Product Type Actions with Rokhlin Properties

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Free actions of topological groups on topological spaces have a number of important regularity properties and allow for several classification theorems. It thus is only natural to try and find suitable analogues in the context of C*-dynamical systems, that is, in the context of actions of locally compact groups on C*-algebras. It turns out that even the most natural generalization, freeness on the primitive ideal space, yields little to no results outside some rather restricted classes of C*-algebras, for which one then defines other notions. In this work, we present two of these notions: the strict and the tracial Rokhlin properties.

We start by introducing several important concepts in the theory of C*-algebras and C*-dynamical systems, such as the enveloping C*-algebra, the inductive limit and the tensor and crossed products. We also define important classes of C*-algebras, namely the UHF algebras. Finally, we include a brief overview of the theory of unitary representations of locally compact groups, and define the K_0 group for unital C*-algebras.

Next, we define the above mentioned Rokhlin properties.

Definition. Let $\alpha : G \to \mathrm{Aut}\,(\mathcal{A})$ be an action of a finite group G on a separable unital C*-algebra $\mathcal{A}$. We say α has the **(strict) Rokhlin property** if for any finite set $F \subseteq \mathcal{A}$ and every $\epsilon > 0$, there is a set of pairwise orthogonal projections $\{p_g\}_{g\in G}$ such that

1. $\|\alpha_g(p_h) - p_{gh}\| < \epsilon$ for all $g, h \in G$.
2. $\|p_g a - a p_g\| < \epsilon$ for all $a \in \mathcal{A}$ and $g \in G$.
3. $\sum_{g\in G} p_g = 1_{\mathcal{A}}$.

Despite being a weaker property, in a certain sense, than that of freeness, the Rokhlin property is strong enough to force some regularity results. This is particularly clear when one considers product type actions, i.e. actions on infinite tensor products given, at every finite stage, by the tensor product of inner actions.

Theorem. *Let $\mathcal{A}$ be the UHF algebra of type $\prod k_n$, seen as the infinite tensor product $\bigotimes_{i=1}^{\infty} \mathcal{M}_{k_i}$ of full matrix algebras, and let p_n and q_n be projections in $\mathcal{M}_{k_n}$ for each $n \in \mathbb{N}$, such that $p_n + q_n = 1$. Let α be the product type action of*

$\mathbb{Z}_2$ generated by the order 2 automorphism

$$\bigotimes_{i=1}^{\infty} \mathrm{Ad}\,(p_n - q_n).$$

Then the following conditions are equivalent:

1. *α has the Rokhlin property.*
2. *$\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ is a UHF algebra.*
3. *$K_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}_2)$ is a totally ordered group.*
4. *For infinitely many $n \in \mathbb{N}$, we have $\mathrm{rank}\,(p_n) = \mathrm{rank}\,(q_n)$.*

The Rokhlin property of product type actions on UHF algebras thus forces structure on both the crossed product and the K-theory, and can also be easily checked via condition 4 of the above theorem. Unfortunately, it is quite rare. In an attempt to weaken the Rokhlin property, one then defines the tracial Rokhlin property.

Definition. Let $\alpha : G \to \mathrm{Aut}\,(\mathcal{A})$ be an action of a finite group G on an infinite dimensional simple separable unital C*-algebra $\mathcal{A}$. We say that α has the **tracial Rokhlin property** if for every finite set $F \subseteq \mathcal{A}$, every $\epsilon > 0$ and every positive element $a \in \mathcal{A}$ of norm 1 there is a set of mutually orthogonal projections $\{p_g\}_{g\in G}$ such that, setting $p \equiv \sum_{g\in G} p_g$, for all $f \in F$ and $g, h \in G$

1. $\|\alpha_g(p_h) - p_{gh}\| < \epsilon$.
2. $\|p_g f - f p_g\| < \epsilon$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of $\mathcal{A}$ generated by a.
4. $\|pap\| > 1 - \epsilon$.

In the case of product type actions on UHF algebras, we again have some regularity and structural results.

Theorem. *Let $\mathcal{A}$ be the UHF algebra of type $\prod k_n$, seen as the infinite tensor product $\bigotimes_{i=1}^{\infty} \mathcal{M}_{k_i}$ of full matrix algebras, and let p_n and q_n be projections in $\mathcal{M}_{k_n}$ for each $n \in \mathbb{N}$, such that $p_n + q_n = 1$. Let α be the product type action of $\mathbb{Z}_2$ generated by the order 2 automorphism*

$$\bigotimes_{i=1}^{\infty} \mathrm{Ad}\,(p_n - q_n).$$

Then the following conditions are equivalent:

1. *α has the tracial Rokhlin property.*

2. $\hat{\alpha}$ *is trivial on the space of tracial states of* $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$.

3. $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ *has a unique tracial state.*

4. *For all* $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \prod_{i=m+1}^{n} \left(\frac{\operatorname{rank}(p_n) - \operatorname{rank}(q_n)}{\operatorname{rank}(p_n) + \operatorname{rank}(q_n)} \right) = 0.$$

The proofs of these results rely on the explicit construction of certain partial isometries, and unfortunately such construction is not trivial to generalize for actions of $\mathbb{Z}_n$ with $n > 2$, where the action is generated by, this time,

$$\bigotimes_{n=1}^{\infty} \operatorname{Ad} \left(\sum_{j=1}^{\gamma} e^{\frac{2\pi i j}{\gamma}} p_j^{(n)} \right),$$

where the set $\{p_j^{(n)}\}$ is, for each $n \in \mathbb{N}$, a partition of unity of mutually orthogonal projections. In this work we present a lemma that may prove useful should one wish to pursue an explicit proof.

Lemma. *Let* $\{p_i\}$ *be a finite partition of unity of mutually orthogonal projections in a unital* C^**-algebra, and let* i_1, i_2, i_3 *be different indices. Suppose there are two partial isometries* v *and* w *such that, setting* $p \equiv p_{i_1}$, $q \equiv p_{i_2}$ *and* $r \equiv p_{i_3}$ *for some different indices* i_1, i_2 *and* i_3, *we have*

- $p = vv^* = ww^*$.
- $q = v^*v$.
- $r = w^*w$.

Then

$$vw = v^*w = vw^* = v^*w^* = 0.$$

General product type actions of arbitrary finite cyclic groups on UHF algebras is still an open topic, requiring further investigation and quite possibly requiring different machinery than the one used in this work.