regarding the underlying nature of time. One is that time is linear: At each moment there is only one possible future. The other is that time has a branching, treelike nature: At each moment, time may split into alternate courses representing different possible futures [3].

The modalities of a temporal logic system usually reflect the semantics regarding the nature of time. Thus, in a logic of linear time, temporal modalities are provided for describing events along a single time path. In contrast, in a logic of branching time, the modalities reflect the branching nature of time, by allowing quantification over possible futures.

For verification purposes we are typically interested in properties that hold for all computation paths. It is thus satisfactory to pick an arbitrary path and reason about it, asserting the existence of alternative computation paths as provided by a branching time

logic like CTL, and so it serves a purpose as model checking.

We are however more concerned with results that will help to verify correctness of concurrent programs, for which a linear time logic like LTL is generally used. This is the reason why LTL is taken as our base logic.

As a side note, there is also the language CTL* that merges linear time with branching time, allowing linear time assertions prefixed by existential path quantifiers [3].

The use of branching temporal logics like CTL and CTL* and even linear temporal logics as been deeply studied for reasoning about concurrent systems [22, 31]. However, there was a need for a precise notion of causality as reflected by the time structures adopted for logics. This led to the study of partial order temporal logics [32].

The idea of a partial order temporal logic is that it adopts a local causal perspective, instead of taking a look at the entire system. This is the case of the logics of partial ordered computations introduced by Printer and Wolper , or of interleaving set temporal logics and of temporal logics for reasoning about distributed transition systems, with product state spaces, occurrence nets and event structures [32].

In particular, n-agent temporal logics are in the latter category. Event structures are enriched with information about its sequential agents as models of concurrent systems.

In this approach each individual agent has a only a partial view of the whole distributed system in a particular time instant, due to the autonomy of each agent, and therefore communication has an important role to link up the system. n-agent logics can explicitly distinguish sequential agents in the system, refer to the local viewpoint of each agent and express communication between agents.

There are several versions of n-agent logics, with different perspectives on how non-local information can be accessed by each agent. Caleiro and Ehrich proposed a logic which assumes that an assertion about another agent is only possible at a communication point [11]. For example some logics assume that at each instant, the actual information about another agent is the one corresponding to the last communication with it.

Other n-agent logics consider the existence of a "present-knowledge" allowing reference to non-local properties of agents.

We follow the approach of Caleiro and Ehlrich when dealing with concurrent systems [11, 5].

Recently there's been an increasing interest in probabilistic logic due to the growing importance of probability in security and in quantum logic. Adding probability features to a base logic has been a research topic for some time [2, 15, 37]. One of the first attempting to combine logic and probability was Carnap [7].

Several approaches to enriching the base languages introduce a probability operator which allows the construction of new formulas and terms from the base formulas e.g. [1, 34]. From a semantic point of view there are two basic approaches. Either the models of the base logic are modified so that they accommodate the probability component (endogenous approach) or the models are kept as they are and the probabilities are assigned outside the models (exogenous approach).

The exogenous approach to enriching logics was introduced by Mateus and Sernadas while developing a new approach for quantum logic [26]. However it is shown that it can be used to enrich any logic [25].

The idea is to add probabilities features to a logic without changing it's base structure. The probability language is defined by taking the base formulas as terms and having a probability operator as a term constructor. This approach will be most convenient when we have to deal with languages that have some delicate formulas, as is the case of formulas for communication between agents.

When using probabilities, there seems to be two kind of statements that we may want to express. One is what Hacking calls the *chance setup* , that is a statement about what one might expect as the result of performing some experiment in a given situation, e.g. " there's s probability of randomly choosing an agent for which formula φ is true" *[14]*.

The other statement captures what has been called a *degree of belief* [4]. These statements are about quantifying an unknown property of a situation, e.g. "there's s probability that formula φ of agent A is true ". This refers to a particular agent, and expresses our belief in whether agent A has the said attribute.

The difference between the two statements is that the first statement assumes a fixed universe, and makes an assertion about that universe. The second statement implicitly assumes the existence of a number of possibilities with some probability over these possibilities [16]. The Statements that concerns us are of the second type, as we want to represent a system's behavior, allowing all the possible realities.

The first statement is connected to the endogenous approach, where the probability is assigned to formulas inside the logic [29, 12]. The second statement is closer to an exogenous concept, which requires that we define a higher-level logic which will be the probability logic. A model for this logic consists of a probability measure on the space of possible valuations of the base logic.

In this paper we take several steps to define an exogenous probabilistic logic for reasoning about concurrent systems. We start by laying the foundations and defining a language for a linear temporal logic LTL with a sound and weakly complete axiomatic system, following the works [19, 35, 22, 31]. Next we follow with the definition of a distributed temporal

logic DTL, based on the works of [23, 11, 8, 6, 5]. This will be an extension of the previous defined logic LTL to include additional time lines for the components that constitute a system. An axiomatic system is also presented, but we don't show results for completeness, although results of this nature have been shown in [5] for a different deductive system for DTL. Not only that but we will also show that there's a reduction from DTL to LTL, that is, if we are dealing with a system of only one agent, it can be represented by LTL.

In the second half of this work we deal with enriching the base logics LTL and DTL for probabilistic reasoning. For this, we start by defining a linear temporal logic with a probability operator pLTL, that only has probability on classical propositional formulas. Temporal formulas, and *actions*, which are multi-valued symbols, won't have a probability assigned to them. We present an axiomatic system, following mainly the work [30]. We also prove some results for formula manipulation. Lastly, we show that if all classical formulas have probability 1, then pLTL can be reduced to LTL.

To enrich DTL with probabilities, we take two steps. The first step is similar to what we done with pLTL, defining a probability operator only for classical propositional formulas. Thus we define a logic $\mathsf{pDTL}_{\mathsf{L}}$ that defines probabilities at a local level, an exogenous approach that considers a probability measure for each time event.

The second step for a probabilistic DTL takes a different approach of what has been done with pLTL and $\mathsf{pDTL}_{\mathsf{L}}$. Instead of a local operator for classical formulas, we follow the works of [24, 8, 27, 25] with an exogenous approach that will include all formulas. Probabilities are defined at a global level over the models of DTL, resulting in the logic $\mathsf{pDTL}_{\mathsf{G}}$. As a consequence of the exogenous approach, the base structure of DTL is kept intact, allowing probabilities on all formulas, which include classical and temporal formulas, as well as communication and actions. The results we present in this section are mainly about probabilistic reasoning and specific manipulation of formulas, with some focus on actions and communication.

Chapter 2

Temporal logic

Temporal Logic is used to describe a situation through the passage of time. While classical logic allow for a description of a static situation, temporal logic depicts that situation as time progresses.

The execution of a program is a series of chained instructions, and therefore may be represented by Temporal Logic. This logic is also a useful tool to describe concurrent systems. It establishes a way to represent sequences of events for each agent without a limitation in how actions occur, without the need of having global sequences of actions, which would cause problems as some actions would have to be put on hold, instead of allowing them to happen simultaneously. As it is, dependencies between agents of another agent action's are easily established, without the need of queuing them to be dealt with. So it seems temporal logic is an acceptable language to show and to reason about parallel programs, and in general reactive systems.

2.1 Linear Temporal Logic

Linear temporal logic adopts the paradigm of linearly ordered, discrete time where temporal modalities are provided for describing events along a single time path.

The linear temporal logic defined here is a propositional temporal logic of linear time for only one agent and serves as a stepping stone to the logics presented in the subsequent chapters.

2.1.1 Syntax and semantics

The alphabet of LTL is a pair $\langle \Pi, Act \rangle$ where Π is a set of propositional symbols and Act is a set of actions, such that $\mathsf{nil} \in Act$.

The language $\mathcal{L}_{\mathsf{LTL}}$ is given by

$$\mathcal{L}_{\mathsf{LTL}} ::= \Pi \mid Act \mid \neg\mathcal{L}_{\mathsf{LTL}} \mid \mathcal{L}_{\mathsf{LTL}} \Rightarrow \mathcal{L}_{\mathsf{LTL}} \mid \mathcal{L}_{\mathsf{LTL}}\mathsf{U}\mathcal{L}_{\mathsf{LTL}} \mid *$$

The $\neg$ and $\Rightarrow$ are the usual *negation* and *implication* symbols. We denote propositional symbols by p and q, and LTL formulas by φ and ψ. The operator U is the usual (strong) *until*. A formula $\varphi\mathsf{U}\psi$ (read φ until ψ) predicts the eventual occurrence of ψ and states that φ holds continuously at least until the (first) occurrence of ψ. The symbol $*$ is a marker for the *initial moment* and will only be true at the first state of a temporal structure.

Other logical connectives ($\top, \bot, \vee, \wedge$, etc.) and temporal operators can be defined as abbreviations in the usual way. For instance:

$\bot$	$\equiv$	$\neg(\varphi \Rightarrow \varphi)$	false
$\mathsf{X}\varphi$	$\equiv$	$\bot\mathsf{U}\varphi$	tomorrow (next)
$\mathsf{F}\varphi$	$\equiv$	$\top\mathsf{U}\varphi$	sometime in the future
$\mathsf{F}_\circ\varphi$	$\equiv$	$\varphi \vee \mathsf{F}\varphi$	now or sometime in the future
$\mathsf{G}\varphi$	$\equiv$	$\neg\mathsf{F}\neg\varphi$	always in the future
$\mathsf{G}_\circ\varphi$	$\equiv$	$\varphi \wedge \mathsf{G}\varphi$	now and always in the future
$\varphi\mathsf{W}\psi$	$\equiv$	$(\mathsf{G}\varphi) \vee (\varphi\mathsf{U}\psi)$	weak until, unless

Temporal formulas will be interpreted over infinite sequences of states. A *state* is just a classic boolean valuation for propositional symbols. In addiction, each state will also include an action (the action that is going to take place next).

To have a sequence of states we represent them with the natural numbers using the function $\lambda : \mathbb{N} \to 2^{\Pi}$ that given a natural number will return a valuation of Π. As an abbreviation we write λ_k instead of $\lambda(k)$, to represent the valuation of Π in state k.

Also for each state there will be an action assigned. That is a single action for each state, by the function $\alpha : \mathbb{N} \to Act$. As an abbreviation we write α_k instead of $\alpha(k)$.

An LTL interpretation structure is a pair $\eta = \langle \lambda, \alpha \rangle$.

We define the *global satisfaction relation* of a formula φ by an interpretation structure η as follows:

- $\eta \Vdash \varphi$ if and only $\eta, k \Vdash \varphi$ for every k.

and the *local satisfaction relation* at local states is defined by:

- $\eta, k \Vdash p$ if $\lambda_k(p) = 1$;
- $\eta, k \Vdash act$ if $\alpha_k = act$;
- $\eta, k \Vdash \neg\varphi$ if $\eta, k \nVdash \varphi$;

- $\eta, k \Vdash \varphi_1 \Rightarrow \varphi_2$ if $\eta, k \nVdash \varphi_1$ or $\eta, k \Vdash \varphi_2$;
- $\eta, k \Vdash \varphi \mathsf{U} \psi$ if there is $k'' \in \mathbb{N}$ with $k < k''$ such that $\eta, k'' \Vdash \psi$ and $\eta, k' \Vdash \varphi$ for every k' with $k < k' < k''$;
- $\eta, k \Vdash *$ if $k = 0$.

We say that η is a *model* of $\Gamma \subseteq \mathcal{L}_{\mathsf{LTL}}$ if η globally satisfies each of the formulas in Γ. Also, a formula φ is a *semantic consequence* of Γ, written $\Gamma \vDash \varphi$, if every model η verifies $\eta \Vdash \varphi$ whenever $\eta \Vdash \gamma$ for every $\gamma \in \Gamma$. Lastly, a formula φ is *valid*, written $\vDash \varphi$ if we have $\eta \Vdash \varphi$ for every model η.

Now we present a simple example to illustrate the use of the previous definitions.

Example 2.1.1. 1. Let $\Gamma = \{\varphi\}$, then $\neg(\top \mathsf{U} \neg\varphi)$ is a semantic consequence of Γ.

2. The formula $\neg(\top \mathsf{U} \neg p)$ is not valid.

For the purpose of reaching a contradiction, suppose that there's a model η of Γ such that $\eta \nVdash \neg(\top \mathsf{U} \neg\varphi)$. Then by negation of the definition of global satisfaction relation, it must be that there's some k such that $\eta, k \nVdash \neg(\top \mathsf{U} \neg\varphi)$. This is, by definition of negation, $\eta, k \Vdash \top \mathsf{U} \neg\varphi$. By definition of U we get that there's $k'' \in \mathbb{N}$ with $k < k''$ such that $\eta, k'' \Vdash \neg\varphi$, and again by definition of negation, $\eta, k'' \nVdash \varphi$.

Since η is a model of Γ, we have that $\eta \Vdash \varphi$, which results in $\eta, k \Vdash \varphi$ for every $k \in \mathbb{N}$. This contradicts the previous conclusion that there must be k'' such that $\eta, k'' \nVdash \varphi$. Hence $\eta \Vdash \neg(\top \mathsf{U} \neg\varphi)$ for every model η of Γ.

To see that $\neg(\top \mathsf{U} \neg p)$ is not a valid formula, consider the model $\eta = \langle \lambda, \alpha \rangle$ such that $\lambda_2(p) = 0$. We show that η does not satisfy the formula, that is, $\eta \nVdash \neg(\top \mathsf{U} \neg p)$. Supposing by contradiction that $\eta \Vdash \neg(\top \mathsf{U} \neg p)$, then $\eta, k \Vdash \neg(\top \mathsf{U} \neg p)$ for every $k \in \mathbb{N}$. That is, $\eta, k \nVdash (\top \mathsf{U} \neg p)$, so by definition, either there is no k'' such that $\eta, k'' \Vdash \neg p$ or there is some k' with $k < k' < k''$ such that $\eta, k' \nVdash \top$. Since the later is impossible, then there is no k'' such that $\eta, k'' \Vdash \neg p$. Therefore we reach a contradiction, since $\lambda_2(p) = 0$, that is, for $k'' = 2$ we have $\eta, k'' \Vdash \neg p$. The formula $\neg(\top \mathsf{U} \neg p)$ is not valid.

□

We also derive a result that will be used in future sections.

Lemma 2.1.2. $\vdash \mathsf{F}\varphi \wedge \mathsf{G}(\varphi \Rightarrow \psi) \Rightarrow \mathsf{F}\psi$.

Proof. Aiming for a contradiction suppose there is some model η such that $\eta \nVdash \mathsf{F}\varphi \wedge \mathsf{G}(\varphi \Rightarrow \psi) \Rightarrow \mathsf{F}\psi$, i.e., for some k we have $\eta, k \Vdash \mathsf{F}\varphi \wedge \mathsf{G}(\varphi \Rightarrow \psi)$ and $\eta, k \nVdash \mathsf{F}\psi$. Since $\eta, k \Vdash \mathsf{F}\varphi$ by (prop), we know that for some k' we have $\lambda, k' \Vdash \varphi$, and from $\lambda, k \Vdash \mathsf{G}(\varphi \Rightarrow \psi)$

we know that $\lambda, k'' \Vdash (\varphi \Rightarrow \psi)$ for every k'', in particular for $k'' = k'$, hence we get $\lambda, k' \Vdash \psi$, which contradicts $\lambda, k \nVdash \mathsf{F}\psi$. □

The semantic of the other temporal operators can now be derived. We present the definitions for some abbreviations, as some might be useful ahead.

Lemma 2.1.3. Let η be an LTL model, and $k \in \mathbb{N}$. Then:

- $\eta, k \Vdash \mathsf{X}\varphi$ *if* $\eta, k+1 \Vdash \varphi$*;*
- $\eta, k \Vdash \mathsf{F}\varphi$ *if* $\eta, i \Vdash \varphi$ *for some* $i > k$*;*
- $\eta, k \Vdash \mathsf{F}_\circ\varphi$ *if* $\eta, i \Vdash \varphi$ *for some* $i \geq k$*;*
- $\eta, k \Vdash \mathsf{G}\varphi$ *if* $\eta, i \Vdash \varphi$ *for every* $i > k$*;*
- $\eta, k \Vdash \mathsf{G}_\circ\varphi$ *if* $\eta, i \Vdash \varphi$ *for every* $i \geq k$*;*
- $\eta, k \Vdash \varphi \mathsf{W} \psi$ *if either* $\eta, i \Vdash \varphi$ *for every* $i > k$ *or there is* $j > k$ *such that* $\eta, j \Vdash \psi$ *and* $\eta, i \Vdash \varphi$ *for every* $j > i > k$*.*

Remark 2.1.4. These notions of global satisfaction consider that temporal specification applies to all states. This is the floating viewpoint, that considers that φ holds for η if it holds for all states. The alternative is the anchored viewpoint, where φ holds for model η if it holds at the initial state. This is most natural for programming. Although we go for the floating viewpoint, it is easy to translate it into the anchored viewpoint, sufficing that the formula holds only if the initial state symbol $*$ holds. Conversely we would translate from the anchored viewpoint to the floating viewpoint by adding a G before the formula.

2.1.2 Axiomatization

The first axiomatic system for linear temporal logic was presented in [20]. There were included axioms concerning past operators and a "symmetric" axiom for the future clauses. The system was more extensive as operators such as G and F were taken as basic for the construction of the language. We however are only concerned with the future operators, and some of the axioms in that work are no longer necessary since our basic constructor is U. We follow [3] and present the formal system $\Sigma_{\mathsf{LTL}}$for the derivation of formulas:

Axioms

taut. All tautologically valid formulas

ltl1. $\neg \mathsf{X}\varphi \Leftrightarrow \mathsf{X}\neg\varphi$

ltl2. $\mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi)$

ltl3. $\mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi$

until1. $\varphi \mathsf{U} \psi \Leftrightarrow \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U}\psi)$

until2. $\varphi \mathsf{U}\psi \Rightarrow \mathsf{X}\mathsf{F}_\circ\psi$

init1. $\mathsf{X}\neg *$

just1. $\dot{\bigvee_{act \in Act}} act$

Rules of inference

mp. $\varphi, \varphi \Rightarrow \psi \vdash \psi$

nex. $\varphi \vdash \mathsf{X}\varphi$

ind. $\varphi \Rightarrow \psi, \varphi \Rightarrow \mathsf{X}\varphi \vdash \varphi \Rightarrow \mathsf{G}_\circ\psi$

init2. $* \Rightarrow \mathsf{G}_\circ\varphi \vdash \varphi$

The "axiom" (taut) covers the axioms for classical propositional logic (we are, however, not interesed here in how tautologically valid formulas can be derived). In the axioms (ltl2), (ltl3) and (until2) we point the fact that these are only implications and not equivalences, even though we can prove the equivalences as valid. The axiom (init1) states that the symbol for the initial state is never valid after the first state. The rule (init2) relates a formula to the initial state. The axiom (just1) states that exactly one action is true at every instant.

The *derivability* of a formula φ from a set Γ of formulas is denoted by $\Gamma \vdash_\Sigma \varphi$, or $\Gamma \vdash \varphi$ when the formal system Σ is understood from the context. We define derivability, in this case for Σ_{LTL}, inductively by:

- $\Gamma \vdash_{\Sigma_{\mathsf{LTL}}} \varphi$ for every axiom,
- $\Gamma \vdash_{\Sigma_{\mathsf{LTL}}} \varphi$ for every $\varphi \in \Gamma$,
- Given a rule $\varphi_1, ..., \varphi_n \vdash \psi$, if $\Gamma \vdash_{\Sigma_{\mathsf{LTL}}} \varphi_i$ for all premises φ_i $(i = 1, ..., n)$, then $\Gamma \vdash \psi$.

A formula φ is called *derivable*, denoted $\vdash_\Sigma \varphi$ or $\vdash \varphi$, if $\emptyset \vdash \varphi$.

Let us now show the soundness of Σ_{LTL} with respect to the semantics of LTL.

Theorem 2.1.5. *Let φ be a formula and Γ a set of formulas. If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Proof. The proof runs by induction on the derivation of φ from Γ. Let η be an arbitrary model.

ltl1. $\neg \mathsf{X}\varphi \Leftrightarrow \mathsf{X}\neg\varphi$. Let us first see that $\neg \mathsf{X}\varphi \Rightarrow \mathsf{X}\neg\varphi$ and for this purpose assume by contradiction that $\eta \nVdash \neg \mathsf{X}\varphi \Rightarrow \mathsf{X}\neg\varphi$. So for some $k \in \mathbb{N}$ we have $\eta, k \nVdash \neg \mathsf{X}\varphi \Rightarrow \mathsf{X}\neg\varphi$ which is defined as $\eta, k \Vdash \neg \mathsf{X}\varphi$ and $\eta, k \nVdash \mathsf{X}\neg\varphi$. The former is by definition $\eta, k+1 \nVdash \varphi$ while the later is $\eta, k+1 \Vdash \varphi$ which is a contradiction. Conversely, suppose $\eta \nVdash \neg \mathsf{X}\varphi \Leftarrow \mathsf{X}\neg\varphi$. So for some $k \in \mathbb{N}$ we have that $\eta, k \nVdash \neg \mathsf{X}\varphi \Leftarrow \mathsf{X}\neg\varphi$ which is in turn $\eta, k \nVdash \neg \mathsf{X}\varphi$ and $\eta, k \Vdash \mathsf{X}\neg\varphi$. Again we reach a contradiction as by definition we get $\eta, k+1 \Vdash \varphi$ and $\eta, k+1 \nVdash \varphi$.

ltl2. $\mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi)$. Assume by absurd that $\eta \nVdash (\mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi))$, and so for some $k \in \mathbb{N}$ we have that $\eta, k \nVdash (\mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi))$. That is $\eta, k \Vdash \mathsf{X}(\varphi \Rightarrow \psi)$ and $\lambda, k \nVdash (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi)$. The former leads by definition to $\eta, k+1 \Vdash \varphi \Rightarrow \psi$ and the later to $\eta, k \Vdash \mathsf{X}\varphi$ and $\eta, k \nVdash \mathsf{X}\psi$ and consequently to $\eta, k+1 \Vdash \varphi$ and $\eta, k+1 \nVdash \psi$. Note that this contradicts the implication in $\eta, k+1 \Vdash \varphi \Rightarrow \psi$. So it must be that $\eta, k \Vdash \mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi)$ for all k.

ltl3. $\mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi$. Again the proof runs by contradiction. Assume that $\eta \nVdash (\mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi)$ and so for some $k \in \mathbb{N}$ we have that $\eta, k \nVdash (\mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi)$. It then follows that $\eta, k \Vdash \mathsf{G}_\circ\varphi$ and $\eta, k \nVdash \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi$. This last expression expands to $\eta, k \nVdash \varphi$ or $\eta, k \nVdash \mathsf{X}\mathsf{G}_\circ\varphi$. Notice that neither of these two is compatible with $\eta, k \Vdash \mathsf{G}_\circ\varphi$ for the fact that $\eta, k \Vdash \mathsf{G}_\circ\varphi$ is by definition $\eta, k' \Vdash \varphi$ for every $k' \geq k$, so in particular $\eta, k \Vdash \varphi$. Also if $\eta, k' \Vdash \varphi$ for every $k' \geq k$, then certainly $\eta, k'' \Vdash \varphi$ for every $k'' \geq k+1$, which is by definition $\eta, k+1 \Vdash \mathsf{G}_\circ\varphi$ and again by definition $\eta, k \Vdash \mathsf{X}\mathsf{G}_\circ\varphi$ and a contradiction is reached. Ergo, $\eta, k \Vdash \mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi$ for every k.

until1. $\varphi \mathsf{U} \psi \Leftrightarrow \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$. Assume for the purpose of a contradiction that $\eta \nVdash \varphi \mathsf{U} \psi \Rightarrow \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$. The definition is that $\eta, k \nVdash \varphi \mathsf{U} \psi \Rightarrow \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$ for some $k \in \mathbb{N}$. So the implication unfolds to $\eta, k \Vdash \varphi \mathsf{U} \psi$ and $\eta, k \nVdash \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$. The definition of the former is that there is an event k'' with $k < k''$ where $\lambda, k'' \Vdash \psi$ and $\eta, k' \Vdash \varphi$ for every k' such that $k < k' < k''$. So considering two cases, either $k'' = k+1$ or $k'' > k+1$.
For the first case, we conclude that $\eta, k+1 \Vdash \psi$ so by definition $\eta, k \Vdash \mathsf{X}\psi$.
For the second case we have that in particular for $k' = k+1$, $\eta, k+1 \Vdash \varphi$, and still that there is a state $k < k''$ where $\eta, k'' \Vdash \psi$ and $\eta, k''' \Vdash \varphi$ for every k''' such that $k+1 < k''' < k''$ which is by definition $\eta, k+1 \Vdash \varphi \mathsf{U} \psi$. In

conclusion, we have $\eta, k+1 \Vdash \varphi \wedge \varphi \mathsf{U} \psi$, which by definition is the same as $\eta, k \Vdash \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$.
Combining the two cases yields $\eta, k \Vdash \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$, which contradicts the initial assumption.
The inverse implication will lead to similar results.

until2. $\varphi \mathsf{U} \psi \Rightarrow \mathsf{XF}_{\circ}\psi$. Assume in view of a contradiction that $\eta \nVdash \varphi \mathsf{U} \psi \Rightarrow \mathsf{XF}_{\circ}\psi$ and so for some $k \in \mathbb{N}$ we get $\eta, k \nVdash \varphi \mathsf{U} \psi \Rightarrow \mathsf{XF}_{\circ}\psi$, or identically that $\eta, k \Vdash \varphi \mathsf{U} \psi$ and $\eta, k \nVdash \mathsf{XF}_{\circ}\psi$. The former is by definition $\eta, k'' \Vdash \psi$ and $\eta, k' \Vdash \varphi$ for $k'' > k' > k$ and the later is $\eta, k''' \nVdash \psi$ for $k''' > k$, which is a contradiction. Therefor $\eta, k \Vdash \varphi \mathsf{U} \psi \Rightarrow \mathsf{XF}_{\circ}\psi$ for any k.

init1. $\mathsf{X}\neg *$. Note that $\eta, k' \Vdash *$ if and only if $k' = 0$. This is the same as saying that $\eta, k'' \nVdash *$ for $k'' \geq 1$.If we write $k'' = k+1$, we get $\eta, k+1 \nVdash *$ for $k \geq 0$ which is by definition $\eta, k \Vdash \mathsf{X}\neg *$.

mp. $\varphi, \varphi \Rightarrow \psi \vdash \psi$. Assume that $\eta \Vdash \varphi$ and $\eta \Vdash \varphi \Rightarrow \psi$ and for the purpose of a contradiction that $\eta \nVdash \psi$. By global satisfaction we have that $\eta, k \Vdash \varphi$ and $\eta, k \Vdash \varphi \Rightarrow \psi$ for $k \in \mathbb{N}$. From the definition of the implication either $\eta, k \Vdash \psi$ or $\eta, k \nVdash \varphi$. Since we have that $\eta, k \Vdash \varphi$ which contradicts $\eta, k \nVdash \varphi$, then it must be that $\eta, k \Vdash \psi$.

nex. $\varphi \vdash \mathsf{X}\varphi$. Assume that $\eta \Vdash \varphi$ and by absurd that $\eta \nVdash \mathsf{X}\varphi$. So by definition $\eta, k \Vdash \varphi$ for any $k \in \mathbb{N}$, in particular for any $k = 1, 2, ...$ The previous expression can be written using $k' + 1 = k$ resulting in $\eta, k' + 1 \Vdash \varphi$ for any $k' + 1 = 1, 2, ..$ which is in fact $k' \in \mathbb{N}$. Hence by definition we have $\eta, k' \Vdash \mathsf{X}\varphi$ for $k' \in \mathbb{N}$, which yields $\eta \Vdash \mathsf{X}\varphi$ and contradicts the initial assumption.

ind. $\varphi \Rightarrow \psi, \varphi \Rightarrow \mathsf{X}\varphi \vdash \varphi \Rightarrow \mathsf{G}_{\circ}\psi$. Assume that $\eta \Vdash \varphi \Rightarrow \psi$ and $\eta \Vdash \varphi \Rightarrow \mathsf{X}\varphi$ and with the objective of reaching a contradiction suppose that $\eta \nVdash \varphi \Rightarrow \mathsf{G}_{\circ}\psi$. Then for some $k \in \mathbb{N}$ we have $\eta, k \Vdash \varphi \Rightarrow \psi$ and $\eta, k \Vdash \varphi \Rightarrow \mathsf{X}\varphi$ but also $\eta, k \nVdash \varphi \Rightarrow \mathsf{G}_{\circ}\psi$. Consider the two former expressions and note that from the implications there are two possible cases:
1) $\eta, k \nVdash \varphi$;
2) $\eta, k \Vdash \psi$ and $\eta, k \Vdash \mathsf{X}\varphi$.
Now back to $\eta, k \nVdash \varphi \Rightarrow \mathsf{G}_{\circ}\psi$ we note that this is by definition $\eta, k \Vdash \varphi$ and $\eta, k' \nVdash \psi$ for $k' \geq k$. We can easily note that the case 1) is incompatible with this expression, due to $\eta, k \nVdash \varphi$. The second case's contradiction comes from the expression $\eta, k \Vdash \psi$ which doesn't match with $\eta, k' \nVdash \psi$ for $k' \geq k$ when

$k' = k$. Hence we reach a contradiction and it must be that we can derive $\eta, k \Vdash \varphi \Rightarrow \mathsf{G}_\circ\psi$ for any $k \in \mathbb{N}$.

init. $* \Rightarrow \mathsf{G}_\circ\varphi \vdash \varphi$. In view of a contradiction assume that $\eta \Vdash * \Rightarrow \mathsf{G}_\circ\varphi$ and $\eta \nVdash \varphi$. So for some $k \in \mathbb{N}$ we must have $\eta, k \Vdash * \Rightarrow \mathsf{G}_\circ\varphi$ and $\eta, k \nVdash \varphi$. From $\eta, k \Vdash * \Rightarrow \mathsf{G}_\circ\varphi$ either $\eta, k \nVdash *$ or $\eta, k \Vdash \mathsf{G}_\circ\varphi$. Note that $\eta, k \nVdash *$happens only for $k \geq 1$. In fact we need only consider the case where $k = 0$, as it will restrict the behavior of $k > 0$. So for $k = 0$ we get $\eta, 0 \Vdash *$ which contradicts $\eta, 0 \nVdash *$, and so from $\eta, 0 \Vdash * \Rightarrow \mathsf{G}_\circ\varphi$ we must take that $\eta, 0 \Vdash \mathsf{G}_\circ\varphi$. Now this is by definition $\eta, k' \Vdash \varphi$ for $k' \geq 0$, which contradicts $\eta, k \nVdash \varphi$ for $k \in \mathbb{N}$. So we can in fact derive $\eta, k \Vdash \varphi$ for all k.

□

We argued above that in derivations within Σ_{LTL} we do not want to bother how to derive tautologically valid formulas; we will simply use them as axioms. Nevertheless, there will still occur purely classical derivation parts where only (taut) and (mp) are used. We will abbreviate such parts by using the following rule

prop. $\varphi_1, ...\varphi_n \vdash \psi$ if ψ is a tautological consequence of $\varphi_1, ...\varphi_n$.

As an example we note the chain rule

$$\varphi \Rightarrow \psi, \psi \Rightarrow \phi \vdash \phi \Rightarrow \varphi$$

which we will apply from now on in derivations, together with many other, as a rule of the kind (prop).

Example 2.1.6. We make use of the presented axiomatization to derive and auxiliary result

xout. $\vdash \mathsf{X}\varphi \wedge \mathsf{X}\psi \Rightarrow \mathsf{X}(\varphi \wedge \psi)$

The expanded form of (xout) is $\neg(\mathsf{X}\varphi \Rightarrow \neg\mathsf{X}\psi) \Rightarrow \mathsf{X}\neg(\varphi \Rightarrow \neg\psi)$ so this is our goal:

(1) $\mathsf{X}(\varphi \Rightarrow \neg\psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\neg\psi)$ (ltl2)
(2) $\mathsf{X}(\varphi \Rightarrow \neg\psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \neg\mathsf{X}\psi)$ (prop),(ltl1),(1)
(3) $\neg(\mathsf{X}\varphi \Rightarrow \neg\mathsf{X}\psi) \Rightarrow \neg\mathsf{X}(\varphi \Rightarrow \neg\psi)$ (prop),(2)
(4) $\neg(\mathsf{X}\varphi \Rightarrow \neg X\psi) \Rightarrow \mathsf{X}\neg(\varphi \Rightarrow \neg\psi)$ (prop),(ltl1),(3)

□

Theorem 2.1.7 (Deduction theorem)**.** *Let φ, ψ be formulas and Γ a set of formulas. If $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \psi$*

Proof. The proof runs by induction on the derivation of ψ from $\Gamma \cup \{\varphi\}$.

1. ψ is an axiom of Σ_{LTL} or $\psi \in \Gamma$: then $\Gamma \vdash \psi$, and $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \psi$ follows with (prop)

2. $\psi \equiv \varphi$: then $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{XG}\varphi$ by (ltl3), and $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \varphi$ follows with (prop)

3. ψ is a conclusion of (mp) with premises ϕ and $\phi \Rightarrow \psi$: We then have both $\Gamma \cup \{\varphi\} \vdash \phi$ and $\Gamma \cup \{\varphi\} \vdash \phi \Rightarrow \psi$ Applying the induction hypothesis, we get $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \phi$ and $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \psi)$ from which $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \psi$ follows with (prop).

4. $\psi \equiv \mathsf{X}\phi$ is a conclusion of (nex) with premise ϕ: Then $\Gamma \cup \{\varphi\} \vdash \phi$, and therefore $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \phi$ by induction hypothesis. We continue the derivation of $\mathsf{G}_\circ\varphi \Rightarrow \phi$ to a derivation of $\mathsf{G}_\circ\varphi \Rightarrow \mathsf{X}\phi$:

(1)	$\mathsf{G}_\circ\varphi \Rightarrow \phi$	induction hypothesis
(2)	$\mathsf{X}(\mathsf{G}_\circ\varphi \Rightarrow \phi)$	(nex),(1)
(3)	$\mathsf{X}(\mathsf{G}_\circ\varphi \Rightarrow \phi) \Rightarrow (\mathsf{XG}_\circ\varphi \Rightarrow \mathsf{X}\phi)$	(ltl2)
(4)	$\mathsf{XG}_\circ\varphi \Rightarrow \mathsf{X}\phi$	(mp),(2),(3)
(5)	$\mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{XG}_\circ\varphi$	(ltl3)
(6)	$\mathsf{G}_\circ\varphi \Rightarrow \mathsf{XG}_\circ\varphi$	(prop),(5)
(7)	$\mathsf{G}_\circ\varphi \Rightarrow \mathsf{X}\phi$	(prop),(4),(6)

5. $\psi \equiv \phi \Rightarrow \mathsf{G}_\circ\eta$ is a conclusion of (ind) with premises $\phi \Rightarrow \eta$ and $\phi \Rightarrow \mathsf{X}\phi$: As above, we get with the induction hypothesis that $\mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \eta)$ and $\mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \mathsf{X}\phi)$ are derivable from Γ, and their derivations can be continued to derive $\mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \mathsf{G}_\circ\eta)$ as follows:

(1)	$\mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \eta)$	induction hypothesis
(2)	$\mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \mathsf{X}\phi)$	induction hypothesis
(3)	$\mathsf{G}_\circ\varphi \wedge \phi \Rightarrow \eta$	(prop),(1)
(4)	$\mathsf{G}_\circ\varphi \wedge \phi \Rightarrow \mathsf{X}\phi$	(prop),(2)
(5)	$\mathsf{G}_\circ\varphi \Rightarrow \mathsf{XG}\varphi$	(prop),(ltl3)
(6)	$\mathsf{G}_\circ\varphi \wedge \phi \Rightarrow \mathsf{XG}\varphi \wedge \mathsf{X}\phi$	(prop),(4),(5)
(7)	$\mathsf{XG}\varphi \wedge \mathsf{X}\phi \Rightarrow \mathsf{X}(\mathsf{G}_\circ\varphi \wedge \phi)$	(xout)
(8)	$\mathsf{G}_\circ\varphi \wedge \phi \Rightarrow \mathsf{X}(\mathsf{G}_\circ\varphi \wedge \phi)$	(prop),(6),(7)
(9)	$\mathsf{G}_\circ\varphi \wedge \phi \Rightarrow \mathsf{G}_\circ\eta$	(ind),(3),(8)
(10)	$\mathsf{G}_\circ\varphi \Rightarrow (\phi \Rightarrow \mathsf{G}_\circ\eta)$	(prop),(9)

6. $\psi \equiv \varphi$ is a conclusion of (init2) with premise $* \Rightarrow \mathsf{G}_\circ\phi$: Then $\Gamma \cup \{\varphi\} \vdash * \Rightarrow \mathsf{G}_\circ\phi$, and therefore $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow (* \Rightarrow \mathsf{G}_\circ\phi)$ by ind. hyp. We continue the derivation of $\mathsf{G}_\circ\varphi \Rightarrow (* \Rightarrow \mathsf{G}_\circ\phi)$ to a derivation of $\mathsf{G}_\circ\varphi \Rightarrow \phi$:

(1)	$\mathsf{G}_\circ\varphi \Rightarrow (* \Rightarrow \mathsf{G}_\circ\phi)$	induction hypothesis
(2)	$\mathsf{G}_\circ\varphi \wedge * \Rightarrow \mathsf{G}_\circ\phi$	(prop),(1)
(3)	$* \Rightarrow (\mathsf{G}_\circ\varphi \Rightarrow \mathsf{G}_\circ\phi)$	(prop),(2)
(4)	$\mathsf{G}_\circ\varphi \Rightarrow \mathsf{G}_\circ\phi$	(init2),(3)
(5)	$\mathsf{G}_\circ\phi \Rightarrow \phi \wedge \mathsf{XG}\phi$	(ltl3)
(6)	$\mathsf{G}_\circ\varphi \Rightarrow \phi$	(prop),(4),(5)

□

Theorem 2.1.8. *Let φ, ψ be formulas, and let Γ be a set of formulas. If $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \psi$ then $\Gamma \cup \{\varphi\} \vdash \psi$.*

Proof. If $\Gamma \vdash \mathsf{G}_\circ\varphi \Rightarrow \psi$ then also $\Gamma \cup \{\varphi\} \vdash \mathsf{G}_\circ\varphi \Rightarrow \psi$. With $\Gamma \cup \{\varphi\} \vdash \varphi$ we get $\Gamma \cup \{\varphi\} \vdash \mathsf{G}_\circ\varphi$ with (nex), (ind) and (prop) and finally $\Gamma \cup \{\varphi\} \vdash \psi$ by applying (mp). □

2.1.3 Completeness

The axiom system was based on the one presented in [3]. In that book there was also a proof for completeness. The difference is that they use G as a basic constructor, instead of U. Our proof of completeness is heavily based in their method, but adapted it to be built around U. We present part of their proof that is somewhat redundant now, since we have proved for the "until" operator and have very little concern for the "always" operator.

The proof lines follow the often called *Henkin-Hasenjager method,* modified in many details for our situation.

Now, to answer questions related to completeness of Σ_{LTL} let us first take into account an elucidating situation. Consider the infinite set

$$\Gamma = \{\varphi \Rightarrow \psi, \varphi \Rightarrow \mathsf{X}\psi, \varphi \Rightarrow \mathsf{XX}\psi, \varphi \Rightarrow \mathsf{XXX}\psi, ...\}$$

of formulas. It is easy to calculate

$$\Gamma \vDash \varphi \Rightarrow \mathsf{G}_\circ\psi.$$

If we were dealing with a sound and complete formal system, then $\varphi \Rightarrow \mathsf{G}_\circ\psi$ would be derivable from Γ. Any derivation in such a system can only have finitely many assumptions of Γ. So assuming a derivation of $\varphi \Rightarrow \mathsf{G}_\circ\psi$ from Γ, the soundness of the system would imply that $\varphi \Rightarrow \mathsf{G}_\circ\psi$ is a consequence of a finite subset of Γ. This is not the case however. The situation presented shows that

$$\Gamma \vDash \varphi \text{ does not imply } \Gamma \vdash \varphi$$

for the formal system Σ_{LTL} (for arbitrary Γ and φ). In fact no sound system can achieve this kind of completeness. The argument for incompleteness was based on F being infinite. This leads us to consider a weaker notion of completeness

Definition 2.1.9. We call a formal system weakly complete if

$$\Gamma \vDash \varphi \text{ implies } \Gamma \vdash \varphi \text{ for finite } \Gamma.$$

We will show that Σ_{LTL} is weakly complete.

Since we are restricted to finite set Γ of assumptions, it suffices to consider cases where $\Gamma = \emptyset$ (taking in consideration Theorem 2.1.7) to prove $\vDash \varphi$ implies $\vdash \varphi$, or conversely that $\nvdash \varphi$ implies $\nvDash \varphi$. We will show that φ is satisfiable by constructing a suitable temporal structure, whenever $\neg\varphi$ cannot be derived in Σ_{LTL}.

Let us begin with introducing some notation. A positive-negative pair (shortly PNP) is a pair $\mathcal{P} = (\Gamma^+, \Gamma^-)$ of two finite sets Γ^+ and Γ^- of formulas. We denote the set $\Gamma^+ \cup \Gamma^-$ by $\Gamma_{\mathcal{P}}$. Furthermore, we will sometimes denote Γ^+ by $pos\,(\mathcal{P})$ and Γ^- by $neg\,(\mathcal{P})$. Finally the formula $\widehat{\mathcal{P}}$ will be the abbreviation

$$\widehat{\mathcal{P}} \equiv \bigwedge_{\varphi \in \Gamma^+} \varphi \wedge \bigwedge_{\psi \in \Gamma^-} \neg\psi$$

where empty conjunctions are identified with the formula $\top$. PNP will be used to represent information about the temporal structure under construction; the intuition is that the formulas in Γ^+ should be true and those in Γ^- should be false at the current state.

A PNP $\mathcal{P}$ is called inconsistent if $\vdash \neg\widehat{\mathcal{P}}$. Otherwise, $\mathcal{P}$ is called *consistent.*$\mathcal{P} = (\Gamma^+, \Gamma^-)$

Lemma 2.1.10. *Let $\mathcal{P} = (\Gamma^+, \Gamma^-)$ be a consistent PNP and φ a formula.*

1. Γ^+ and Γ^- are disjoint
2. $(\Gamma^+ \cup \{\varphi\}, \Gamma^-)$ or $(\Gamma^+, \Gamma^- \cup \varphi)$ is a consistent PNP

Proof. 1. Assume that Γ^+ and Γ^- are not disjoint and pick $\varphi \in \Gamma^+ \cap \Gamma^-$. Then $\widehat{\mathcal{P}}$ is of the form $... \wedge \varphi \wedge ... \wedge \neg\varphi \wedge ...$ which implies that $\neg\widehat{\mathcal{P}}$ is tautologically derivable. So $\vdash \neg\widehat{\mathcal{P}}$ which means that $\mathcal{P}$ is inconsistent and a contradiction is reached. Hence, Γ^+ and Γ^- must be disjoint.

2. If $\varphi \in \Gamma^+$ or $\varphi \in \Gamma^-$ then we have $(\Gamma^+ \cup \{\varphi\}, \Gamma^-) = (\Gamma^+, \Gamma^-)$ or $(\Gamma^+, \Gamma^- \cup \{\varphi\}) = (\Gamma^+, \Gamma^-)$, respectively, and the assertion follows by the premise that (Γ^+, Γ^-) is consistent. If instead $(\Gamma^+ \cup \{\varphi\}, \Gamma^-)$ and $(\Gamma^+, \Gamma^- \cup \{\varphi\})$ are consistent, then it reverts to case a) where we can obtain a pair that is not disjoint. Otherwise, assuming that both pairs under

consideration are inconsistent implies $\vdash \neg\left(\widehat{\mathcal{P}} \wedge \varphi\right)$ and $\vdash \neg\left(\widehat{\mathcal{P}} \wedge \neg\varphi\right)$. It is easy to obtain $\vdash \neg\widehat{\mathcal{P}}$ with (prop), which again contradicts the consistency of $\mathcal{P}$. Hence (at least) one of the pairs must be consistent. □

Lemma 2.1.11. *Let $\mathcal{P} = (\Gamma^+, \Gamma^-)$ be a consistent PNP and φ and ψ formulas.*

a) $\bot \notin \Gamma^+$

b) If $\varphi, \psi, \varphi \Rightarrow \psi \in \Gamma_{\mathcal{P}}$ then: $\varphi \Rightarrow \psi \in \Gamma^+$ if and only if $\varphi \in \Gamma^-$ or $\psi \in \Gamma^+$.

c) If $\vdash \varphi \Rightarrow \psi$, with $\varphi \in \Gamma^+$ and $\psi \in \Gamma_{\mathcal{P}}$, then $\psi \in \Gamma^+$.

Proof. a) Assume $\bot \in \Gamma^+$. Then $\vdash \widehat{\mathcal{P}} \Rightarrow \bot$ (taut) which is just $\vdash \neg\widehat{\mathcal{P}}$, and contradicts the consistency. This proves $\bot \notin \Gamma^+$.

b) Assume that $\varphi \Rightarrow \psi \in \Gamma^+$ but $\varphi \notin \Gamma^-$ and $\psi \notin \Gamma^+$. Since $\varphi, \psi \in \mathcal{F}_{\mathcal{P}}$ we get $\varphi \in \Gamma^+$ and $\psi \in \Gamma^-$. Then $\vdash \widehat{\mathcal{P}} \Rightarrow (\varphi \Rightarrow \psi) \wedge \varphi \wedge \neg\psi$ and this yields $\vdash \neg\widehat{\mathcal{P}}$ which is a contradiction. Hence, $\varphi \in \Gamma^-$ or $\psi \in \Gamma^+$. On the other hand, assume that $\varphi \in \Gamma^-$ or $\psi \in \Gamma^+$. If $\varphi \Rightarrow \psi \notin \Gamma^+$ we must have $\varphi \Rightarrow \psi \in \Gamma^-$, and we get $\vdash \widehat{\mathcal{P}} \Rightarrow \neg(\varphi \Rightarrow \psi) \wedge \neg\varphi$ or $\vdash \widehat{\mathcal{P}} \Rightarrow \neg(\varphi \Rightarrow \psi) \wedge \psi$. In both cases we again obtain the contradiction $\vdash \neg\widehat{\mathcal{P}}$; hence $\varphi \Rightarrow \psi \in \Gamma^+$.

c) Assume that $\psi \notin \Gamma^+$. Then $\psi \in \Gamma^-$ because $\psi \in \Gamma_{\mathcal{P}}$, and with $\varphi \in \Gamma^+$ we get $\vdash \widehat{\mathcal{P}} \Rightarrow \varphi \wedge \neg\psi$, and since $\vdash \varphi \Rightarrow \psi$, then $\vdash \widehat{\mathcal{P}} \Rightarrow (\varphi \Rightarrow \psi)$, and combining these with (prop) we get$\vdash \widehat{\mathcal{P}} \Rightarrow \varphi \wedge \neg\psi \wedge (\varphi \Rightarrow \psi)$, furthermore we get $\vdash \neg\widehat{\mathcal{P}}$. This contradicts the consistency of $\mathcal{P}$; hence $\psi \in \Gamma^+$. □

Let φ be a formula. With φ we associate a set $\tau(\varphi)$ of formulas, inductively defined as follows:

1. $\tau(p) = \{p\}$ for $p \in \Pi$.
2. $\tau(act) = \{act\}$ for $act \in Act$.
3. $\tau(\bot) = \{\bot\}$.
4. $\tau(\varphi \Rightarrow \psi) = \{\varphi \Rightarrow \psi\} \cup \tau(\varphi) \cup \tau(\psi)$.
5. $\tau(\mathsf{X}\varphi) = \{\mathsf{X}\varphi\}$
6. $\tau(\mathsf{G}_\circ\varphi) = \{\mathsf{G}_\circ\varphi\} \cup \tau(\varphi)$
7. $\tau(\varphi \mathsf{U} \psi) = \{\varphi \mathsf{U} \psi\} \cup \{\mathsf{X}\varphi\} \cup \{\mathsf{X}\psi\}$

In result $\tau(\varphi)$ is the set of "sub-formulas" of φ. However note that $\mathsf{X}\varphi$ formulas are treated as "indivisible".

For a set of formulas Γ we let

$$\tau(\Gamma) = \{\psi : \psi \in \tau(\varphi), \varphi \in \Gamma\}.$$

Obviously, $\tau(\tau(\Gamma)) = \tau(\Gamma)$ and $\tau(\Gamma_{\mathcal{P}})$ is finite for every PNP $\mathcal{P}$ (since $\mathcal{F}_{\mathcal{P}}$ is finite). We call a PNP $\mathcal{P}$ *complete if* $\tau(\Gamma_{\mathcal{P}}) = \Gamma_{\mathcal{P}}$.

Lemma 2.1.12. *Let $\mathcal{P}$ be a consistent PNP. There is a consistent and complete PNP $\mathcal{P}^*$ with $pos(\mathcal{P}) \subseteq pos(\mathcal{P}^*)$ and $neg(\mathcal{P}) \subseteq neg(\mathcal{P}^*)$.*

Proof. Starting from $\mathcal{P}$, $\mathcal{P}^*$ is constructed by successively adding φ to $pos(\mathcal{F}_{\mathcal{P}})$ or to $neg(\mathcal{F}_{\mathcal{P}})$ for every $\varphi \in \tau(\mathcal{F}_{\mathcal{P}})$, depending on which of these is consistent. By Lemma 2.1.10.2 this is always possible and it evidently yields some consistent and complete PNP $\mathcal{P}^*$. □

Given a consistent PNP $\mathcal{P}$, we call any PNP $\mathcal{P}^*$ that satisfies the conditions of Lemma 2.1.12 a *completion* of $\mathcal{P}$. In general, different completions of a given $\mathcal{P}$ are possible, but only finitely many.

Lemma 2.1.13. *Let $\mathcal{P}_1^*, ..., \mathcal{P}_n^*$ be all different completions of a consistent PNP $\mathcal{P}$. Then $\vdash \widehat{\mathcal{P}} \Rightarrow \widehat{\mathcal{P}_1^*} \vee ... \vee \widehat{\mathcal{P}_n^*}$.*

Proof. We first prove an auxiliary assertion: let Γ be some finite set of formulas and let $Q_1, ..., Q_m$ be all different PNP Q with $\Gamma_Q = \tau(\Gamma)$ and such that $pos(Q)$ and $neg(Q)$ are disjoint. Because $\tau(\Gamma_Q) = \tau(\tau(\Gamma)) = \tau(\Gamma) = \Gamma_Q$ holds for any such Q, all $Q_1, ..., Q_m$ are complete and we show bu induction on the number of formulas in $\tau(\Gamma)$ that
$(*) \vdash \bigvee_{i=1}^{m} \widehat{Q_i}$.
If $\tau(\Gamma) = \emptyset$ then $m = 1$, $Q_1 = (\emptyset, \emptyset)$, and $\widehat{Q_1} = \top$, so $(*)$ holds by (taut). Assume now that $\tau(\Gamma) = \{\varphi_1, ..., \varphi_k\}$ for some $k \geq 1$. Clearly there must be some j (where $1 \leq j \leq k$) such that $\varphi_j \notin \tau(\{\varphi_1, ..., \varphi_{j-1}, \varphi_{j+1}, ..., \varphi_k\})$, i.e., φ_j is a "most complex" formula in $\tau(\Gamma)$; let $\Gamma' = \tau(\Gamma) \setminus \{\varphi_j\}$. I particular, it follows that $\tau(\Gamma') = \Gamma'$. Let $Q_1', ...Q_l'$ be all PNP constructed for Γ' as described. Then $m = 2l$ and the PNP $Q_1, ..., Q_m$ are obtained form $Q_1', ...Q_l'$ as follows:

$Q_1 = (pos(Q_1') \cup \{\varphi_j\}, neg(Q_1'))$,
$\vdots$
$Q_l = (pos(Q_l') \cup \{\varphi_j\}, neg(Q_l'))$,
$Q_{l+1} = \left(pos\left(Q_{l+1}'\right), neg\left(Q_{l+1}'\right) \cup \{\varphi_j\}\right)$,

$\vdots$

$Q_m = (pos\,(Q'_m)\,, neg\,(Q'_m) \cup \{\varphi_j\})$.

By the induction hypothesis we have $\vdash \bigvee_{i=1}^{m} \widehat{Q_i}$ which yields

$\vdash \bigvee_{i=1}^{l} \left(\widehat{Q_i} \wedge \varphi_j\right) \vee \bigvee_{i=1}^{l} \left(\widehat{Q_i} \wedge \neg\varphi_j\right)$,

i.e., $(*)$ by (prop).

Let now $\mathcal{P}$ be a consistent PNP, and let $\mathcal{P}'_1, ..., \mathcal{P}'_m$ be all different PNP $\mathcal{P}'$ with $\mathcal{F}_{\mathcal{P}} = \tau\,(\mathcal{F}_{\mathcal{P}})$ and such that $pos\,(\mathcal{P}')$ and $neg\,(\mathcal{P}')$ are disjoint. The completions $\mathcal{P}^*_1, ..., \mathcal{P}^*_n$ are just those $\mathcal{P}'_i$ which are consistent and for which $pos\,(\mathcal{P}) \subseteq pos\,(\mathcal{P}'_i)$ and $neg\,(\mathcal{P}) \subseteq neg\,(\mathcal{P}'_i)$. without loss of generality, we may suppose that these are $\mathcal{P}'_1, ..., \mathcal{P}'_n$ which means that for $i > n$,

(i) $\mathcal{P}'_i$ is inconsistent

or

(ii) $pos\,(\mathcal{P}) \nsubseteq pos\,(\mathcal{P}'_i)$ or $neg\,(\mathcal{P}) \nsubseteq neg\,(\mathcal{P}'_i)$.

We obtain $\vdash \neg\widehat{\mathcal{P}'_i}$ in case (i) and $pos\,(\mathcal{P}) \cap neg\,(\mathcal{P}'_i) \neq \emptyset$ or $neg\,(\mathcal{P}) \cap pos\,(\mathcal{P}'_i) \neq \emptyset$ and therefore $\vdash \neg\left(\widehat{\mathcal{P}} \wedge \widehat{\mathcal{P}'_i}\right)$ by (taut) in case (ii). In either case, we may conclude $\vdash \widehat{\mathcal{P}} \Rightarrow \neg\widehat{\mathcal{P}'_i}$ with (prop), and this holds for every $i > n$. With $(*)$ we obtain $\vdash \bigvee_{i=1}^{m} \widehat{\mathcal{P}'_i}$ and with (prop) we the get $\vdash \widehat{\mathcal{P}} \Rightarrow \bigvee_{i=1}^{m} \widehat{\mathcal{P}'_i}$ which is just the desired assertion. □

When we build a completion $\mathcal{P}^*$ of a consistent PNP $\mathcal{P}$ what we are doing is that the sub-formulas of formulas appearing in $\mathcal{F}_{\mathcal{P}}$ that should be true or false in some state are collected in $pos\,(\mathcal{P}^*)$ and $neg\,(\mathcal{P}^*)$, respectively, and so all formulas of $pos\,(\mathcal{P})$ are true and all formulas of $neg\,(\mathcal{P})$ are false in that state. Let us illustrate this idea with a little example.

Example 2.1.14. Suppose $\varphi \equiv ((p_1 \Rightarrow p_2) \Rightarrow \mathsf{G}_\circ p_3)$, $\psi \equiv (p_3 \Rightarrow \mathsf{X}p_2)$ with $p_1, p_2, p_3 \in \Pi$ and $\mathcal{P} = (\{\varphi\}, \{\psi\})$. One possible completion of $\mathcal{P}$ is

$$\mathcal{P}^* = (\{\varphi, p_1 \Rightarrow p_2, (\mathsf{G}_\circ p_3)\,, p_2, p_3\}\,, \{\psi, p_1, (\mathsf{X}p_2)\})\,.$$

If all sub-formulas of φ and ψ in $pos\,(\mathcal{P}^*)$ evaluate to true and those in $neg\,(\mathcal{P}^*)$ evaluate to false, then φ becomes true and ψ becomes false and, moreover, such valuation is in fact possible because of the consistency of $\mathcal{P}^*$. However, some of this information focused on one state may also have implications for other states. In our example, $\mathsf{G}_\circ p_3$ becomes true in a state only if p_3 is true in that state which is already the case as p_3 belongs to $pos\,(\mathcal{P}^*)$ and p_3 is also true in every future state or, equivalently, $\mathsf{G}_\circ p_3$ is true in the next state. To

make $\mathsf{X}p_2$ false we have to make p_2 false in the next state.

□

To "transfer" this information from one state to the next is the purpose of the next construction.

For a PNP $\mathcal{P} = (\Gamma^+, \Gamma^-)$ we define the following four sets of formulas

$\sigma_1(\mathcal{P}) = \{\varphi : \mathsf{X}\varphi \in \Gamma^+\}$,

$\sigma_2(\mathcal{P}) = \{\mathsf{G}_\circ\varphi : \mathsf{G}_\circ\varphi \in \Gamma^+\}$,

$\sigma_3(\mathcal{P}) = \{\varphi\mathsf{U}\psi : \varphi\mathsf{U}\psi \in \Gamma^+ \text{ and } \mathsf{X}\varphi \in \Gamma^+ \text{and } \mathsf{X}\psi \in \Gamma^-\}$,

$\sigma_4(\mathcal{P}) = \{\varphi : \mathsf{X}\varphi \in \Gamma^-\}$,

$\sigma_5(\mathcal{P}) = \{\mathsf{G}_\circ\varphi : \mathsf{G}_\circ\varphi \in \Gamma^- \text{ and } \varphi \in \Gamma^+\}$

$\sigma_6(\mathcal{P}) = \{\varphi\mathsf{U}\psi : \varphi\mathsf{U}\psi \in \Gamma^- \text{ and } \mathsf{X}\varphi \in \Gamma^+ \text{and } \mathsf{X}\psi \in \Gamma^-\}$,

and the PNP

$\sigma(\mathcal{P}) = (\sigma_1(\mathcal{P}) \cup \sigma_2(\mathcal{P}) \cup \sigma_3(\mathcal{P}), \sigma_4(\mathcal{P}) \cup \sigma_5(\mathcal{P}) \cup \sigma_6(\mathcal{P}))$.

Note 2.1.15. The function σ will only be applied to consistent and complete PNPs $\mathcal{P}^*$. This is the reason why σ_3 is defined as it is. Although it might be true that $\varphi\mathsf{U}\psi \in \Gamma^+$, we don't have to consider $\mathsf{X}\psi \in \Gamma^+$ because it would trivially be processed by σ_1 since $\mathcal{P}^*$ is complete.

There are eight different PNPs $\mathcal{P} = (\Gamma^+, \Gamma^-)$ for $\varphi\mathsf{U}\psi \in \mathbf{\Gamma}$, only five consistent and of those five only two are non-trivial and to our concern, and thus we treat them with σ_3 and σ_6.

Example 2.1.16. Back to the example 2.1.14, we would have $\sigma(\mathcal{P}^*) = (\{\mathsf{G}_\circ p_3\}, \{p_2\})$, which notes what has to be true or false for the next state in accordance with $\mathcal{P}^*$ and the future states.

□

Lemma 2.1.17. *Let $\mathcal{P}$ be a PNP.*

a) $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\widehat{\sigma(\mathcal{P})}$

b) If $\mathcal{P}$ is consistent then $\sigma(\mathcal{P})$ is consistent.

Proof. (1) We show that $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\varphi$ if $\varphi \in \sigma_1(\mathcal{P}) \cup \sigma_2(\mathcal{P}) \cup \sigma_3(\mathcal{P})$ and that $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\neg\varphi$ if $\varphi \in \sigma_4(\mathcal{P}) \cup \sigma_5(\mathcal{P}) \cup \sigma_6(\mathcal{P})$. The assertion (1) then follows immediately with (prop) and the law $\mathsf{X}\psi_1 \wedge \mathsf{X}\psi_2 \Rightarrow \mathsf{X}(\psi_1 \wedge \psi_2)$. We distinguish the four cases of $\varphi \in \sigma_i$, $i = 1, 2, 3, 4, 5, 6$.

1. If $\varphi \in \sigma_1$ then $\mathsf{X}\varphi \in pos(\mathcal{P})$ and therefore $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\varphi$ by (prop).
2. If $\varphi \equiv \mathsf{G}_\circ\psi \in \sigma_2(\mathcal{P})$ then $\mathsf{G}_\circ\psi \in pos(\mathcal{P})$ and therefore $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{G}_\circ\psi$ by (prop), from which we get $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\mathsf{G}_\circ\varphi$ with (ltl3) and (prop).

3. If $\varphi \equiv \phi \mathsf{U} \psi \in \sigma_3(\mathcal{P})$ then $\phi \mathsf{U} \psi \in pos(\mathcal{P})$ and $\mathsf{X}\phi \in pos(\mathcal{P})$ and $\mathsf{X}\psi \in neg(\mathcal{P})$ and therefore $\vdash \widehat{\mathcal{P}} \Rightarrow \phi \mathsf{U} \psi \wedge \mathsf{X}\phi \wedge \neg \mathsf{X}\psi$ by (prop), from which we get $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\phi \mathsf{U} \psi$ with (until1) and (prop).

4. If $\varphi \in \sigma_4(\mathcal{P})$ then $\mathsf{X}\varphi \in neg(\mathcal{P})$ and therefore$\vdash \widehat{\mathcal{P}} \Rightarrow \neg \mathsf{X}\varphi$ by (prop) from which we get $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\neg\varphi$ with (ltl1) and (prop).

5. if $\varphi \equiv \mathsf{G}_\circ \psi \in \sigma_5(\mathcal{P})$ then $\mathsf{G}_\circ \psi \in neg(\mathcal{P})$ and $\psi \in pos(\mathcal{P})$ and therefore $\vdash \widehat{\mathcal{P}} \Rightarrow \varphi \wedge \neg \mathsf{G}_\circ \psi$ by (prop), from which we get $\vdash \widehat{\mathcal{P}} \Rightarrow \neg \mathsf{X}\mathsf{G}_\circ \varphi$ with (ltl3) and (prop) and finally $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\neg \mathsf{G}_\circ \varphi$ with (ltl1) and (prop).

6. If $\varphi \equiv \phi \mathsf{U} \psi \in \sigma_6(\mathcal{P})$ then $\phi \mathsf{U} \psi \in neg(\mathcal{P})$ and $\mathsf{X}\phi \in pos$ and $\mathsf{X}\psi \in neg$ and therefore $\vdash \widehat{\mathcal{P}} \Rightarrow \neg(\phi \mathsf{U} \psi) \wedge \mathsf{X}\phi \wedge \neg \mathsf{X}\psi$ by (prop), from which we get $\vdash \widehat{\mathcal{P}} \Rightarrow \neg \mathsf{X}\phi \mathsf{U} \psi$ with (until1) and (prop) and finally $\vdash \widehat{\mathcal{P}} \Rightarrow \mathsf{X}\neg \phi U \psi$.

(2) Assume that $\sigma(\mathcal{P})$ is inconsistent, i.e., $\vdash \neg\widehat{\sigma(\mathcal{P})}$. Using (next) it follows that $\vdash \mathsf{X}\neg\widehat{\sigma(\mathcal{P})}$; hence also $\vdash \neg \mathsf{X}\widehat{\sigma(\mathcal{P})}$ with (ltl1) and (prop). Together with (1) we infer $\vdash \neg\widehat{\mathcal{P}}$ by (prop), implying that $\mathcal{P}$ would be inconsistent. □

According to the idea of the proof above, in order to satisfy the formulas of $pos(\mathcal{P})$ and falsify those of $neg(\mathcal{P})$ of a given consistent PNP $\mathcal{P}$, respectively, in a state, the infinite sequence

$$\mathcal{P}^*, \sigma(\mathcal{P}^*)^*, \sigma(\sigma(\mathcal{P}^*)^*)^* ...$$

should now carry the complete information about how the sub-formulas should evaluate in their state and the ones after that. There is, however, two remaining problems: for some element $\mathcal{P}_i$ of this sequence there could be a $\mathsf{G}_\circ \varphi \in neg(\mathcal{P}_i)$ which means that φ should become false in the corresponding state or in a future state. But either forced by the consistency constraint or just by having chosen a "bad" completion, $\varphi \in pos(\mathcal{P}_j)$ could hold for all elements $\mathcal{P}_j$, $j \geq i$, of the sequence. A similar obstacle arises with $\phi \mathsf{U} \psi \in pos(\mathcal{P}_i)$ where ψ should become true for some future state, but it could be the case that $\psi \in neg(\mathcal{P}_j)$ holds for all elements $\mathcal{P}_j$, $j \geq i$. In order to overcome these last difficulties we consider all possible completions in every step from one state to the next.

Formally, let $\mathcal{P}$ be a consistent and complete PNP. We define an infinite tree $\mathcal{K}_\mathcal{P}$:

- The root of $\mathcal{K}_\mathcal{P}$ is $\mathcal{P}$.
- If Q is a node of $\mathcal{K}_\mathcal{P}$ then the successor nodes of Q are all different completions of $\sigma(Q)$.

According to our remarks and results above, every node of $\mathcal{K}_\mathcal{P}$ is a consistent and complete PNP. If Q is a node then the sub-tree of $\mathcal{K}_\mathcal{P}$ with root Q is just $\mathcal{K}_\mathcal{Q}$.

Lemma 2.1.18. *Let $\mathcal{P}$ be a consistent and complete PNP*

1. $\mathcal{K}_{\mathcal{P}}$ has only finitely many different nodes $Q_1, ..., Q_n$.
2. $\vdash \bigvee_{i=1}^{n} \widehat{Q_i} \Rightarrow \mathsf{X} \bigvee_{i=1}^{n} \widehat{Q_i}$.

Proof. (1) From the definitions of σ and τ operations it follows immediately that all formulas that occur in some node of $\mathcal{K}_{\mathcal{P}}$ are sub-formulas of the formulas contained in $\mathcal{F}_{\mathcal{P}}$, of which there are only finitely many. This implies that there can be only finitely many different nodes in $\mathcal{K}_{\mathcal{P}}$.

(2) Lemma 2.1.17.1 shows that we have $\vdash \widehat{\mathcal{Q}_i} \Rightarrow \mathsf{X}\widehat{\sigma(\mathcal{Q}_i)}$ for every $i = 1, ..., n$. Let $\mathcal{Q}'_{i_1}, ..., \mathcal{Q}'_{i_m}$ be all different completions of $\sigma(\mathcal{Q}_i)$; then Lemma 2.1.13 proves $\vdash \widehat{\sigma(\mathcal{Q}_i)} \Rightarrow \bigvee_{j=1}^{m} \widehat{\mathcal{Q}'_j}$. The definition of $\mathcal{K}_{\mathcal{P}}$ implies $\mathcal{Q}'_{i_j} \in \{\mathcal{Q}_1, ..., \mathcal{Q}_n\}$; hence $\vdash \widehat{\mathcal{Q}'_{i_j}} \Rightarrow \bigvee_{k=1}^{n} \widehat{Q_k}$, for every $j = 1, ..., m$. So we get $\vdash \widehat{\sigma(\mathcal{Q}_i)} \Rightarrow \bigvee_{k=1}^{n} \widehat{Q_k}$ with the (nex) and (ltl2) and hence $\vdash \widehat{\mathcal{Q}_i} \Rightarrow \mathsf{X} \bigvee_{k=1}^{n} \widehat{Q_k}$ for every $i = 1, ..., n$. From this, assertion (2) follows with (prop). □

A finite path (from $\mathcal{P}_1$ to $\mathcal{P}_k$) in $\mathcal{K}_{\mathcal{P}}$ is a sequence $\mathcal{P}_1, ..., \mathcal{P}_k$ of nodes such that $\mathcal{P}_{i+1}$ is a successor node of $\mathcal{P}_i$ for every $i = 1, ..., k-1$. An infinite path is defined analogously.

Lemma 2.1.19. *Let $\mathcal{P}$ be a consistent and complete PNP, $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, ...$ and infinite path in $\mathcal{K}_{\mathcal{P}}$, $i \in \mathbb{N}$, and φ a formula.*

1. If $\mathsf{X}\varphi \in \mathcal{F}_{\mathcal{P}i}$ then $\mathsf{X}\varphi \in pos(\mathcal{P}_i)$ if and only if $\varphi \in pos(\mathcal{P}_{i+1})$.
2. $\mathsf{G}_\circ\varphi \in pos(\mathcal{P}_i)$ implies that $\varphi \in pos(\mathcal{P}_j)$ for every $j \geq i$

Proof. (1) Assume that $\mathsf{X}\varphi \in \mathcal{F}_{\mathcal{P}i}$. If $\mathsf{X}\varphi \in pos(\mathcal{P}_i)$ then $\varphi \in pos(\sigma(\mathcal{P}_i))$; hence $\varphi \in pos(\mathcal{P}_{i+1})$. If $\mathsf{X}\varphi \notin pos(\mathcal{P}_i)$ then $\mathsf{X}\varphi \in neg(\mathcal{P}_i)$; hence $\varphi \in neg(\sigma(\mathcal{P}_i))$, and therefore $\varphi \in pos(\mathcal{P}_{i+1})$ and $\varphi \notin pos(\mathcal{P}_{i+1})$ with Lemma 2.1.10.1.

(2) Assume that $\mathsf{G}_\circ\varphi \in pos(\mathcal{P}_i)$. Then $\varphi \in \mathcal{F}_{\mathcal{P}i}$, because of $\varphi \in \tau(\mathsf{G}_\circ\varphi)$ and the completeness of $\mathcal{P}_i$. We get $\varphi \in pos(\mathcal{P}_i)$ with Lemma 2.1.11.3 and $\vdash \mathsf{G}_\circ\varphi \Rightarrow \varphi$ which follows from (ltl3). Moreover, $\mathsf{G}_\circ\varphi \in pos(\sigma(\mathcal{P}_i))$; hence $\mathsf{G}_\circ\varphi \in pos(\mathcal{P}_{i+1})$. By induction we may conclude that $\varphi \in pos(\mathcal{P}_j)$ for every $j \geq i$. □

An infinite path in $\mathcal{K}_{\mathcal{P}}$ is just a sequence of PNPs as in our informal explanation above, However, as explained there, we have to find such path where every occurrence of some formula $\mathsf{G}_\circ\varphi$ in some $neg(\mathcal{P}_i)$ is eventually followed by a negative occurrence of φ and likewise for every occurrence of some formula $\varphi\mathsf{U}\psi$ in some $pos(\mathcal{P}_i)$is eventually followed

by a occurrence of a positive ψ . Formally, let us call an infinite path $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, ...$ in $\mathcal{K}_\mathcal{P}$ *complete* if $\mathcal{P}_0 = \mathcal{P}$ and the following condition holds for every $i \in \mathbb{N}$:

If $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{P}_i)$ then $\varphi \in neg\,(\mathcal{P}_j)$ for some $j \geq i$.

Lemma 2.1.19 and this definition will be seen to ensure the existence of a temporal structure satisfying $\widehat{\mathcal{P}}$. It remains to guarantee that a complete path really exists whenever $\mathcal{P}$ is consistent and complete.

Lemma 2.1.20. *Let $\mathcal{P}$ be a consistent and complete PNP. There is a complete path $\mathcal{K}_\mathcal{P}$.*

Proof. We first show:

(1*) If $\mathcal{Q}$ is some node of $\mathcal{K}_\mathcal{P}$ and φ is some formula such that $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q})$ then there is a node $\mathcal{Q}'$ such that $\varphi \in neg\,(\mathcal{Q}')$.

(2*) If $\mathcal{Q}$ is some node of $\mathcal{K}_\mathcal{P}$ and ψ is some formula such that $\phi\mathsf{U}\psi \in pos\,(\mathcal{Q})$ then there is a node $\mathcal{Q}'$ such that $\mathsf{X}\psi \in pos\,(\mathcal{Q}')$.

Assume that $\varphi \notin neg\,(\mathcal{Q}')$ for every node of $\mathcal{Q}'$ of $\mathcal{K}_\mathcal{P}$. Because of $\varphi \in \tau\,(\mathsf{G}_\circ\varphi)$ we then have $\varphi \in pos\,(\mathcal{Q})$ and therefore $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q}')$ for all successor nodes $\mathcal{Q}'$ of $\mathcal{Q}$ according to the construction σ. Continuing inductively, we find that $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q}')$, $\varphi \in pos\,(\mathcal{Q}')$, and hence $\vdash \widehat{\mathcal{Q}'} \Rightarrow \varphi$ for every node $\mathcal{Q}'$ of $\mathcal{K}_\mathcal{P}$. Let $\mathcal{Q}'_1, ..., \mathcal{Q}'_n$ be all nodes of $\mathcal{K}_\mathcal{P}$. Then $\vdash \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i} \Rightarrow \varphi$. Furthermore, by Lemma 2.1.18.2 we have $\vdash \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i} \Rightarrow \mathsf{X}\bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i}$; so with (ind) we obtain $\vdash \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i} \Rightarrow \mathsf{G}_\circ\varphi$. Because of $\mathcal{Q} \in \{\mathcal{Q}'_1, ..., \mathcal{Q}'_n\}$ we also have $\vdash \widehat{\mathcal{Q}} \Rightarrow \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i}$ and so we get $\vdash \widehat{\mathcal{Q}} \Rightarrow \mathsf{G}_\circ\varphi$ by (prop). Because of $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q})$, i.e., $\vdash \widehat{\mathcal{Q}} \Rightarrow \neg\mathsf{G}_\circ\varphi$, this implies $\vdash \neg\widehat{\mathcal{Q}}$ by (prop) which means that $\mathcal{Q}$ is inconsistent. This is a contradiction; thus (1*) is proved.

Assume that $\psi \notin pos\,(\mathcal{Q}')$ for every node of $\mathcal{Q}'$ of $\mathcal{K}_\mathcal{P}$. Because of $\mathsf{X}\psi \in \tau\,(\phi\mathsf{U}\psi)$ we then have $\mathsf{X}\psi \in neg\,(\mathcal{Q})$ and therefore $\phi\mathsf{U}\psi \in pos\,(\mathcal{Q}')$ for all successor nodes $\mathcal{Q}'$ of $\mathcal{Q}$ according to the construction σ. Continuing inductively, we find that $\phi\mathsf{U}\psi \in pos\,(\mathcal{Q}')$, $\mathsf{X}\psi \in neg\,(\mathcal{Q}')$, and hence $\vdash \widehat{\mathcal{Q}'} \Rightarrow \neg\mathsf{X}\psi$ for every node $\mathcal{Q}'$ of $\mathcal{K}_\mathcal{P}$. Let $\mathcal{Q}'_1, ..., \mathcal{Q}'_n$ be all nodes of $\mathcal{K}_\mathcal{P}$. Then $\vdash \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i} \Rightarrow \neg\mathsf{X}\psi$. Furthermore, by Lemma 2.1.18.2 we have $\vdash \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i} \Rightarrow \mathsf{X}\bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i}$; so with (ind) we obtain $\vdash \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i} \Rightarrow \mathsf{G}_\circ\neg\mathsf{X}\psi$. Because of $\mathcal{Q} \in \{\mathcal{Q}'_1, ..., \mathcal{Q}'_n\}$ we also have $\vdash \widehat{\mathcal{Q}} \Rightarrow \bigvee_{i=1}^{n} \widehat{\mathcal{Q}'_i}$ and so we get $\vdash \widehat{\mathcal{Q}} \Rightarrow \mathsf{G}_\circ\neg\mathsf{X}\psi$ by (prop) and since $\mathsf{G}_\circ\neg\mathsf{X}\psi \Leftrightarrow \neg\mathsf{F}\psi$ we get $\vdash \widehat{\mathcal{Q}} \Rightarrow \neg\mathsf{F}\psi$ with (prop). Furthermore we obtain $\vdash \widehat{\mathcal{Q}} \Rightarrow \neg\,(\phi\mathsf{U}\psi)$ by (until2) and (prop). Because of $\phi\mathsf{U}\psi \in pos\,(\mathcal{Q})$, i.e., $\vdash \widehat{\mathcal{Q}} \Rightarrow \phi\mathsf{U}\psi$, this implies $\vdash \neg\widehat{\mathcal{Q}}$ by (prop) which means that $\mathcal{Q}$ is inconsistent. This is a contradiction; thus (2*) is proved.

From Lemma 2.1.18.1 we know that $\mathcal{K}_\mathcal{P}$ contains only finitely many different nodes. Since $neg\,(\mathcal{Q})$ and $pos\,(\mathcal{Q})$ are finite sets of formulas for every node $\mathcal{Q}$, there can only be finitely many formulas such that $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q})$ and finitely many such that $\phi\mathsf{U}\psi \in pos\,(\mathcal{Q})$ for some node $\mathcal{Q}$ on $\mathcal{K}_\mathcal{P}$. Choose some fixed enumeration $\varphi_0, ..., \varphi_{m-1}$ of all such formulas (including both the $\mathsf{G}_\circ\varphi$ and $\phi\mathsf{U}\psi$ formulas). In order to construct a complete path in $\mathcal{K}_\mathcal{P}$ we now define a succession $\pi_0, \pi_1, ...$ of finite and non-empty paths in $\mathcal{K}_\mathcal{P}$ such that π_i is a proper prefix of π_{i+1}:

- Let $\pi_0 = \mathcal{P}$ consist only of the root $\mathcal{P}$ of $\mathcal{K}_\mathcal{P}$.
- Inductively, assume that $\pi_i = \mathcal{Q}_0, \mathcal{Q}_1, ..., \mathcal{Q}_k$ has already been defined.
  - If $\varphi_{i \bmod m}$ is of the form $\mathsf{G}_\circ\varphi$ we distinguish two cases: if $\mathsf{G}_\circ\varphi \notin neg\,(\mathcal{Q}_k)$ or $\varphi \in neg\,(\mathcal{Q}_k)$ then π_{i+1} is obtained from $\pi_i$by appending some successor node $\mathcal{Q}'$ of $\mathcal{Q}_k$ in $\mathcal{K}_\mathcal{P}$. (Lemmas 2.1.12 and 2.1.17 imply that $\mathcal{Q}_k$ has at least one successor node.)
    If $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q}_k)$ and $\varphi \notin neg\,(\mathcal{Q}_k)$ then, by (1*), $\mathcal{K}_{\mathcal{Q}_k}$contains some node $\mathcal{Q}'$ such that $\varphi \in neg\,(\mathcal{Q}')$. Choose such $\mathcal{Q}'$ (which must obviously be different from $\mathcal{Q}_k$), and let π_{i+1} be obtained by appending the path from $\mathcal{Q}_k$ to $\mathcal{Q}'$ to the path π_i.
  - If $\varphi_{i \bmod m}$ is of the form $\phi\mathsf{U}\psi$ we distinguish two cases: if $\phi\mathsf{U}\psi \notin pos\,(\mathcal{Q}_k)$ or $\mathsf{X}\psi \in pos\,(\mathcal{Q}_k)$ then π_{i+1} is obtained from $\pi_i$by appending some successor node $\mathcal{Q}'$ of $\mathcal{Q}_k$ in $\mathcal{K}_\mathcal{P}$. (Lemmas 2.1.12 and 2.1.17 imply that $\mathcal{Q}_k$ has at least one successor node.)
    If $\phi\mathsf{U}\psi \in pos\,(\mathcal{Q}_k)$ and $\mathsf{X}\psi \in neg\,(\mathcal{Q}_k)$ then, by (2*), $\mathcal{K}_{\mathcal{Q}_k}$contains some node $\mathcal{Q}'$ such that $\mathsf{X}\psi \in pos\,(\mathcal{Q}')$. Choose such $\mathcal{Q}'$ (which must obviously be different from $\mathcal{Q}_k$), and let π_{i+1} be obtained by appending the path from $\mathcal{Q}_k$ to $\mathcal{Q}'$ to the path π_i.

The succession $\pi_0, \pi_1, ...$ uniquely determines an infinite path $\pi_i = \mathcal{Q}_0, \mathcal{Q}_1, ...$ with $\mathcal{Q}_0 = \mathcal{P}$ in $\mathcal{K}_\mathcal{P}$. To see that π is complete, assume that $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q}_i)$ for some i but that $\varphi \notin neg\,(Q_{i'})$ for all $i' \geq i$. As in the proof of (1*), it follows that $\mathsf{G}_\circ\varphi \in neg\,(\mathcal{Q}_{i'})$ for every $i' \geq i$. The formula φ occurs in the enumeration of all formulas of this kind fixed above, say, as φ_l. Now choose $j \in \mathbb{N}$ such that $\pi_{j*m+l} = \mathcal{Q}_0, \mathcal{Q}_1, ..., \mathcal{Q}_k$ where $k \geq i$; in particular it follows that $\mathsf{G}_\circ\varphi_l \in neg\,(\mathcal{Q}_k)$. But the construction of π_{i+1}ensures that π_{i+1}, which is a finite prefix of π, ends with some node $\mathcal{Q}'$ such that $\varphi \equiv \varphi_l \in neg\,(\mathcal{Q}')$, and a contradiction is reached. We get an analogous contradiction for formulas $\phi U \psi$. We have thus found a complete path $\pi = \mathcal{Q}_0, \mathcal{Q}_1, ...$ in $\mathcal{K}_\mathcal{P}$. □

Now we have in fact all means for proving a theorem which is a rather trivial transcription of the desired completeness result.

Theorem 2.1.21. *(Satisfiability Theorem for Σ_{LTL}). For every consistent PNP $\mathcal{P}$, the formula $\widehat{\mathcal{P}}$ is satisfiable.*

Proof. Let $\mathcal{P}$ be a consistent PNP, $\mathcal{P}^*$ be a completion of $\mathcal{P}$, and $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, ...$ a complete path in $\mathcal{K}_{\mathcal{P}^*}$ according to Lemma 2.1.20. We define a LTL model $\eta = \langle \alpha, \lambda \rangle$ where the labellings are $\lambda_i(p) = 1$ if and only if $p \in pos(\mathcal{P}_i)$ for every $p \in \Pi$ and $\alpha_i = act$ if and only if $act \in pos(\mathcal{P}_i)$, $i \in \mathbb{N}$.
We will prove below that for every formula φ and every $i \in \mathbb{N}$:

(*) If $\varphi \in \mathcal{K}_{\mathcal{P}_i}$ then: $\lambda, i \Vdash \varphi$ if and only if $\varphi \in pos(\mathcal{P}_i)$.

Before we prove this, let us show that (*) implies the satisfiability of $\widehat{\mathcal{P}}$: because of $pos(\mathcal{P}) \subseteq pos(\mathcal{P}_0)$, $neg(\mathcal{P}) \subseteq neg(\mathcal{P}_0)$ and $pos(\mathcal{P}_0) \cap neg(\mathcal{P}_0) = \emptyset$ we get

$$\lambda, 0 \Vdash \widehat{\mathcal{P}} \text{ or expanded } \lambda,0 \Vdash \bigwedge_{\varphi \in pos(\mathcal{P})} \varphi \wedge \bigwedge_{\psi \in neg(\mathcal{P})} \neg\psi$$

from (*). In particular, $\widehat{\mathcal{P}}$ is satisfiable.
The proof of (*) runs by structural induction on the formula φ.

- $\varphi \equiv p \in \Pi$; $\lambda, i \Vdash p$ if and only $p \in pos(\mathcal{P}_i)$ by definition.
- $\varphi \equiv act \in Act$; $\lambda, i \Vdash act$ if and only $act \in pos(\mathcal{P}_i)$ by definition.
- $\varphi \equiv act \in Act$; $\lambda, i \Vdash act$ if and only if
- $\varphi \equiv \phi \Rightarrow \psi$; If $\phi \Rightarrow \psi \in \Gamma_{\mathcal{P}_i}$, then also $\phi \in \Gamma_{\mathcal{P}_i}$ and $\psi \in \Gamma_{\mathcal{P}_i}$ because $\mathcal{P}_i$ is a complete PNP, and therefore:
  $\lambda, i \Vdash \phi \Rightarrow \psi$ if and only if $\lambda, i \nVdash \phi$ or $\lambda, i \Vdash \psi$, which by the induction hypothesis results in $\phi \notin pos(\mathcal{P}_i)$ or $\psi \in pos(\mathcal{P}_i)$, which by Lemma 2.1.10.2 is results in $\phi \in neg(\mathcal{P}_i)$ or $\psi \in pos(\mathcal{P}_i)$ and by Lemma 2.1.11.2 leads to $\phi \Rightarrow \psi \in pos(\mathcal{P}_i)$.
- $\varphi \equiv \mathsf{X}\psi$; From $\mathsf{X}\psi \in \Gamma_{\mathcal{P}_i}$ we obtain $\varphi \in \Gamma_{\mathcal{P}_{i+1}}$ and therefore:
  $\lambda, i \Vdash \mathsf{X}\psi$ by definition $\lambda, i+1 \Vdash \psi$ and by ind. hyp. $\varphi \in pos(\mathcal{P}_{i+1})$ which by Lemma 2.1.19.1 is $\mathsf{X}\psi \in pos(\mathcal{P}_i)$.
- $\varphi \equiv \phi \mathsf{U} \psi$; If $\phi \mathsf{U} \psi \in pos(\mathcal{P}_i)$ from the definition of a complete path and Lemma 2.1.10.1 ensure $\mathsf{X}\psi \in pos(\mathcal{P}_j)$ for some $j \geq i$. Since it may not be so obvious, $\mathsf{X}\phi \in pos(\mathcal{P}_k)$ for all $i \leq k < j$ from (prop) and (until1) and form the consistency of

$\mathcal{P}$. By the induction hypothesis we get $\lambda, j+1 \Vdash \psi$ and $\lambda, k+1 \Vdash \phi$ for $j+1 > i$ and $i \leq k+1 < j+1$ and by definition $\lambda, i \Vdash \phi \mathsf{U} \psi$.

□

Before we finally deduce our main result from this theorem we will still mention that a close look at its proof provides another interesting corollary called *finite model property* (of LTL):

Corollary 2.1.22. *Every satisfiable formula is satisfiable by a temporal structure which has only finitely many different states.*

To see this fact, assume that a formula φ is satisfiable. From the definition it follows that $\neg\neg\varphi$ is satisfiable; hence $\neg\varphi$ is not valid and not derivable in Σ_{LTL} by Theorem 2.1.2. So, by definition, the PNP $(\{\varphi\}, \emptyset)$ is consistent and therefore φ is satisfiable by a temporal structure λ according to (the proof of) Theorem 2.1.13. By construction and Lemma 2.1.18.1, λ has only finitely many different states.

Theorem 2.1.23. *(Weak Completeness Theorem for Σ_{LTL}). Σ_{LTL} is weakly complete, i.e., for every finite set Γ of formulas and formulas φ, if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$. In particular, if $\vDash \varphi$ then $\vdash \varphi$.*

Proof. We prove the claim first for $\Gamma = \emptyset$: if $\vDash \varphi$ then $\neg\varphi$ is not satisfiable and hence the PNP $(\emptyset, \{\varphi\})$ is inconsistent by theorem 2.1.13. This means $\vdash \neg\neg\varphi$ by definition and implies $\vdash \varphi$ using (prop).

Let now $\Gamma = \{\varphi_1, ..., \varphi_n\} \neq \emptyset$. We then have $\Gamma \vDash \varphi$ and this implies

$\varphi_1 ... \varphi_{n-1} \vDash \mathsf{G}_\circ \varphi_n \Rightarrow \varphi$	with Theorem 2.1.7
$\vdots$	
$\vDash \mathsf{G}_\circ \varphi_1 \Rightarrow (\mathsf{G}_\circ \varphi_2 \Rightarrow ... \Rightarrow (\mathsf{G}_\circ \varphi_n \Rightarrow \varphi) ...)$	with Theorem 2.1.7
$\vdash \mathsf{G}_\circ \varphi_1 \Rightarrow (\mathsf{G}_\circ \varphi_2 \Rightarrow ... \Rightarrow (\mathsf{G}_\circ \varphi_n \Rightarrow \varphi) ...)$	with the result proved above
$\varphi_1 \vdash \mathsf{G}_\circ \varphi_2 \Rightarrow (\mathsf{G}_\circ \varphi_3 \Rightarrow ... \Rightarrow (\mathsf{G}_\circ \varphi_n \Rightarrow \varphi) ...)$	with Theorem 2.1.8
$\vdots$	
$\Gamma \vdash \varphi$	with Theorem 2.1.8

□

Summarizing, we know from soundness and the Weak Completeness Theorems that

$$\Gamma \vDash \varphi \text{ if and only if } \Gamma \vdash \varphi \text{for finite } \Gamma$$

in particular that

$$\vDash \varphi \text{ if and only if } \vdash \varphi.$$

2.1.4 Decidability

We will present a short description on weather the validity of LTL is decidable. In fact we will point for a method to decide the *satisfiability problem* used in [3].

The decision procedure for satisfiability would be based on the same idea of the proof of the Weak Completeness Theorem 2.1.23. It's essence was to construct a satisfying temporal structure for any finite and consistent set of formulas. The tree of PNPs and the operations involved to construct that tree were carefully chosen to ensure there were only finitely many different nodes. The information contained in the infinite tree can therefore be represented in a finite graph, a *tableau*. We would just need to identify the identical nodes and represent cycles between them. Eventually there will be nodes that are closed, i.e., that result in some sort of contradiction. This will lead to the notion of a tableau being successful.

Ultimately it must be shown that a tableau for a PNP $\mathcal{P}$ is successful if and only if $\widehat{\mathcal{P}}$ is satisfiable. We won't show this.

Anyway, the mentioned result would provide an algorithmic decision procedure for satisfiability and allow us to prove that *The satisfiability and validity problems for* $\mathcal{L}_{\mathsf{LTL}}$ *are decidable.*

2.2 Distributed Temporal Logic

The distributed temporal logic DTL is a logic for reasoning about temporal properties of distributed systems. The agents that constitute the system have a sequential execution, and interact with each other by means of synchronous communication. The definition of a DTL model and its semantics were heavily based on [5].

For the previous presented language of LTL we were only concerned with the development of a simple situation as time progressed. Now DTL introduces several agents in a system. The situation expands to include the intricacies of all the agents, how they each evolve and affect the others as time progresses .

2.2.1 Syntax and Semantics

The syntax of distributed temporal logic is defined over a distributed signature $\Sigma = \langle Id, \{\Pi_i\}_{i\in Id}, \{Act_i\}_{i\in Id}\rangle$ of a system, where Id is a finite set of agents and, for each $i \in Id$, Π_i is a set of local state propositions and Act_i is a set of local actions. The language $\mathcal{L}_{\mathrm{DTL}}$ of distributed temporal logic DTL is defined by:

$$\mathcal{L} ::= @_i\,[\mathcal{L}_i] \mid \mathcal{L} \Rightarrow \mathcal{L} \mid \neg\mathcal{L}$$

for $i \in Id$, where the local languages $\mathcal{L}_i$ are defined as

$$\mathcal{L}_i ::= \Pi_i \mid Act_i \mid \neg\mathcal{L}_i \mid \mathcal{L}_i \Rightarrow \mathcal{L}_i \mid \mathcal{L}_i \mathsf{U} \mathcal{L}_i \mid ©_j\,[\mathcal{L}_j] \mid *$$

with $j \in Id$.

The global formula $@_i[\varphi]$ means that the formula φ holds at current local state of agent i. A formula of the form $©_j\,[\varphi] \in \mathcal{L}_i$ is called a communication formula, and means that agent i has just communicated with agent j for whom φ holds.

To extend our concept of temporal structure lets consider life-cycles.

A local life-cycle of an agent i is a discrete countable total order $\lambda_i = \langle E_i, \leq_i \rangle$ where E_i is the set of local events and $\leq_i$ the local order of causality. We define the corresponding local successor relation $\rightarrow_i \subseteq E_i \times E_i$ to be the relation such that $e \rightarrow_i e'$ if $e \leq_i e'$ and there is no e'' such that $e \leq_i e'' \leq_i e'$. As a consequence, we have that $\leq_i = \rightarrow_i^*$, i.e. $\leq_i$ is the reflexive and transitive closure of $\rightarrow_i$.

A distributed life-cycle is a family $\lambda = \{\lambda_i\}_{i \in Id}$ of local life-cycles such that $\leq = \left(\bigcup_{i \in Id} \leq_i\right)^*$ defines a partial order of global causality on the set of all events $E = \bigcup_{i \in Id} E_i$. Note that communication is modeled by event sharing and thus for some event e we may have $e \in E_i \cap E_j$, for $i \neq j$.

A local state of an agent i is a finite set $\xi \subseteq E_i$ such that if $e \leq_i e'$ and $e' \in \xi$, then $e \in \xi$. The set Ξ_i of all local states of agent i is totally ordered by inclusion and has $\emptyset$ as the minimal element. Each non-empty local state ξ of agent i is reached, by the occurrence of the event we call $last_i(\xi)$, from the local state $\xi \setminus \{last_i(\xi)\}$. The local states of each agent are totally ordered as a consequence of the total order on local events. Since they are discrete and well-founded, we enumerate them as follows: $\emptyset$ is the 0^{th} state; $\{e\}$, where e is the minimum of $\langle E_i, \leq_i \rangle$, is the first state; and in general, if ξ is the k^{th}state of agent i and $last_i(\xi) \rightarrow_i e'$, then $\xi \cup \{e'\}$ is the $(k+1)^{th}$ state of agent i. We denote ξ_i^k the k^{th} state of agent i. Note that $\xi_i^0 = \emptyset$ is the initial state and ξ_i^k is the state reached from the initial state after the occurrence of the first k events, so two states are related by inclusion $\xi^m \subseteq \xi^n$ if and only if $m \leq n$. Also, ξ_i^k is the only state of agent i that contains k elements, i.e., where $\left|\xi_i^k\right| = k$. Given $e \in E_i$, observe that $(e \downarrow i) = \{e' \in E_i \mid e' \leq_i e\}$ is always a local state.

An interpretation structure $\eta = \langle \lambda, \sigma, \alpha \rangle$ consists of a distributed life-cycle λ and families $\sigma = \{\sigma_i\}_{i \in Id}$ and $\alpha = \{\alpha_i\}_{i \in Id}$ of labeling functions. For each $i \in Id$, $\sigma_i : \Xi_i \rightarrow p(\Pi_i)$ associates a set of local state propositions to each local state, where the function $\alpha_i : Ev_i \rightarrow Act_i$ associates a local action to each event. We denote $\langle \lambda_i, \sigma_i, \alpha_i \rangle$ by η_i and define the *global satisfaction relation* by

- $\eta \Vdash_{\mathsf{DTL}} @_i [\varphi]$ if and only if $\eta_i \Vdash_i \varphi$ if and only if $\eta_i, \xi_i \Vdash_i \varphi$ for every $\xi_i \in \Xi_i$;
- $\eta \Vdash_{\mathsf{DTL}} \neg\varphi$ if and only if $\eta \nVdash_{\mathsf{DTL}} \varphi$;
- $\eta \Vdash_{\mathsf{DTL}} \varphi \Rightarrow \psi$ if and only if $\eta \nVdash_{\mathsf{DTL}} \varphi$ or $\eta \Vdash_{\mathsf{DTL}} \psi$.

The local satisfaction relations at local states is defined by

- $\eta_i, \xi_i \Vdash_i p$ if $p \in \sigma_i(\xi)$;
- $\eta, \xi_i \Vdash_i act$ if $\xi_i \neq \emptyset$ and $\alpha_i(last(\xi_i)) = act$;
- $\eta_i, \xi_i \Vdash_i \neg\varphi$ if $\eta_i, \xi_i \nVdash_i \varphi$;
- $\eta_i, \xi_i \Vdash_i \varphi \Rightarrow \psi$ if $\eta_i, \xi_i \nVdash_i \varphi$ or $\eta_i, \xi_i \Vdash_i \psi$;
- $\eta_i, \xi_i \Vdash_i \varphi \mathsf{U} \psi$ if there exists $\xi_i' \in \Xi_i$ such that $|\xi_i| < |\xi_i'|$ and $\eta_i, \xi_i' \Vdash_i \psi$ and $\eta_i, \xi_i'' \Vdash_i \varphi$ for every $|\xi_i| < |\xi_i''| < |\xi_i'|$
- $\eta_i, \xi_i \Vdash_i ©_i [\varphi]$ if $|\xi_i| > 0$ and $last_i(\xi_i) \in E_j$, and $\mu_j, (last_i(\xi_i) \downarrow j) \Vdash_j \varphi$;
- $\eta_i, \xi_i \Vdash_i *$ if $\xi_i = \emptyset$.

We define that $\Gamma \vDash_{\mathsf{DTL}} \varphi$ if any model η verifies $\eta \Vdash_{\mathsf{DTL}} \varphi$ whenever $\eta \Vdash_{\mathsf{DTL}} \gamma$ for every $\gamma \in \Gamma$. Also, local semantic consequence is defined as $\Gamma \vDash_i \varphi$ if any model η verifies $\eta_i \Vdash_i \varphi$ whenever $\eta_i \Vdash_i \gamma$ for every $\gamma \in \Gamma$ for some agent $i \in Id$. A formula φ is locally valid, $\vDash_i \varphi$ or equivalently $\vDash @_i [\varphi]$ if $\eta_i \Vdash \varphi$ for any model η for some agent $i \in Id$.

When the agent is understood from the context, we will write $\Vdash$ instead of $\Vdash_i$.

2.2.2 Axiomatization

We present an axiomatiozation for DTL, mainly by adjusting the axioms we already have for LTL to the new notation. The previous LTL axioms are valid only locally, hence the need to add the symbol $@_i$ to now assign an agent and shift them to a global level. We don't prove any soundness or completeness results. For results on those subjects, check [5] where it is provided a sound and complete labeled tableaux system for this logic.

Axioms

For every $i \in Id$ we have the following set of axioms:

dtl1.	$@_i [\neg \mathsf{X}\varphi \Leftrightarrow \mathsf{X}\neg\varphi]$
dtl2.	$@_i [\mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi)]$
dtl3.	$@_i [\mathsf{G}_\circ\varphi \Rightarrow \varphi \wedge \mathsf{X}\mathsf{G}_\circ\varphi]$
untild1.	$@_i [\varphi \mathsf{U} \psi \Leftrightarrow \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U}\psi)]$
untild2.	$@_i [\varphi \mathsf{U} \psi \Rightarrow \mathsf{X}\mathsf{F}_\circ\psi]$
initd1.	$@_i [\mathsf{X}\neg *]$

Rules of inference

mpd.	$@_i [\varphi], @_i [\varphi \Rightarrow \psi] \vdash @_i [\psi]$
nexd.	$@_i [\varphi] \vdash @_i [\mathsf{X}\varphi]$
indd.	$@_i [\varphi \Rightarrow \psi], @_i [\varphi \Rightarrow \mathsf{X}\varphi] \vdash @_i [\varphi \Rightarrow \mathsf{G}_\circ\psi]$
initd2.	$@_i [* \Rightarrow \mathsf{G}_\circ\varphi] \vdash @_i [\varphi]$

Future sections will be based of DTL language and will have some examples where it is more practical to deal with an anchored viewpoint. Therefore we present some results that will be useful for those sections.

Lemma 2.2.1. *Given any* DTL *model* η *and any agent* i *it is true that* $\eta \Vdash @_i [* \Rightarrow \varphi]$ *if and only if* $\eta, \xi_i^0 \Vdash \varphi$.

Proof. Suppose first that $\eta, \xi_i^0 \Vdash \varphi$ and note that also $\eta, \xi_i^0 \Vdash *$ since by definition $\xi^0 = \emptyset$. With some propositional calculus results we can conclude that $\eta, \xi_i^0 \Vdash * \Rightarrow \varphi$.

Now note that for any $\xi_i \neq \emptyset$ it is true that $\eta, \xi_i \Vdash * \Rightarrow \varphi$ since the antecedent is always false, that is $\eta, \xi_i \nVdash *$. In conclusion, for any ξ_i we have $\eta, \xi_i \Vdash * \Rightarrow \varphi$ and consequentially $\eta \Vdash @_i [* \Rightarrow \varphi]$.
Conversely suppose $\eta \Vdash @_i [* \Rightarrow \varphi]$. This is by definition, $\eta, \xi_i \Vdash * \Rightarrow \varphi$ for every ξ_i, in particular for $\xi_i = \emptyset$. Since $\eta, \xi_i^0 \Vdash *$ and $\eta, \xi_i^0 \Vdash * \Rightarrow \varphi$ we then get that $\eta, \xi_i^0 \Vdash \varphi$. □

Lemma 2.2.2. *Given any* DTL *model* η *and any agent* $i \in Id$ *we have that if* $\eta, \xi_i^k \Vdash \varphi$ *for some* ξ_i^k *then* $\eta \Vdash @_i [* \Rightarrow \mathsf{F}\varphi]$.

Proof. Note that $\eta \Vdash @_i [* \Rightarrow \mathsf{F}\varphi]$ if and only if $\eta, \xi_i^0 \Vdash \mathsf{F}\varphi$ by the previous Lemma. The definition is that $\eta, \xi^0 \Vdash \mathsf{F}\varphi$ if there is $\xi > \xi^0$ such that $\eta, \xi \Vdash \varphi$, which is exactly our case, hence the result. □

As we stated before, a DTL system is an expansion from LTL, whereas we had one agent, we now have several that communicate with each other. So if we restrain a DTL model to only one agent, its behavior would be of a LTL model.

To show this we have to deal with the communication formulas that are not present in LTL. The simple way is to include them in the set of propositional formulas as if just another formula. So consider the set $\Pi_{©_i} = \Pi_i \bigcup \left\{ ©_j [\varphi] : \varphi \in \mathcal{L}_j \right\}$ for some agent i.

To translate one model from a language into the other we will use a function $\delta : \{\eta_{\mathsf{DTL}} : \eta_{\mathsf{DTL}} \text{ is a } \mathsf{DTL} \text{ model}\} \to \{\eta_{\mathsf{LTL}} : \eta_{\mathsf{LTL}} \text{ is a } \mathsf{LTL} \text{ model}\}$ that maps $\delta : \langle \lambda, \alpha, \sigma \rangle \longmapsto \langle \lambda', \alpha \rangle$.

The local life-cycle λ in the DTL model has a correspondent set of states $\mathbb{N}$ in the LTL model. The first DTL state, ξ^0, will be the first LTL state 0, and the state ξ' such that $\xi^0 \to_i \xi'$ will be state 1; state ξ''such that $\xi' \to_i \xi''$ will be state 2 and so on.

The crucial detail here are the labeling functions. We define $\lambda' : \mathbb{N} \to 2^{\Pi_{©_i}}$ such that $\lambda'(k) = \left\{ p : p \in \sigma(\xi^k) \right\} \cup \left\{ ©_j[\varphi] : \eta, \xi^k \Vdash ©_j[\varphi] \right\}$. This way the communication formulas are included in $\mathcal{L}_{\mathsf{LTL}}$as if they were another propositional formula.

The actions suffer no alteration from one model to the other.

Proposition 2.2.3. *Given $\varphi \in \mathsf{LTL}_{©}$ and a DTL model η then $\eta \Vdash_{\mathsf{DTL}} \varphi$ if and only if $\delta(\eta) \Vdash_{\mathsf{LTL}_{©}} \varphi$.*

Proof. The proof follows by induction on the the structure of a local formula φ. Let i be an agent and ξ_i any local state for that agent.

For the basis of induction let φ be a propositional formula $p \in \Pi_{©_i}$. Starting from $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} \beta(p)$ which is the same as $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} p$ we know that $p \in \sigma_i\left(\xi^k\right)$ and by definition $p \in \lambda'(k)$ so we have $\delta(\eta), k \Vdash_{\mathsf{LTL}_{©}} p$. Also if $p \in \lambda'(k)$, then $p \in \sigma_i\left(\xi^k\right)$, and so if $\delta(\eta), k \Vdash_{\mathsf{LTL}_{©}} p$ then $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} p$.

If φ is $p \in \Pi_{©_i}$ but of the form $©_j[\varphi]$. Then $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} ©_j[\varphi]$ and by definition $©_j[\varphi] \in \lambda'(k)$ so $\delta(\eta), k \Vdash_{\mathsf{LTL}_{©}} ©_j[\varphi]$. Also if $©_j[\varphi] \in \lambda'(k)$, then $\eta, \xi^k \Vdash ©_j[\varphi]$ by definition, and so if $\delta(\eta), k \Vdash_{\mathsf{LTL}_{©}} ©_j[\varphi]$ then $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} ©_j[\varphi]$.

Also if φ is $act \in Act_i$ then $\eta, \xi_i^n \Vdash_{\mathsf{DTL}} act$ if and only if $\delta(\eta), k \Vdash_{\mathsf{DTL}} act$ since $\delta(\alpha) = \alpha$.

For the induction step consider that:

- φ is $\neg\varphi$. $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} \neg\varphi$ then by definition $\eta, \xi_i^k \nVdash_{\mathsf{DTL}} \varphi$ if and only if by induction hypothesis $\delta(\eta), k \nVdash_{\mathsf{LTL}_{©}} \varphi$ that is $\delta(\eta), k \Vdash_{\mathsf{LTL}_{©}} \neg\varphi$.

- φ is $\varphi \Rightarrow \psi$. Supposing that $\eta, \xi_i^k \nVdash_{\mathsf{DTL}} \varphi \Rightarrow \psi$, it follows that $\eta, \xi_i^k \Vdash_{\mathsf{DTL}} \varphi$ and $\eta, \xi_i^k \nVdash_{\mathsf{DTL}} \psi$, so by induction hypothesis and the previous step, $\eta, k \Vdash_{\mathsf{DTL}} \varphi$ and $\delta(\eta), k \nVdash_{\mathsf{DTL}} \psi$, yielding that $\delta(\eta), k \nVdash_{\mathsf{LTL}} \varphi \Rightarrow \psi$.

- φ is $\varphi \mathsf{U} \psi$. The definition of $\eta, \xi_i^k \Vdash_i \varphi \mathsf{U} \psi$ is that there is $\xi_i^n \in \Xi_i$ such that $\xi^k \subset \xi^n$ and $\eta, \xi_i^n \Vdash_i \psi$ and $\eta, \xi_i^m \Vdash_i \varphi$ for every ξ^m such that $\xi^k \subset \xi^m \subset \xi^n$. Note that by definition ξ^x is the state reached after the occurrence of x events, and so, $k < m < n$. So by induction hypothesis we state that $\delta(\eta), n \Vdash_i \psi$ and $\delta(\eta), m \Vdash_i \varphi$ and since $k < m < n$, by definition $\delta(\eta), k \Vdash \varphi \mathsf{U} \psi$. $\square$

Chapter 3

Probabilistic temporal logic

Probabilistic thinking has been seen as a rather useful tool in the fields of Computer Science and Artificial Intelligence. Either to generate verbal explanations when the reasoning is numeric, for decision making in unknown environments in AI or for modeling the non-deterministic phenomena, as run time for algorithms or queuing requests in a database.

Probability comes as a way to express uncertain outcomes. We may argue that we have, for example, the sometime operator F which gives some uncertainty, but it's use is limited as it won't allow us to express a great deal of ideas. We would like to express information of the type "the probability of sometime in the future φ being true is s".

In this chapter we propose to extend the temporal logics presented previously with a probabilistic dimension. In doing so we adopt an exogenous approach following the ideas presented in [25, 27].

3.1 Probabilistic linear temporal logic

We start by enriching the language of LTL to include a probability operator. The approach is to only consider the case of probability in classical formulas. This will allow us to express simple statements of the form "tomorrow the probability that φ is true is at least s". What we would like to express, but cannot with this approach, is the statement "the probability that tomorrow φ is true is at least s". The two statements are very similar, so the limitation seems acceptable and at first glance we don't seem to be losing too much expressive power.

The probability is given at a local level, with the advantage of having one probability function for each time state resulting in that the probabilities of a state are independent of the probabilities of previous and future states. Information can be transferred from one state to another via temporal formulas. Note that temporal formulas represent classical formulas that are to be evaluated at some other time state. We don't give probability to

temporal formulas and can use them to carry probabilities to other states.

3.1.1 Syntax and semantics

We start of by giving the language of a classical propositional logic $\mathcal{C}$ over the set Π of propositional symbols, defined by the grammar

$$\mathcal{C} ::= \Pi \mid \neg\mathcal{C} \mid \mathcal{C} \Rightarrow \mathcal{C}.$$

The alphabet of a language $\mathcal{L}_{\mathsf{pLTL}}$ for Propositional Linear Temporal Logic $\mathcal{L}_{\mathsf{pLTL}}$ is defined by:

$$\mathcal{L}_{\mathsf{pLTL}} ::= act \mid \textstyle\int_{\geq s}\mathcal{C} \mid \mathcal{L}_{\mathsf{pLTL}} \Rightarrow \mathcal{L}_{\mathsf{pLTL}} \mid \neg\mathcal{L}_{\mathsf{pLTL}} \mid \mathcal{L}_{\mathsf{pLTL}}\mathsf{U}\mathcal{L}_{\mathsf{pLTL}}$$

where $\left\{\int_{\geq s}\right\}_{s\in[0,1]_{\mathbb{Q}}}$ is a family of unary probabilistic operators where $[0,1]_Q$ is the set of rational numbers from the interval $[0,1]$. All previous abbreviations are still in use with the addition of the following: $\int_{\geq s}$

$$\begin{array}{lll}
\text{D1.} & \int_{<s} p & \equiv \neg \int_{\geq s} p \\
\text{D2.} & \int_{\leq s} p & \equiv \int_{\geq 1-s} \neg p \\
\text{D3.} & \int_{>s} p & \equiv \neg \int_{\geq 1-s} \neg p \\
\text{D4.} & \int_{=s} p & \equiv \int_{\geq s} p \wedge \int_{\leq} p
\end{array}$$

We use the operator $\int$ for probability. The formula $\int_{\geq s} p$ reads "the probability of p is at least s".

Let $\mathcal{F}$ be a set of probability spaces $\left(\Pi, 2^{\Pi}, \mu_k\right)$ where Π as the sample space and 2^{Π} the σ-algebra of all subsets of Π. The probability functions are defined by $\mu_k : 2^{\Pi} \to [0,1]$, where 2^{Π} is the set of valuations for the propositional constants. We now define the labeling function $\lambda : \mathbb{N} \to \mathcal{F}$ that assigns to each state a probability space.

By associating a probability space to each state we obtain the pretended transposition from LTL to pLTL. Recall that the corresponding labeling function λ on LTL would match a valuation to each state. Now on pLTL it matches a probability on valuations. The result is that, instead of a valuation in a state, we will have the probability of each possible valuation for that state.

Example 3.1.1. Suppose that $\Pi = \{p\}$ and $Act = \{\mathsf{nil}\}$. If we are only concerned with state k, there are only two types of LTL models η. Either $\eta, k \Vdash p$ or $\eta, k \nVdash p$ for $k \in \mathbb{N}$, that is, there's only two valuations, $\lambda_k(p) = 1$ or $\lambda_k(p) = 0$.

A pLTL model η' will give a probability to each of these valuations, for example $\mu_k(\lambda_k(p) = 1) = s$ and $\mu_k(\lambda_k(p) = 0) = 1 - s$.

We will no longer be concerned whether p is true or false in state k, what we want to know is the probability of p being true or false. The expressions $\eta, k \Vdash p$ are replaced by $\eta', k \Vdash \int_{\geq s} p$. A model will satisfy "the probability of p being true is s" instead of satisfying "p". □

In the example we limited the set of propositional constants to have only one symbol. In the general case where Π has more than one element, there will be more than one valuation v that satisfies a propositional constant p. What this means is that we will be measuring sets of valuations that satisfy p. The sets will be represented by $\{v : v \Vdash_{\mathcal{C}} p\}$.

Example 3.1.2. Suppose $\Pi = \{p, q\}$ and $Act = \{\mathsf{nil}\}$. For a state $k \in \mathbb{N}$ there are four possible valuations v_i for the pair (p, q): $v_1 = (0, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 0)$, $v_4 = (1, 1)$.

For instance, the set $\{v : v \Vdash p\}$ is represented by $\{v_3, v_4\}$. Suppose we have a probability for $\{v : v \Vdash p\}$, e.g. $\mu_k(\{v : v \Vdash p\}) = s$. By writing this we mean that the set of valuations in state k for which p is true has probability s. □

A pLTL model $\eta = \langle \lambda, \alpha \rangle$ will consist of two labeling functions, $\lambda$and α, where the first was defined above, and $\alpha : \mathbb{N} \to Act$ assigns an action to each state, as for plain LTL, wihout probabilities.

We define the satisfaction relation for pLTL. The *global satisfaction* relation is defined as $\eta \Vdash \varphi$ if and only if $\eta, k \Vdash \varphi$ for all $k \in \mathbb{N}$.

Local satisfaction is defined by:

- $\eta, k \Vdash \int_{\geq s} p$ if $\mu_k(\{v : v \Vdash_{\mathcal{C}} p\}) \geq s$;
- $\eta, k \Vdash act$ if $\alpha(k) = act$;
- $\eta, k \Vdash \neg\varphi$ if $\eta, k \nVdash \varphi$;
- $\eta, k \Vdash \varphi \mathsf{U} \psi$ if there is $k'' \in \mathbb{N}$ with $k < k''$ such that $\eta, k'' \Vdash \psi$ and $\eta, k' \Vdash \varphi$ for every k' with $k < k' < k''$;

Entailment is defined as $\Gamma \vDash \varphi$ if for every model pLTLη verifies $\eta \Vdash \varphi$ whenever $\eta \Vdash \gamma$ for $\gamma \in \Gamma$.

Lemma 3.1.3. *We now prove the probability abreviatures.*

D1. $\int_{<s} p \equiv \neg \int_{\geq s} p$

D2. $\int_{\leq s} p \equiv \int_{\geq 1-s} \neg p$

D3. $\int_{>s} p \equiv \neg \int_{\geq 1-s} \neg p$

D4. $\int_{=s} p \equiv \int_{\geq s} p \wedge \int_{\leq} p$

Analougously we could obtain D1': $\int_{>s} p \equiv \neg \int_{\leq s} p$, *and also* D2' *and* D3'.

Proof. Let η pLTL model.

D1. Suppose that $\eta \Vdash \int_{<s} p$. Then for all $k \in \mathbb{N}$ $\eta, k \Vdash \int_{<s} p$ if and only if $\mu_k(\{v : v \Vdash p\}) < s$. Consequently, $\mu_k(\{v : v \Vdash p\}) \geq s$ is not true, and we get $\eta, k \nVdash \int_{\geq s} p$ which results in $\eta, k \Vdash \neg \int_{\geq s} p$ for all k

D2. Suppose that $\eta \Vdash \int_{\leq s} p$. Then for all $k \in \mathbb{N}$ $\eta, k \Vdash \int_{\leq s} p$ if $\mu_k(\{v : v \Vdash_{\mathcal{C}} p\}) \leq s$, which is true if and only if $\mu_k(\{v : v \Vdash \neg p\}) \geq 1 - s$ since $\{v : v \Vdash_{\mathcal{C}} p\}$ is the complementary set to $\{v : v \Vdash \neg p\}$. Therefore we get that $\eta, k \Vdash \int_{\geq 1-s} \neg p$ for all k.

D3. Suppose that $\eta \Vdash \int_{>s} p$. Then for all $k \in \mathbb{N}$ $\eta, k \Vdash \int_{>s} p$. Using D1' we get $\eta, k \Vdash \neg \int_{\leq s} p$ which is $\eta, k \nVdash \int_{\leq s} p$ and it is false that $\mu(\{v : v \Vdash p\}) \leq s$. The complementary of this set is $\mu(\{v : v \Vdash \neg p\})$ so it must also be false that $\mu(\{v : v \Vdash \neg p\}) \geq 1 - s$, leading to $\eta, k \nVdash \int_{\geq 1-s} \neg p$ resulting in $\eta, k \Vdash \neg \int_{\geq 1-s} \neg p$ for all k.

D4. Suppose $\eta \Vdash \int_{\geq s} p \wedge \int_{\leq} p$. Then for all $k \in \mathbb{N}$ $\eta, k \Vdash \int_{\geq s} p$ and $\eta, k \Vdash \int_{\leq} p$, which means that $\mu(\{v : v \Vdash p\}) \geq s$ and $\mu(\{v : v \Vdash p\}) \leq s$ which leads to $\mu(\{v : v \Vdash p\}) = s$ and $\eta, k \Vdash \int_{=s} p$ for all k

□

3.1.2 Axiomatization

The axiomatization system defined below was based on [30].

Axioms

C1. All tautologically valid formulas

C2. $\neg \mathsf{X}\varphi \Leftrightarrow \mathsf{X}\neg\varphi$

C3. $\mathsf{X}(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{X}\varphi \Rightarrow \mathsf{X}\psi)$

C4. $\varphi \mathsf{U} \psi \Leftrightarrow \mathsf{X}\psi \vee \mathsf{X}(\varphi \wedge \varphi \mathsf{U} \psi)$

C5. $\varphi \mathsf{U} \psi \Rightarrow \mathsf{F}\psi$

C6. $\mathsf{X}\neg *$

C7. $\int_{\geq 0} \varphi$, $\varphi \in \Pi$

C8. $\int_{\leq s} \varphi \Rightarrow \int_{<t} \varphi$, $t > s$, $\varphi \in \Pi$

C9. $\int_{<s} \varphi \Rightarrow \int_{\leq s} \varphi$, $\varphi \in \Pi$

C10. $\left(\int_{\geq s} \varphi \wedge \int_{\geq r} \psi \wedge \int_{\geq 1} (\neg\varphi \vee \neg\psi)\right) \Rightarrow \int_{\geq min(1,s+r)} (\varphi \vee \psi)$, $\varphi, \psi \in \Pi$

C11. $\left(\int_{\leq s} \varphi \wedge \int_{<r} \psi\right) \Rightarrow \int_{<s+r} (\varphi \vee \psi)$, $s + r \leq 1$, $\varphi, \psi \in \Pi$

Rules of inference

R1. $\varphi, \varphi \Rightarrow \psi \vdash \psi$

R2. $\varphi \vdash \mathsf{X}\varphi$

R3. $\varphi \Rightarrow \psi, \varphi \Rightarrow \mathsf{X}\varphi \vdash \varphi \Rightarrow \mathsf{G}_{\circ}\psi$

R4. $* \Rightarrow \mathsf{G}_{\circ}\varphi \vdash \varphi$

Note that the axiom system can be divided into three parts dealing with propositional, temporal and probabilistic reasoning, respectively.

Axioms C1-c6 were already covered in LTL, dealing with the classical propositional logic, and the basic *next* and *until* operators of temporal logic.

The axioms C7-C11 take the probabilistic aspect of pLTL. The axiom C7 announces that every formula is satisfied by a set of valuations of measure at least 0.

Axioms C8 and C9 represent some inequality properties, and are equivalent to (C8') $\int_{\geq t}\varphi \Rightarrow \int_{>s}\varphi$, $t > s$, $\varphi \in \Pi$ and (C9') $\int_{>s}\varphi \Rightarrow \int_{\geq s}\varphi$, $\varphi \in \Pi$.

The axioms C10 and C11 correspond to the finite addictivity of probabilities. The rules R1 and R2 are Modus Ponens and Necessitation, respectively.

Theorem 3.1.4. *Let ϕ be a formula and Γ a set of formulas. If $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$.*

Proof. The proof runs by induction on the derivation of ϕ from Γ. Let η be an arbitrary model.

C7 ϕ is $\int_{\geq 0}\psi$, $\psi \in \Pi$. By definition, $\eta \Vdash \int_{\geq 0}\varphi$ if for all $k \in \mathbb{N}$ we have $\eta, k \Vdash \int_{\geq 0}\varphi$ and this is by local satisfaction definition $\mu(\{v : v \Vdash_{\mathcal{C}} \varphi\}) \geq 0$. This is always true since the probability measure is defined as $\mu : 2^{\Pi} \to [0, 1]$.

C8 ϕ is $\int_{\leq s}\varphi \Rightarrow \int_{<t}\varphi$, $t > s$, $\varphi \in \Pi$. Suppose by contradiction that $\eta \nVdash \int_{\leq s}\varphi \Rightarrow \int_{<t}\varphi$. This means that there's some $k \in \mathbb{N}$ such that $\eta, k \nVdash \int_{\leq s}\varphi \Rightarrow \int_{<t}\varphi$, and this is by definition $\eta, k \Vdash \int_{\leq s}\varphi$ and $\eta, k \nVdash \int_{<t}\varphi$. This is by definition$\mu(\{\{v : v \Vdash_{\mathcal{C}} \varphi\}\}) \leq s$ and $\mu(\{\{v : v \Vdash_{\mathcal{C}} \varphi\}\}) > t$. This leads to a contradiction since $t > s$. Therefore for every k we have $\eta, k \Vdash \int_{\leq s}\varphi \Rightarrow \int_{<t}\varphi$

C9 ϕ is $\int_{<s}\varphi \Rightarrow \int_{\leq s}\varphi$, $\varphi \in \Pi$. Suppose by contradiction that $\eta \nVdash \int_{<s}\varphi \Rightarrow \int_{\leq s}\varphi$. This means that there's some $k \in \mathbb{N}$ such that $\eta, k \nVdash \int_{<s}\varphi \Rightarrow \int_{\leq s}\varphi$, and this is by definition $\eta, k \Vdash \int_{<s}\varphi$ and $\eta, k \nVdash \int_{\leq s}\varphi$. This is by definition$\mu(\{\{v : v \Vdash_{\mathcal{C}} \varphi\}\}) < s$ and $\mu(\{\{v : v \Vdash_{\mathcal{C}} \varphi\}\}) > s$ which makes no sense. Therefore for every k we have $\eta, k \Vdash \int_{\leq s}\varphi \Rightarrow \int_{<t}\varphi$

C10 ϕ is $\left(\int_{\geq s}\varphi \wedge \int_{\geq r}\psi \wedge \int_{\geq 1}(\neg\varphi \vee \neg\psi)\right) \Rightarrow \int_{\geq min(1,s+r)}(\varphi \vee \psi)$, $\varphi, \psi \in \Pi$. Suppose by contradiction that $\eta \nVdash \left(\int_{\geq s}\varphi \wedge \int_{\geq r}\psi \wedge \int_{\geq 1}(\neg\varphi \vee \neg\psi)\right) \Rightarrow \int_{\geq min(1,s+r)}(\varphi \vee \psi)$.

This means that there's some $k \in \mathbb{N}$ such that $\eta, k \nVdash \left(\int_{\geq s} \varphi \wedge \int_{\geq r} \psi \wedge \int_{\geq 1} (\neg\varphi \vee \neg\psi)\right) \Rightarrow \int_{\geq min(1,s+r)} (\varphi \vee \psi)$, and this is by definition $\eta, k \Vdash \left(\int_{\geq s} \varphi \wedge \int_{\geq r} \psi \wedge \int_{\geq 1} (\neg\varphi \vee \neg\psi)\right)$ and $\eta, k \nVdash \int_{\geq min(1,s+r)} (\varphi \vee \psi)$. This results in $\mu(\{v : v \Vdash_{\mathcal{C}} \varphi\}) \geq s$, and also in $\mu(\{v : v \Vdash_{\mathcal{C}} \psi\}) \geq r$ and in $\mu(\{v : v \Vdash_{\mathcal{C}} (\neg\varphi \vee \neg\psi)\}) \geq 1$. We need to take notice of some details here. $\mu(\{v : v \Vdash_{\mathcal{C}} (\neg\varphi \vee \neg\psi)\}) \geq 1$ means that st-he negation of $(\neg\varphi \vee \neg\psi)$, which is $\varphi \wedge \psi$, has probability 0. Since $\{v : v \Vdash_{\mathcal{C}} (\varphi \vee \psi)\}$ is the union of the disjoint sets $\{v : v \Vdash_{\mathcal{C}} (\varphi \wedge \neg\psi)\}$, $\{v : v \Vdash_{\mathcal{C}} (\neg\varphi \vee \psi)\}$ and $\{v : v \Vdash_{\mathcal{C}} (\varphi \wedge \psi)\}$ only the first two have any weight in the probability, since the last one has probability 0. We need only notice now that $\{v : v \Vdash_{\mathcal{C}} \psi\}$ is the union of disjoint sets $\{v : v \Vdash_{\mathcal{C}} \psi \wedge \neg\varphi\}$ and $\{v : v \Vdash_{\mathcal{C}} \psi \wedge \varphi\}$. This means that $\mu(\{v : v \Vdash_{\mathcal{C}} \psi\}) = \mu(\{v : v \Vdash_{\mathcal{C}} \psi \wedge \neg\varphi\}) = r$ since, again $\{v : v \Vdash_{\mathcal{C}} \psi \wedge \varphi\}$ has probability 0. This is analogous for $\{v : v \Vdash_{\mathcal{C}} \psi\}$. The final conclusion is that $\mu(\{v : v \Vdash_{\mathcal{C}} (\varphi \vee \psi)\}) \geq s + r$. The measure as a maximum limit of 1, so $\mu(\{v : v \Vdash_{\mathcal{C}} (\varphi \vee \psi)\}) \geq \min(1, s + r)$. This contradicts $\eta, k \nVdash \int_{\geq min(1,s+r)} (\varphi \vee \psi)$, hence the initial supposition is wrong.

C11 ϕ is $\left(\int_{\leq s} \varphi \wedge \int_{<r} \psi\right) \Rightarrow \int_{<s+r} (\varphi \vee \psi)$, $s + r \leq 1$, $\varphi, \psi \in \Pi$. Suppose by contradiction that $\eta \nVdash \left(\int_{\leq s} \varphi \wedge \int_{<r} \psi\right) \Rightarrow \int_{<s+r} (\varphi \vee \psi)$. This means that there's some $k \in \mathbb{N}$ such that $\eta, k \nVdash \left(\int_{\leq s} \varphi \wedge \int_{<r} \psi\right) \Rightarrow \int_{<s+r} (\varphi \vee \psi)$, and this is by definition $\eta, k \Vdash \left(\int_{\leq s} \varphi \wedge \int_{<r} \psi\right)$ and $\eta, k \nVdash \int_{<s+r} (\varphi \vee \psi)$. This is similar to C10, the difference is that we no longer have the restriction that $\{v : v \Vdash_{\mathcal{C}} \psi \wedge \varphi\}$ has probability 0. This means that since $\{v : v \Vdash_{\mathcal{C}} \psi\}$ is not disjoint of $\{v : v \Vdash_{\mathcal{C}} \varphi\}$ because they overlap in $\{v : v \Vdash_{\mathcal{C}} \psi \wedge \varphi\}$, then we can't get a direct measure for $\{v : v \Vdash_{\mathcal{C}} \psi \vee \varphi\}$. Instead, note that $\mu(\{v : v \Vdash_{\mathcal{C}} \psi\}) + \mu(\{v : v \Vdash_{\mathcal{C}} \varphi\}) = \mu(\{v : v \Vdash_{\mathcal{C}} \psi \wedge \neg\varphi\}) + \mu(\{v : v \Vdash_{\mathcal{C}} \neg\psi \wedge \varphi\}) + 2\mu(\{v : v \Vdash_{\mathcal{C}} \psi \wedge \varphi\})$.
Since
$\mu(\{v : v \Vdash_{\mathcal{C}} \psi \vee \varphi\}) = \mu(\{v : v \Vdash_{\mathcal{C}} \psi \wedge \neg\varphi\}) + \mu(\{v : v \Vdash_{\mathcal{C}} \neg\psi \wedge \varphi\}) + \mu(\{v : v \Vdash_{\mathcal{C}} \psi \wedge \varphi\})$
then $\mu(\{v : v \Vdash_{\mathcal{C}} \psi\}) + \mu(\{v : v \Vdash_{\mathcal{C}} \varphi\}) > \mu(\{v : v \Vdash_{\mathcal{C}} \psi \vee \varphi\})$,
hence the conclusion $\mu(\{v : v \Vdash_{\mathcal{C}} \psi \vee \varphi\}) < s + r$. This contradicts $\eta, k \nVdash \int_{<s+r} (\varphi \vee \psi)$, and so the initial supposition leads to a contradiction.

□

Axiom (C7) gives a lower bound for the probability of a formula.Also, an upper bound was implicitly defined with (C7). To see this, let φ be $\neg\psi$ in axiom (C7), thus obtaining $\int_{\geq 0} \neg\psi$. If we write this formula as $\int_{\geq 1-s} \neg\psi$, for $s = 1$, using definition (D2) we get $\int_{\leq 1} \psi$.

This gives us the upper bound, as each formula is forced to be satisfied by a valuation set of measure at most 1. This goes along with the previous definition of probability measure, $\mu : 2^{\Pi} \to [0,1]$.

Let us now look at some simple examples to make use of the axiomatic system.

Example 3.1.5. We want to show that if we have $\Gamma = \left\{\int_{<s} \varphi \wedge \int_{<r} \psi\right\}$ then, $\Gamma \vdash \neg \mathsf{X} \int_{\geq s+r} (\varphi \vee \psi)$ with $s + r < 1$.

Starting by the only premise:

(1)	$\int_{<s} \varphi \wedge \int_{<r} \psi$	premise
(2)	$\int_{<s} \varphi \Rightarrow \int_{\leq} \varphi$	(C9)
(3)	$\int_{\leq s} \varphi$	(Prop),(1),(2)
(4)	$\int_{\leq s} \varphi \wedge \int_{<r} \psi$	(Prop),(1),(3)
(5)	$\int_{\leq s} \varphi \wedge \int_{<r} \psi \Rightarrow \int_{<s+r} (\varphi \vee \psi)$	(C11)
(6)	$\int_{<s+r} (\varphi \vee \psi)$	(Prop),(4),(5)
(7)	$\neg \int_{\geq s+r} (\varphi \vee \psi)$	(D1)
(8)	$\mathsf{X} \neg \int_{\geq s+r} (\varphi \vee \psi)$	(R2)
(9)	$\mathsf{X} \neg \int_{\geq s+r} (\varphi \vee \psi) \Leftrightarrow \neg \mathsf{X} \int_{\geq s+r} (\varphi \vee \psi)$	(C2)
(10)	$\neg \mathsf{X} \int_{\geq s+r} (\varphi \vee \psi)$	(Prop),(8),(9)

□

Example 3.1.6. We show that $\left\{\int_{=s} \rho\right\} \vdash \int_{=1-s} \neg\rho$ where ρ is a propositional formula.

(1)	$\int_{=s} \rho$	premise
(2)	$\int_{\leq s} \rho \wedge \int_{\geq s} \rho$	(D4),(1)
(3)	$\int_{\leq s} \rho$	(Prop),(2)
(4)	$\int_{\geq 1-s} \neg\rho$	(D2),(3)
(5)	$\int_{\geq s} \rho$	(Prop),(2)
(6)	$\int_{\geq 1-(1-s)} \neg\neg\rho$	(Prop),(5)
(7)	$\int_{\leq 1-s} \neg\rho$	(D2),(6)
(8)	$\int_{=1-s} \neg\rho$	(D4),(4),(7)

□

We present some simple relations between logic and probability.

Lemma 3.1.7. The following holds

1. $\vDash \int_{=0}(p \wedge q) \Rightarrow \left(\int_{=s}(p \vee q) \Rightarrow \left(\int_{=r} p \Rightarrow \int_{=s-r} q\right)\right)$
2. $\vDash \int_{=1}(p \Rightarrow q) \Rightarrow \left(\int_{=r} p \Rightarrow \int_{\geq r} q\right)$
3. $\vDash \left(\int_{\geq s} p \wedge \int_{=1} q\right) \Rightarrow \int_{\geq s} p \Leftrightarrow q$

Proof. Let η be a pLTL model. Both relations follow by contradiction.

1. To prove this by contradiction suppose that $\eta \nVdash \int_{=0}(p \wedge q) \Rightarrow \left(\int_{=s}(p \vee q) \Rightarrow \left(\int_{=r} p \Rightarrow \int_{=s-r} q\right)\right)$. So there is a k such that $\eta, k \Vdash \int_{=0}(p \wedge q)$, $\eta, k \Vdash \int_{=s}(p \vee q)$, $\eta, k \Vdash \int_{=r} p$ and $\eta, k \nVdash \int_{=s-r} q$. For the set $\{p, q\}$ there are four possible valuations, $v_1 = (0,0)$, $v_2 = (0,1)$,$v_3 = (1,0)$ and $v_4 = (1,1)$. Since $\eta, k \Vdash \int_{=0}(p \wedge q)$, we know $\mu_k(\{v : v \Vdash p \wedge q\}) = \mu_k(\{v_4\}) = 0$. And since $\eta, k \Vdash \int_{=r} p$ then $\mu_k(\{v : v \Vdash p\}) = \mu_k(\{v_3, v_4\}) = \mu_k(v_3) + \mu_k(v_4) = r$, since $\mu_k(v_4) = 0$, then $\mu_k(v_3) = r$. Now from $\eta, k \Vdash \int_{=s}(p \vee q)$ we know that $\mu_k(\{v : v \Vdash p \vee q\}) = \mu_k(v_2) + \mu_k(v_3) + \mu_k(v_4) = s$ which implies that $\mu_k(v_2) = s - (\mu_k(v_3) + \mu_k(v_4)) = s - r$. Considering now that $\mu_k(\{v : v \Vdash q\}) = \mu_k(\{v_2, v_4\}) = \mu_k(v_2) + \mu_k(v_4) = s - r$, by definition we have that $\eta, k \Vdash \int_{=s-r} q$, which is a contradiction. So it must be true that for every k, $\eta, k \Vdash \int_{=0}(p \wedge q) \Rightarrow \left(\int_{=s}(p \vee q) \Rightarrow \left(\int_{=r} p \Rightarrow \int_{=s-r} q\right)\right)$.

2. Aiming for a contradiction, suppose that that$\eta \nVdash \int_{=1}(p \Rightarrow q) \Rightarrow \left(\int_{=r} p \Rightarrow \int_{\geq r} q\right)$. So there is a k such that $\eta, k \Vdash \int_{=1}(p \Rightarrow q)$, $\eta, k \Vdash \int_{=r} p$ and $\eta, k \nVdash \int_{\geq r} q$. Considering valuations $v_1 = (0,0)$, $v_2 = (0,1)$,$v_3 = (1,0)$ and $v_4 = (1,1)$, from $\eta, k \Vdash \int_{=1}(p \Rightarrow q)$ we know that $\mu_k(\{v : v \Vdash p \Rightarrow q\}) = \mu_k(\{v_1, v_2 v_4, \}) = \mu_k(v_1) + \mu_k(v_2) + \mu_k(v_4) = 1$, and since μ_k is a probability measure, $\mu_k(v_3) = 0$. From $\eta, k \Vdash \int_{=r} p$, we know that $\mu_k(v_3) + \mu_k(v_4) = r$ and so $\mu_k(v_4) = r$. Now $\eta, k \nVdash \int_{\geq r} q$ which is the same as $\eta, k \Vdash \neg \int_{\geq r} q$ or abbreviated $\eta, k \Vdash \int_{<r} q$ and so $\mu_k(\{v_2, v_4\}) < r$ if and only if $\mu_k(v_2) + \mu_k(v_4) < r$. But $\mu_k(v_4) = r$, so $\mu_k(v_2) + r < r$ and we get a contradiction. Therefor $\eta, k \Vdash \int_{=1}(p \Rightarrow q) \Rightarrow \left(\int_{=r} p \Rightarrow \int_{\geq r} q\right)$ for every k.

3. Given a model η suppose $\eta \Vdash \left(\int_{\geq s} p \wedge \int_{=1} q\right)$. So for every k we have $\eta, k \Vdash \left(\int_{\geq s} p \wedge \int_{=1} q\right)$, that is $\eta, k \Vdash \int_{\geq s} p$ and $\eta, k \Vdash \int_{=1} q$. By definition of local satisfaction we have that $\mu_k(\{v : v \Vdash p\}) \geq s$ and $\mu_k(\{v : v \Vdash q\}) = 1$. Let $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (0,1)$ and $v_4 = (1,1)$ be all possible valuations for the pair (p, q). Since $\{v : v \Vdash q\}$ is a certain event we have that $\mu_k(\{v_1, v_2\}) = 0$ and $\mu_k(\{v_3, v_4\}) = 1$. Plus we have that $\{v : v \Vdash p\}$ is composed of $\{v_2, v_4\}$. It follows that $\mu_k(v_4) \geq s$ since $\mu_k(v_2) = 0$. The set $\{v : v \Vdash p \Leftrightarrow q\}$ is the set $\{v_1, v_4\}$, which leads us to conclude that $\mu_k(\{v : v \Vdash p \Leftrightarrow q\}) \geq s$, and by definition $\eta, k \Vdash \int_{\geq s} p \Leftrightarrow q$. This proves the implication.

□

Our presentation of pLTL is just a step to get into a distributed probabilized system. To conclude this section we will be looking at a simple example that conjugates all the possibilities of this language.

Example 3.1.8. Consider the following situation. An agent wants to encrypt a one bit binary message m. At the start she tosses a perfect coin with half of probability for each face. After this, she encrypts the message m by the *xor* operation with the value e obtained from the toss. The result is stored in the cyphered message c. We shall prove that the cyphered message has one half of probability of being the original message. For this, consider $\Pi = \{m, e, c\}$, $Act = \{toss, enc, \mathsf{nil}\}$ and the set Γ of formulas

1. $* \Rightarrow \mathsf{F}toss$
2. $* \Rightarrow \mathsf{G}_\circ \int_{=1} m$
3. $toss \Rightarrow \mathsf{X} \int_{=\frac{1}{2}} e$
4. $toss \Rightarrow \mathsf{G}\neg toss$
5. $toss \Rightarrow \mathsf{X}enc$
6. $enc \Rightarrow \mathsf{G}\neg enc$
7. $enc \Rightarrow \int_{=1} (c \Leftrightarrow m \oplus e)$.

We can conclude that $\Gamma \vDash \left(* \Rightarrow \mathsf{F} \int_{=\frac{1}{2}} (c \Leftrightarrow m)\right)$.

Consider the eight possible valuations for $\{m, e, c\}$, $v_1 = (0,0,0)$, $v_2 = (0,1,0)$, $v_3 = (1,0,0)$, $v_4 = (1,1,0)$, $v_5 = (0,0,1)$, $v_6 = (0,1,1)$, $v_7 = (1,0,1)$ and $v_8 = (1,1,1)$.

Consider a model η that satisfies Γ. From (2) and since $\eta, 0 \Vdash *$ we take that $\eta, 0 \Vdash \mathsf{G_0} \int_{=1} m$, so for any $k \geq 0$ $\eta, k \Vdash \int_{=1} m$, and note that $\mu_k(\{v : v \Vdash m\}) = \mu_k(v_3) + \mu_k(v_4) + \mu_k(v_7) + \mu_k(v_8) = 1$, so for any k also $\mu_k(v_1) + \mu_k(v_2) + \mu_k(v_5) + \mu_k(v_6) = 0$ since μ is a probability measure. Taking (1) and $\eta, 0 \Vdash *$ we know that $\eta, 0 \Vdash \mathsf{F}toss$, so for some n, it will be true that $\eta, n \Vdash toss$, and since the model satisfies that (3), then $\eta, n \Vdash \mathsf{X} \int_{=\frac{1}{2}} e$, that is $\lambda, n+1 \Vdash \int_{\frac{1}{2}} e$. Note that $\mu_{n+1}(\{v : v \Vdash e\}) = \mu_{n+1}(v_2) + \mu_{n+1}(v_4) + \mu_{n+1}(v_6) + \mu_{n+1}(v_8) = \frac{1}{2}$ and $\mu_{n+1}(v_2) + \eta_{n+1}(v_6) = 0$ (we have seen that this is true for any state k), so it must be that $\mu_{n+1}(v_4) + \mu_{n+1}(v_8) = \frac{1}{2}$ and consequently $\mu_{n+1}(v_3) + \mu_{n+1}(v_7) = \frac{1}{2}$. Now note that $\mu_{n+1}(\{v_2, v_3, v_6, v_7\}) = \frac{1}{2}$ and that $\{v_2, v_3, v_6, v_7\}$ is the set $\{v : v \Vdash \neg(m \Leftrightarrow e)\}$, which means that $\mu_{n+1}(\{v : v \Vdash \neg(m \Leftrightarrow e)\}) = \frac{1}{2}$ and therefore $\eta, n+1 \Vdash \int_{=\frac{1}{2}} \neg(m \Leftrightarrow e)$, which is abbreviated to $\eta, n+1 \Vdash \int_{=\frac{1}{2}} m \oplus e$. Now from (5) we take that $\eta, n \Vdash \mathsf{X}enc$, by definition $\eta, n+1 \Vdash enc$, and also considering (7) we conclude $\eta, n+1 \Vdash \int_{=1} (c \Leftrightarrow m \oplus e)$. From $\eta, n+1 \Vdash \int_{=\frac{1}{2}} m \oplus e$ we know that $\eta, n+1 \Vdash \int_{=\frac{1}{2}} c$. We now would need to consider four more valuations for the tuple $\{m, c\}$, $v_5 = (0,0)$, $v_6 = (0,1)$, $v_7 = (1,0)$ and $v_8 = (1,1)$. Since $\mu_{n+1}(\{v_4, v_8\}) = \frac{1}{2}$ and $\mu_{n+1}(\{v_1, v5\})$ and noting that $\{v_1, v_4, v_5, v_8\}$ is the set $\{v : v \Vdash m \Leftrightarrow c\}$, we get $\mu_{n+1}(\{v : v \Vdash m \Leftrightarrow c\}) = \mu_{n+1}(\{v_1, v_4, v_5, v_8\}) = \frac{1}{2}$

This is true for some $n+1$, and so by definition of F, $\lambda, 0 \Vdash \mathsf{F} \int_{=\frac{1}{2}} (c \Leftrightarrow m)$.

□

3.2 Probabilistic distributed temporal logic (local) $\mathsf{pDTL_L}$

We now address the problem of extending probabilities to DTL. We do this in two steps. First we will only allow probability on propositional formulas, excluding temporal formulas and actions. This will result in the logic $\mathsf{pDTL_L}$. Later we will define a new logic, $\mathsf{pDTL_G}$, that will allows for reasoning with probabilities about any DTL formula.

When we introduced probabilities in pLTL, we only covered the mentioned first step. We didn't have any great difficulties in pLTL, but the fact that actions are not probabilized in $\mathsf{pDTL_L}$ will lead to some limitations. The main problem is that the communication between agents through actions will not carry probability from one agent to another. We will try to come around this limitation by reasoning through it.

From now on we will be only interested in the semantics of the languages, and won't address the deductive counterpart.

3.2.1 Syntax and semantics

The language for $\mathsf{pDTL_L}$ is an extension of pLTL just like DTL is for LTL. The syntax is distributed over the same signature $\Sigma = \langle Id, \{\Pi_i\}_{i \in Id}, \{Act_i\}_{i \in Id} \rangle$. The alphabet of a language for $\mathsf{pDTL_L}$ is defined as

$$\mathcal{L}_{\mathsf{pDTL}} ::= @_i [\mathcal{L}_i] \mid \mathcal{L} \Rightarrow \mathcal{L} \mid \neg \mathcal{L}$$

for $i \in Id$, where the local languages are defined as

$$\mathcal{L}_i ::= Act_i \mid \int_{\geq s} \mathcal{C}_i \mid \neg \mathcal{L}_i \mid \mathcal{L}_i \Rightarrow \mathcal{L}_i \mid \mathcal{L}_i \mathsf{U} \mathcal{L}_i \mid ©_j [\mathcal{L}_j] \mid *$$

where $\mathcal{C}_i$ is the classical propositional language of agent i

$$\mathcal{C}_i ::= \Pi_i \mid \neg \mathcal{C}_i \mid \mathcal{C}_i \Rightarrow \mathcal{C}_i.$$

For $\mathsf{pDTL_L}$ we referred that we have a measure for each instant. The natural progression to $\mathsf{pDTL_L}$ is to have a measure for each local state of each agent, and this is the choice we implement in $\mathsf{pDTL_L}$, although this leads to some difficulties in synchronization between agents.

The interpretation structure $\eta = \langle \lambda, \alpha, \sigma \rangle$ consists of a distributed life-cycle λ, a family $\alpha = \{\alpha_i\}_{i \in Id}$ of local labeling functions where for each $i \in Id$ the function $\alpha_i : Ev_i \to Act_i$

associates a local action to each event, and a family $\sigma = \{\sigma_i\}_{i \in Id}$ of local labeling functions such that $\sigma_i : Ev_i \to \mathcal{F}_i$ where $\mathcal{F}_i$ is a set of probability spaces. For each agent and each local state ξ_i we then have a probability space defined by $\left(\Pi_i, 2^{\Pi_i}, \mu\left(\xi_i\right)\right)$. When we want to refer to the probability measure of local state ξ_i we may only write μ_{ξ_i}. This is a measure on Π that satisfy the Kolmogorv's axioms and is defined as $\mu_{\xi_i} : 2^{\Pi} \to [0,1]$ where 2^{Π} is the set of classical valuations for a formula. Recall that we are only considering probability over classical formulas, that is, we exclude the probability to actions and temporal formulas.

We define the global satisfaction relation by

- $\eta \Vdash @_i\left[\varphi\right]$ if and only if $\eta \Vdash_i \varphi$ if and only if $\eta, \xi \Vdash_i \varphi$ for every $\xi \in \Xi$.

The local satisfaction relations at local states are defined by:

- $\eta, \xi_i \Vdash_i act$ if $\xi_i \neq \emptyset$ and $\alpha_i(last(\xi_i)) = act$;
- $\eta, \xi_i \Vdash_i \int_{\geq s} p$ if $\mu_{\xi_i}\left(\{v : v \Vdash_{Class_i} p\}\right) \geq s$;
- $\eta, \xi_i \Vdash_i \neg\varphi$ if $\eta, \xi_i \nVdash_i \varphi$;
- $\eta, \xi_i \Vdash_i \varphi \Rightarrow \psi$ if $\eta, \xi_i \nVdash_i \varphi$ or $\eta, \xi_i \Vdash_i \psi$;
- $\eta, \xi_i \Vdash_i \varphi \mathsf{U} \psi$ if there exists $\xi_i' \in \Xi_i$ such that $|\xi_i| < |\xi_i'|$ and $\eta, \xi_i' \Vdash_i \psi$ and $\eta, \xi_i'' \Vdash_i \varphi$ for every $|\xi_i| < |\xi_i''| < |\xi_i'|$;
- $\eta, \xi_i \Vdash_i ©_i\left[\varphi\right]$ if $|\xi_i| > 0$ and $last_i\left(\xi_i\right) \in E_j$, and $\eta, \left(last_i(\xi_i) \downarrow j\right) \Vdash_j \varphi$;
- $\eta_i, \xi_i \Vdash_i *$ if $\xi_i = \emptyset$.

The local semantic consequence is defined as $\Gamma \vDash_i \varphi$ if any model η verifies $\eta \Vdash_i \varphi$ whenever $\eta \Vdash_i \gamma$ for every $\gamma \in \Gamma$ for some agent $i \in Id$. A formula φ is locally valid, $\vDash_i \varphi$ or equivalently $\vDash @_i\left[\varphi\right]$ if $\eta_i \Vdash \varphi$ for every model η for some agent $i \in Id$.

Let's look at a simple example to illustrate a $\mathsf{pDTL_L}$ system.

Example 3.2.1. Let the system be composed of two agents, $Id = \{A, B\}$ such that $\Pi_A = \emptyset$, $\Pi_B = \{b\}$, $Act_A = \{init, \mathsf{nil}\}$, $Act_B = \{beep, \mathsf{nil}\}$.

In this simulation agent A will initiate a system, and agent B will beep in reply and try to initiate component b, succeeding with probability $\frac{1}{2}$. This protocol is represented $\Gamma \vDash @_B\left[* \Rightarrow \mathsf{F} \int_{=\frac{1}{2}} b\right]$ where Γ is the set of formulas:

1. $@_A\left[* \Rightarrow \mathsf{X} init\right]$
2. $@_A\left[init \Rightarrow ©_B\left[beep\right]\right]$

3. $@_B\,[beep \Rightarrow ©_A\,[init]]$

4. $@_B\left[beep \Rightarrow \int_{=\frac{1}{2}} b\right]$

Consider a model η that satisfies Γ. By definition $\eta, \xi_A^0 \Vdash *$ and from (1) we take that $\eta, \xi_A^0 \Vdash \mathsf{X} init$, and so $\eta, \xi_A^1 \Vdash init$. From (2) $\eta, \xi_A^1 \Vdash ©_B\,[beep]$, so by definition there is $\xi_B^n = \left(last\left(\xi_A^1\right) \downarrow B\right)$ such that $\eta, \xi_B^n \Vdash beep$. The last formula of Γ will yield $\eta, \xi_B^n \Vdash \int_{=\frac{1}{2}} b$. By definition of F we have that $\eta, \xi_B^0 \Vdash \mathsf{F} \int_{=\frac{1}{2}} b$ and by Lemma 2.2.1 $\eta \Vdash_B * \Rightarrow \mathsf{F} \int_{=\frac{1}{2}} b$ which is by definition $\Gamma \vDash @_B\left[* \Rightarrow \mathsf{F} \int_{=\frac{1}{2}} b\right]$.

□

3.2.1.1 Extending probability to actions

We will now be addressing the problem of not having probabilized actions. To better understand what we are facing, let us look at an example

Example 3.2.2. Suppose we a system such that, $Id = \{A, B\}$ such that $\Pi_A = \{a\}$, $\Pi_B = \{b\}$, $Act_A = \{init, \mathsf{nil}\}$, $Act_B = \{beep, \mathsf{nil}\}$ with Γ:

1. $@_A\left[* \Rightarrow \mathsf{X} \int_{=s} a\right]$

2. $@_A\left[\int_{=s} a \Rightarrow init\right]$

3. $@_A\,[init \Rightarrow ©_B\,[beep]]$

4. $@_B\,[beep \Rightarrow ©_A\,[init]]$

5. $@_B\,[beep \Rightarrow b]$

Actually formula (5) is syntactically incorrect, as b must have a probability operator.

But this system represents what we would like to express: that the probability of b depends on the occurrence of *beep*. We would also like to have formula (2) "transmit" the probability from a to *init*. Formulas (3) and (4) are another set back to communicate the probability from *init* to *beep*.

□

To try and solve this problem we will use the following conventions. Assume that we have two action symbols $a\,(\top)$ and $a\,(\bot)$ with the underlying intended meaning of $a\,(\bot)$ occurring with a parameter value $\bot$ and $a\,(\top)$ occurring with parameter value $\top$.

Also, given a local propositional formula φ of agent A we may want to write $a\,(\varphi)$to say that $a\,(\top)$ will occur whenever φ is true and $a\,(\bot)$occurs when φ is false. Hence, we

make an abuse of notation to say that $a(\top)$ will occur with "probability $\mu(\{v : v \Vdash \varphi\})$" and $a(\bot)$ will occur with "probability $1 - \mu(\{v : v \Vdash \varphi\})$".

Furthermore, let $act_A(ph)$ be a formula of some agent A and $act_B(ph)$ be a formula of an agent B where ph is a "placeholder" for a formula. An interaction formula of the form $@_A[act_A(ph) \Rightarrow ©_B[act_B(ph)]]$ is intended to carry out the value of A to B maintaining the probability of the action. Given the previous formula we say that $act_B(\top)$ will occur when $act_A(\top)$ occurs and the "probability" will carry out from one to another.

Let us look at an example of how this unfolds in practice:

Example 3.2.3. Suppose

- $Id = \{A, B\}$,
- $\Pi_A = \{a\}$, $Act_A = \{act_A(\top), act_A(\bot), \mathsf{nil}\}$,
- $\Pi_B = \{b\}$, $Act_B = \{act_B(\top), act_B(\bot), \mathsf{nil}\}$

and Γ is the set containing the formulas:

- $@\left[\int_{\geq s} \varphi\right]$
- $@_A[act_A(\varphi)]$
- $@_A[act_A(ph) \Rightarrow ©_B[act_B(ph)]]$
- $@_B[act_B(ph) \Rightarrow ©_A[act_A(ph)]]$
- $@_B[act_B(\psi)]$

The conclusion is that $\Gamma \vDash @_B\left[\int_{\geq s} \psi\right]$. Let us see this.

Let η be a PDTL1 model. $\eta \Vdash @_A\left[\int_{\geq s} \varphi\right]$ and $\eta \Vdash @_A[act_A(\varphi)]$ so by our convention $\mu(\{v : v \Vdash \varphi\}) = \mu(act_A(\top)) \geq s$. Furthermore, since $@_A[act_A(ph) \Rightarrow ©_B[act_B(ph)]]$ and $@_B[act_B(ph) \Rightarrow ©_A[act_A(ph)]]$, then $\mu(act_A(\top)) = \mu(act_B(\top))$ and again by the first convention, $\mu(\{v : v \Vdash \psi\}) = \mu(act_B(\top))$. Hence the result $\eta \Vdash @_B\left[\int_{\geq s} \psi\right]$.

□

Using the mentioned convention we would be able to reason through some more complex examples, as shown next.

Example 3.2.4. Let us now look at a pDTL_L system, consisting of two agents and a channel, $Id = [A, B, Ch]$. For this example, A will have a message m that he wishes to send to B. To do so, he will have to send the message through the channel. The channel

has some noise so the message will get scrambled. There will be a $\frac{1}{4}$ chance that the channel will send to B the opposite value of a bit, instead of its true value. We will conclude that the chance is $\frac{3}{4}$ for each bit b to be the same as the bit from the original message m.

Formalizing the signature, we have:

- $\Pi_A = \{m\}$; $Act_A = \{send\,(\top)\,, send\,(\bot)\}$;
- $\Pi_{Ch} = \{m, a\}$;$Act_{Ch} = \{in\,(\top)\,, in\,(\bot)\,, scr\,(\top)\,, scr\,(\bot)\,, out\,(\top)\,, out\,(\bot)\}$;
- $\Pi_B = \{a\}$;$Act_B = \{receive\,(\top)\,, receive\,(\bot)\}$.

Consider for the set Γ consisting of the following formulas:

1. $@_A\left[\int_{=1} m\right]$
2. $@_A\left[* \Rightarrow \mathsf{F} send(m)\right]$
3. $@_A\left[send(ph) \Rightarrow ©_{Ch}\left[in(ph)\right]\right]$
4. $@_{Ch}\left[in(ph) \Rightarrow ©_A\left[send(ph)\right]\right]$
5. $@_{Ch}\left[in(m) \Rightarrow \mathsf{X} scr(m)\right]$
6. $@_{Ch}\left[scr(m) \Rightarrow \int_{=\frac{3}{4}} (m \Leftrightarrow a)\right]$
7. $@_{Ch}\left[scr(m) \Rightarrow \mathsf{X} out(a)\right]$
8. $@_{Ch}\left[out(ph) \Rightarrow ©_B\left[receive(ph)\right]\right]$
9. $@_B\left[receive(ph) \Rightarrow ©_{Ch}\left[out(ph)\right]\right]$
10. $@_A\left[send(m) \Rightarrow \mathsf{G}\neg send\,(m)\right]$
11. $@_{Ch}\left[in(m) \Rightarrow \mathsf{G}\neg in\,(m)\right]$
12. $@_{Ch}\left[scr(m) \Rightarrow \mathsf{G}\neg scr\,(m)\right]$
13. $@_{Ch}\left[out(a) \Rightarrow \mathsf{G}\neg out\,(a)\right]$

We want to conclude that $\Gamma \vDash @_B\left[* \Rightarrow \mathsf{F}\int_{=\frac{3}{4}} a\right]$.

To see this, consider a model η that satisfy Γ.

If $\eta \Vdash @_A\left[\int_{=1} m\right]$ then for all local state ξ_A^k, $\eta, \xi_A^k \Vdash \int_{=1} m$. From (2) we know that $\eta, \xi_A^n \Vdash send(m)$ for some n. From (3), (4) and the previous formula, using the convention we say that the action $in\,(\top)$ will happen with probability 1. Now using formulas (2) and (3) we know that for some local state the action $in\,(m)$ will occur, and in fact that local

state is $\xi_{Ch}^{n'} \equiv (last_A(\xi_A^n) \downarrow Ch)$. Since $in\,(\top)$ will happen at $\xi_{Ch}^{n'}$ with probability 1, by the convention this results in $\mu_{\xi_{Ch}^{n'}}(\{v : v \Vdash m\}) = 1$. In resume we now have $\eta, \xi_{Ch}^{n'} \Vdash in(m)$ and $\eta, \xi_{Ch}^{n'} \Vdash \int_{=1} m$.

From $\eta, \xi_{Ch}^{n'} \Vdash in(m)$ and (5) we take that $\eta, \xi_{Ch}^{n'} \Vdash \mathsf{X} scr(k)$, so $\eta, \xi_{Ch}^{n'+1} \Vdash scr(k)$. Now from (6) we get that $\eta, \xi_{Ch}^{n'+1} \Vdash \int_{=\frac{3}{4}} (m \Leftrightarrow a)$.

We denote by v_i, $i = 1, 2, 3, 4$ the valuation for the pair (m, a) where $v_1 = (0, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 0)$, $v_4 = (1, 1)$; since $\eta, \xi_A \Vdash \int_{=1} k$, $\mu_{\xi_{Ch}^{n'+1}}(\{v_3, v_4\}) = 1$ and then $\eta, \xi_{Ch}^{n'+1} \Vdash \int_{=\frac{3}{4}} (k \Leftrightarrow a)$ will result in $\mu_{\xi_{Ch}^{n'+1}}(\{v_1, v_4\}) = \mu_{\xi_{Ch}^{n'+1}}(v_4) = \frac{3}{4}$, since $\mu_{\xi_{Ch}^{n'+1}}(\{v_1, v_2\}) = 0$. This leads to $\mu_{\xi_{Ch}^{n'+1}}(\{v : v \Vdash_{Class} a\}) = \mu_{\xi_{Ch}^{n'+1}}(\{v_2, v_4\}) = \frac{3}{4}$. So $\eta_i, \xi_{Ch}^{n'+1} \Vdash \int_{=\frac{3}{4}} a$.

From $\eta, \xi_{Ch}^{n'+1} \Vdash scr(k)$ and (7), we get $\eta, \xi_{Ch}^{n'+2} \Vdash out\,(a)$. This combined with the formulas (8) and (9), by the convention we must consider two cases:

- at state $\xi_{Ch}^{n'+2}$ we have $a = \top$.
  If that's the case, $out\,(\top)$ and $receive\,(\top)$ will happen with probability $\frac{3}{4}$. Now using formula (8) and $\eta, \xi_{Ch}^{n'+2} \Vdash out\,(a)$ we know that for some local state the action $receive\,(a)$ will occur, and in fact that local state is $\xi_B^{n''} \equiv (last_{Ch}(\xi_{Ch}^{n'+2}) \downarrow B)$. Since $receive\,(\top)$ will happen at $\xi_B^{n''}$ with probability $\frac{3}{4}$, by the convention this results in $\mu_{\xi_B^{n''}}(\{v : v \Vdash a\}) = \frac{3}{4}$. In resume we now have $\eta, \xi_B^{n''} \Vdash receive(a)$ and $\eta, \xi_B^{n''} \Vdash \int_{=\frac{3}{4}} a$.

- at state $\xi_{Ch}^{n'+2}$ we have $a = \bot$.
  If that's the case, $out\,(\bot)$ and $receive\,(\bot)$ will happen with probability $\frac{1}{4}$. Now using formula (8) and $\eta, \xi_{Ch}^{n'+2} \Vdash out\,(a)$ we know that for some local state the action $receive\,(a)$ will occur, and in fact that local state is $\xi_B^{n''} \equiv (last_{Ch}(\xi_{Ch}^{n'+2}) \downarrow B)$. Since $receive\,(\bot)$ will happen at $\xi_B^{n''}$ with probability $\frac{1}{4} = 1 - \frac{3}{4}$, by the convention this results in $\mu_{\xi_B^{n''}}(\{v : v \Vdash a\}) = \frac{3}{4}$. In resume we now have $\eta, \xi_B^{n''} \Vdash receive(a)$ and $\eta, \xi_B^{n''} \Vdash \int_{=\frac{3}{4}} a$.

For either cases the result is that $\eta, \xi_B^{n''} \Vdash \int_{=\frac{3}{4}} a$.

Note that $\eta \Vdash @_B \left[* \Rightarrow \mathsf{F} \int_{=\frac{3}{4}} a\right]$ is vacuously true for any $\xi_B \neq \emptyset$. For $\xi_B = \emptyset$ the mentioned formula results in $\eta, \emptyset_B \Vdash \mathsf{F} \int_{=\frac{3}{4}} a$, that is, by definition for some $\xi_B^{'}$ it will be $\eta, \xi_B^{'} \Vdash \int_{=\frac{3}{4}} a$.

But we have just proved that for $\xi_B^{'} = \xi_B^{n''}$ we have $\eta, \xi_B^{n''} \Vdash \int_{=\frac{3}{4}} a$, so $\Gamma \vDash @_B \left[* \Rightarrow \mathsf{F} \int_{=\frac{3}{4}} a\right]$. □

When a propositional formula is a certain event having a probability value of 1, there won't be a chance for it to be false. This is the case for a regular propositional formula in

a DTL model. If we were to restrain the $\mathsf{pDTL_L}$ language to formulas with probability 1 a model for $\mathsf{pDTL_L}$ would represent the same systems as a DTL model.

Let $\mathsf{pDTL_{=1}}$ be the subset of $\mathsf{pDTL_L}$ for which the language will be restricted to formulas of the form $\int_{=s} \mathcal{C}_i$ with $s \in \{0,1\}$ (where without the restriction all formulas $\int_{\geq s} \mathcal{C}_i$ for $s \in [0,1]_{\mathbb{Q}}$ are allowed).

Note that the main difference between DTL and $\mathsf{pDTL_L}$ languages is that one has valuations for propositional formulas, whereas the other probabilizes them.

To prove the equivalence of the restricted language let us consider a function $\beta : \mathcal{L}_{\mathsf{DTL}} \to \mathcal{L}_{\mathsf{pDTL}}$. Given formula $\varphi \in \mathcal{L}_{DTL}$ either φ is a propositional symbol, an action, or a function of propositional symbols and actions. The function β will have the following values:

- φ is $@_i[p]$ where $p \in \Pi_i$, then $\beta(\varphi) = @_i\left[\int_{=1} p\right]$ where $p \in \Pi_i$;
- φ is $@_i[a]$ where $a \in Act_i$, then $\beta(\varphi) = @_i[a]$ where $a \in Act_i$
- φ is $@_i[f(\psi_1, ..., \psi_n)]$ where $\psi_1, ..., \psi_n \in \mathcal{L}_{\mathsf{DTL}}$ and f is one of the language operators, then $\beta(\varphi) = @_i[f(\beta(\psi_1), ..., \beta(\psi_n))]$.

To translate one model from a language into the other we will use a function $\delta : \{\eta : \eta \text{ is a } \mathsf{pDTL_{=1}}\text{model}\} \to \{\eta' : \eta' \text{ is a } \mathsf{DTL}\text{model}\}$ that maps $\delta : \langle \lambda, \alpha, \sigma \rangle \mapsto \langle \lambda, \alpha, \sigma' \rangle$.

Both models will have the same life cycle and actions, that is $\delta(\lambda) = \lambda$ and $\delta(\alpha) = \alpha$, and note that the set of events will also be the same.

The family of probability measures on $\mathsf{pDTL_{=1}}$ will have a corresponding family of valuations for DTL. Given a local state, the set of propositional formulas that have probability 1 for that state will be the set of valid formulas in the DTL model, written $\sigma'_i(\xi_i) = \{p : \mu_{\xi_i}(\{v : v \Vdash p\}) = 1\}$, and $\sigma' = \{\sigma'_i\}_{i \in Id}$.

Proposition 3.2.5. *Given $\varphi \in \mathsf{DTL}$ and a $\mathsf{pDTL_{=1}}$ model η then $\eta \Vdash_{\mathsf{pDTL_{=1}}} \beta(\varphi)$ if and only if $\delta(\eta) \Vdash_{\mathsf{DTL}} \varphi$.*

Proof. The proof follows by induction on the the structure of a local formula φ. Let i be an agent and ξ_i any local state for that agent.

For the basis of induction let φ be $p \in \Pi_i$. Starting from $\eta \Vdash_{\mathsf{pDTL_{=1}}} @_i[\beta(p)]$ This is which is the same as $\eta \Vdash_{\mathsf{pDTL_{=1}}} @_i\left[\int_{=1} p\right]$, that is, for every local state ξ_i^n we get $\eta, \xi_i^n \Vdash_{\mathsf{pDTL_{=1}}} \int_{=1} p$. We know that $\mu_{\xi_i^n}(\{v : v \Vdash p\}) = 1$ and by definition $p \in \sigma_i\left(\xi_i^k\right)$ so we have $\eta', \xi_i^n \Vdash_{\mathsf{DTL}} p$. Also if $p \in \sigma(\xi_i^n)$, then $\mu_{\xi_i}(\{v : v \Vdash p\}) = 1$, and so if $\delta(\eta), \xi_i^n \Vdash_{\mathsf{DTL}} p$ then $\eta, \xi_i^n \Vdash_{\mathsf{pDTL_{=1}}} \int_{=1} p$. The life cycles are the same for both models, so $\eta, \xi_i^n \Vdash_{\mathsf{pDTL_{=1}}} \int_{=1} p$ is true for all ξ_i^n which results in $\eta \Vdash_{\mathsf{pDTL_{=1}}} @_i\left[\int_{=1} p\right]$.

Also if φ is $@_i[act]$, $act \in Act_i$ then $\eta \Vdash_{\mathsf{pDTL}_{=1}} @_i[act]$ is $\eta \Vdash_{\mathsf{pDTL}_{=1}} @_i[act]$ for every ξ_i^n and this happens if and only if $\delta(\eta), \xi_i^n \Vdash_{\mathsf{DTL}} act$ since $\delta(\alpha) = \alpha$, for every ξ_i^n, and so $\delta(\eta) \Vdash_{\mathsf{DTL}} @_i[act]$.

For the induction step consider that:

- φ is $@_i \neg\psi$. $\eta \Vdash_{\mathsf{pDTL}_1} @_i[\neg\psi]$ so for every state ξ_i^n we get $\eta, \xi_i^n \Vdash_{\mathsf{pDTL}_1} \neg\psi$ then by definition $\eta, \xi_i^n \nVdash_{\mathsf{pDTL}_{=1}} \psi$ if and only if, by induction hypothesis, $\delta(\eta), \xi_i^n \nVdash_{\mathsf{DTL}} \psi$ by and that is $\delta(\eta), \xi_i^n \Vdash_{\mathsf{DTL}} \neg\psi$, for every ξ_i^n, so $\delta(\eta) \Vdash_{\mathsf{DTL}} @_i[\neg\psi]$.

- φ is $@_i[\varphi \Rightarrow \psi]$. Supposing that $\eta \nVdash_{\mathsf{DTL}} @_i[\varphi \Rightarrow \psi]$ then for some state ξ_i^n we get $\eta, \xi_i^n \nVdash_{\mathsf{DTL}} \varphi \Rightarrow \psi$. It follows that $\eta, \xi_i^n \Vdash_{\mathsf{DTL}} \varphi$ and $\delta(\eta), \xi_i^n \nVdash_{\mathsf{DTL}} \psi$, if and only if, by induction hypothesis and the previous step, $\delta(\eta), \xi_i^k \Vdash_{\mathsf{pDTL}_{=1}} \varphi$ and $\delta(\eta), \xi_i^n \nVdash_{\mathsf{pDTL}_{=1}} \psi$, yielding that $\delta(\eta), \xi_i^n \nVdash_{\mathsf{pDTL}_{=1}} \varphi \Rightarrow \psi$ for some state ξ_i^n, so $\delta(\eta) \nVdash_{\mathsf{pDTL}_{=1}} @_i[\varphi \Rightarrow \psi]$.

- φ is $\varphi \mathsf{U} \psi$. $\eta \Vdash_{\mathsf{pDTL}_{=1}} @_i[\varphi \mathsf{U} \psi]$ so for every state ξ_i^k we get $\eta, \xi_i^k \Vdash_{\mathsf{pDTL}_{=1}} \varphi \mathsf{U} \psi$. The definition of $\eta, \xi_i^k \Vdash_{\mathsf{pDTL}_{=1}} \varphi \mathsf{U} \psi$ is that there is $\xi_i^n \in \Xi_i$ such that $\xi^k \subset \xi^n$ and $\eta, \xi_i^n \Vdash_{\mathsf{pDTL}_{=1}} \psi$ and $\eta, \xi_i^m \Vdash_{\mathsf{pDTL}_{=1}} \varphi$ for every ξ^m such that $\xi^k \subset \xi^m \subset \xi^n$. This is true if and only if, by induction hypothesis, $\delta(\eta), \xi_i^n \Vdash_{\mathsf{DTL}} \psi$ and $\delta(\eta), \xi_i^m \Vdash_{\mathsf{DTL}} \varphi$ which is by definition $\delta(\eta), \xi_i^k \Vdash_{\mathsf{DTL}} \varphi \mathsf{U} \psi$, which is true for every ξ_i^k since the life-cycles are the same, and so $\delta(\eta) \Vdash_{\mathsf{DTL}} @_i[\varphi \mathsf{U} \psi]$

- φ is $@_i\left[©_j[\varphi]\right]$. From $\eta \Vdash_{\mathsf{pDTL}_{=1}} @_i\left[©_j[\varphi]\right]$, for every ξ_i we have $\eta, \xi_i \Vdash_{\mathsf{pDTL}_{=1}} ©_j[\varphi]$. By definition this happens if $|\xi_i| > 0$ and $last_i(\xi_i) \in E_j$, and $\eta, (last_i(\xi_i) \downarrow j) \Vdash_{\mathsf{pDTL}_{=1}} \varphi$ if and only if, by the hypothesis, $\delta(\eta), (last_i(\xi_i) \downarrow j) \Vdash_{\mathsf{DTL}} \varphi$ which is the definition of $\delta(\eta), \xi_i \Vdash_{\mathsf{DTL}} ©_j[\varphi]$, for every ξ_i^n so $\delta(\eta) \Vdash_{\mathsf{DTL}} @_i\left[©_j[\varphi]\right]$

□

3.3 Probabilistic distributed temporal logic (global) pDTL_G

In this chapter we take a full exogenous approach to probabilities. Until now we have been inserting probabilities inside the models of the base language, modifying the models to accommodate the probability operators. This allowed us to make statements such as "tomorrow the probability of p is s".

Now we take a different approach. We leave the structure of the base language intact. Instead of adding probability operators, we instead introduce a probability measure on the models of the base language. This means that we will have sets of models, and to

those sets we will assign a probability. This will allow us to make statements such as "The probability of tomorrow p is s".

Since we are not changing the base language, the problems we had with actions and communication formulas no longer exist. Formulas will have probability as a whole, instead of having probability just in the classical propositional formulas, which will provide a smooth transition of probabilities between agents.

3.3.1 Syntax and semantics

The language for $\mathsf{pDTL_G}$ is an extension of pLTL. The syntax is distributed over the signature $\Sigma = \langle Id, \{\Pi_i\}_{i\in Id}, \{Act_i\}_{i\in Id}\rangle$. The alphabet of a language for $\mathsf{pDTL_G}$ is defined as

$$\mathcal{L}_{\mathsf{pDTL_G}} ::= \int_{\geq s}\mathcal{L}_{prob} \mid \neg\mathcal{L}_{\mathsf{pDTL}} \mid \mathcal{L}_{\mathsf{pDTL}} \Rightarrow \mathcal{L}_{\mathsf{pDTL}}$$

where $\mathcal{L}_{prob}$ is a DTL language defined by:

$$\mathcal{L}_{prob} ::= @_i\,[\mathcal{L}_i] \mid \mathcal{L}_{prob} \Rightarrow \mathcal{L}_{prob} \mid \neg\mathcal{L}_{prob}$$

for $i \in Id$, where the local languages $\mathcal{L}_i$, are defined as

$$\mathcal{L}_i ::= \Pi_i \mid Act_i \mid \neg\mathcal{L}_i \mid \mathcal{L}_i \Rightarrow \mathcal{L}_i \mid \mathcal{L}_i \mathsf{U} \mathcal{L}_i \mid ©_j\,[\mathcal{L}_j] \mid *$$

Basically we will have DTL expressions with a probability operator applied to them. This will allow for an easy transfer of probability and information between agents.

We will be measuring sets of DTL models, such as e.g., the set that contains all models $\{\eta : \eta \text{ is a } \mathsf{DTL}\text{model}\}$. We will have a global probability measure that assigns a probability to each set of models. What we mean by global is that we don't have a probability measure for each event, like we had before in $\mathsf{pDTL_L}$. We just need one probability measure to define a $\mathsf{pDTL_G}$ model.

Back in Chapter 2, when we were given a DTL model η and a formula φ, we were concerned whether $\eta \Vdash \varphi$ or $\eta \nVdash \varphi$. Our concern with $\mathsf{pDTL_G}$ will be the measure of the set of models that satisfy φ, that is, what will be the measure of the set $\{\eta : \eta \Vdash \varphi \text{ where } \eta\text{is a } \mathsf{DTL}\text{model}\}$. This is the exogenous approach, maintaining the base structure of DTL "$\eta \Vdash \varphi$" and measuring the models η.

A $\mathsf{pDTL_G}$ model consists only of the probability measure represented by μ. Let $\mathcal{M}_{\mathsf{DTL}}$ be the set of all DTL models. The probability space is defined by $\left(\mathcal{M}_{\mathsf{DTL}}, 2^{\mathcal{M}_{\mathsf{DTL}}}, \mu\right)$ where $\mu : 2^{\mathcal{M}_{\mathsf{DTL}}} \to [0,1]$.

The satisfaction relation is defined by:

- $\mu \Vdash \int_{\geq s} @_i [\varphi]$ if $\varphi \in \Pi_i$ and $\mu(\{\eta : \eta_i \Vdash_{DTL} \varphi\}) \geq s$;
- $\mu \Vdash \neg\varphi$ if $\mu \nVdash \varphi$;
- $\mu \Vdash \varphi \Rightarrow \psi$ if $\mu \nVdash_i \varphi$ or $\mu \Vdash_i \psi$;

We define that $\Gamma \vDash_{\mathsf{pDTL_G}} \varphi$ if any model μ verifies $\mu \Vdash_{\mathsf{pDTL_G}} \varphi$ whenever $\mu \Vdash_{\mathsf{pDTL_G}} \gamma$ for every $\gamma \in \Gamma$. A formula φ is valid, $\vdash \varphi$ if $\mu \Vdash \varphi$ for any model μ.

Throughout this section we will fix a time event using the *first event* operator. We do this because otherwise we would have sets such as $\{\eta : \eta \Vdash act\}$, where an action would be true for all events, excluding the possibility of other actions, which wouldn't allow for some interesting examples. We will make use of $*$ and present results of sets such as $\{\eta : \eta \Vdash * \Rightarrow act\}$ so that we can estabilish the event in which the action occurs.

We follow with a lemma that has useful results concerning fixed time events.

Lemma 3.3.1. *Given a* $\mathsf{pDTL_G}$ *model* μ*,*

1. $\mu(\{\eta : \eta \Vdash * \Rightarrow \varphi\}) = s$ *if and only if* $\mu\left(\left\{\eta : \eta, \xi^0 \Vdash \varphi\right\}\right) = s$
2. *if we have two sets of* DTL *models,* E *and* D *and* $D \subseteq E$*, then* $\mu(D) \leq \mu(E)$
3. *Given a fixed* ξ*, if* $\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = s$ *and* $\mu(\{\eta : \eta, \xi \Vdash \varphi\}) = 1$ *then* $\mu(\{\eta : \eta, \xi \Vdash \psi\}) = s$
4. *Given a fixed* ξ*, if* $\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = 1$ *and* $\mu(\{\eta : \eta, \xi \Vdash \varphi\}) = s$ *then* $\mu(\{\eta : \eta, \xi \Vdash \psi\}) \geq s$*.*

Proof.

1. We show that $\eta, \xi^0 \Vdash \varphi$ if and only if $\eta \Vdash * \Rightarrow \varphi$.
   Suppose first that $\eta, \xi^0 \Vdash \varphi$ and note that $\eta, \xi^0 \Vdash * \eta, \xi^0 \Vdash * \Rightarrow \varphi$ since by definition $\eta, \xi^0 \Vdash *$. With Lemma 2.2.1 we conclude that $\eta, \xi \Vdash * \Rightarrow \varphi$ for every ξ, and so by definition $\eta \Vdash * \Rightarrow \varphi$.
   Conversely suppose $\eta \Vdash * \Rightarrow \varphi$. By definition, $\eta, \xi \Vdash * \Rightarrow \varphi$ for every ξ, in particular for $\xi = \emptyset$. Since $\eta, \xi^0 \Vdash *$ and $\eta, \xi^0 \Vdash * \Rightarrow \varphi$ we then get that $\eta, \xi^0 \Vdash \varphi$.
2. This is a trivial result, since $D \subseteq E$ then $E = D \cup C$ where $C \cap D = \emptyset$ and $\mu(E) = \mu(D) + \mu(C)$ because μ is countably additive, and we have $\mu(C) \geq 0$ which leads to $\mu(E) \geq \mu(D)$.

3. Suppose we have $\mu(\{\eta : \eta, \xi \Vdash \varphi\}) = 1$. By countable addictivity of μ we know that $\mu(\{\eta : \eta, \xi \nVdash \varphi\}) = 0$. Now suppose $\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = s$, and note that $\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\} = \{\eta : \eta, \xi \Vdash \psi\} \cup (\{\eta : \eta, \xi \nVdash \varphi\} \setminus \{\eta : \eta, \xi \Vdash \psi\})$. Again, μ is a probability measure so
$\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = \mu(\{\eta : \eta, \xi \Vdash \psi\}) + \mu(\{\eta : \eta, \xi \nVdash \varphi\} \setminus \{\eta : \eta, \xi \Vdash \psi\})$, as both sets are disjoint. By the previous alinea, $\mu(\{\eta : \eta, \xi \nVdash \varphi\} \setminus \{\eta : \eta, \xi \Vdash \psi\}) \leq \mu(\{\eta : \eta, \xi \nVdash \varphi\}) = 0$, hence the result $\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = \mu(\{\eta : \eta, \xi \nVdash \varphi\}) + 0 = s$.

4. Suppose that $\mu(\{\eta : \eta, \xi \Vdash \varphi\}) = s$. We then conclude that $\mu(\{\eta : \eta, \xi \nVdash \varphi\}) = 1 - s$. Suppose now that $\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = 1$. This time we write the set as $\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\} = \{\eta : \eta, \xi \nVdash \varphi\} \cup \{\eta : \eta, \xi \Vdash \psi\} \setminus \{\eta : \eta, \xi \nVdash \varphi\}$. This will lead to

$$\mu(\{\eta : \eta, \xi \Vdash \varphi \Rightarrow \psi\}) = \mu(\{\eta : \eta, \xi \nVdash \varphi\}) + \mu(\{\eta : \eta, \xi \Vdash \psi\} \setminus \{\eta : \eta, \xi \nVdash \varphi\})$$

so $1 = 1 - s + \mu(\{\eta : \eta, \xi \Vdash \psi\} \setminus \{\eta : \eta, \xi \nVdash \varphi\})$ simplified to $\mu(\{\eta : \eta, \xi \Vdash \psi\} \setminus \{\eta : \eta, \xi \nVdash \varphi\}) = s$. Again by point (2), $\{\eta : \eta, \xi \Vdash \psi\} \setminus \{\eta : \eta, \xi \nVdash \varphi\} \subseteq \{\eta : \eta, \xi \Vdash \psi\}$ so $s \leq \mu(\{\eta : \eta, \xi \Vdash \psi\})$.

□

To make use of the previous Lemma we present an example.

Example 3.3.2. Let us describe the situation we wish to represent. After the first moment of the system occurs a *push*. Whenever there's a *push*, the next state will have a *pull* half the times. When a *pull* occurs, p becomes true. We will show that p has more than half probability of being true in the third state (after the push and the pull).

The signature is $Id = \{A\}$, $\Pi_A = \{p\}$, $Act_A = \{push, pull, \mathsf{nil}\}$.

1. $@_A\left[\int_{=1} * \Rightarrow \mathsf{X}push\right]$
2. $@_A\left[\int_{=\frac{1}{2}} * \Rightarrow \mathsf{G}(push \Rightarrow Xpull)\right]$
3. $@_A\left[\int_{=1} * \Rightarrow \mathsf{G}(pull \Rightarrow p)\right]$

We show that $\Gamma \vDash @_A\left[\int_{\geq\frac{1}{2}} * \Rightarrow \mathsf{XX}p\right]$.
Suppose μ satisfies Γ. From (1) we get $\mu \Vdash @_A\left[\int_{=1} * \Rightarrow \mathsf{X}push\right]$ and equivalently $\mu(\{\eta : \eta_A \Vdash * \Rightarrow \mathsf{X}push\}) = 1$. By Lemma 3.3.1(.1) we know $\mu\left(\left\{\eta : \eta_A, \xi_A^0 \Vdash \mathsf{X}push\right\}\right) = 1$
From (2) we get $\mu \Vdash @_A\left[\int_{=\frac{1}{2}} * \Rightarrow \mathsf{G}(push \Rightarrow Xpull)\right]$,
and that is $\mu\left(\left\{\eta : \eta_A, \xi_A^0 \Vdash \mathsf{G}(push \Rightarrow Xpull)\right\}\right) = \frac{1}{2}$. We have that $\left\{\eta : \eta_A, \xi_A^0 \Vdash \mathsf{G}(push \Rightarrow Xpull)\right\} \subseteq \left\{\eta : \eta, \xi_A^1 \Vdash push \Rightarrow Xpull\right\}$ so by Lemma 3.3.1(.2) we get $\mu\left(\left\{\eta : \eta_A, \xi_A^1 \Vdash push \Rightarrow Xpull\right\}\right) \geq \frac{1}{2}$. We had $\mu\left(\left\{\eta : \eta_A, \xi_A^0 \Vdash \mathsf{X}push\right\}\right) = \mu\left(\left\{\eta : \eta_A, \xi_A^1 \Vdash push\right\}\right) = 1$, and now using Lemma 3.3.1(.3) we get $\mu\left(\left\{\eta : \eta, \xi_A^1 \Vdash Xpull\right\}\right) = \mu\left(\left\{\eta : \eta, \xi_A^2 \Vdash pull\right\}\right) \geq \frac{1}{2}$.

From (3) we get $\mu \Vdash @_A \left[\int_{=1} * \Rightarrow \mathsf{G}\,(\mathit{pull} \Rightarrow p)\right]$, by definition $\mu\left(\{\eta : \eta_A \Vdash * \Rightarrow \mathsf{G}\,(\mathit{pull} \Rightarrow p)\}\right) = 1$. Using Lemma 3.3.1(.1) and 3.3.1(.2) we get $\mu\left(\left\{\eta : \eta, \xi_A^2 \Vdash (\mathit{pull} \Rightarrow p)\right\}, \right) = 1$. Finally, using Lemma 3.3.1(.4) we get $\mu\left(\left\{\eta : \eta, \xi_A^2 \Vdash p\right\}\right) \geq \frac{1}{2}$.

Hence, by definition we get $\mu\left(\left\{\eta : \eta, \xi_A^0 \Vdash \mathsf{XX}p\right\}, \right) \geq \frac{1}{2}$, and that is $\mu \Vdash @_A \left[\int_{\geq \frac{1}{2}} * \Rightarrow \mathsf{XX}p\right]$ by Lemma 3.3.1(.1).

□

The previous example achieved something that was not possible in the previous section. What we wanted to show was similar to the objective of Example 3.2.2. Back then in $\mathsf{pDTL_L}$, we had no way of influencing the probability of a propositional formula from an action (since actions didn't even have a probability assigned!). Now we are able to do this easily. We just showed an example where the probability of p was a consequence of the probability of *pull*.

3.3.1.1 Actions and probability in $\mathsf{pDTL_G}$

In a DTL scenario, propositional formulas can only be true or false. For one of such formulas, we know exactly what it's negation is. Therefore, from the probability of a propositional formula we can get the probability of it's negation. This is not so simple once we look at an action formula.

The negation of an action in a state corresponds to the possibility of all other actions for that state. I.e., given a set of actions $Act = \{a_1, a_2, a_3\}$, if in some state $a_1$is false, then a_2 or a_3 are true. This follows from the fact that for each state there must always be one and only one action that is true.

We know that, if ρ is a propositional formula and has probability s, then $\neg\rho$ has probability $1 - s$ (shown for pLTL in example 3.1.6). The sum of probabilities of a formula and it's negation must equal 1.

The same must be the case with an action and it's negation, as we show in the next Lemma.

Lemma 3.3.3. Given a set of actions $Act_i = \{act_1, ..., act_n\}$ for an agent i and a fixed local state ξ_i, the sum of the probabilities of the sets which model satisfies each different action, must be 1, i.e.,

$$\sum_{act \in Act_i} \mu\left(\{\eta : \eta_i, \xi_i \Vdash act\}\right) = 1.$$

Proof. Consider the set of actions $Act = \{act_1, ..., act_n\}$. Let us fix a local state ξ_i and let $\mu\left(\{\eta : \eta, \xi_i \Vdash act_1\}\right) = s$, so by addictivity we conclude that $\mu\left(\{\eta : \eta, \xi_i \nVdash act_1\}\right) = 1 - s$, as the sets considered are each others complement.

We should now check what the set $\{\eta : \eta, \xi_i \nVdash act_1\}$ represents. By definition $\eta, \xi_i \Vdash act_1$ if $\xi_i \neq \emptyset$ and $\alpha_i(last(\xi_i)) = act$, and taking $\xi_i \neq \emptyset$, if we have $\eta, \xi_i \nVdash act_1$ it must be that $\alpha_i(last(\xi_i)) \neq act_1$. With the information we have, $\alpha_i(last(\xi_i))$ could be any $act \neq act_1$, $act \in Act$. Ergo, the set $\{\eta : \eta, \xi_i \nVdash act_1\}$ is a union of disjoint sets $\bigcup_{i=2}^{n} \{\eta : \eta, \xi_i \Vdash act_i\}$. Therefore, the sum of probabilities of the various actions equals 1, $\sum_{act \in Act} \mu(\{\eta : \eta, \xi_i \Vdash act\}) = 1$. □

Actions are formulas that are mutually exclusive in a state. An interesting result of having formulas that are mutually exclusive is that once we define the probability of some formulas, the probability of the other formulas gets implicitly conditioned and restrained by a lower bound.

To see this we present an example:

Example 3.3.4. Consider a system η with only one agent A and only two possible actions for a local state ξ. Let these actions be $left$ and $right$, with $\mu(\{\eta : \eta, \xi \Vdash left\}) = \frac{1}{3}$ and $\mu(\{\eta : \eta, \xi \Vdash right\}) = \frac{2}{3}$.

Given any formula φ, the probability of $left \Rightarrow \varphi$ is then bounded,

$\frac{2}{3} \leq \mu(\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}) \leq 1$.

Let us take a closer look at this.

The set $\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}$ can be represented as the union of the disjoint sets $\{\eta : \eta, \xi \nVdash left\}$ and $\{\eta : \eta, \xi \Vdash \varphi\} \cap \{\eta : \eta, \xi \Vdash left\}$. By addictivity of μ we get

$$\mu(\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}) = \mu(\{\eta : \eta, \xi \nVdash left\}) + \mu(\{\eta : \eta, \xi \Vdash \varphi\} \cap \{\eta : \eta, \xi \Vdash left\})$$

But as we noted, actions are exclusive, i.e. if $left$ is false then $right$ is true, and we get $\{\eta : \eta, \xi \nVdash left\} = \{\eta : \eta, \xi \Vdash right\}$.
This is where we get the lower bound, from the formula

$$\mu(\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}) = \mu(\{\eta : \eta, \xi \nVdash left\}) + \mu(\{\eta : \eta, \xi \Vdash \varphi\} \cap \{\eta : \eta, \xi \Vdash left\})$$

replacing $\{\eta : \eta, \xi \Vdash right\}$

$$\mu(\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}) = \mu(\{\eta : \eta, \xi \Vdash right\}) + \mu(\{\eta : \eta, \xi \Vdash \varphi\} \cap \{\eta : \eta, \xi \Vdash left\})$$

$$\Leftrightarrow$$

$$\mu(\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}) = \frac{2}{3} + \mu(\{\eta : \eta, \xi \Vdash \varphi\} \cap \{\eta : \eta, \xi \Vdash left\})$$

and $\mu(\{\eta : \eta, \xi \Vdash \varphi\} \cap \{\eta : \eta, \xi \Vdash left\}) \geq 0$, hence

$$\mu(\{\eta : \eta, \xi \Vdash left \Rightarrow \varphi\}) \geq \frac{2}{3}$$

Analogously for any formula φ, the probability of $right \Rightarrow \varphi$ is
$\frac{1}{3} \leq \mu\left(\{\eta : \eta, \xi \Vdash right \Rightarrow \varphi\}\right) \leq 1$.

□

To further extend the notions presented in Lemma 3.3.3 and Example 3.3.4 we include an example where three actions are possible after the first event. Each of this actions affects component p in a different way. The final probability of p is a result of these influences The interesting bit is that the result is somewhat counter-intuitive if we are just looking at the probabilities in the implication formulas (4), (5), and (6)..

Example 3.3.5. The system is composed of a sole agent, $Id = \{A\}$. The possible actions are $Act_A = \{green, blue, red, \mathsf{nil}\}$. The set of propositional symbols is $\Pi_A = \{p\}$. To present this in a more appealing way, let us set a scenario: There are three levers, one *green*, one *blue* and one *red*. We will pull one of the levers to try and win a prize p. The chance to choose any lever is the same, $\frac{1}{3}$. However, pulling the *green* lever always gets us the prize. Pulling the *red* rewards no prize. The blue lever has an arbitrary chance of awarding us the prize.

We show that the chance to get the prize is greater than $\frac{1}{3}$ plus s.

1. $@_A \left[\int_{=\frac{1}{3}} * \Rightarrow \mathsf{X} green\right]$
2. $@_A \left[\int_{=\frac{1}{3}} * \Rightarrow \mathsf{X} blue\right]$
3. $@_A \left[\int_{=\frac{1}{3}} * \Rightarrow \mathsf{X} red\right]$
4. $@_A \left[\int_{=1} * \Rightarrow \mathsf{G}\left(green \Rightarrow p\right)\right]$
5. $@_A \left[\int_{=\frac{2}{3}+s} * \Rightarrow \mathsf{G}\left(blue \Rightarrow p\right)\right]$, $s \in \left[0, \frac{1}{3}\right]$
6. $@_A \left[\int_{=\frac{2}{3}} * \Rightarrow \mathsf{G}\left(red \Rightarrow p\right)\right]$

We show that $\Gamma \vDash @\left[\int_{\geq \frac{1}{3}+s} * \Rightarrow \mathsf{X} p\right]$.

To see this, let μ be an arbitrary model.

Using the (1), $\mu \Vdash @_A \left[\int_{=\frac{1}{3}} * \Rightarrow \mathsf{X} green\right]$ so by definition we get $\mu\left(\{\eta : \eta \Vdash * \Rightarrow \mathsf{X} green\}\right) = \frac{1}{3}$ and by Lemma 3.3.1(.1) and we get $\mu\left(\left\{\eta : \eta, \xi^0 \Vdash \mathsf{X} green\right\}\right) = \frac{1}{3}$ which is $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash green\right\}\right) = \frac{1}{3}$. Analogously we get $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash blue\right\}\right) = \frac{1}{3}$ and $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash red\right\}\right) = \frac{1}{3}$.

From (4), (5) and (6) using Lemma 3.3.1(.1) and 3.3.1(.2) we get

- $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash green \Rightarrow p\right\}\right) \geq 1$
- $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash blue \Rightarrow p\right\}\right) \geq \frac{2}{3} + s$

- $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash red \Rightarrow p\right\}\right) \geq \frac{2}{3}$

Note that $\left\{\eta : \eta, \xi^1 \Vdash green \Rightarrow p\right\} = \left\{\eta : \eta, \xi^1 \nVdash green\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash p\right\} \cap \left\{\eta : \eta, \xi^1 \Vdash green\right\}$. Also note that the opposite of the action *green* taking place is when it either happens *blue* or *red*, that is $\left\{\eta : \eta, \xi^1 \nVdash green\right\} = \left\{\eta : \eta, \xi^1 \Vdash blue\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash red\right\}$ and these last two are disjoint sets. Also we have that $\left\{\eta : \eta, \xi^1 \Vdash p\right\} \cap \left\{\eta : \eta, \xi^1 \Vdash green\right\}$ is in fact $\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash green\right\}$.
With this in mind, we have that
$\mu\left(\left\{\eta : \eta, \xi^1 \Vdash green \Rightarrow p\right\}\right) =$
$= \mu\left(\left\{\eta : \eta, \xi^1 \Vdash blue\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash red\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash green\right\}\right)$
and all the sets we are measuring are disjoint, therefore we have
$\mu\left(\left\{\eta : \eta, \xi^1 \Vdash green \Rightarrow p\right\}\right) = \frac{1}{3} + \frac{1}{3} + \mu\left(\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash green\right\}\right) \geq 1$,
which leads to
$\mu\left(\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash green\right\}\right) \geq \frac{1}{3}$.
Analogously we derive that $\left\{\eta : \eta, \xi^1 \Vdash blue \Rightarrow p\right\} =$
$= \left\{\eta : \eta, \xi^1 \nVdash blue\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash p\right\} \setminus \left\{\eta : \eta, \xi^1 \nVdash blue\right\}$
and write this as
$\left\{\eta : \eta, \xi^1 \Vdash blue \Rightarrow p\right\} =$
$= \left\{\eta : \eta, \xi^1 \Vdash green\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash red\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash blue\right\}$.
The measure for these sets is
$\mu\left(\left\{\eta : \eta, \xi^1 \Vdash blue \Rightarrow p\right\}\right) =$
$= \mu\left(\left\{\eta : \eta, \xi^1 \Vdash green\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash red\right\} \cup \left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash blue\right\}\right)$
and that is$\mu\left(\left\{\eta : \eta, \xi^1 \Vdash blue \Rightarrow p\right\}\right) = \frac{1}{3} + \frac{1}{3} + \mu\left(\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash blue\right\}\right) \geq \frac{2}{3} + s$,
so we get $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash blue\right\}\right) \geq s$.
Lastly and in the same way we reach that
$\mu\left(\left\{\eta : \eta, \xi^1 \Vdash red \Rightarrow p\right\}\right) = \frac{1}{3} + \frac{1}{3} + \mu\left(\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash red\right\}\right) \geq \frac{2}{3}$
and this means that $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash red\right\}\right) \geq 0$.
If we combine all these sets, $\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash green\right\}$, $\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash blue\right\}$ and $\left\{\eta : \eta, \xi^1 \Vdash p \text{ and } \eta, \xi^1 \Vdash red\right\}$we get the union $\left\{\eta : \eta, \xi^1 \Vdash p\right\}$, and since they are all disjoint, $\mu\left(\left\{\eta : \eta, \xi^1 \Vdash p\right\}\right) \geq \frac{1}{3} + s + 0$, that is $\mu\left(\left\{\eta : \eta, \xi^0 \Vdash \mathsf{X}p\right\}\right) =\geq \frac{1}{3} + s$ and by definition and Lemma 3.3.1(.1) we get $\mu \Vdash @_A\left[\int_{\geq \frac{1}{3}+s} * \Rightarrow \mathsf{X}p\right]$.

If at first glance the result seemed a bit counter-intuitive, now we see that this was due to having probabilities in a implication conditioned by other formulas. In fact, if we were to write e.g. formula (6) in the form $@_A\left[\int_{=r} * \Rightarrow \mathsf{G}\left(red \Rightarrow p\right)\right]$ then it must be that $r \in \left[\frac{2}{3}, 1\right]$. The lower bound for r is conditioned by the first formulas, since the implication in (6) is true if we have either *green* or *blue*, which is an event with $\frac{2}{3}$ probability. It only remains $\frac{1}{3}$ for an implication $red \Rightarrow p$ where *red* is true . □

3.3.1.2 Communication in $\mathsf{pDTL_G}$

As we see it, the way to proceed with communication is to have a channel that transfers the messages between agents. The reason for this is that it allow us to represent communication with noise (distortion of the message) by having a probability inside the channel. E.g., channel Ch receives message m form agent A and has probability s of sending m to B.

The "alternative" that we reject, is to have a probability that differs from 1 for the communication formula, e.g. $@_A \left[\int_{=s} act1 \Rightarrow ©_B [act2]\right]$. This raises another set of problems. For instance, since we can't guarantee that $act2$ occurs in B when $act1$ occurs in A, we would reach situations where there was $act2$ without $act1$ ever occurring. This would ruin some of the concepts of communication. So we leave it as a sure event, always with probability 1 for a communication formula, and insert the distortion inside the channel.

To end our presentation of $\mathsf{pDTL_G}$ we provide a last example.

Example 3.3.6. This is a simple scenario. An action act happens with half chance at the start of events of agent A. If this action happens, then A communicates with the channel, which in turn communicates with agent B to assign the value 1 to formula p. Since action act has $\frac{1}{2}$ chances, then in the end p will be true for B with at least $\frac{1}{2}$ chances.

For this system let $Id = \{A, B, Ch\}$, and

- $\Pi_A = \emptyset$, $Act_A = \{act, \mathsf{nil}\}$
- $\Pi_B = \{p\}$, $Act_B = \{finish, \mathsf{nil}\}$
- $\Pi_{Ch} = \emptyset$, $Act_{Ch} = \{queue, send, \mathsf{nil}\}$

And let Γ be the set containing the formulas:

1. $@_A \left[\int_{=\frac{1}{2}} * \Rightarrow \mathsf{X}act\right]$
2. $@_A \left[\int_{=1} act \Rightarrow ©_{Ch} [queue]\right]$
3. $@_{Ch} \left[\int_{=1} queue \Rightarrow ©_A [act]\right]$
4. $@_{Ch} \left[\int_{=1} queue \Rightarrow \mathsf{X}send\right]$
5. $@_{Ch} \left[\int_{=1} send \Rightarrow ©_B [finish]\right]$
6. $@_B \left[\int_{=1} finish \Rightarrow ©_{Ch} [send]\right]$
7. $@_B \left[\int_{=1} finish \Rightarrow p\right]$
8.

We wish to show that $\Gamma \vDash @_B \left[\int_{\geq \frac{1}{2}} * \Rightarrow \mathsf{F}p\right]$.

Suppose μ satisfies Γ. From (1) we get $\mu \Vdash @_A \left[\int_{=\frac{1}{2}} * \Rightarrow \mathsf{X}act\right]$, by definition $\mu\left(\{\eta : \eta_A \Vdash * \Rightarrow \mathsf{X}act\}\right) = \frac{1}{2}$.

Also by Lemma 3.3.1.1 we know that $\mu\left(\left\{\eta : \eta_A, \xi_A^1 \Vdash act\right\}\right) = \frac{1}{2}$.

From (2) we get $\mu\left(\{\eta : \eta_A \Vdash act \Rightarrow ©_{Ch}\,[queue]\}\right) = 1$ by definition and with Lemma 3.3.1(.2) we get $\mu\left(\left\{\eta : \eta_A, \xi_A^1 \Vdash act \Rightarrow ©_{Ch}\,[queue]\right\}\right) = 1$. Using Lemma 3.3.1(.4) we then get $\mu\left(\left\{\eta : \eta_A, \xi_A^1 \Vdash ©_{Ch}\,[queue]\right\}\right) \geq \frac{1}{2}$ which by definition yields $\mu\left(\{\eta : \eta_{Ch}, \xi_{Ch}^n \Vdash queue\}\right) \geq \frac{1}{2}$ for some ξ^n.

Using formula (4) $\mu \Vdash @_{Ch}\left[\int_{=1} queue \Rightarrow \mathsf{X}send\right]$ is by definition $\mu\left(\{\eta : \eta_{Ch} \Vdash queue \Rightarrow \mathsf{X}send\}\right) = 1$, and by Lemma 3.3.1(.2) $\mu\left(\{\eta : \eta_{Ch}, \xi_{Ch}^n \Vdash queue \Rightarrow \mathsf{X}send\}\right) = 1$. This will lead to $\mu\left(\{\eta : \eta_{Ch}, \xi_{Ch}^n \Vdash \mathsf{X}send\}\right) = \mu\left(\left\{\eta : \eta_{Ch}, \xi_{Ch}^{n+1} \Vdash send\right\}\right) \geq \frac{1}{2}$ using Lemma 3.3.1(.4).

Using Lemma 3.3.1(.2) with (5) we get $\mu\left(\left\{\eta : \eta_{Ch}, \xi_{Ch}^{n+1} \Vdash send \Rightarrow ©_B\,[finish]\right\}\right) = 1$ and again by 3.3.1(.4) we get $\mu\left(\left\{\eta : \eta_{Ch}, \xi_{Ch}^{n+1} \Vdash ©_B\,[finish]\right\}\right) \geq \frac{1}{2}$ if by definition $\mu\left(\{\eta : \eta_B, \xi_B^m \Vdash finish\}\right) \geq \frac{1}{2}$ for some ξ^m.

Finally from (7) we get $\mu\left(\{\eta : \eta_B, \xi_B^m \Vdash finish \Rightarrow p\}\right) = 1$ with Lemma 3.3.1(.2) and $\mu\left(\{\eta : \eta_B, \xi_B^m \Vdash p\}\right) \geq \frac{1}{2}$ with 3.3.1(.4). Now if we notice that the set $\left\{\eta : \eta_B, \xi_B^0 \Vdash \mathsf{F}p\right\}$ contains the set $\{\eta : \eta_B, \xi_B^m \Vdash p\}$, then we can relate their measures, $\mu\left(\left\{\eta : \eta_B, \xi_B^0 \Vdash \mathsf{F}p\right\}\right) \geq \mu\left(\{\eta : \eta_B, \xi_B^m \Vdash p\}\right) \geq \frac{1}{2}$, and finally using Lemma 3.3.1(.1) we get that $\mu\left(\{\eta : \eta_B \Vdash * \Rightarrow \mathsf{F}p\}\right) \geq \frac{1}{2}$ which is by definition $\mu \Vdash @_B \left[\int_{\geq \frac{1}{2}} * \Rightarrow \mathsf{F}p\right]$, proving the result.

□

Chapter 4

Conclusion

In this work we have started by defining a linear temporal logic. We presented results for soundness and completeness that were similar to others [19] but had some new input since we used different base temporal operators.

In the section of distributed temporal logic, the contribution of this work was to show that there is a reduction from DTL to LTL.

When defining pLTL we took a somewhat naive approach to adding probabilities to LTL by having only propositional formulas with probabilities. This limited the expressive power of the logic.

The first section of probabilities and DTL, where we presented pDTL1, served to understand the difficulties of inserting probabilities in a distributed system, the problems that arise when there are agents interacting with each other, what happens when there's a probability on a formula of one agent and how probabilities would travel through the communication between the agents.

We choose to have probabilities just in propositional formulas to try and keep everything under control, whit out having to deal with probabilities inside probabilities. If we had temporal formulas with probabilities, this would have implications on probabilities of simple propositional formulas on different states, not mentioning that keeping an endogenous approach would allow probabilities on temporal level and propositional level, which means there would be nested probability operators.

So in any case, an higher level would be needed to insert the probabilities.

This is why we took the final section with an exogenous concept. It limits the expressive power of the language, allowing only statements that are more general in nature. We are not able to fix one world and make assertions about that specific world.

The advantages are the antithesis to the problems that arose when we first tried the language of pDTL_L. We get a clean language, with no complications about chained prob-

abilities and no communication problems.

Now we know that the initial approach to pLTL should have been similar to that of pDTL_G, with probabilities on the models, and then work up to pDTL from there.

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