Modelling Multiple Uncertainties in Real Options

Evaluating the investment in the high speed train

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Abstract. In this work, we study the problem of determining an optimum investment policy. We assume that the demand for a certain good or service follows a Brownian geometric motion with Poisson jumps and that the agent must determine the moment on which the investment will be made. This investment moment is a result of a trade off between the investment costs and the expected future earnings from that investment. In particular, we analyze the optimal investment moment for the high speed transportation (TGV). This case study is performed by using real option analysis, assuming a stochastic context. Determining the optimal investment moment can be regarded as an optimal stopping problem, and as such, we will use dynamic programming. At first, we assume a single stochastic process involved (the demand) and later we reason about the case where the investment expenses themselves are modeled by a stochastic process. The latter situation - closer to the financial reality - allows us to consider interesting questions, like the correlation between the processes. The theoretical results are presented with the aid of simulations, allowing us, in some cases, to draw conclusions about the optimal investment policy in regard to the moment when it should be executed and the expenses associated with such decision.

1 Introduction

Nowadays, financial markets are heavily influenced by uncertainty. Such uncertainty unavoidably affects the Decision Process (process through which one chooses an alternative among a set of potential actions) of any investment.

In this work, we evaluate the option of investing *versus* delaying an investment in an uncertain environment (modeled stochastically) using Real Option Analysis (ROA). This kind of analysis offers new perspectives about the impact of uncertainty in the valuation of a project. The idea is that *naive* approach of comparing *a priori* the expected benefits with the expected costs of a project and committing to a decision in the initial moment should be revised, since it disregards the environment's uncertainty that may make the project profitable sometime in the future.

ROA considers that the option to invest should take into account that the investment incurs an irreversible cost and that there is a possibility of postponing the investment in view of the uncertainty. Despite being conceptually simple, ROA is of great use in the analysis of projects heavily influenced by different sources of uncertainty.

The decision process of an investment project may need to consider several kinds of real options. Naturally, the ones that are most used are the ones that more accurately model reality due to their flexibility. Among these are options like Delaying Options, Expansion Options or Abandonment.

In this dissertation, we will only consider Delaying Options, that is, the opportunity to wait before investing. Put simply, our objective is to decide, in each moment, if we should invest or delay the investment - Delaying Option. Ultimately, what we wish to know is the moment in which we should take the investment decision; in decision theory language, we want to solve an Optimal Stopping Problem, where the moment of optimal investment is the Optimal Stopping Time.

The problem is tackled in a stochastic environment, considering either one or two uncertainty sources and using Dynamic Programming, a technique that is widely used in Economies and Finance.

We apply the theoretical results presented in this dissertation in the real-world case study of investment in the new high speed train service (TGV). We shall assume two uncertainty environments: the demand for the TGV is the only source of uncertainty and, in addition to the demand for the TGV, the investment costs are also a source of uncertainty.

Uncertainty is modeled with either a Geometric Brownian Motion or with a Geometric Browninan Motion with Jumps. We will find, analytically, the value of the delaying option and, using the data in [11] and [5], compute numerically, optimal stopping values and times.

2 Formal definition of the problem

In this work, we analyze the optimal decision procedure for investment in the high speed train service. This is a case study amenable to analysis through Delaying Options: It's a project with a high and irreversible cost (since the costs of investment in the high speed rail cannot be converted in investments in the conventional service, at least with current technology), influenced by several sources of uncertainty that affect its long term profitability, and flexible (in the sense that the investment can be delayed until optimal conditions are gathered).

The analysis we perform here is mathematical in nature, in line with those performed in [11]. This means that this analysis in not exhaustive, since it does not deal with competitiveness between services, social/political/environmental impact, or the consequences of relocation of industries related to train services. The investment is studied *per si* , that is, considering only profit generation by passengers. For a more detailed discussion of these considerations, we suggest the reading of [11]. We consider that the main benefit associated with the implementation of the TGV is the decrease in travel time. We also assume that a user only opts for using the new service if it's more convenient to him personally than using the conventional service. Should he choose to use it, the service will have for each passenger a positive impact (by reducing the travel time) and a negative impact (since the cost of the ticket is greater then otherwise). Therefore, using the TGV service is the result of a balance between these factors.

We shall assume, at first, that the uncertainty of the project originates from a single source, the *demand* for the TGV, which we shall model by a continuous time stochastic process, denoted $\{X_t, t \geq 0\}$. To define the optimal investment policy, we need to find:

- Critical demand level, x^*: this is the value of the uncertainty (demand, in this case) that justifies the decision of investing in this project. It's a function of the investment cost (that we assume constant at this point, but we will also assume later to be random) and the expected future earnings from the increase in the demand itself.
- Optimal investment moment, t^*: this is the moment in time when, for the first time, the demand level is greater or equal to x^*. This means that t^* denotes the moment when the investment decision should be made.

AS a result, $t^* = \inf\{t : X_t \geq x^\star\}$ is therefore a stopping time for the stochastic process $\{X_t, t \geq 0\}$.

Determining t^* analytically is not always possible. Because of this, in this work, we will estimate this quantity through simulation.

We will assume two models for the stochastic process $\{X_t, t \geq 0\}$:

- Geometric Brownian Motion (process with continuous trajectories);
- Geometric Brownian Motion with jumps (process with trajectories that are eventually discontinuous).

In the first case, the underlying model is

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1}$$

where $\{W_t, t \geq 0\}$ is the standard Brownian motion, and μ and σ represent the drift and diffusion coefficients, both assumed known. We use the notation from Financial Mathematics, namely by (abusively) using the differential d.

In the second case, the underlying model is

$$dX_t = \mu X_t dt + \sigma X_t dW_t + U X_t dN_t \tag{2}$$

where $\{N_t, t \geq 0\}$ is a Poisson Process with rate λ, independent of $\{W_t\}$, that models the rate of jumps, and U denotes the value of those jumps. Notice that this equation implies that, if there are jumps in the interval $(t, t+h)$ then, disregarding the continuous part, $X_{t+h} = (1+U)X_t$.

In this work we will follow the assumptions on [11] and [5], as well as some of the mathematical expositions in these references.

Later in the work, we assume another source of uncertainty, modeling the investment expense. This is a realistic assumption, since the investment expense is function of a lot of factors, leading to its random behavior. Therefore, besides the dynamics for demand $\{X_t, t \geq 0\}$, we shall also assume that the investment expenses, denoted by $\{\Gamma_t, t \geq 0\}$, is a stochastic process with similar characteristics to the demand.

In both cases, we deduce x^*, but we only determine t^* through simulation.

We shall assume that the time between the decision making and its implementation is constant and equal to n, that is, n represents the time that it takes to build the TGV.

We also need to define the costs associated with time of travel and cost of tickets. For $s \leq t^* + n$, we shall assume that the cost of each trip is given by ψ, obtained as

$$\psi = \eta + p$$

where η is the value of time spent in a trip and p is the value of the price of a ticket (value measured in abstract utility, not necessarily in currency), both function of the demand as

$$\eta(X_s) = \beta X_s^{\delta_\beta} \tag{3}$$

$$p(X_s) = \alpha X_s^{\delta_\alpha} \tag{4}$$

where δ_β and δ_α are usually called elasticity (δ_β is the elasticity between the value of time η and demand X_s and δ_α the elasticity between the value of the price of tickets p and demand X_s). We shall assume that $\delta_\beta = \delta_\alpha = \delta$.

The previous equation counts the travel time under the total cost, and is thus different in the period of time prior and posterior to the execution of the project. Such difference is captured by the parameter β that takes two different values, β_1 and β_2, such that their different reflects the impact in reducing traveling time. Therefore, the total cost of the trip is different if its made through the current service (ψ_1) or the TGV service (ψ_2), such that

$$\psi_1(X_s) = \beta_1 X_s^\delta + \alpha X_s^\delta \tag{5}$$

$$\psi_2(X_s) = \beta_2 X_s^\delta \tag{6}$$

for $s \leq t^* + n$

Notice that in (6) we do not include the cost associated with the price of the tickets, that is traditionally included in the investment expense. We believe this decision may not be the best, since it wouldn't be too difficult to handle this cost explicitly, like in (5) but for the sake of continuity with [11], we've decided to keep this assumption.

We shall now find the optimal investment policy under these hypothesis. We want to find a rule that allows us to decide for which demand and investment expense levels one should invest. In dynamic Programming, this is called an Optimal Stopping Time problem.

3 Dynamic Programming

In our problem, in each moment, we can only take one of two actions: either we choose to *stop* (i.e. we implement the TGV system in that moment) or to *continue* (i.e. we keep waiting).

For a moment t, let x_t be the value of the random variable X_t, assumed known. Additionally,

- $\pi(x_t)$ denotes the cash flow (difference between earnings and expenses in moment t if in that moment one chooses to *continue*) when demand equals x_t;
- $F(x_t)$ denotes future profits (difference between all earnings and expenses from moment t on);
- ρ denotes the interest rate (which we assume constant to ease the computations).

If the decision is to *continue*, the future profit in t, $F(t)$, is the cash flow in that instant $\pi(x_t)$ plus the expected value of the future profit in $t+1$, dully actualized, i.e.:

$$\pi(x_t) + \frac{1}{1+\rho} E\left[F(X_{t+1})|x_t\right] \tag{7}$$

In the literature, the value $E\left[F(X_{t+1})|x_t\right]$ is usually called *Continuity Value.*

Let $\Omega(x_t)$be the profit associated with the decision to *stop* (i.e. implementing the TGV). Then, since our goal is to maximize profit, if $\pi(x_t)+\frac{1}{1+\rho}E\left[F(X_{t+1})|x_t\right] > \Omega(x_t)$, then it is more beneficial to *continue* (waiting) and otherwise, it is more beneficial to *stop* (and invest). We keep on making this comparison until $\Omega(x_t)$ exceeds the continuation profit. Therefore

$$F(x_t) = \max\left\{\Omega(x_t), \pi(x_t) + \frac{1}{1+\rho} E\left[F(X_{t+1})|x_t\right]\right\}$$

Definition 1 (Optimal Decision, Continuation Region, Stopping Region).
In moment t, let

$$C = \{x : \pi(x) + \frac{1}{1+\rho} E\left[F(x_{t+1})|x\right] > \Omega(x)\}$$

Then, if $x \in C$, we say that the Optimal Decision *is to continue. Otherwise we say that the Optimal Decision is stopping.*

The set C is called Continuation Region *and its complement is called* Stopping Region.

All the presented definitions are formalized in [6]. If $\overline{C} \neq \emptyset$, then there is at least one boundary value that we will call *critical value* or *optimal stopping time.*

We can therefore redefine t^* as

$$t^* = \inf\left\{t : \Omega(X_t) > \pi(X_t) + \frac{1}{1+\rho} E\left[F(X_{t+1})|X_t\right]\right\}$$

and define the optimal stopping time as a function of t^*.

Definition 2 (Optimal Stopping Time). *The* optimal stopping time*, x^*, is the value for which the decision of continuing is optimal for $X_t < x^*$ and the decision of stopping is optimal if $X_t \geq x^*$.*

In the continuous case, from Bellman's equation ([6, 13]), the profit in the continuation region is given by

$$F(x_t) = \pi(x_t)dt + \frac{1}{1+\rho dt} E\left[F(x_t + dX_t)|x_t\right]$$

and, with some manipulation,

$$(1-\rho dt)dF(X_t) = dF(X_t) - \rho dt dF(X_t) = dF(X_t).$$

On the other hand, in the stopping region

$$F(x_t) = \Omega(x_t) \tag{8}$$

Combining both theses expressions and applying Bellman's optimality principle, the profit function F is the solution of the equation

$$F(x_t) = \max\left\{\Omega(x_t), \pi(x_t)dt + (1-\rho dt)F(x_t) + E\left[dF(X_t)|x_t\right]\right\} \tag{9}$$

The objective is now to find a function F that satisfies this equation. By Bellman's optimality principle, by finding F we are also finding the optimal policy, since for each t we can determine the optimal decision.

Rewriting Equation (9), expanding in Taylor Series and using properties of the Brownian motion we get a second order differential equation

$$\frac{1}{2}F''(x)\sigma_x^2 x^2 + F'(x)\mu_x x = \rho F(x)$$

with solution given by

$$F(X_t) = AX_t^a + BX_t^b + C \tag{10}$$

where A, B and C depend on the case we are dealing with.

These three values can be found by analysis of frontier conditions. In particular we want to impose that $F(0) = 0$, that F should be continuous (and therefore $F(x^*) = \Omega(x^*)$) and that F should meet some regularity at the border of C (and therefore $F'(x^*) = \Omega(x^*)$).

4 Derivations of x^* in specific cases

We consider four cases:

- case 1: Constant investment expense γ
  - subcase 1.1 : Geometric Brownian motion
  - subcase 1.2: Geometric Brownian motion with jumps
- case 2: Stochastic investment expense
  - subcase 2.1 : Geometric Brownian motion
  - subcase 2.2: Geometric Brownian motion with jumps

Case 1.1

Let

$$dX_t = X_t\mu_x dt + X_t\sigma_x dW_t^x \tag{11}$$

be the equation that models the demand for the TGV.

The solution for this partial differential equation is given by

$$X_t = x \exp\left\{\left(\mu_x - \frac{\sigma_x^2}{2}\right)t + \sigma_x W_t^x\right\} \tag{12}$$

where $X_0 = x$.

Some manipulation of this equation allows us to conclude that the values for A, B and C in F, in this case, are

$$A = \frac{2(\beta_1 - \beta_2)\exp\left\{\left(\mu_x\theta + \frac{1}{2}\theta(\theta-1)\sigma_x^2 - \rho\right)n\right\}}{2\rho - 2\mu_x\theta - \theta^2\sigma_x^2 + \theta\sigma_x^2} \tag{13}$$

$$B = \frac{2\alpha\ \exp\left\{\left(\mu_x\theta + \frac{1}{2}\theta(\theta-1)\sigma_x^2 - \rho\right)n\right\}}{2\rho - 2\mu_x\theta - \theta^2\sigma_x^2 + \theta\sigma_x^2} \tag{14}$$

$$C = -\frac{\varphi e^{-\rho n} + \gamma}{\rho} \tag{15}$$

where φ denotes the (fixed and known) operational cost.

From these we get that

$$x^* = \exp\left\{\frac{1}{\theta}\ln\left\{\frac{-r_1 C}{(A+B)(r_1-\theta)}\right\}\right\} \tag{16}$$

where r_1 is the positive root of $\frac{1}{2}\sigma_x^2 r(r-1) + \mu_x r - \rho = 0$.

Case 1.2

Let

$$dX_t = X_t\mu_x dt + X_t\sigma_x dW_t^x + X_t dN_t \tag{17}$$

be the equation that models the demand for the TGV.

The solution for this partial differential equation is given by

$$X_t = x\exp\left\{\left(\mu_x - \frac{\sigma_x^2}{2}\right)t + \sigma_x W_t^x\right\}(1+U)^{N_t} \tag{18}$$

Some manipulation of this equation allows us to conclude that the values for A, B and C in F, in this case, are

$$A = \frac{2(\beta_1 - \beta_2)\exp\left\{\left(\mu_x\theta + \frac{1}{2}\theta(\theta-1)\sigma_x^2 + \lambda(1+U)^\theta - \lambda - \rho\right)\right\}n}{2\rho - 2\mu_x\theta - \theta^2\sigma_x^2 + \theta\sigma_x^2 + 2\lambda(1+U)^\theta + 2\lambda} \tag{19}$$

$$B = \frac{2\alpha\ \exp\left\{\left(\mu_x\theta + \frac{1}{2}\theta(\theta-1)\sigma_x^2 + \lambda(1+U)^\theta - \lambda - \rho\right)n\right\}}{2\rho - 2\mu_x\theta - \theta^2\sigma_x^2 + \theta\sigma_x^2 + 2\lambda(1+U)^\theta + 2\lambda} \tag{20}$$

$$C = -\frac{\varphi e^{-\rho n} + \gamma}{\rho} \tag{21}$$

From these we get that

$$x^* = \exp\left\{\frac{1}{\theta}\ln\left\{\frac{-r_1 C}{(A+B)(r_1-\theta)}\right\}\right\} \tag{22}$$

where r_1 is the positive root of $\frac{1}{2}\sigma_x^2 r(r-1) + \mu_x r - (\rho+\lambda) + \lambda(1+U)^r = 0$.

Case 2.1

Let

$$dX_t = X_t\mu_x dt + X_t\sigma_x dW_t^x \tag{23}$$

be the equation that models the demand for the TGV and

$$d\mathit{\Gamma}_t = \mathit{\Gamma}_t\mu_\gamma dt + \mathit{\Gamma}_t\sigma_\gamma dW_t^\gamma \tag{24}$$

be the equation that models the investment cost. We do not assume anything about the correlation between processes.

The method previously employed is not adequate for this situation, since the analytic manipulations quickly become very complex. In order to proceed, we will assume that, in the initial moment, $C = l\gamma$, that is, the exploration costs are proportional to the investment cost. Additionally, we shall assume that the profit function F satisfies $F(kX_t, k\mathit{\Gamma}_t) = kf(X_t^\theta, \mathit{\Gamma}_t)$ for some function f. This means that the project's profit is affected, in the same proportion, by the earnings (function of X_t) and the investment expenses ($\mathit{\Gamma}_t$).

Now we have

$$F(x,\gamma) = F(\gamma\frac{x}{\gamma}, \gamma\frac{\gamma}{\gamma}) = \gamma f\left(\left(\frac{x}{\gamma}\right)^\theta, 1\right) = \gamma f(q)$$

where $q = \left(\frac{x}{\gamma}\right)^\theta$.

Using some properties of Cauchy-Euler equations and some symbolic manipulation we conclude that

$$f(q) = \left[{q^*}^{1-s_1}(A+B) - (l+1){q^*}^{-s_1}\right] q^{s_1}$$

where $q^* = \left(\frac{x^*}{\gamma}\right)^\theta$.

Now we can use this expression to get the value of x^*:

$$x^* = \left[\frac{l+1}{A+B}\frac{s_1}{s_1-1}\gamma^\theta\right]^{\frac{1}{\gamma}} \tag{25}$$

where A and B are defined, respectively, as (13) and (14) and s_1 is the positive root of the equation

$$\frac{1}{2}\left[\sigma_x^2\theta^2 + \sigma_\gamma^2 - 2\sigma_x\sigma_\gamma corr(x,\gamma)\theta\right] s(s-1) + \left[\mu_x\theta + \frac{1}{2}\sigma_x^2\theta(\theta-1) - \mu_\gamma\right] s - (\rho-\mu_\gamma) = 0.$$

Case 2.2

Let

$$dX_t = X_t\mu_x dt + X_t\sigma_x dW_t^x + X_t dN_t \tag{26}$$

be the equation that models the demand for the TGV and

$$d\Gamma_t = \Gamma_t\mu_\gamma dt + \Gamma_t\sigma_\gamma dW_t^\gamma + \Gamma_t dN_t \tag{27}$$

be the equation that models the investment cost. The jump process regulates both the demand and the investment expense. The following results are extensible to the case where the processes are independent, with due alterations. They do not extend to non-trivially correlated processes. However, in economic settings, it is rare to assume such correlations (either the same process or independent processes are usually considered).

Like before, we will assume that, in the initial moment, $C = l\gamma$ and that the profit function F satisfies $F(kX_t, k\Gamma_t) = kf(X_t^\theta, \Gamma_t)$ for some function f.

A very similar reasoning to the previous case allows us to conclude that

$$x^* = \left[\frac{l+1}{A+B}\frac{s_1}{s_1-1}\gamma^\theta\right]^{\frac{1}{\gamma}} \tag{28}$$

where A and B are given, respectively, by (20) and (21) and s_1 is the positive root of

$$\frac{1}{2}\left[\sigma_x^2\theta_\gamma^2 - 2\sigma_x\sigma_\gamma corr(x,\gamma)\theta\right] s(s-1) + \left[\mu_x\theta + \frac{1}{2}\sigma_x^2\theta(\theta-1) - \mu_\gamma\right] s$$
$$-(\rho+\lambda-\mu_\gamma)\lambda(1+U)^{2s} = 0$$

5 Numerical Results

We used numerical simulation both to illustrate the previous results and to estimate numerically the value of t^*, $\widehat{t^*}$. Due to space constraints, we do not present the results here, but they are publicly available in http://isabelle.math.ist.utl.pt/∼l54145/. The computations and simulations were based on a base-case scenario and consisted in varying several parameters in relation to that base case. The values of the parameters for the base case are presented in Table 1. These values were extracted from real world data present in [5] and [11].

Parameters	x	γ	η_1	η_2	p	φ	ρ	μ_x
Value	3	5000	30	10	25	90	0.009	0.1
Parameters	σ_x	n	δ	μ_γ	σ_γ	λ	u	l
Value	0.22	5	0.5	0.001	0.01	0.1	0.1	0.5

Table 1. Data for the base case

x is measured in millions of passengers, t in years, $\gamma, \eta_1, \eta_2, p, \psi$ in millions of Euro.

We present here only the results for the base case under different uncertainty models.

Model	MGB 1 incert	MGBS 1 incert	MGB 2 incert c = -0.5	MGBS 2 incert c = -0.5	MGB 2 incert c = 0	MGBS 2 incert c = 0	MGB 2 incert c = 0.5	MGBS 2 incert c = 0.5
x^*	6.895	7.384	6.917	7.263	6.490	6.950	6.076	6.507
t^*	10.912	10.634	14.594	7.407	10.235	9.968	8.362	5.379

6 Conclusions

From the simulations, it is clear that the values of x^* are greater when we have a model with jumps. This is to be expected since jumps increase the randomness of the processes. It is known from Real Options theory that increases in randomness lead almost surely to decisions in higher critical levels than otherwise. As for t^*, keeping in mind that these results were obtained by simulation, we conclude that t^* is smaller for models with jumps. This is more visible in the cases where we assume two uncertainties, since we have a greater random component. This observations are also in line with the predictions of Real Options Analysis.

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