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Product Type Actions with Rokhlin Properties

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Resumo

Apresentam-se vários conceitos e resultados da teoria de álgebras-C*, assim como construções especiais tais como a álgebra-C* envolvente, o limite indutivo e os produtos tensorial e cruzado. Abordam-se igualmente as bases da teoria de grupos topológicos e representações unitárias de grupos localmente compactos. Apresentam-se, como noções relacionadas com o conceito de "acção livre de um grupo numa álgebra-C*", as noções centrais neste trabalho: as de acção de um grupo com a propriedade de Rokhlin e de acção de grupo com a propriedade tracial de Rokhlin.

Mostra-se que para acções tipo produto de grupos finitos cíclicos em álgebras UHF as propriedades de Rokhlin têm diversas caracterizações. Em particular, a propriedade de Rokhlin é equivalente ao produto cruzado ser igualmente uma álgebra UHF, o que por sua vez é equivalente a certas condições nas projecções subjacentes. A propriedade tracial de Rokhlin pode ser descrita em termos dos estados traciais do produto cruzado, em termos da acção dual, ou ainda em termos das projecções envolvidas.

Palavras-chave: álgebra-C*, sistema dinâmico-C*, acção tipo produto, propriedade de Rokhlin, propriedade tracial de Rokhlin.

Abstract

We present several important definitions and results in the theory of C*-algebras, as well as some special constructions such as the enveloping C*-algebra, the inductive limit and the tensor and crossed products. We also present the stepping stones of the theory of topological groups and unitary representations of locally compact groups. We introduce, as notions related to the concept of "free action of a group on a C*-algebra", the main topic of this work: group actions with the Rokhlin or tracial Rokhlin properties.

We show that for product type actions of finite cyclic groups on UHF algebras, the Rokhlin properties can be expressed in several equivalent ways. In this scenario, the Rokhlin property is equivalent to the associated crossed product also being a UHF algebra, which in turn is equivalent to certain conditions on the ranks of the underlying projections. Similarly, the tracial Rokhlin property can be expressed in terms of tracial states on the crossed product, in terms of the dual action, or, once more, in terms of the ranks of the projections involved.

Keywords: C*-algebra, C*-dynamical system, product type action, Rokhlin property, tracial Rokhlin property.

Contents

Glossary

$0 \leq a$	a is a positive element, 7
1_A	unit (multiplicative identity) of $\mathcal{A}$, 5
$G^{\frown}$	dual group of G, 24
H_x	Heaviside function, 13
$K_0(\mathcal{A})$	the K_0 group of $\mathcal{A}$, 27
S^1	circle group, 23
$T(\mathcal{A})$	tracial states of $\mathcal{A}$, 12
$U(\mathcal{A})$	unitary elements of $\mathcal{A}$, 6
$V(\mathcal{A})$	equivalence classes of $\mathcal{A} \otimes \mathcal{K}$, 27
$\mathcal{A} \oplus \mathcal{B}$	direct sum of $\mathcal{A}$ and $\mathcal{B}$, 13
$\mathcal{A} \otimes \mathcal{B}$	tensor product of $\mathcal{A}$ and $\mathcal{B}$, 20
$\mathcal{A}_+$	positive elements of $\mathcal{A}$, 7
$\operatorname{Aut}(\mathcal{A})$	automorphisms of $\mathcal{A}$, 5
$\mathcal{M}_n$	C*-algebra of $n \times n$ complex matrices, 15
$\hat{\mathcal{A}}$	spectrum of $\mathcal{A}$, 12
$\mathcal{K}$	compact operators on L^2, 27
$\rtimes$	integrated form (of a covariant representation), 31
$\rtimes_\alpha$	crossed product, 30
$\operatorname{sp}(x)$	spectrum of x, 12
tr_n	tracial state of $\mathcal{M}_n$, 15
$\varinjlim G_n$	inductive limit of groups, 26
$\widehat{G}$	dual group of G, 24
$*$	involution, 4
$a \geq 0$	a is a positive element, 7
$b \geq a$	$b - a$ is a positive element, 8
$p \overset{u}{\sim} q$	p is unitarily equivalent to q, 7
$p \perp q$	p and q are orthogonal, 6
$p \sim q$	p and q are Murray-von Neumann equivalent, 7
Ad	adjoint automorphism, 43

Introduction

With the birth of quantum mechanics, it became apparent that certain mathematical structures, coined "algebras of observables" by physicists, were of great importance. These were used to model algebras of physically observable quantities, hence the name. In 1943, Gelfand and Naimark formulated a precise axiomatic definition of these algebras, which is now the standard definition of C*-algebra, but it was not until 1947 that Segal actually named them C*-algebras. At that point, Segal was considering norm-closed subalgebras of algebras of bounded linear operators on Hilbert spaces, and the 'C' stood for 'closed'. It is worth mentioning that all C*-algebras as defined by Gelfand and Naimark can be considered as C*-algebras as defined by Segal and vice-versa.

Actions of certain topological groups on C*-algebras quickly entered the mainstream of the theory, as a way to generalize classical dynamical systems. This gave rise to the notion of crossed product C*-algebra, a systematic construction to encode information of the group action, first introduced in its now standard form in 1966, in a paper by Doplicher, Kastler and Robinson. As actions of groups proved their importance, mathematicians sought generalizations of standard useful topological and algebraic notions, in particular that of a free action of a group on a set or topological space. Unfortunately, this proved to be quite hard, as there is no natural analogous notion in the context of C*-algebras.

The purpose of this work is to, starting from the very beginning, present two concepts that can, to some extent, be considered as possible generalizations of freeness: the Rokhlin and the tracial Rokhlin properties. Furthermore, we will be interested in seeing exactly how these properties can be expressed via the previously mentioned constructions in the case of product type actions of finite cyclic groups on UHF algebras. There was also considerable effort into making this work reasonably self-contained, but unfortunately some of the proofs had to be omitted.

In Chapter 1 we present the most basic of definitions, such as C*-algebra (naturally), ideals, C*-subalgebras and the enveloping C*-algebra, as well as fundamental constructions like the inductive limit and the tensor product. We end Chapter 1 with a brief overview of topological groups, unitary representations of locally compact groups and the K_0-group. In Chapter 2 we dedicate ourselves to the study of the crossed product and its associated representation theory, and introduce three notions related with the concept of "free action of a group on a C*-algebra", in particular the Rokhlin and the tracial Rokhlin properties. Finally, in Chapter 3, we take a close look at the consequences of the Rokhlin properties for product type actions of finite cyclic groups, especially $\mathbb{Z}_2$, on UHF algebras. For these, the Rokhlin property determines the structure of the crossed product. Indeed, the action has the Rokhlin property if and only if the crossed product is also a UHF algebra. Furthermore, when the action is seen as a representation of the group on the UHF algebra, these properties are equivalent to a simple condition on the ranks of the underlying projections. Similarly, the formally weaker notion of tracial Rokhlin property also forces some structure on the crossed product: a product type action has the tracial Rokhlin property if and only if the crossed product has a unique tracial state, which is also equivalent to the dual action being trivial on the space of tracial states of the crossed product. In turn, these three properties can also be expressed via a simple condition on the ranks of the underlying projections.

Although there appears to be no reason for the two previously mentioned results to be false for actions of arbitrary finite cyclic groups, proving them to be true is not immediate. In the final section of this work, we mention partial results, and give a simple criterion which may prove useful for constructing a generalized proof. However, the careful study of Rokhlin properties on this general setting, and understanding how they affect the structure of the crossed product and its respective K-theory is a topic for further investigation.

Chapter 1

Basic Concepts

In the study of operator algebras, there are two extremely important concepts: those of *normed algebra* and *-*algebra*. Combining both allows for the powerful and rich algebraic and topological structure of $*$-*normed algebra*. The natural requirement of completeness along with a simple postulate then gives rise to the notion of *C*-algebra*.

In Section 1.1 we start by introducing the above mentioned structures, and proceed to give several basic definitions. We develop some theory for especially important elements of *C*-algebras* called *projections*, we define and point out some properties of *ideals*, and we give a quick overview of the *representation theory* of *C*-algebras*. We also state the far reaching Gelfand-Naimark theorems, and end the section with two simple constructions: the *direct sum* and the *enveloping C*-algebra*.

In Section 1.2, we take a close look at *matrix algebras*, and advance to more complex constructions: the *inductive limit* and the *tensor product*. We here also introduce two classes of C*-algebras, the *AF* and *UHF algebras*, and state a major classification theorem for the latter, due to Glimm.

Finally, Section 1.3 is devoted to some group theory. We introduce the concept of a *topological group*, quickly survey their associated *representation theory*, and finish with the definition of the K_0-group for C*-algebras.

1.1 C*-algebras: fundamental properties

1.1.1 Definitions and examples

Definition 1.1.1. Let $\mathcal{A}$ be a complex algebra. An **involution** on $\mathcal{A}$ is a unary operation * on $\mathcal{A}$ such that

1. $(a^*)^* = a$.
2. $(z_1 a + z_2 b)^* = \overline{z_1} a^* + \overline{z_2} b^*$.
3. $(ab)^* = b^* a^*$.

for any $a, b \in \mathcal{A}$ and $z_1, z_2 \in \mathbb{C}$. An algebra endowed with an involution is called a ***-algebra**.

Note that $0^* = 0$, for given any $a \in \mathcal{A}$,

$$a + 0^* = (a^* + 0)^* = (a^*)^* = a.$$

The morphisms in the category of *-algebras are, of course, just algebra homomorphisms that preserve the involution (i.e., morphisms f such that $f(x^*) = (f(x))^*$). We call these ***-homomorphisms**, where the word homomorphism may be replaced by *isomorphism*, *endomorphism* or *automorphism* depending on the underlying algebra morphism type. The set of automorphisms of a *-algebra $\mathcal{A}$ is a group (under composition) denoted by $\mathrm{Aut}\,(\mathcal{A})$.

The structure of *-algebra is insufficient for our purposes. We enhance it by defining on it a metric.

Definition 1.1.2. A **normed algebra** is an algebra $\mathcal{A}$ endowed with a norm $\|\cdot\|$ such that

$$\|ab\| \leq \|a\|\|b\|$$

for any $a, b \in \mathcal{A}$.

Definition 1.1.3. We call **normed *-algebra** to any complex normed algebra $(\mathcal{A}, \|\cdot\|)$ equipped with an involution * that satisfies $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ is complete, we say $\mathcal{A}$ is a **Banach *-algebra**, and if in addition

$$\|a^*a\| = \|a\|^2 \tag{1.1}$$

for all $a \in \mathcal{A}$, $\mathcal{A}$ is said to be a **C*-algebra**. A C*-algebra $\mathcal{A}$ is said to be **separable** if it has a countable dense subset, and **unital** if the underlying algebra structure is unital, in which case we denote the unit by $1_{\mathcal{A}}$ or 1 when no confusion arises.

Equation 1.1 is called the **C*-axiom**.

We also make a simple remark regarding the unit. Should it exist, $1^* = 1$, for

$$1^* \cdot a = (a^* \cdot 1)^* = a = (1 \cdot a^*)^* = a \cdot 1^*.$$

Despite its apparent simplicity, the C*-axiom is extremely strong, as illustrated by the following.

Proposition 1.1.4 (cf. [4], Corollary II.1.6.6)**.** *Any *-homomorphism ϕ from a *-Banach algebra to a C*-algebra is norm-decreasing. In particular ϕ is continuous and $\|\phi\| \leq 1$.*

In the study of C*-algebras, it is fruitful to consider a couple of classes of elements. We list those that will be useful to us in the future.

Definition 1.1.5. An element a of a C*-algebra $\mathcal{A}$ is said to be

- **Normal** if $a^*a = aa^*$.
- **Unitary** if $\mathcal{A}$ is unital and $a^*a = aa^* = 1$.
- **Self-adjoint** or **hermitian** if $a = a^*$ (in general, a subset S of $\mathcal{A}$ is said to be **self-adjoint** if $S^* \equiv \{s^* \in \mathcal{A} : s \in S\} \subseteq S$).
- A **projection** if $a = a^* = a^2$.
- A **partial isometry** if a^*a is a projection.

Two projections $p, q \in \mathcal{A}$ are said to be **orthogonal**, written $p \perp q$, if $pq = 0$.

Note that the set of unitary elements of $\mathcal{A}$ forms a group, which we denote by $U(\mathcal{A})$.

Also note that orthogonal projections commute, for if $pq = 0$, then

$$qp = q^*p^* = (pq)^* = 0^* = 0.$$

A useful result is the following.

Lemma 1.1.6. *The sum of two projections p and q in a C^*-algebra is again a projection if and only if $p \perp q$.*

Proof. ($\Rightarrow$) If $p + q$ is a projection then

$$(p + q)^2 = p + q,$$

so $p + pq + qp + q = p + q$. We show that

$$pq + qp = 0 \Rightarrow p \perp q.$$

Indeed,

$$\begin{aligned} & pq + qp = 0 \\ \Rightarrow \quad & q(pq + qp)q = 0 \\ \Rightarrow \quad & 2qpq = 0 \\ \Rightarrow \quad & 2(pq)^*(pq) = 0, \end{aligned}$$

so $pq = 0$.

($\Leftarrow$) If $p \perp q$, then $pq + qp = 0$ and the result follows. □

Partial isometries allow us to establish an important equivalence relation in the set of projections. We need just a preliminary result and a definition.

Lemma 1.1.7. *Given an element w of a C^*-algebra, the following are equivalent:*

1. *w is a partial isometry.*
2. *w^* is a partial isometry.*
3. *$w = ww^*w$.*

Proof. We prove that ($1 \Leftrightarrow 3$), and the rest will follow.

($\Rightarrow$) If w is a partial isometry, w^*w is a projection, by definition. Consider the element $x \equiv w - ww^*w$. Then

$$\begin{aligned} x^*x &= (w - ww^*w)^*(w - ww^*w) \\ &= (w^* - w^*ww^*)(w - ww^*w) \\ &= w^*w - w^*ww^*w - w^*ww^*w + w^*ww^*ww^*w \\ &= w^*w - w^*w - w^*w + w^*w \\ &= 0. \end{aligned}$$

Thus $x = 0$ by the C*-axiom.

($\Leftarrow$) If $w = ww^*w$, then

$$(w^*w)^2 = w^*ww^*w = w^*w$$

and

$$(w^*w)^* = w^*w,$$

so w is a partial isometry. □

Definition 1.1.8. Let w be a partial isometry. Then w^*w is called the **initial projection** and ww^* the **final projection** of w.

We are now ready to introduce the above mentioned relation, which will come in use in Chapters 2 and 3.

Definition 1.1.9. Two projections p and q on a C*-algebra $\mathcal{A}$ are said to be **Murray-von Neumann equivalent**, written $p \sim q$, if there exists a partial isometry $w \in \mathcal{A}$ such that p is the initial and q the final projection of w.

This is indeed an equivalence relation: reflexivity is immediate ($p = pp^* = p^*p$) and so is symmetry. Transitivity is given by use of Lemma 1.1.7: suppose $p = vv^*$, $q = v^*v = ww^*$ and $e = w^*w$. Then $vww^* = v$ and $v^*vw = w$, so p and e are, respectively, the initial and final projections of vw.

A similar concept is that of *unitary equivalence.*

Definition 1.1.10. Two projections p and q in a unitary C*-algebra $\mathcal{A}$ are said to be **unitarily equivalent**, written $p \overset{u}{\sim} q$ if there exists a unitary $w \in \mathcal{A}$ such that

$$wpw^* = q.$$

It is easy to see that $p \overset{u}{\sim} q$ implies $p \sim q$. Indeed p and q are, respectively, the initial and final projections of up.

Proposition 1.1.11 (cf. [4], Proposition II.3.3.4)**.** *Let $\mathcal{A}$ be a unital C*-algebra, and let p and q be two projections in $\mathcal{A}$. If $\|p - q\| < 1$, then $p \overset{u}{\sim} q$.*

Another class of elements not listed in Definition 1.1.5 is the class of *positive elements* of a C*-algebra. These elements share a number of good properties, and problems or definitions are often reduced to problems or definitions within the class of positive elements.

Definition 1.1.12. Let $\mathcal{A}$ be a C*-algebra. A self-adjoint element $a \in \mathcal{A}$ is said to be **positive** if there exists $x \in \mathcal{A}$ such that $a = x^*x$, in which case we write $a \geq 0$ (or $0 \leq a$). The set of positive elements of $\mathcal{A}$ is denoted $\mathcal{A}_+$.

There is a natural partial order in $\mathcal{A}$: we define b to be greater than a, written $b \geq a$ (or $a \leq b$) if $b - a$ is a positive element.

Murray-von Neumann equivalence and the above order relation play a big part in the theory of C*-algebras, namely in the classification of C*-algebras by *type*. Although we won't be exploring this concept, we need one of the basic definitions.

Definition 1.1.13. A unital C*-algebra $\mathcal{A}$ is said to be **finite** if $1 \sim p \leq 1$ implies $p = 1$ for any projection $p \in \mathcal{A}$.

The possible lack of a unit in a C*-algebra leads to the next concept, which will prove of particular importance in the study of *representations*.

Definition 1.1.14. A net $\{e_i\}$ of elements in a C*-algebra $\mathcal{A}$ is called a **left** (resp. right) **approximate unit** if:

1. $\underset{i}{\forall} \|e_i\| \leq 1$.
2. $\underset{a\in\mathcal{A}}{\forall} \|e_i a - a\| \to 0$ (resp. $\|ae_i - a\| \to 0$).

An **approximate unit** is a net $\{e_i\}$ that is both a left and a right approximate unit.

We proceed with additional definitions and structures. First recall that a subset X of a set Y is said to be **stable** for a function $f : Y \to Y$ if $f(X) \subseteq X$. We may also say X is **invariant** under f, or **f-invariant**.

Definition 1.1.15. A subalgebra of a *-algebra $\mathcal{A}$, stable under involution, is called a ***-subalgebra** of $\mathcal{A}$. If $\mathcal{A}$ is a C*-algebra, a closed *-subalgebra of $\mathcal{A}$ is called a **C*-subalgebra**.

An intersection of *-subalgebras (resp. C*-subalgebras) is again a *-subalgebra (resp. C*-subalgebra), so we can define what is meant by a subalgebra being *generated* by given elements.

Definition 1.1.16. Let X be a subset of a *-algebra (resp. C*-algebra) $\mathcal{A}$. The *-subalgebra (resp. C*-subalgebra) **generated by** X is the intersection of all *-subalgebras (resp. C*-subalgebras) of $\mathcal{A}$ containing X. We may also call it the **smallest** *-subalgebra (resp. C*-subalgebra) of $\mathcal{A}$ containing X.

Of particular importance in what is to come are the *hereditary* subalgebras.

Definition 1.1.17. A C*-subalgebra $\mathcal{B}$ of a C*-algebra $\mathcal{A}$ is said to be a **hereditary** if for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$

$$0 \leq a \leq b \Rightarrow a \in \mathcal{B}.$$

Given any element $x \in \mathcal{A}$, it's easy to describe the hereditary C*-subalgebra generated by x, and should $\mathcal{A}$ be separable, this description covers all hereditary C*-subalgebras of $\mathcal{A}$.

Proposition 1.1.18 (cf. [4], Proposition II.3.4.2)**.** *Let $\mathcal{A}$ be a C*-algebra, let $x \in \mathcal{A}$ and let $\mathcal{B}$ be a hereditary C*-subalgebra. Then:*

1. $\overline{x^*\mathcal{A}x}$ *is the smallest hereditary C*-subalgebra of $\mathcal{A}$ containing x^*x.*
2. *If $\mathcal{A}$ is separable there exists $h \in \mathcal{A}_+$ such that $\mathcal{B} = \overline{h\mathcal{A}h}$.*

Note now that, since a C*-algebra is, in essence, an enhanced ring, the quotient by a C*-subalgebra is only well-defined for ideals. We recall that definition here.

Definition 1.1.19. A **left** (resp. **right**) **ideal** I of a C*-algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ with the following property:

$$a \cdot i \in I \quad (\text{resp. } i \cdot a \in I)$$

for all $a \in \mathcal{A}$ and $i \in I$. An ideal which is both a left and right ideal is called a **two-sided ideal**.

Since we will be concerned with two-sided ideals only, the word ideal without any further qualification will always mean two-sided ideal.

As is usual in algebraic theories, having the least possible number of ideals is quite a nice property to have.

Definition 1.1.20. A C*-algebra $\mathcal{A}$ is said to be **simple** if its only closed ideals are $\{0\}$ and itself.

The property of being simple carries over to large C*-algebras with relative ease.

Proposition 1.1.21 (cf. [12], Theorem 6.1.3)**.** *Let $\mathcal{S}$ be a non-empty directed set of simple C*-subalgebras of some C*-algebra $\mathcal{A}$. If $\bigcup \mathcal{S}$ is dense in $\mathcal{A}$, then $\mathcal{A}$ is simple.*

Kernels of *-homomorphisms are important examples of ideals.

Proposition 1.1.22. *The kernel of any *-homomorphism between *-algebras is a self-adjoint ideal.*

Proof. Given a *-homomorphism $\rho : \mathcal{A} \to \mathcal{B}$ between *-algebras, any $x_1, x_2 \in \ker(\rho)$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have:

- $\pi(x_1 + x_2) = \pi(x_1) + \pi(x_2) = 0 + 0 = 0$
- $\pi(\lambda x_1) = \lambda\pi(x_1) = \lambda \cdot 0 = 0$
- $\pi(x_1^*) = (\pi(x_1))^* = 0^* = 0$

- $\pi(x_1 a) = \pi(x_1)\pi(a) = 0 = \pi(a)\pi(x_1) = \pi(a x_1)$,

which proves the proposition. □

Given two ideals I and J, their intersection $I \cap J$ is again an ideal, and their product IJ is defined to be the set of all finite sums of products of an element of I and an element of J, and is also an ideal. We have, of course, a trivial inclusion: $IJ \subseteq I \cap J$.

Definition 1.1.23. An ideal I of a *-algebra $\mathcal{A}$ is said to be **proper** if $I \neq \mathcal{A}$.

A proper ideal M is called **maximal** if for any ideal I such that $M \subseteq I$, we have $I = M$ or $I = \mathcal{A}$.

A proper ideal P is called **prime** if for any two ideals I and J such that $IJ \subseteq P$, we have $I \subseteq P$ or $J \subseteq P$.

The notion of ideal only settles things down for the ring structure of a C*-algebra. In order to have the quotient of a C*-algebra by an ideal be again a C*-algebra, one must obviously require that ideal to be self-adjoint and closed, so as to ensure well-defined norm and involution in the quotient as well as completeness. It just so happens that closedness suffices.

Proposition 1.1.24 (cf. [7], Proposition 1.8.2)**.** *Every closed ideal of a C*-algebra is self-adjoint.*

It is also useful to consider quotients of *-algebras, in which case we only need to require the ideal to be self-adjoint.

1.1.2 States, representations and the Gelfand transform

An extremely useful concept in all branches of mathematics, and in particular in the study of normed *-algebras, is that of *representation*. The idea is to study given (complicated) structures by considering them as subspaces of a simple space. In the case of normed *-algebras, such a simple space is a space of bounded linear operators on a complex Hilbert space.

Definition 1.1.25. A *-homomorphism $\pi : \mathcal{A} \to B(H)$ from a normed *-algebra $\mathcal{A}$ to the normed *-algebra $B(H)$ of bounded linear operators on a complex Hilbert space H is called a **representation** of $\mathcal{A}$ (on H). A representation is said to be **faithful** if it is injective.

For convenience, we may sometimes abuse notation and refer to the pair (H, π) itself as a representation. This should cause no confusion.

Note that there is always at least one representation: the **trivial representation** $\pi = 0$. Of course this is just a pathological case, completely useless, and we accordingly rule it out most of the time.

The rest of this section will focus on representations of C*-algebras, although most definitions and results extend to the general case.

Definition 1.1.26. Two representations (H_1, π_1) and (H_2, π_2) of a C*-algebra $\mathcal{A}$ are said to be **unitarily equivalent** if there exists a unitary operator $U : H_1 \to H_2$ such that $U\pi_1(a)U^* = \pi_2(a)$ for all $a \in \mathcal{A}$.

Given a representation of a C*-algebra $\mathcal{A}$ on a Hilbert space H, we can consider the closed subspace K generated by all elements of the form $\pi(a)\xi$ (with $a \in \mathcal{A}$ and $\xi \in H$), to obtain the canonical decomposition $H = K \oplus K^\perp$. It so happens that $K^\perp$ has a very simple description: it is the closed subspace generated by the collection of elements ν such that $\pi(a)\nu = 0$ for all $a \in \mathcal{A}$. This gives rise to te following definition.

Definition 1.1.27. A representation (H, π) of a C*-algebra $\mathcal{A}$ is said to be **nondegenerate** if, given $\xi \in H$, $\pi(a)\xi = 0$ for all $a \in \mathcal{A}$ implies $\xi = 0$.

Proposition 1.1.28 (cf. [13], Proposição 1.1.18)**.** *Let $\mathcal{A}$ be a C*-algebra, and let (H, π) be a representation of $\mathcal{A}$. Then π is nondegenerate if and only if $\pi(e_i)$, where $\{e_i\}$ is an approximate unit of $\mathcal{A}$, converges pointwise to the identity operator I in $B(H)$.*

Another important notion is that of irreducibility. This has a deep connection, as we shall see, with the concept of *free action*, which is the heart of this work.

Definition 1.1.29. A representation (H, π) of a C*-algebra $\mathcal{A}$ is said to be **irreducible** if the only closed subspaces of H invariant under $\pi(a)$ for all $a \in \mathcal{A}$ are $\{0\}$ and H.

It can be shown (cf. [7], Corollaire 2.8.4) that closedness of the subspaces in the previous definition is an unnecessary condition. It is not, however, a trivial result.

The kernels of irreducible representations are self-adjoint ideals (cf. Proposition 1.1.22) of particular significance.

Definition 1.1.30. An ideal P of a C*-algebra $\mathcal{A}$ is called **primitive** if there exists a non-trivial irreducible representation (H, π) of $\mathcal{A}$ such that $\ker \pi = P$. The set of all primitive ideals of $\mathcal{A}$ is called the **primitive spectrum** of $\mathcal{A}$ and denoted by $\mathrm{Prim}\,(\mathcal{A})$.

Non-triviality forces any primitive ideal to be proper. This notion of primitive ideal will serve its purpose in Chapter 2. We here only prove an auxiliary result.

Lemma 1.1.31. *Every primitive ideal of a C*-algebra is prime.*

Proof. Let $\mathcal{A}$ be a C*-algebra, let P be the kernel of a non-trivial irreducible representation (H, π) of $\mathcal{A}$, and let I and J be two ideals of $\mathcal{A}$ such that neither $I \subseteq P$ nor $J \subseteq P$. Consider

$$K \equiv \{\xi \in H : \pi(I)\xi = 0\}.$$

Fix $a \in \mathcal{A}$, $i \in I$ and $\xi \in K$ to obtain

$$\pi(i)\pi(a)(\xi) = \pi(i \cdot a)(\xi) = \pi(i')(\xi) = 0$$

for some $i' \in I$. K is thus $\pi(\mathcal{A})$-invariant. Since π is non-trivial by definition, $K \neq H$, so there exists $\gamma \neq 0$ such that $\pi(I)\gamma \neq 0$. Now $\pi(I)\gamma$ is $\pi(\mathcal{A})$-invariant (same reasoning as to prove K is $\pi(\mathcal{A})$-invariant), so $\pi(I)\gamma = H$. Again using the ideal structure, we obtain $\pi(I)\pi(J)H = H$, and consequently $\pi(IJ) \neq 0$.

Since $I \not\subseteq P$ and $J \not\subseteq P$, we have $\pi(IJ) \neq 0$, hence $IJ \not\subseteq P$. □

If $\mathcal{A}$ is separable, every closed prime ideal is primitive (cf. [14], Proposition 4.3.6). This is not true in general, as recently shown by Weaver in [24].

Representations can be constructed via certain linear functionals called *states*.

Definition 1.1.32. A linear functional ϕ on a C*-algebra $\mathcal{A}$ is said to be **positive** if $\phi(\mathcal{A}^+) \subseteq [0, \infty[$. A positive linear functional is called a **state** if it has norm 1. A state τ is said to be **tracial** if for any $a, b \in \mathcal{A}$ we have $\tau(ab) = \tau(ba)$. The set of tracial states on $\mathcal{A}$ is denoted $T(\mathcal{A})$.

There is a particularly elegant definition of state in the case of unital C*-algebras, given by the following.

Proposition 1.1.33 (cf. [27], Corollary 13.6)**.** *A positive linear functional ϕ on a unital C*-algebra $\mathcal{A}$ is a state if and only if $\phi(1_{\mathcal{A}}) = 1$.*

A major result in the theory of C*-algebras is the classification theorem of commutative C*-algebras known as the Gelfand-Naimark theorem, which we now state.

Definition 1.1.34. The set of nonzero homomorphisms from a C*-algebra $\mathcal{A}$ to $\mathbb{C}$ is called the **spectrum** of $\mathcal{A}$ and written $\hat{\mathcal{A}}$.

There is a naturally induced map from $\mathcal{A}$ to $C_0(\hat{\mathcal{A}})$, taking a to $\hat{a}$, given by setting

$$\hat{a}(\phi) \equiv \phi(a)$$

for all $\phi \in \hat{\mathcal{A}}$. This map is called the **Gelfand transform** of $\mathcal{A}$.

Theorem 1.1.35 (Gelfand-Naimark I)**.** *If $\mathcal{A}$ is a commutative C*-algebra, its Gelfand transform is an isometric *-isomorphism.*

Its far reaching generalization bears the same name.

Theorem 1.1.36 (Gelfand-Naimark II - cf. [12], Theorem 3.4.1)**.** *For any C*-algebra $\mathcal{A}$ there exists a Hilbert space H such that $\mathcal{A}$ is isometrically *-isomorphic to a subalgebra of $B(H)$.*

Thus, in essence, every C*-algebra is an algebra of operators. In particular, concepts such as the rank of an operator can be carried over to elements of an arbitrary C*-algebra.

The Gelfand transform also allows one to define a continuous functional calculus for normal elements on unital C*-algebras. In fact, the following is easily proved.

Theorem 1.1.37. *Let $\mathcal{A}$ be a unital C*-algebra, and $a \in \mathcal{A}$ a normal element. Then there exists a unique map π, sending a continuous function f on the spectrum of a, $\operatorname{sp}(a)$, to $f(a)$, such that $\pi(1) = 1$ and $\pi(\iota) = a$ where ι is the identity function on $\operatorname{sp}(a)$.*

We here state a functional calculus result we shall be needing. First, we fix some nomenclature: we call **Heaviside function centered at** x to the function H_x given by

$$H_x(t) = \begin{cases} 0, & t < x \\ 1, & t > x \end{cases}.$$

Lemma 1.1.38. *Let a be a Hermitian element on a C*-algebra, and $r \in]0, \frac{1}{2}[$. If $\operatorname{sp}(a) \subseteq [-r, r] \cup [1-r, 1+r]$, then $H_{\frac{1}{2}}(a)$ is a projection satisfying*

$$\|H_{\frac{1}{2}}(a) - a\| < r.$$

Furthermore, $H_{\frac{1}{2}}(a)$ is in the C-algebra generated by a.*

1.1.3 The direct sum and the enveloping C*-algebra

This short paragraph is dedicated to two simple constructions one can do to generate new C*-algebras from previously known ones. The first is a well-known operation: the *direct sum*.

Given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, their Cartesian product $\mathcal{A} \times \mathcal{B}$ has a natural *-algebra structure, obtained by considering the pointwise operations. Furthermore, if one defines the norm

$$\|(x_1, x_2)\| \equiv \max\{\|x_1\|, \|x_2\|\}$$

where $x_1 \in \mathcal{A}$ and $x_2 \in \mathcal{B}$, the resulting *-algebra is a C*-algebra.

Definition 1.1.39. In the construction above, the resulting C*-algebra, denoted by $\mathcal{A} \oplus \mathcal{B}$, is called the **direct sum** of $\mathcal{A}$ and $\mathcal{B}$.

This one was a trivial generalization of an otherwise known concept. The next one, however, is a bit trickier, but will soon prove to be fundamental.

Definition 1.1.40. A norm (resp. seminorm) $\|\cdot\|$ on a *-algebra $\mathcal{A}$ is said to be a **C*-norm** (resp. **C*-seminorm**) if

1. $\|a_1 a_2\| \leq \|a_1\| \|a_2\|$
2. $\|a^*\| = \|a\|$
3. $\|a^* a\| = \|a\|^2$

It is routine to check that the inverse image N of $\{0\}$ by a C*-seminorm is a self-adjoint ideal of the underlying *-algebra $\mathcal{A}$. Therefore we have a well-defined quotient $\mathcal{A}/N$, and can consider its completion, so as to turn it into a C*-algebra.

Definition 1.1.41. Given a *-algebra $\mathcal{A}$ and a C*-seminorm (resp. C*-norm) $\|\cdot\|$ on $\mathcal{A}$, let $N \equiv \{a \in \mathcal{A} : \|a\| = 0\}$. We call the completion of $\mathcal{A}/N$ the **enveloping C*-algebra** (resp. **C*-completion**) of $(\mathcal{A}, \|\cdot\|)$.

We end this short section with an important definition, after the following preliminary result, whose proof is again routine.

Proposition 1.1.42 (cf. [13], Teorema 1.1.30)**.** *Let $(\mathcal{B}, \|\cdot\|_1)$ be a normed *-algebra. The map $\|\cdot\|_u : \mathcal{B} \to \mathbb{R}$ defined by*

$$\|b\|_u \equiv \sup\{\|\pi(b)\| : \pi \text{ is a representation of } \mathcal{B}\}$$

is a C-seminorm on $\mathcal{B}$ such that*

$$\|b\|_u \leq \|b\|_1.$$

We call $\|\cdot\|_u$ the **universal norm** of $\mathcal{B}$. The corresponding enveloping C*-algebra is particularly important.

Definition 1.1.43. The **universal enveloping C*-algebra** of a normed *-algebra $\mathcal{B}$ is the enveloping C*-algebra of $(\mathcal{B}, \|\cdot\|_u)$.

1.2 Matrix algebras and special constructions

We now proceed with the study of a particular type of C*-algebras: *matrix algebras*, and continue towards two major constructions: the *inductive limit* and the *tensor product* of C*-algebras. The purpose and usefulness of each of them will be made clear from the examples and theorems to follow.

1.2.1 Matrix algebras

Matrix algebras are among the simplest yet most interesting examples of C*-algebras available, and several more advanced constructions arise from them.

Definition 1.2.1. The space $\mathcal{M}_n$ of $n \times n$ complex matrices is a C*-algebra for the usual matrix multiplication, involution (conjugate transpose) and operator norm, called a **(full) matrix algebra**.

Matrix algebras are the building blocks of any finite-dimensional C*-algebra, and the model of finite-dimensional simple C*-algebras.

Theorem 1.2.2 (cf. [12], Theorem 6.3.8)**.** *A non-zero finite-dimensional C*-algebra is *-isomorphic to a finite direct sum of matrix algebras.*

Theorem 1.2.3 (cf. [12], Remark 6.2.1)**.** *A non-zero finite-dimensional C*-algebra is simple if and only if it is *-isomorphic to a matrix algebra.*

These are very powerful decomposition theorems that show some of the importance of matrix algebras within the study of general C*-algebras. For future reference, we make the following remark.

Corollary 1.2.4. *Every finite-dimensional simple C*-algebra is unital.*

Next are the two most used results about matrix algebras in this work, both related to the tracial states in these algebras.

Proposition 1.2.5 (cf. [12], Example 6.2.1)**.** *Given any* $n \in \mathbb{N}$*,* $\mathcal{M}_n$ *has a unique tracial state, denoted by* tr_n*.*

Proposition 1.2.6. *Given* $n, m \in \mathbb{N}$*,* $\tau : \mathcal{M}_n \oplus \mathcal{M}_m \to \mathbb{C}$ *is a tracial state on* $\mathcal{M}_n \oplus \mathcal{M}_m$ *if and only if there exists* $r \in [0, 1]$ *such that*

$$\tau(a, b) = r\operatorname{tr}_n(a) + (1 - r)\operatorname{tr}_m(b).$$

Proof. ($\Rightarrow$) Consider the restriction $\hat{\tau}$ of τ to elements of the form $(a, 0)$, with $a \in \mathcal{M}_n$. It is trivial that $\hat{\tau} : \mathcal{M}_n \to \mathbb{C}$ is a positive linear functional with norm $0 \leq r \leq 1$. We thus have $\hat{\tau} = r\operatorname{tr}_n$ by uniqueness of tr_n. Similarly, the analogously defined $\tilde{\tau} : \mathcal{M}_m \to \mathbb{C}$ is equal to $s\operatorname{tr}_m$ for some $0 \leq s \leq 1$. Now by Proposition 1.1.33 any state takes the value 1 at the identity, so, using linearity of τ, we obtain

$$\tau(1_{\mathcal{M}_n \oplus \mathcal{M}_m}) = \hat{\tau}(1_{\mathcal{M}_n}) + \tilde{\tau}(1_{\mathcal{M}_m}) = r + s = 1,$$

finishing the proof.

($\Leftarrow$) Conversely, given $r \in [0,1]$, we prove that $\tau : \mathcal{M}_n \oplus \mathcal{M}_m \to \mathbb{C}$ defined by

$$\tau(a,b) \equiv r\mathrm{tr}_{\,n}(a) + (1-r)\mathrm{tr}_{\,n}(b)$$

is a tracial state on $\mathcal{M}_n \oplus \mathcal{M}_m$. Linearity and positivity follow from the fact that $\mathrm{tr}_{\,n}$ and $\mathrm{tr}_{\,m}$ are positive linear functionals. Furthermore

$$\begin{aligned} \sup\{|\tau(a,b)| : \|(a,b)\| = 1\} &= \sup\{|r\mathrm{tr}_{\,n}(a) + (1-r)\mathrm{tr}_{\,m}(b)| : \|(a,b)\| = 1\} \\ &\leq r + (1-r) \\ &= 1 \end{aligned}$$

so $\|\tau\| \leq 1$, and since

$$\tau(1_{\mathcal{M}_n}, 1_{\mathcal{M}_m}) = r + (1-r) = 1,$$

we get that $\|\tau\| = 1$, which finishes the proof. □

Of course this can easily be generalized for an arbitrary finite direct sum of matrix algebras.

Proposition 1.2.7. *Given $N \in \mathbb{N}$ and a sequence (k_n) of natural numbers, $\tau : \bigoplus_{n=1}^{N} \mathcal{M}_{k(n)} \to \mathbb{C}$ is a tracial state on $\bigoplus_{n=1}^{N} \mathcal{M}_{k(n)}$ if and only if there exist $r_1, r_2, \cdots r_N \in [0,1]$ such that:*

1. $\tau(a_1, \ldots, a_N) = \sum_{n=1}^{N} r_n \mathrm{tr}_{\,k(n)}(a_n)$;
2. $\sum_{n=1}^{N} r_n = 1$.

1.2.2 The inductive limit

The *inductive limit* of an *inductive system* of C*-algebras is a quite simple construction that gives rise to several interesting families of C*-algebras.

Definition 1.2.8. Let $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ be a sequence of C*-algebras and $\{\phi_{i,j} : \mathcal{A}_i \to \mathcal{A}_j\}_{\{(i,j):i\leq j\}}$ a familiy of *-homomorphisms (called **connecting maps** or **connecting homomorphisms**) obeying the following properties:

1. $\phi_{i,i} = I_{\mathcal{A}_i}$;
2. $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$,

for all $i \leq j \leq k \in \mathbb{N}$. Then the collection of pairs $\{(\mathcal{A}_i, \phi_{i,j})\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$, shortened to $(\mathcal{A}_i, \phi_{i,j})$ or schematically

$$\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \cdots$$

(where $\phi_i \equiv \phi_{i,(i+1)}$), is called an **inductive** or **direct system** (or **sequence**) of C*-algebras (and *-homomorphisms).

Now given an inductive system of C*-algebras $(\mathcal{A}_i, \phi_{ij})$, we can consider the product

$$\mathcal{P} \equiv \prod_{i=1}^{\infty} \mathcal{A}_i,$$

which, under the usual pointwise operations, is a *-algebra. Consider the following *-subalgebra of $\mathcal{P}$:

$$\mathcal{A} \equiv \{(a_1, a_2, \cdots) \in \mathcal{P} : \underset{N\in\mathbb{N}}{\exists} \underset{k>N}{\forall} \phi_k(a_k) = a_{k+1}\}. \tag{1.2}$$

Since the ϕ_k are all norm-decreasing (cf. Proposition 1.1.4), $\|a_{k+1}\| \leq \|a_k\|$ for large enough k, and we can thus define a C*-seminorm $\|\cdot\|_{\text{ind}}$ on $\mathcal{A}$ by setting

$$\|(a_1, a_2, \cdots)\|_{\text{ind}} \equiv \lim_{i\to\infty} \|a_k\|.$$

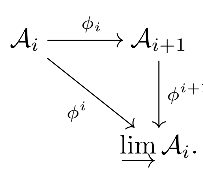
Definition 1.2.9. In the above construction, the enveloping C*-algebra of $(\mathcal{A}, \|\cdot\|_{\text{ind}})$ is called the **inductive** or **direct limit** of the inductive system $(\mathcal{A}_i, \phi_{ij})$. When no confusion arises, we denote it by $\varinjlim \mathcal{A}_i$.

Direct limits have two basic properties.

Proposition 1.2.10. *Given the direct limit $\varinjlim \mathcal{A}_i$ of a direct system*

$$\mathcal{A}_1 \overset{\phi_1}{\to} \mathcal{A}_2 \overset{\phi_2}{\to} \cdots$$

of C-algebras, there exists, for each $i \in \mathbb{N}$, a *-homomorphism $\phi^i : \mathcal{A}_i \to \varinjlim \mathcal{A}_i$ such that the following diagrams commute*



We also have

$$\overline{\bigcup_{i=1}^{\infty} \phi^i(\mathcal{A}_i)} = \varinjlim \mathcal{A}_i.$$

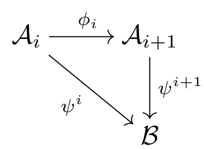
Proof. Let $\mathcal{A}$ be as in (1.2). For each i, define $f_i : \mathcal{A}_i \to \mathcal{A}$ componentwise by

$$[f_i(a)]_k \equiv \begin{cases} 0 & k < i \\ \phi_{ik}(a) & k \geq i \end{cases},$$

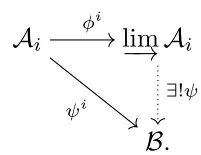
where $[f_i(a)]_k$ is the k'th component of $f_i(a)$. In other words, f_i sends a to the element of $\mathcal{A}$ of the form $(0, \ldots, 0, a, \phi_{i(i+1)}(a), \phi_{i(i+2)}(a), \ldots)$, with $i-1$ zeroes. The required *-homomorphisms are then easily seen to be $\phi^i \equiv \pi \circ f_i$ where π is the quotient map from $\mathcal{A}$ to $\varinjlim \mathcal{A}_i$.

The density result is trivial. $\square$

Theorem 1.2.11 (Universal property of the inductive limit of C*-algebras - cf. [12], Theorem 6.1.2)**.** *Let* $\varinjlim \mathcal{A}_i$ *be the inductive limit of an inductive sequence* $(\mathcal{A}_i, \phi_{ij})$ *of C*-algebras. Adopting the notation of Proposition 1.2.10, for every C*-algebra* $\mathcal{B}$ *and collection of *-homomorphisms* $\psi^i : \mathcal{A}_i \to \mathcal{B}$ *such that the diagrams*



commute (for all $i \in \mathbb{N}$*), there exists a unique *-homomorphism* $\psi : \varinjlim \mathcal{A}_i \to \mathcal{B}$ *making the following diagrams commute (for all* $i \in \mathbb{N}$*)*



This universal property of course defines the inductive limit up to isomorphism.

As one could expect, several properties that might be shared by C*-algebras in an inductive sequence carry over to their respective inductive limit, in particular the properties of being simple and having a unique tracial state.

Theorem 1.2.12. *The inductive limit of an inductive sequence of simple C*-algebras is simple.*

Proof. This is a simple corollary of Proposition 1.1.21 and the density result in Proposition 1.2.10, by considering the directed set $\{\phi^i(\mathcal{A}_i)\}$. □

Proposition 1.2.13. *If* $\mathcal{A}$ *is unital and is the inductive limit of an increasing sequence* $(\mathcal{A}_n, \phi_{ij})$ *such that all* ϕ_{ij} *are injective, and each* $\mathcal{A}_n$ *has a unique tracial state* τ_n *and contains the unit of* $\mathcal{A}$*, then* $\mathcal{A}$ *has a unique tracial state.*

Proof. Define τ on $\bigcup_{n=0}^{\infty} \mathcal{A}_n$ by setting

$$\tau(a) \equiv \tau_n(a) \tag{1.3}$$

if $a \in \mathcal{A}_n$. Note that this is well defined since the restriction of any τ_{n+1} to $\mathcal{A}_n$ is τ_n by uniqueness. It is immediate that τ is a norm-decreasing linear functional and can thus be uniquely extended to a bounded linear functional on $\mathcal{A}$, which is the required tracial state. □

We now turn our attention to a special inductive system construction: the *AF (Approximately Finite) algebras.* These C*-algebras are, as mentioned, extremely simple to obtain from the machinery we've developed, yet exhibit a rich structure.

Definition 1.2.14. The inductive limit of a direct system $(\mathcal{A}_i, \phi_{ij})$, where all $\mathcal{A}_i$ are finite-dimensional, is called an **AF algebra**.

Another possible definition is given by the following theorem.

Theorem 1.2.15. *A C^*-algebra $\mathcal{A}$ is an AF algebra if and only if there exists an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ of finite-dimensional C^*-subalgebras of $\mathcal{A}$ such that*

$$\mathcal{A} = \overline{\bigcup_{i=1}^{\infty} \mathcal{A}_i}.$$

Proof. ($\Rightarrow$) One needs only consider the sequence $(\phi^i(\mathcal{A}_i))_{i\in\mathbb{N}}$. It is increasing and its union is dense by Proposition 1.2.10, and each $\phi^i(\mathcal{A}_i)$ is finite dimensional since each $\mathcal{A}_i$ is finite dimensional.

($\Leftarrow$) This is just a straightforward application of Theorem 1.2.11. $\square$

A closely related but stronger concept is that of *UHF (Uniformily HyperFinite)* algebras.

Definition 1.2.16. A **UHF algebra** is the inductive limit of an inductive system $(\mathcal{A}_i, \phi_{ij})$ where all $\mathcal{A}_i$ are finite-dimensional and simple, and all ϕ_{ij} are unital embeddings.

In view of Theorems 1.2.3 and 1.2.11, a UHF algebra $\mathcal{A}$ may always be regarded as an inductive limit of matrix algebras, where the connecting maps are unital embeddings. Thus, any UHF algebra is (isomorphic to) the inductive limit of a system of the form

$$\mathcal{M}_{k_1} \overset{\phi_1}{\to} \mathcal{M}_{k_2} \overset{\phi_2}{\to} \cdots$$

where (k_n) is a given sequence in $\mathbb{N}$. Now, since each ϕ_n is assumed to be a unital embedding, we necessarily have $k_n | k_{n+1}$, and can thus construct a formal product

$$\delta(\mathcal{A}) \equiv \prod_p p^{t_p}$$

over the set of prime numbers p, where $t_p \in \mathbb{N}_0 \cup \{\infty\}$ is given by

$$t_p \equiv \sup\{m : p^m | k_n \text{ for some } n\}.$$

The number $\delta(\mathcal{A})$ is called the **supernatural number** corresponding to $\mathcal{A}$. The supernatural numbers form a complete invariant of UHF algebras, as shown by Glimm.

Theorem 1.2.17 (cf. [8])**.** *Two UHF algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $\delta(\mathcal{A}) = \delta(\mathcal{B})$.*

In light of this result it is common to refer to a UHF algebra by its supernatural number, i.e., refer to $\mathcal{A}$ as above as the $\prod p^{t_p}$ UHF algebra, or the UHF algebra of **type** $\prod p^{t_p}$, e.g.: the 2^∞ UHF algebra (also called the **CAR algebra** - Canonical Anticommutation Relations algebra) is the inductive limit of the system

$$\mathcal{M}_2 \to \mathcal{M}_4 \to \mathcal{M}_8 \to \cdots$$

where the connecting maps are $a \mapsto \mathrm{diag}(a, a)$.

1.2.3 The tensor product

The tensor product of C*-algebras is an adapted version of the more common tensor product of algebras. Indeed, given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, we start by considering their algebraic tensor product (as algebras over $\mathbb{C}$), which we denote by $\mathcal{A} \odot \mathcal{B}$. $\mathcal{A} \odot \mathcal{B}$ is easily turned into a *-algebra by setting

$$(a \odot b)^* \equiv a^* \odot b^*.$$

This construction is characterized by the following universal property.

Lemma 1.2.18 (Universal property of the tensor product of *-algebras)**.** *If $\phi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are *-homomorphisms from *-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively to the *-algebra $\mathcal{C}$, such that $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then there is a unique *-homomorphism $\sigma : \mathcal{A} \odot \mathcal{B} \to \mathcal{C}$ such that*

$$\sigma(a \odot b) = \phi(a)\psi(b).$$

Proof. The map

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{B} & \to & \mathcal{C} \\ (a, b) & \mapsto & \phi(a)\psi(b) \end{array}$$

is a bilinear map. It thus extends to a linear map on the tensor product, and this extension has the desired properties. ☐

Defining the actual tensor product of C*-algebras then comes down to defining a C*-norm on $\mathcal{A} \odot \mathcal{B}$.

Definition 1.2.19. Given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, and a cross C*-norm $\|\cdot\|$ (cross meaning $\|a \otimes b\| = \|a\|\|b\|$) on $\mathcal{A} \odot \mathcal{B}$, the **tensor product** $\mathcal{A} \otimes_{\|\cdot\|} \mathcal{B}$ (of $\mathcal{A}$ and $\mathcal{B}$ with respect to $\|\cdot\|$) is the C*-completion of $(\mathcal{A} \odot \mathcal{B}, \|\cdot\|)$.

For the above definition to be of any use, we need to prove existence of a C*-norm on $\mathcal{A} \odot \mathcal{B}$. We will do this via representation theory. First recall that the tensor product $H_1 \otimes H_2$ of two Hilbert spaces H_1 and H_2 is the completion of their algebraic tensor product $H_1 \odot H_2$ (as vector spaces) with respect to the following inner product

$$\langle \xi_1 \odot \xi_2, \eta_1 \odot \eta_2 \rangle \equiv \langle \xi_1, \eta_1 \rangle_{H_1} \langle \xi_2, \eta_2 \rangle_{H_2}$$

for all $\xi_1, \eta_1 \in H_1$ and $\xi_2, \eta_2 \in H_2$, extended by sesquilinearity. Given two bounded linear operators on H_1 and H_2 respectively, we can also construct a particular operator on the tensor product.

Lemma 1.2.20 (cf. [12], Lemma 6.3.2)**.** *Let T_1 and T_2 be two bounded linear operators on two Hilbert spaces H_1 and H_2 respectively. Then there exists a unique operator $T_1 \otimes T_2 \in B(H_1 \otimes H_2)$ such that*

$$(T_1 \otimes T_2)(\xi \otimes \eta) = T_1(\xi) \otimes T_2(\eta)$$

for all $\xi \in H_1$ and $\eta \in H_2$, and we have

$$\|T_1 \otimes T_2\| = \|T_1\|\|T_2\|.$$

If in Lemma 1.2.18 we take $\mathcal{C}$ to be $B(H)$ for some Hilbert space H, we obtain a systematic way of producing representations of $\mathcal{A} \odot \mathcal{B}$ from representations of $\mathcal{A}$ and $\mathcal{B}$, which leads to the following more general result.

Lemma 1.2.21 (cf. [12], Theorem 6.3.3)**.** *Given two representations $(H_{\mathcal{A}}, \pi_{\mathcal{A}})$ and $(H_{\mathcal{B}}, \pi_{\mathcal{B}})$ of C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, there is a unique representation $\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}$ of $\mathcal{A} \odot \mathcal{B}$ on $H_{\mathcal{A}} \otimes H_{\mathcal{B}}$ such that*

$$(\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}})(a \odot b) \equiv \pi_{\mathcal{A}}(a) \otimes \pi_{\mathcal{B}}(b).$$

Furthermore, if both $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are faithful, then so is $\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}$.

This result allows us to define a C*-norm $\|\cdot\|_{\min}$ on $\mathcal{A} \odot \mathcal{B}$, called the **minimal** or **spatial norm** (on $\mathcal{A} \odot \mathcal{B}$), by setting

$$\left\|\sum_{i=1}^{n} a_i \odot b_i\right\|_{\min} \equiv \sup \left\|(\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}) \left(\sum_{i=1}^{n} a_i \odot b_i\right)\right\|$$

where the supremum is taken over all representations $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$ respectively.

Definition 1.2.22. The tensor product of two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ with respect to the minimal norm is called the **minimal** or **spatial tensor product** (of $\mathcal{A}$ and $\mathcal{B}$), and denoted $\mathcal{A} \otimes_{\min} \mathcal{B}$.

The spatial tensor product has quite a number of good properties, of which we will need the following.

Theorem 1.2.23. *The spatial tensor product of two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is simple if and only if both $\mathcal{A}$ and $\mathcal{B}$ are simple.*

If we could prove the minimal norm is unique we would have a perfectly well defined notion of tensor product. Alas, there are in general more than just one C*-norm on $\mathcal{A} \odot \mathcal{B}$, in particular the *maximal norm* $\|\cdot\|_{\max}$, that shares the following property with the minimal norm: given any C*-norm $\|\cdot\|$ on $\mathcal{A} \odot \mathcal{B}$,

$$\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max},$$

hence the names. This leads to the following important concept.

Definition 1.2.24. A C*-algebra $\mathcal{A}$ is said to be **nuclear** if for any C*-algebra $\mathcal{B}$, there is a unique C*-norm on $\mathcal{A} \odot \mathcal{B}$.

Matrix algebras are examples of nuclear C*-algebras.

Theorem 1.2.25 (cf. [4], II.9.4.2)**.** *For all $n \in \mathbb{N}$, $\mathcal{M}_n$ is a nuclear C*-algebra.*

Corollary 1.2.26. *Given $j, k \in \mathbb{N}$,*

$$\mathcal{M}_j \otimes \mathcal{M}_k \cong \mathcal{M}_{jk}.$$

Proof. The *-isomorphism between the *-algebras $\mathcal{M}_j \odot \mathcal{M}_k$ and $\mathcal{M}_{jk}$ is trivially given by the Kronecker product. Now, by the previous theorem, $\mathcal{M}_j$ is nuclear, so the *-isomorphism must be isometric. □

If $\mathcal{A}$ is nuclear, we drop the subscripts on the tensor product notation, and just write, for any C*-algebra $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$ for their tensor product with respect to any C*-norm.

This settles the question for the tensor product of two and hence any finite number of C*-algebras. But what about infinite tensor products?

Since we will only be needing countable infinite tensor products, we will only consider, as in the inductive limit, the case where we are given a countable collection $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ of C*-algebras, and we this time require all of them to be unital and nuclear. For each $n \in \mathbb{N}$, set

$$\mathcal{B}_n \equiv \bigotimes_{i=1}^{n} \mathcal{A}_i.$$

There is an obvious injection ι_n of each $\mathcal{B}_n$ to $\mathcal{B}_{n+1}$, sending $b \in \mathcal{B}_n$ to $b \otimes 1_{\mathcal{A}_{n+1}} \in \mathcal{B}_{n+1}$, giving rise to the inductive system

$$\mathcal{B}_1 \xrightarrow{\iota_1} \mathcal{B}_2 \xrightarrow{\iota_2} \cdots . \tag{1.4}$$

Definition 1.2.27. The **infinite tensor product** of a family of unital nuclear C*-algebras $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ is the inductive limit of the corresponding inductive system (1.4) and we write $\bigotimes_{i=1}^{\infty} \mathcal{A}_i$ instead of $\varinjlim \bigotimes_{i=1}^{n} \mathcal{B}_i$.

1.3 Group theory and the K_0-group

1.3.1 Topological groups and their representations

This work is mainly dedicated to studying *continuous actions* of *topological groups* on C*-algebras, notions which require some previous work. Recall that an **action of a group** G on a set X is just a group homomorphism from G to the group of permutations of X (or the analogous of Definition 1.3.4 - see for instance [10] for a more in-depth study), which, in the context of algebraic structures, generalizes to a homomorphism to the group of automorphisms instead. This is insufficient, since we will also be needing a topological aspect.

Definition 1.3.1. A **topological group** G is a group $(G, \cdot)$ together with a Hausdorff topology under which the map

$$(x, y) \mapsto xy^{-1}$$

is continuous.

We here list some examples.

Examples 1.3.2.
1. A trivial example is a group G suited with the discrete topology, in which case we say that G is a **discrete group**.
2. The **circle group** $S^1 \equiv \{z \in \mathbb{C} : |z| = 1\}$ is a topological group when endowed with the induced topology.

Of particular importance are the *locally compact* groups, since they have a well-developed theory of integration. Recall that a topological space (or its topology) is said to be **locally compact** if every point has a compact neighbourhood.

Definition 1.3.3. A topological group is said to be a **locally compact group** if its topology is locally compact.

Since this work deals mainly with finite groups, equipped with the discrete topology, we will not be developing the above mentioned theory of integration on general locally compact groups. All our integrations will be the familiar summations. For the general theory, we refer the reader to [5].

We now generalize the usual notion of *action* of a group on a set to topological groups *acting* on topological spaces. This is a central notion all throughout the rest of this work.

Definition 1.3.4. A **(continuous) action** of a group G on a topological space X is a continuous function $\cdot$ from $G \times X$ to X such that:

1. $1 \cdot x = x$,
2. $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$

for all $g_1, g_2 \in G$ and $x \in X$. If, in addition, $g \cdot x \neq x$ for all $x \in X$ and $g \in G \setminus \{1\}$, the action is said to be **free**.

We take the time to present a central duality result in the study of topological groups. It was Fourier analysis that led to the construction of a duality theory for abelian groups developped by Pontryagin. The theory was eventually generalized and now plays center roles in several branches of mathematics, in particular within the study of actions of groups on C*-algebras.

Definition 1.3.5. Let G be a locally compact abelian group. A **character** of G is a continuous group homomorphism from G to S^1.

It is trivial to check that the set of all characters of G is again an abelian group under pointwise multiplication. It is furthermore locally compact when endowed with the compact open topology (regarding it as a subspace of all continuous functions from G to S^1). We call it the **character group** or the **(Pontryagin) dual group** of G and denote it by $\widehat{G}$ or $G\widehat{\ }$.

Theorem 1.3.6 (Pontryagin duality)**.** *For any given locally compact abelian group G,*

$$G \cong (\widehat{G})\widehat{\ }.$$

The isomorphism is natural, sending $g \in G$ to $\tau \mapsto \tau(g)$ for $\tau \in \widehat{G}$.

We now present some basic results of representations of topological groups, in particular finite abelian discrete groups.

Definition 1.3.7. A **(unitary) representation** of a topological group G on a Hilbert space H is a continuous group homomorphism $G \to U(H)$, where $U(H)$ is equipped with the strong operator topology, and we usually write U_g instead of $U(g)$. We call $\dim(H)$ the **degree** of the representation U.

To avoid countless repetitions, we fix some notation: throughout the remainder of this section, G will be used to denote an arbitrary topological group, and H and H' arbitrary Hilbert spaces, unless stated otherwise.

Definition 1.3.8. We say a unitary representation U of G on H is (unitarily) equivalent to a unitary representation V on H' if there exists a unitary operator $W : H \to H'$ such that

$$V_g = WU_gW^*$$

for all $g \in G$.

For discrete groups, we will be interested in a particularly important representation: the *regular representation.*

Definition 1.3.9. Let G be a discrete group. The unitary representation λ of G on $\ell^2(G)$, the space of square-summable complex-valued functions on G, given by

$$\lambda_g(f)(h) \equiv f(g^{-1}h)$$

is called the **(left) regular representation** of G.

Definition 1.3.10. Let U be a unitary representation of G on H, and let H' be a closed linear subspace of H. We say that H' is **invariant** under U, or U-invariant if for all $g \in G$ we have

$$U_g(H') \subseteq H'.$$

Given a U-invariant subspace H', we say that the representation $U_{|H'}$ (the restriction of U to H') is a **subrepresentation** of U. Finally, we say that U is **irreducible** if the only U-invariant closed subspaces of H are $\{0\}$ and H.

Definition 1.3.11. Let U be a unitary representation of G on H, and suppose H is the direct sum of a family $\{H_i\}$ of U-invariant closed subspaces of H. Then U is said to be the **direct sum** of the family of subrepresentations $\{U_{|H_i}\}$.

We are finally able to state the result we need, which is a very particular case of the Peter-Weyl theorem (cf. [23], Theorem 3.3).

Theorem 1.3.12. *Let G be a finite abelian discrete group. Then*

$$\ell^2(G) = \bigoplus_{\sigma \in \widehat{G}} \mathbb{C} \cdot \sigma.$$

The regular representation of a finite abelian group is thus a finite direct sum of characters of G.

1.3.2 The inductive limit of groups

The *inductive limit* of groups is in many ways analogous to the inductive limit of C*-algebras. We will need only be needing the basic definitions.

Definition 1.3.13. Let $\{G_n\}_{n\in\mathbb{N}}$ be a collection of groups and $\{\phi_{i,j} : G_i \to G_j\}_{i\leq j}$ a family of group homomorphisms obeying the following properties:

1. $\phi_{i,i} = I_{G_i}$,
2. $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$

for all $i \leq j \leq k$. Then the collection of pairs $(G_n, \phi_{n,m})$, also written schematically as

$$G_1 \overset{\phi_1}{\to} G_2 \overset{\phi_2}{\to} \cdots,$$

where $\phi_n \equiv \phi_{n,n}$, is called an **inductive** or **direct system** (or **Sequence**) of groups (and group homomorphisms).

Given an inductive system $(G_n, \phi_{n,m})$ of groups, we can consider the product

$$G_0 \equiv \prod_{n=1}^{\infty} G_n$$

which, under the usual pointwise operations, is again a group. The set G of elements $(g_n) \in G_0$ for which there exists $N \in \mathbb{N}$ such that for all $k > N$

$$\phi_k(g_k) = g_{k+1}$$

is a subgroup of G_0. Consider the normal subgroup H of G, formed by elements (e_n) for which there exists $N \in \mathbb{N}$ such that for all $k > N$

$$e_k = e_{G_k}.$$

Definition 1.3.14. In the previous construction, the quotient G/H is called the **inductive** or **direct limit** of the system $(G_n, \phi_{n,m})$, and written $\varinjlim G_n$.

Again, one can define natural homomorphisms $\phi^n : G_n \to \varinjlim G_n$, which make the following diagrams commute for all $n \in \ N$

$$\begin{array}{ccc} G_n & \xrightarrow{\phi_n} & G_{n+1} \\ & \searrow^{\phi^n} & \downarrow{\scriptstyle \phi^{n+1}} \\ & & \varinjlim G_n. \end{array}$$

The inductive limit of groups has, of course, the following (categorical) universal property.

Theorem 1.3.15 (Universal property of the inductive limit of groups - cf, [12], Theorem 6.1.1)**.** *Let $(G_n, \phi_{n,m})$ be an inductive system of groups, G a group, and $\psi^n : G_n \to G$ a collection of group homomorphisms such that the following diagrams commute for all $n \in \mathbb{N}$*

$$\begin{array}{ccc} G_n & \xrightarrow{\phi_n} & G_{n+1} \\ & \searrow^{\psi^n} & \downarrow{\scriptstyle \psi^{n+1}} \\ & & G. \end{array}$$

Then there exists a unique homomorphism $\rho : \varinjlim G_n \to G$ making the following diagrams commute for all $n \in \mathbb{N}$

$$\begin{array}{ccc} G_n & \xrightarrow{\phi^n} & \varinjlim G_n \\ & \searrow^{\psi^n} & \downarrow{\scriptstyle \rho} \\ & & G'. \end{array}$$

Once more, this universal property uniquely defines the inductive limit of groups up to isomorphism.

1.3.3 The K_0-group

We end this section with a typical construction of a group from an abelian semigroup.

Definition 1.3.16. Let H be an abelian semigroup and define the following equivalence relation $\sim$ on $H \times H$: $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists $z \in H$ such that

$$x_1 + y_2 + z = x_2 + y_1 + z.$$

Then $(H \times H)/\sim$ is a group, called the **Grothendieck group** of H.

The Grothendieck group of an abelian semigroup H can be regarded as the group of (classes of) formal differences of elements of H. The prototype example is the construction of $\mathbb{Z}$ out of $\mathbb{N}$.

Now let $\mathcal{A}$ be a C*-algebra, $\mathcal{K}$ the (nuclear) C*-algebra of compact operators on L^2, and $V(\mathcal{A})$ the set of Murray-von Neumann equivalence classes of projections in $\mathcal{A} \otimes \mathcal{K}$.

Lemma 1.3.17 (cf. [4], Section V.I.I)**.** *With $V(\mathcal{A})$ as above, given two classes $[p]$ and $[q]$ in $V(\mathcal{A})$, there exist projections $p', q' \in \mathcal{A} \otimes \mathcal{K}$ such that $p' \in [p]$, $q' \in [q]$ and*

$$p' \perp q'.$$

This allows us to turn $V(\mathcal{A})$ into an abelian semigroup $(V(\mathcal{A}), +)$, by setting

$$[p] + [q] \equiv [p' + q'],$$

where p' and q' are given by the previous lemma.

Definition 1.3.18. Let $\mathcal{A}$ be a unital C*-algebra, and $V(\mathcal{A})$ be as above. The Grothendieck group of $V(\mathcal{A})$ is called the K_0 **group** of $\mathcal{A}$, and written $K_0(\mathcal{A})$.

Note that for non-unital C*-algebras, the definition is slightly more complicated, but we won't be needing it here.

We list two important properties of the K_0 group.

Theorem 1.3.19 (cf. [25], Examples 6.1.4 and Proposition 6.2.1)**.** *For any $n \in \mathbb{N}$ and any unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$:*

1. $K_0(\mathcal{M}_n) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$;
2. $K_0(\mathcal{A} \oplus \mathcal{B}) \cong K_0(\mathcal{A}) \oplus K_0(\mathcal{B})$.

We will also be interested in K_0 as an *ordered group*.

Definition 1.3.20. An **ordered group** is an abelian group G, together with a subsemigroup G_+ containing the identity 1, called the **positive cone** of G, having the following properties:

1. $G_+ - G_+ = G$.

2. $G_+ \cap (-G_+) = \{1\}$.

IF it is clear from context what the positive cone G_+ is, we refer to the ordered group simply by G.

The partial order $\leq$ on G is then induced by G_+ in the usual way: we say $x \leq y$ if $x - y \in G_+$.

Given a C*-algebra, it would be natural to choose the image $K_0(\mathcal{A})_+$ of $V(\mathcal{A})$ as the positive cone of $K_0(\mathcal{A})$. It turns out $K_0(\mathcal{A})_+$ does not usually satisfy the positive cone requirements.

Definition 1.3.21. A C*-algebra $\mathcal{A}$ is said to be **stably finite** if for every $n \in \mathbb{N}$, the C*-algebra $\mathcal{M}_n(\mathcal{A})$ of $n \times n$ matrices with entries in $\mathcal{A}$, is finite.

Proposition 1.3.22 (cf. [3], Proposition 6.3.3)**.** *If $\mathcal{A}$ is a stably finite C*-algebra, $K_0(\mathcal{A})$ is an ordered group.*

We can finally state our main result, whose proof can be found in Chapter 7 of [3].

Theorem 1.3.23. *If $\mathcal{A}$ is a UHF algebra, then $K_0(\mathcal{A})$ is totally ordered.*

We here only presented the very very basics of K-theory, a vast branch of mathematics. For further details on the theory, see for instance [3] or [25].

Chapter 2

Crossed Products and Freeness of Group Actions on C*-algebras

C-dynamical systems* are natural generalizations of standard dynamical systems, and can also be viewed as actions of groups on C*-algebras. They consist of a triple: a C*-algebra, a group and the action itself, and one can think of applying representation theory to this triple, which culminates in the notion of *covariant representation*. This quickly leads to a construction of a new C*-algebra, called the *crossed product*, which encodes information of the C*-algebra, the group and the action, in the sense that its representation theory can be identified with the *covariant representation theory* of the triple.

In Section 2.1, we define *dynamical system* and *crossed product*, and study the deep connection between their representation theories.

In Section 2.2, we turn our attention back to group actions on C*-algebras at a more basic level, and try to adapt the usual notion of a free action of a group on a set to our theory of dynamical systems.

2.1 The crossed product

Definition 2.1.1. An **action** of a locally compact group G on a C*-algebra $\mathcal{A}$ is a continuous homomorphism $\alpha : G \to \mathrm{Aut}\,(\mathcal{A})$ in the strong operator topology. We call the triple $(\mathcal{A}, G, \alpha)$ a **C*-dynamical system** (or just **dynamical system** for short). We often write α_s instead of $\alpha(s)$ to designate the image of an element $s \in G$ through α.

From now on we shall mainly consider discrete groups, where continuity is automatic. We may also refer to dynamical systems as **discrete** dynamical systems as a way to say the group at play is discrete.

As is common, it is useful and easier to study a given dynamical system by considering it as a subspace of a simple, well-understood space, i.e. by *representing* it. We've already seen how to represent both a group and a C*-algebra individually. What we now need is a new notion of representation able to describe the action of a group on a C*-algebra.

Definition 2.1.2. A **covariant representation** of a dynamical system $(\mathcal{A}, G, \alpha)$ is a pair (π, U) consisting of a representation π of $\mathcal{A}$ on a Hilbert space H, and a unitary representation U of G on the same Hilbert space H such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*. \tag{2.1}$$

The previous equation is called the **covariance relation** between π and U. We say a covariant representation is **nondegenerate** if π is nondegenerate.

It would be very desirable, instead of working with an action directly and the corresponding pair of representations, to work with a single object (the crossed product) powerful enough to encode the covariant representations. To do so, we start by considering the space $C_c(G, \mathcal{A})$ of compactly supported functions from the discrete group G to the C*-algebra $\mathcal{A}$, and turning it into a *-normed algebra by defining the following product (henceforth called convolution), involution and norm

- $(f * g)(t) \equiv \sum_{s \in G} f(s)\alpha_s(g(s^{-1}t))$.
- $f^*(t) \equiv \alpha_t(f(t^{-1})^*)$.
- $\|f\|_1 \equiv \sum_{s \in G} \|f(s)\|$.

It can be checked (cf. [13], Proposição 2.3.4 and Proposição 2.3.5) that the previous three relations do indeed define a multiplication, involution and norm, which is not surprising since they have roots on previously well-defined operations of same nature. $C_c(G, \mathcal{A})$ is thus a *-normed algebra, and we can now consider its universal enveloping C*-algebra.

Definition 2.1.3. Given a dynamical system $(\mathcal{A}, G, \alpha)$, the universal enveloping C*-algebra of $C_c(G, \mathcal{A})$ is called the **(full) crossed product** of $\mathcal{A}$ by G, and denoted $\mathcal{A} \rtimes_\alpha G$.

When we say crossed product from here onwards, we will always be referring to the previous definition, that is, the full crossed product.

We take the chance to introduce an important element in the crossed product, which will be useful in the future.

Definition 2.1.4. Let $(\mathcal{A}, G, \alpha)$ be a discrete dynamical system with $\mathcal{A}$ unital. Given $g \in G$, the unitary element $u_g \in C_c(G, \mathcal{A})$ defined by

$$u_g(h) \equiv \delta_{gh}$$

is called the **standard** or **canonical unitary** associated to $g \in G$. We use the same notation and terminology for the image of u_g in the crossed product $\mathcal{A} \rtimes_\alpha G$.

Up until this point the link between $C_c(G, \mathcal{A})$ or $\mathcal{A} \rtimes_\alpha G$ and the dynamical system $(\mathcal{A}, G, \alpha)$ is not at all obvious. However, it turns out that covariant representations of $(\mathcal{A}, G, \alpha)$ and representations $C_c(G, \mathcal{A})$ or $\mathcal{A} \rtimes_\alpha G$ are very closely related, which justifies this construction.

Theorem 2.1.5. *Let (π, U) be a covariant representation of a discrete dynamical system $(\mathcal{A}, G, \alpha)$ on a Hilbert space H. Then the map $\pi \rtimes U : C_c(G, \mathcal{A}) \to B(H)$ (called the **integrated form** of (π, U)) given by*

$$(\pi \rtimes U)(f) \equiv \sum_{s \in G} \pi(f(s)) U_s$$

is a representation of $C_c(G, \mathcal{A})$. Furthermore, $\pi \rtimes U$ is nondegenerate if π is nondegenerate.

Proof. We need to prove that $\pi \rtimes U$ is a continuous *-homomorphism. We start by observing that it is linear since π, being a representation itself, is linear. The preservation of both the convolution and involution is a matter of simple calculations. In fact, for any $f, g \in C_c(G, \mathcal{A})$

$$\begin{aligned}
(\pi \rtimes U)(f * g) &= \sum_{s \in G} \pi((f * g)(s)) U_s \\
&= \sum_{s \in G} \pi \left(\sum_{t \in G} f(t) \alpha_t(g(t^{-1}s)) \right) U_s \\
&= \sum_{s \in G} \sum_{t \in G} \pi(f(t)) \pi(\alpha_t(g(t^{-1}s))) U_s \\
&= \sum_{s \in G} \sum_{t \in G} \pi(f(t)) U_t \pi(g(t^{-1}s)) U_t^* U_s \\
&= \sum_{s \in G} \sum_{t \in G} \pi(f(t)) U_t \pi(g(s)) U_s \\
&= [\pi \rtimes U(f)][\pi \rtimes U(g)]
\end{aligned}$$

and

$$\begin{aligned}
(\pi \rtimes U)(f^*) &= \sum_{s \in G} \pi(f^*(s)) U_s \\
&= \sum_{s \in G} \pi(\alpha_s(f(s^{-1})^*)) U_s \\
&= \sum_{s \in G} U_s \pi(f(s^{-1})^*)) U_s^* U_s \\
&= \sum_{s \in G} U_s \pi(f(s^{-1}))^* \\
&= \sum_{s \in G} (\pi(f(s)) U_s)^* \\
&= [\pi \rtimes U(f)]^*.
\end{aligned}$$

Finally,

$$\begin{aligned}\|\pi \rtimes U(f)\| &= \left\|\sum_{s\in G} \pi(f(s))U_s\right\| \\ &\leq \sum_{s\in G} \|\pi(f(s))U_s\| \\ &\leq \sum_{s\in G} \|f(s)\| \\ &= \|f\|_1,\end{aligned}$$

which proves $\pi \rtimes U$ is continuous, and thus a representation of $C_c(G, \mathcal{A})$.

We now check $\pi \rtimes U$ is nondegenerate whenever π is nondegenerate. Let $\{e_i\}$ be an approximate unit of $\mathcal{A}$. We need only show the net $\pi \rtimes U(f_i)$, where $f_i \in C_c(G, \mathcal{A})$ is defined by

$$f_i(s) \equiv \begin{cases} e_i, & s = 1 \\ 0, & s \neq 1 \end{cases},$$

converges pointwise to the identity operator on $B(H)$. This is easy. Indeed we get

$$\begin{aligned}\pi \rtimes U(f_i) &= \sum_{s\in G} \pi(f_i(s))U_s \\ &= \pi(e_i),\end{aligned}$$

and the result follows from nondegeneracy of π. ☐

One might question the name integrated form, since there's apparently no integrals being taken at all. The name comes from the general framework, where one does not necessarily consider discrete groups and thus has to develop a theory of integration on groups (more specifically locally compact groups) to account for all summation signs we've been using with ease.

Just as any nondegenerate covariant representation of a dynamical system induces a nondegenerate continuous representation of $C_c(G, \mathcal{A})$, so is every nondegenerate continuous representation of $C_c(G, \mathcal{A})$ induced by a covariant representation of the associated system.

Theorem 2.1.6. *Given a discrete group G and a C^*-algebra $\mathcal{A}$, every nondegenerate continuous representation of $C_c(G, \mathcal{A})$ is the integrated form of a nondegenerate covariant representation of a discrete dynamical system $(\mathcal{A}, G, \alpha)$.*

Proof. Let ρ be a nondegenerate continuous representation of $C_c(G, \mathcal{A})$ on H. Consider the map

$$\begin{aligned}\pi : \quad \mathcal{A} &\to B(H) \\ a &\mapsto \rho(f_a)\end{aligned},$$

where $f_a \in C_c(G, \mathcal{A})$ is the function taking the value a at 1 and 0 elsewhere. We check that π is a nondegenerate representation of $\mathcal{A}$. Choosing $a, b \in \mathcal{A}$ and λ a scalar,

$$\pi(a+b) = \rho(f_{a+b}) = \rho(f_a + f_b) = \rho(f_a) + \rho(f_b) = \pi(a) + \pi(b),$$

$$\pi(ab) = \rho(f_{ab}) = \rho(f_a f_b) = \rho(f_a)\rho(f_b) = \pi(a)\pi(b),$$

$$\pi(\lambda a) = \rho(f_{\lambda a}) = \rho(\lambda f_a) = \lambda\rho(f_a) = \lambda\pi(a)$$

and

$$\pi(a^*) = \rho(f_{a^*}) = \rho((f_a)^*) = (\rho(f_a))^* = (\pi(a))^*,$$

so π is a *-homomorphism. Finally, nondegeneracy of ρ implies nondegeneracy of π.

We now need the representation of G. Again for each $r \in G$ and $g \in C_c(G, \mathcal{A})$, define $\beta_{r,g} \in C_c(G, \mathcal{A})$ by $\beta_{r,g}(s) \equiv \alpha_r(g(r^{-1}s))$, and note that

$$\beta_{rs,g} = \beta_{r,g}\beta_{s,g}.$$

Let $\{e_i\}$ be an approximate unit of $\mathcal{A}$. A simple computation shows that $\beta_{r,f_{e_i}} * g = \beta_{r,(f_{e_i} * g)}$, so

$$\rho(\beta_{r,(f_{e_i})})\rho(g) = \rho(\beta_{r,(f_{e_i} * g)}) \to \rho(\beta_{r,g}).$$

We conclude that $\rho(\beta_{r,f_{e_i}})$ converges pointwise to an operator U_r on H, easily seen to be a linear contraction, and such that

$$U_r\rho(g) = \rho(\beta_{r,g}).$$

We show that U (the map $r \mapsto U_r$) is a unitary representation of G. Letting $s, t \in G$

$$U_{st}\rho(g) = \rho(\beta_{st,g}) = \rho(\beta_{s,g}\beta_{t,g}) = \rho(\beta_{s,g})\rho(\beta_{t,g}) = U_s\rho(g)U_t\rho(g)$$

proves U_r is a group homomorphism, since ρ is nondegenerate. Furthermore, U_1 is the identity operator on $B(H)$, so all U_s are invertible. Being contractions whose inverses are contractions, they're unitary.

As for the covariance relation between π and U, we have

$$\begin{aligned}
U_s\pi(a)U_{s^{-1}}\rho(g) &= U_s\pi(a)\rho(\beta_{s^{-1},g}) \\
&= U_s\rho(f_a\beta_{s^{-1},g}) \\
&= \rho(\beta_{s,f_a\beta_{s^{-1},g}}) \\
&= \rho(\alpha_s(a)g) \\
&= \pi(\alpha_s(a))\rho(g).
\end{aligned}$$

There's only one thing left to check: the integrated form of (π, U) is indeed ρ. Indeed

$$\begin{aligned}
\pi \rtimes U(k)\rho(g) &= \sum_{s\in G} \pi(k(s))U_s\rho(g) \\
&= \sum_{s\in G} \pi(k(s))\rho(\beta_{s,g}) \\
&= \sum_{s\in G} \rho(f_{k(s)}\beta_{s,g}) \\
&= \rho\left(\sum_{s\in G} k(s)\beta_{s,g}\right) \\
&= \rho(k * g) \\
&= \rho(k)\rho(g),
\end{aligned}$$

which ends the proof. □

The two theories of representations being considered are thus isomorphic: studying a covariant representation of a dynamical system, which involves a pair of representations and a covariant relation, is nothing more than studying a representation (in the usual, simpler way) of a C*-algebra: a reassuring thought.

However, despite these strong connecting results, it is seldom possible to exhibit all covariant representations of a dynamical system, and so computing the universal norm is a rather difficult task. If one is to study the crossed product of a given system, one might want to look at and work with a weaker version, called the *reduced crossed product*, instead of the one just presented. It so happens that under some mild conditions, both notions are equivalent, as we're about to see.

Definition 2.1.7. Let $(\mathcal{A}, G, \alpha)$ be a discrete dynamical system and π a representation of $\mathcal{A}$ on a Hilbert space H. The pair (π_α, λ), of representations of $\mathcal{A}$ and G, respectively, on $\ell^2(G, H)$, defined by

$$\pi_\alpha(x)\xi(t) \equiv \pi(\alpha_t^{-1}(x))(\xi(t))$$

$$\lambda(t)\xi(s) \equiv \xi(t^{-1}s)$$

is called a **regular representation** of $(\mathcal{A}, G, \alpha)$ (associated to π).

It is easy to see that any regular representation is a covariant representation. It will now be the set of regular representations which will play the role of the covariant representations when defining the reduced crossed product. In fact, we now need only mimic and adapt Proposition 1.1.42.

Proposition 2.1.8. *Given a discrete dynamical system $(\mathcal{A}, G, \alpha)$, the map $\|\cdot\|_r : \mathbb{C}_c(G, \mathcal{A}) \to \mathbb{R}$ defined by*

$$\|a\|_r \equiv \sup\{\|(\pi_\alpha \rtimes \lambda)(f)\| : (\pi_\alpha, \lambda) \textit{ is a regular representation of } (\mathcal{A}, G, \alpha)\}$$

is a norm on $C_c(G, \mathcal{A})$ that satisfies

1. $\|f * f\|_r = \|f\|_r^2$.

2. $\|\cdot\|_r \leq \|\cdot\|_u$.

We may now define the following weaker version of crossed product.

Definition 2.1.9. Given a discrete dynamical system $(\mathcal{A}, G, \alpha)$, we call the completion of $(C_c(G, \mathcal{A}), \|\cdot\|_r)$ the **reduced crossed product** of $\mathcal{A}$ by G, and we write $\mathcal{A} \rtimes_\alpha^r G$.

Since $\|\cdot\|_r \leq \|\cdot\|_u$, the natural inclusion mapping $(C_c(G, \mathcal{A}), \|\cdot\|_u) \hookrightarrow \mathcal{A} \rtimes_\alpha^r G$ is continuous and thus extends to a surjective homomorphism ρ from the full crossed product to the reduced crossed, and so

$$(\mathcal{A} \rtimes_\alpha G)/\ker \rho \cong \mathcal{A} \rtimes_\alpha^r G.$$

It turns out that some mild conditions on the group are enough to guarantee $\ker \rho = \{0\}$.

Definition 2.1.10. Given a discrete group G, a state μ on $\ell^\infty(G)$ is called a **mean** (on $\ell^\infty(G)$). A mean is said to be **left-invariant** if for all $g \in G$ and $f \in \ell^\infty(G)$

$$\mu(L_g f) = \mu(f),$$

where L_g is defined by $L_g f(s) \equiv f(g^{-1}s)$.

Definition 2.1.11. A discrete group G is called **amenable** if there exists a left-invariant mean on $\ell^\infty(G)$.

Amenable groups include all finite, all abelian and all compact groups, for instance. It is indeed a weak condition to ask for in a group. In fact, for quite a few years, the only known nonamenable groups were groups that contained $\mathbb{F}_2$ (the free group on two generators) as a subgroup (in particular $\mathbb{F}_2$ itself). Examples have, however, been produced, of nonamenable groups which do not contain $\mathbb{F}_2$ as a subgroup.

Theorem 2.1.12. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system with G amenable. Then $\mathcal{A} \rtimes_\alpha G \cong \mathcal{A} \rtimes_\alpha^r G$.*

We now turn our attention to a technical result linking the crossed product and the inductive limit together. We're going to prove that, under certain conditions, the crossed product of an inductive limit is the inductive limit of the crossed products, so describing $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ can be done by describing the inductive system of crossed products, which in turn is fairly simple to do.

Definition 2.1.13. If $(\mathcal{A}_1, G, \alpha^{(1)})$ and $(\mathcal{A}_2, G, \alpha^{(2)})$ are dynamical systems, a *-homomorphism $\eta : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be an **equivariant homomorphism** if for all $s \in G$

$$\alpha_s^{(2)} \circ \eta = \eta \circ \alpha_s^{(1)}.$$

Lemma 2.1.14. *If* $(\mathcal{A}_1, G, \alpha_1)$ *and* $(\mathcal{A}_2, G, \alpha_2)$ *are dynamical systems, and* $\eta : \mathcal{A}_1 \to \mathcal{A}_2$ *is an equivariant homomorphism, there exists a *-homomorphism* $\Gamma : C_c(G, \mathcal{A}_1) \to C_c(G, \mathcal{A}_2)$ *such that*

$$\Gamma(f)(s) = \eta(f(s)).$$

Theorem 2.1.15. *Let* $(\mathcal{A}, G, \alpha)$ *be a discrete dynamical system, where* $\mathcal{A}$ *is the inductive limit of an inductive system* $(\mathcal{A}_i, \phi_{ij})$ *and each* ϕ_{ij} *is an equivariant homomorphism. Considering the inductive system*

$$\mathcal{A}_1 \rtimes_{\alpha_1} G \overset{\overline{\phi}_1}{\to} \mathcal{A}_2 \rtimes_{\alpha_2} G \overset{\overline{\phi}_2}{\to} \cdots$$

where each $\overline{\phi}_{ij}$ *is obtained by applying Lemma 2.1.14, we have*

$$\mathcal{A} \rtimes_\alpha G \cong \varinjlim \left[\mathcal{A}_i \rtimes_{\alpha_i} G\right].$$

Proof. Recall (Proposition 1.2.10) that there are canonical *-homomorphisms ϕ^i from each $\mathcal{A}_i$ to $\mathcal{A}$, and thus there are canonical *-homomorphisms γ^i from each $\mathcal{A}_i \rtimes_{\alpha_i} G$ to $\mathcal{A} \rtimes_\alpha G$. We thus need only prove that $\mathcal{A} \rtimes_\alpha G$, together with the maps γ^i, satisfies the universal property of Theorem 1.2.11.

Let $\mathcal{B}$ be a C*-algebra, which we may assume to be a nondegenerate *-subalgebra of $L(H)$ for some Hilbert space H by the Gelfand-Naimark theorem, and let $\psi^i : \mathcal{A}_i \rtimes_{\alpha_i} G \to \mathcal{B}$ be *-homomorphisms such that the diagrams

$$\begin{array}{ccc} \mathcal{A}_i \rtimes_{\alpha_i} G & \xrightarrow{\phi_i} & \mathcal{A}_{i+1} \rtimes_{\alpha_{i+1}} G \\ & \searrow^{\psi^i} & \downarrow \psi^{i+1} \\ & & \mathcal{B} \end{array}$$

commute. In light of Theorem 2.1.6, each ψ^i is the integrated form of a nondegenerate covariant representation (π_i, U_i) of $\mathcal{A}_i \rtimes_{\alpha_i} G$ on H, and the previous commutative diagram implies that each ψ^i is also the integrated form of $(\pi_{i+1} \circ \phi_i, U_{i+1})$. Since nondegenerate covariant representations of the dynamical system are in one-to-one correspondence with nondegenerate representations of the crossed product, $U_i = U_{i+1}$. We are thus reduced to single unitary representation U of G. Now $\{\mathcal{A}_i, \phi_i\}_{i \in \mathbb{N}}$ forms an inductive system by hypothesis, so there exists a representation π of $\mathcal{A}$ in H making the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}_i & \xrightarrow{\pi_i} & \mathcal{A} \\ & \searrow^{\phi^i} & \downarrow \pi \\ & & \mathcal{B}. \end{array}$$

Since the (π_i, U) are all nondegenerate covariant representations, (π, U) is also a nondegenerate covariant representation. Its integrated form ψ is thus a representation of $\mathcal{A} \rtimes_\alpha G$ on H making the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}_i \rtimes_{\alpha_i} G & \xrightarrow{\phi_i} & \mathcal{A} \\ & \searrow^{\psi_i} & \downarrow{\scriptstyle \psi} \\ & & \mathcal{B}. \end{array}$$

The bijection between nondegenerate covariant representations of the dynamical system and nondegenerate representations of the crossed product then ensures uniqueness of ψ, and the proof is complete. □

2.2 Freeness on the primitive ideal space

Up until now we've been dealing with group actions on C*-algebras without requiring any particular property of them. General actions though are hard to work with, and one eventually demands that they meet certain requirements. When dealing with group actions on sets, or continuous group actions on topological spaces, a basic property with outstanding consequences is that of *freeness*, as defined in Section 1.2. It is thus tempting to see what this translates to in the context of C*-algebras. Attempts to find suitable definitions of freeness in such context have given birth to the notions of *freeness on the primitive ideal space*, the *strict* and the *tracial Rokhlin properties* among others, each of which with its own strengths and drawbacks. There is no actual generalization of freeness for actions on C*-algebras, but instead several different notions with different purposes. Our focus here will be to study the above mentioned notions of freeness, with particular emphasis on the last two. This short section, however, will be devoted to the first one, where the commutative case will provide guidance.

The Gelfand-Naimark theorem (Theorem 1.1.35) asserts that every commutative C*-algebra $\mathcal{A}$ is of the form $C_0(\hat{\mathcal{A}})$, and the spectrum - $\hat{\mathcal{A}}$ - is a locally compact Hausdorff space. If we're given a locally compact transformation group (with G discrete), we can define an action α of G on $C_0(X)$ as follows:

$$\alpha_g(f)(x) \equiv f(g^{-1}x).$$

It is immediate to check that the above formula does indeed define an action of G on $\mathcal{A}$. Conversely, any dynamical system of the form $(C_0(X), G, \alpha)$ with X locally compact arises from a locally compact transformation group (G, X) (cf. [26], Proposition 2.7). It is but natural, then, to define α as being a free action on $\mathcal{A}$ if the corresponding action on $\hat{\mathcal{A}}$ is free. Unfortunately, this is only useful in the commutative case.

To generalize freeness to the non-commutative case, we start by considering the primitive spectrum $\text{Prim}\,(\mathcal{A})$ of $\mathcal{A}$ (if $\mathcal{A}$ is commutative $\hat{A}$ is homeomorphic to $\text{Prim}\,(\mathcal{A})$), and endowing it with a suitable topology. Let X be a nonempty set of primitive ideals of a C*-algebra. We define its **hull** $\overline{X}$ as

$$\overline{X} \equiv \{I \in \text{Prim}\,(\mathcal{A}) : I \supseteq \ker(X)\} \tag{2.2}$$

where $\ker(X) \equiv \bigcap_{P \in X} P$. The use of the prefix ker in this section, and this section alone, will always have this meaning, and should not be confused with the kernel of a morphism.

Lemma 2.2.1. *The hull-closure operation 2.2 satisfies the Kuratowski closure axioms, and as thus defines a unique topology, called the* ***Jacobson*** *or* ***hull-kernel topology*** *on* $\text{Prim}\,(\mathcal{A})$.

Proof. Recall that the Kuratowski closure axioms are:

1. $\overline{\emptyset} = \emptyset$,
2. $X \subseteq \overline{X}$,

3. $\overline{\overline{X}} = \overline{X}$,

4. $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$

for any two subsets X, Y of $\mathrm{Prim}\,(\mathcal{A})$.

1. $\overline{\emptyset} = \emptyset$ since no primitive ideal contains $\mathrm{Prim}\,(\mathcal{A})$.

2. Immediate: any set contains the intersection of itself with another set.

3. Fix $J \in \overline{\overline{X}}$. Then

$$J \supseteq \ker(\overline{X}) = \ker(\{I : I \supseteq \ker(X)\}) \supseteq \ker(X)$$

and $J \in \overline{X}$ by definition. The reverse inclusion is just the previous property.

4. We've shown (Lemma 1.1.31) that any primitive ideal is prime. Thus, if a primitive ideal contains the intersection of two ideals (which in turn contains their product), it must contain one or the other. The result then follows from the obvious fact that $\ker(X \cup Y) = \ker(X) \cap \ker(Y)$.

□

Now given a dynamical system $(\mathcal{A}, G, \alpha)$, there is a naturally induced action of G on $\mathrm{Prim}\,(\mathcal{A})$, given by

$$g \cdot P = \alpha_g(P). \tag{2.3}$$

It is readily seen this is indeed an action (of a group on a set), since $g \mapsto \alpha_g$ is a group homomorphism. Furthermore, since G is assumed to be discrete, continuity of the above map follows from the continuity of each α_g, and so equation 2.3 defines a continuous action of G on $\mathrm{Prim}\,(\mathcal{A})$. In fact, this is a continuous action even when G is not assumed to be discrete (a proof can be found in [21]).

Definition 2.2.2. We say that an action α of a group on a C*-algebra is **free on the primitive ideal space (of $\mathcal{A}$)** if its induced action (equation 2.3) is free.

Freeness on the primitive ideal space is a very strong condition that allows for some interesting results. For instance, in the realm of type I C*-algebras (where $\hat{\mathcal{A}} = \mathrm{Prim}\,(\mathcal{A})$), freeness on the primitive ideal space is closely related to other types of freeness, as extensively studied in [16].

In invertibility theory, in particular in local-trajectory methods, this type of freeness also proves to be a fundamental requirement (cf. [2]). In this case one works with the *center* of a given C*-algebra (a commutative *-subalgebra), where freeness on the primitive (this case maximal) ideal space coincides with freeness of the corresponding action, as discussed in the beginning of this section.

2.3 The strict and the tracial Rokhlin properties

Outside the realm of commutative or other restrictive classes of C*-algebras, freeness on the primitive ideal space is too strong a condition to ask for (see [18] for a detailed analysis). Therefore, several other notions of freeness have been developed. We here introduce two: the *strict Rokhlin property* and the *tracial Rokhlin property.* These cannot, however, be defined for arbitrary C*-algebras or arbitrary discrete groups, as is clear from the definition and the example.

Definition 2.3.1. Let $\alpha : G \to \mathrm{Aut}\,(\mathcal{A})$ be an action of a finite group G on a separable unital C*-algebra $\mathcal{A}$. We say α has the **(strict) Rokhlin property** if for any finite set $F \subseteq \mathcal{A}$ and every $\epsilon > 0$, there is a set of pairwise orthogonal projections $\{p_g\}_{g\in G}$, called **Rokhlin** or **Rokhlin ϵ-projections** for F, such that

1. $\|\alpha_g(p_h) - p_{gh}\| < \epsilon$ for all $g, h \in G$.
2. $\|p_g a - a p_g\| < \epsilon$ for all $a \in \mathcal{A}$ and $g \in G$.
3. $\sum_{g\in G} p_g = 1_{\mathcal{A}}$.

We start by working out a very simple example.

Example 2.3.2. Consider the infinite tensor product C*-algebra

$$\mathcal{A} \equiv \bigotimes_{i=1}^{\infty} \mathcal{M}_2$$

and the automorphism

$$\alpha(X) \equiv \bigotimes_{i=1}^{\infty} (WXW^*)$$

where

$$W \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By definition, $\mathcal{A}$ is an inductive limit

$$\mathcal{A} = \varinjlim \mathcal{A}_k,$$

where

$$\mathcal{A}_k \equiv \bigotimes_{i=1}^{k} \mathcal{M}_2 \quad \text{and} \quad \alpha_k(X) \equiv \bigotimes_{i=1}^{k} (WXW^*).$$

It is immediate that $\alpha^2 = I$, for W is a unitary matrix. As such, one obtains a natural induced action of $\mathbb{Z}_2$ on $\mathcal{A}$:

$$\begin{aligned} \mathbb{Z}_2 &\to \mathrm{Aut}\,(\mathcal{A}) \\ 0 &\mapsto I \\ 1 &\mapsto \alpha. \end{aligned}$$

We check that this action has the strict Rokhlin property. Given $\epsilon > 0$ and a finite subset F of the dense subalgebra $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ of $\mathcal{A}$, fix k large enough to have $F \subseteq \mathcal{A}_k$, and consider

$$P \equiv \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$p_0 \equiv I_{\mathcal{A}_k} \otimes P \qquad\qquad p_1 \equiv I_{\mathcal{A}_k} \otimes (I - P).$$

Now P is a projection in $\mathcal{M}_2$, and consequently so are p_0 and p_1 in $\mathcal{A}_{k+1}$. It is also easy to see that $p_0 \perp p_1$, and we have $WPW^* = I - P$, which leads to

$$\begin{aligned}
\alpha_{k+1}(p_0) &= \bigotimes_{i=1}^{k+1} \mathrm{Ad}\,(W)(p_0) \\
&= \left(\bigotimes_{i=1}^{k+1} W\right)(I_{\mathcal{A}_k} \otimes P)\left(\bigotimes_{i=1}^{k+1} W^*\right) \\
&= \left(\bigotimes_{i=1}^{k} WW^*\right) \otimes WPW^* \\
&= I_{\mathcal{A}_k} \otimes (I - P) \\
&= p_1.
\end{aligned}$$

A similar computation shows $\alpha_{k+1}(p_1) = p_0$, implying $\alpha(p_0) = p_1$ and $\alpha(p_1) = p_0$, which in turn guarantees the first axiom in the definition of the strict Rokhlin property. It is trivial that both p_0 and p_1 commute with every element of $\mathcal{A}_k$, and hence every element of F, whence the second axiom. Finally, one has $p_0 + p_1 = I_{\mathcal{A}_{k+1}}$ by construction and the proof is finished.

The above example is quite simple, and unfortunately is one of very few examples with the strict Rokhlin property. Although this notion may be applied to a broader class of C*-algebras (within the separable and unital family) when compared to freeness in the primitive ideal space, it is still too strong a condition to ask for. A weakened version has then been suggested (cf. [15]): the *tracial Rokhlin property.*

Definition 2.3.3. [20] Let $\alpha : G \to \mathrm{Aut}\,(\mathcal{A})$ be an action of a finite group G on an infinite dimensional simple separable unital C*-algebra $\mathcal{A}$. We say that α has the **tracial Rokhlin property** if for every finite set $F \subseteq \mathcal{A}$, every $\epsilon > 0$ and every positive element $a \in \mathcal{A}$ of norm 1 there is a set of mutually orthogonal projections $\{p_g\}_{g \in G}$ such that, setting $p \equiv \sum_{g \in G} p_g$, for all $f \in F$ and $g, h \in G$

1. $\|\alpha_g(p_h) - p_{gh}\| < \epsilon$.
2. $\|p_g f - f p_g\| < \epsilon$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of $\mathcal{A}$ generated by a.

4. $\|pap\| > 1 - \epsilon$.

Condition (4) in the above definition is redundant if the algebra is finite ([cf. [20], Lemma 1.16]). Note that when studying the Rokhlin properties, we only consider finite C*-algebras in this work.

The following example, of same nature as the previous, illustrates why it may be more tempting to work with the tracial Rokhlin property instead.

Example 2.3.4. The action of $\mathbb{Z}_2$ generated (just like in example 2.3.2) by

$$\alpha(X) \equiv \bigotimes_{i=1}^{\infty} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} X \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)$$

on

$$\mathcal{A} \equiv \bigotimes_{i=1}^{\infty} \mathcal{M}_3$$

has the tracial Rokhlin property. However, it does not have the strict Rokhlin property. The proof of these results will be given in Chapter 3.

We are now ready to explore these two properties more closely by relating them with many of the concepts that have been defined up until now, and constructing further examples.

Chapter 3

The Rokhlin Properties for Product Type Actions

We have seen three notions related with free actions, and in this chapter study the Rokhlin properties in more detail. Section 3.1 is devoted to constructing a *product type action* of $\mathbb{Z}_2$ on an arbitrary UHF algebra, and determining exactly when such an action has the Rokhlin or the tracial Rokhlin properties. It turns out these properties are elegantly described by the structure of the crossed product, or the projections involved, a result that can be considered as a classification theorem for product type actions on UHF algebras.

In Section 3.2, we mention how these two results behave if we consider an arbitrary finite cyclic group $\mathbb{Z}_n$ in place of $\mathbb{Z}_2$. Some of the results are easily seen to be true, but others require a more extensive analysis.

3.1 Product type actions of order 2

We will be studying to some extent a certain class of actions on C*-algebras called *product type actions*. These are constructed by means of another class of actions called *inner actions* and both of these rest upon the notion of *adjoint automorphism*.

Definition 3.1.1. Given a unitary u in a unital C*-algebra $\mathcal{A}$, we call **adjoint automorphism** generated by u, and we write $\mathrm{Ad}\,(u)$, the automorphism of $\mathcal{A}$ given by

$$\begin{array}{rccc} \mathrm{Ad}\,(u): & \mathcal{A} & \to & \mathcal{A} \\ & a & \mapsto & uau^* \end{array}$$

Definition 3.1.2. An action α of a discrete group G on a C*-algebra $\mathcal{A}$ is said to be **inner** if there exists a group homomorphism $\sigma : G \to U(\mathcal{A})$ such that

$$\alpha(g) = \mathrm{Ad}\,(\sigma(g)).$$

Let G be a discrete group and $\mathcal{A}$ the UHF algebra of type $\prod k_n$. $\mathcal{A}$ is, by Corollary 1.2.26, the direct limit of

$$\mathcal{M}_{k_1} \xrightarrow{\phi_1} \mathcal{M}_{k_1} \otimes \mathcal{M}_{k_2} \xrightarrow{\phi_2} \cdots .$$

Set $\mathcal{A}_n \equiv \bigotimes_{i=1}^n \mathcal{M}_{k_i}$, so that $\mathcal{A} = \varinjlim \mathcal{A}_n$, and let $\beta^{(n)}$ be, for each n, an inner action of G on $\mathcal{M}_{k_n}$. Define then, for each n, an action $\alpha^{(n)}$ of G on $\mathcal{A}_n$ by setting

$$\alpha_g^{(n)}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \equiv \beta_g^{(1)}(a_1) \otimes \beta_g^{(2)}(a_2) \otimes \cdots \otimes \beta_g^{(n)}(a_n).$$

It is trivial that $\phi_n \circ \alpha_g^{(n)} = \alpha_g^{(n+1)} \circ \phi_n$ for all n and $g \in G$, so there is an induced action of G on the direct limit $\mathcal{A}$. More often than not we write $\bigotimes_{i=1}^n \beta_g^{(i)}$ in place of $\alpha_g^{(n)}$, and $\bigotimes_{i=1}^\infty \beta_g^{(i)}$ for the direct limit action.

Definition 3.1.3. Every action of a discrete group on a UHF algebra $\mathcal{A}$ obtained as above is called a **product type action**. Product type actions can be defined, with a bit more care, for general topological groups.

Definition 3.1.4. If $\alpha \in \mathrm{Aut}\,(\mathcal{A})$ is an automorphism of order n on a C*-algebra $\mathcal{A}$ (i.e. $\alpha^n = 1_{\mathcal{A}}$ and $\alpha^p \neq 1_{\mathcal{A}}$ for all $1 \leq p \leq n-1$), then the action γ of $\mathbb{Z}_n$ on $\mathcal{A}$ given by

$$\gamma(i) \equiv \alpha^i$$

is said to be **induced** by α.

3.1.1 Main construction

We will now construct a simple class of product type actions to analyze. Before proceeding, however, we warn that throughout the rest of Section 3.1 we deviate a bit from the notational standards used up until now. All sorts of notation in this subsection, once introduced, will remain fixed until the end of the entire Section 3.1 unless otherwise stated.

We start by considering a sequence (k_n) of natural numbers, and the corresponding UHF algebra $\mathcal{A}$ of type $\prod k_n$. As before, we may view $\mathcal{A}$ as the inductive limit of the inductive sequence

$$\mathcal{A}_1 \to \mathcal{A}_2 \to \cdots$$

where

$$\mathcal{A}_n \equiv \bigotimes_{i=1}^n \mathcal{M}_{k(i)},$$

or, setting $t(n) \equiv \prod_{i=1}^n k(n)$, as the inductive limit of the inductive sequence

$$\mathcal{M}_{t(1)} \to \mathcal{M}_{t(2)} \to \cdots .$$

Now for each n, consider two projections $p_n, q_n \in \mathcal{M}_{k(n)}$ such that

$$p_n + q_n = 1.$$

By Lemma 1.1.6, $p_n \perp q_n$ for all $n \in \mathbb{N}$, so we can construct a product type action α of order 2 on $\mathcal{A}$ by setting

$$\alpha \equiv \bigotimes_{n=1}^{\infty} \mathrm{Ad}\,(p_n - q_n).$$

Naturally, α is the direct limit action obtained from the sequence of actions

$$\alpha_N \equiv \bigotimes_{n=1}^{N} \mathrm{Ad}\,(p_n - q_n)$$

on the inductive sequence.

Lemma 3.1.5. *Let $\mathcal{B}$ be a C^*-algebra, and let $b \in \mathcal{B}$. Then, for any $\lambda \in S^1$, we have*

$$\mathrm{Ad}\,(\lambda b) = \mathrm{Ad}\,(b).$$

Proof. For all $x \in \mathcal{B}$ we have

$$\begin{aligned} \mathrm{Ad}\,(\lambda b)(x) &= (\lambda b)x(\lambda b)^* \\ &= \lambda b x \overline{\lambda} b^* \\ &= (\lambda\overline{\lambda}) b x b^* \\ &= b x b^* \\ &= \mathrm{Ad}\,(b)(x). \end{aligned}$$

□

Exchanging p_n and q_n when necessary, without any loss of generality we shall hence always assume $\mathrm{rank}\,(p_n) \geq \mathrm{rank}\,(q_n)$, and we define

$$\lambda_n \equiv \frac{\mathrm{rank}\,(p_n) - \mathrm{rank}\,(q_n)}{\mathrm{rank}\,(p_n) + \mathrm{rank}\,(q_n)} \geq 0$$

for all $n \in \mathbb{N}$. For $m \leq n$ we also set

$$\Lambda(m,n) \equiv \prod_{i=m+1}^{n} \lambda_i \quad \text{and} \quad \Lambda(m,\infty) \equiv \lim_{n\to\infty} \Lambda(m,n).$$

We also set, for each $n \in \mathbb{N}$:

$$T_n \equiv \begin{pmatrix} \mathrm{rank}\,(p_n) & \mathrm{rank}\,(q_n) \\ \mathrm{rank}\,(q_n) & \mathrm{rank}\,(p_n) \end{pmatrix}.$$

We once again recall that this notation will remain fixed throughout the rest of Section 3.1 unless otherwise stated.

3.1.2 The strict Rokhlin property for product type actions of order 2

We proceed with more specific remarks in order to determine necessary and sufficient conditions for α to have the strict Rokhlin property.

Lemma 3.1.6. *If p_n and q_n are, respectively, the initial and final projections of a partial isometry v_n, then*

$$(v_n)^2 = 0.$$

Proof. This is a straightforward application of Lemma 1.1.7:

$$\begin{array}{rrcl} & q_n p_n & = & 0 \\ \Leftrightarrow & v_n^* v_n v_n v_n^* & = & 0 \\ \Rightarrow & v_n v_n^* v_n v_n v_n^* v_n & = & 0 \\ \Leftrightarrow & v_n v_n = 0 & & \end{array}$$

□

Lemma 3.1.7 ([17], Lemma 2.3)**.** *Given $N \in \mathbb{N}$, there exist projections $p, q \in \mathcal{M}_{t(N)}$ such that $p + q = 1$ and $\operatorname{rank}(p) \geq \operatorname{rank}(q)$, and there exists an isomorphism $\Gamma : \mathcal{A}_N \to \mathcal{M}_{t(N)}$ such that*

$$\Gamma \circ \alpha_N = \operatorname{Ad}(p - q) \circ \Gamma.$$

Furthermore:

1. *The ranks of p and q are given by*

$$\begin{pmatrix} \operatorname{rank}(p) & \operatorname{rank}(q) \\ \operatorname{rank}(q) & \operatorname{rank}(p) \end{pmatrix} = T_N T_{N-1} \cdots T_1.$$

2. *If $\operatorname{rank}(p_n) = \operatorname{rank}(q_n)$ for some n, then $\operatorname{rank}(p) = \operatorname{rank}(q)$.*

3. *If $\operatorname{rank}(p_n) > \operatorname{rank}(q_n)$ for all n, then $\operatorname{rank}(p) > \operatorname{rank}(q)$.*

4. *The following holds*

$$\frac{\operatorname{rank}(p) - \operatorname{rank}(q)}{\operatorname{rank}(p) + \operatorname{rank}(q)} = \prod_{n=1}^{N} \frac{\operatorname{rank}(p_n) - \operatorname{rank}(q_n)}{\operatorname{rank}(p_n) + \operatorname{rank}(q_n)}.$$

Proof. Set $N = 2$ and define

$$p \equiv p_1 \otimes p_2 + q_1 \otimes q_2 \quad \text{and} \quad q \equiv p_1 \otimes q_2 + q_1 \otimes p_2.$$

It is then immediate to check that $p + q = 1$ and $p - q = (p_1 - q_1) \otimes (p_2 - q_2)$. Finally, by Corollary 1.2.26

$$\operatorname{rank}(p) - \operatorname{rank}(q) = (\operatorname{rank}(p_1) - \operatorname{rank}(q_1))(\operatorname{rank}(p_2) - \operatorname{rank}(q_2)) \geq 0.$$

For $N > 2$ repeat the process $N - 1$ times. □

Lemma 3.1.8 ([17], Lemma 2.1)**.** *For all $n \in \mathbb{N}$,*

$$\mathcal{A}_n \rtimes_{\alpha_n} \mathbb{Z}_2 \cong \mathcal{M}_{t(n)} \oplus \mathcal{M}_{t(n)}.$$

We can view $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ as the direct limit of the inductive system

$$\mathcal{M}_{t(1)} \oplus \mathcal{M}_{t(1)} \xrightarrow{\psi_1} \mathcal{M}_{t(2)} \oplus \mathcal{M}_{t(2)} \xrightarrow{\psi_2} \cdots$$

where

$$\psi_n(a,b) \equiv (a \otimes p_n + b \otimes q_n, b \otimes p_n + a \otimes q_n).$$

Furthermore, the dual action $\hat{\alpha}_n$ is given by the flip

$$(a,b) \mapsto (b,a).$$

Proof. Consider the map $\Gamma : \mathcal{A}_n \rtimes_{\alpha_n} \mathbb{Z}_2$ defined by

$$\Gamma(a) \equiv (a,a) \text{ for all } a \in \mathcal{M}_{t(n)}$$

$$\Gamma(u_{-1}) \equiv (v, -v)$$

where u_{-1} is the canonical unitary of the crossed product (see Definition 2.1.4), and

$$v \equiv \bigotimes_{i=1}^{n} (p_i - q_i).$$

One then checks Γ is the required isomorphism. □

Corollary 3.1.9. *For all $n \in \mathbb{N}$,*

$$K_0(\mathcal{A}_n \rtimes_{\alpha_n} \mathbb{Z}_2) \cong \mathbb{Z}^2,$$

and $K_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}_2)$ is the inductive limit of

$$\mathbb{Z}^2 \xrightarrow{T_1} \mathbb{Z}^2 \xrightarrow{T_2} \cdots .$$

This will allow us to study $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ via direct sums of matrix algebras, for which we have already developed some theory, leading us to the following.

Theorem 3.1.10. *The following conditions are equivalent:*

1. *α has the strict Rokhlin property.*
2. *There are infinitely many $n \in \mathbb{N}$ such that $\lambda_n = 0$.*
3. *$\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ is a UHF algebra.*
4. *$K_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}_2)$ is totally ordered.*

Proof. $(2 \Rightarrow 1)$ We will prove the Rokhlin property for the dense subalgebra $\bigcup_{i=1}^{\infty} \mathcal{A}_i$. Let F be a finite set in $\bigcup_{i=1}^{\infty} \mathcal{A}_i$. Then there exists $N \in \mathbb{N}$ such that $F \subseteq \mathcal{A}_N$. By Lemma 3.1.7 we can consider just the case where $\operatorname{rank}(p_n) = \operatorname{rank}(q_n)$ for all $n \in \mathbb{N}$. By hypothesis, p_n and q_n are, respectively, the initial and final projections of a partial isometry v_n. Define then $f \in \mathcal{M}_{k(N+1)}$ to be

$$f \equiv \frac{1}{2}(1 + v_{N+1} + v_{N+1}^*).$$

We have (dropping the subscripts for the sake of readability)

$$\begin{aligned} f^* &= \left[\frac{1}{2}(1 + v + v^*)\right]^* \\ &= \frac{1}{2}(1^* + v^* + (v^*)^*) \\ &= \frac{1}{2}(1 + v^* + v) \\ &= f \end{aligned}$$

and

$$\begin{aligned} f^2 &= \frac{1}{4}(1 + v + v^*)^2 \\ &= \frac{1}{4}(1 + v + v^* + v + vv + vv^* + v^* + v^*v + v^*v^*) \\ &= \frac{1}{4}(1 + p + q + 2v + 2v^*) \\ &= \frac{1}{2}(1 + v + v^*) \\ &= f \end{aligned}$$

so f is a projection. Furthermore

$$\begin{aligned} \operatorname{Ad}(p-q)(f) &= (p-q)f(p-q) \\ &= pfp - pfq - qfp + qfq \\ &= 2(pfp + qfq) - qfp - pfq - pfp - qfq \\ &= 2(pfp + qfq) - (p+q)f(p+q) \\ &= (p + q + pqp + qpq + pvp + pv^*p + qvq + qv^*q) - f \\ &= (1 + (pq + qp)(p+q)) - f \\ &= 1 - f. \end{aligned}$$

Consider then

$$e_0 \equiv 1_{\mathcal{A}_N} \otimes f \quad \text{and} \quad e_1 \equiv 1_{\mathcal{A}_N} \otimes (1 - f). \tag{3.1}$$

Computations analogous to the ones made in Example 2.3.2 show that these are the required projections for the strict Rokhlin property to hold.

(1 $\Rightarrow$ 2) Suppose now that α has the strict Rokhlin property, and let F be a finite subset in the dense subalgebra $\bigcup_{i=1}^{\infty} \mathcal{A}_i$. Then there exists N such that $F \subseteq \bigotimes_{n=1}^{N-1} \mathcal{M}_{k(n)}$. We want to first construct Rokhlin projections in $\bigotimes_{n=1}^{N-1} \mathcal{M}_{k(n)}$. This requires some work.

First, let X be a set of unitary matrices spanning $\mathcal{M}_{t(N-1)}$, and set

$$F_0 \equiv F \cup X.$$

Also let

$$R \equiv \max_{x \in F_0} \|x\|$$

and define

$$r \equiv \min\{\frac{\epsilon}{5}, \frac{\epsilon}{4R_0 + 1}\}.$$

Choose $r \geq \delta > 0$ small enough and set

$$\delta_0 = \min\{1, r, \frac{\delta}{4R_0 + 2}\}.$$

Let $\hat{e}_0$ and $\hat{e}_1$ be Rokhlin δ_0-projections for F_0, and let $d_j \in \bigotimes_{n=N}^{M} \mathcal{M}_{k(n)}$ be such that, for $j = 0, 1$,

$$\|\hat{e}_j - d_j\| < \delta_0.$$

Then $a_j \equiv \frac{1}{2}(d_j + d_j^*)$ are self-adjoint elements satisfying the same estimate, and since $\delta_0 \leq 1$, we have

$$\|a_j\| \leq 2.$$

It follows that

$$\begin{aligned} \|a_j x - x a_j\| &\leq 4R_0\delta_0 + \|\hat{e}_j x - x\hat{e}_j\| \\ &< 4R_0\delta_0 + \delta_0 \\ &= (4R_0 + 1)\delta_0 \end{aligned}$$

for all $x \in F_0$.

Now set

$$b_j \equiv \frac{1}{\#X} \sum_{u \in X} u a_j u^*.$$

These are positive elements with norm at most 1 that commute with all elements of W. We also note that

$$b_j \in \bigotimes_{n=1}^{N-1} M_{k(n)}.$$

Furthermore, these elements satisfy

$$\begin{aligned} \|b_j - a_j\| &\leq \max_{u \in W} \|u a_j u^* - a_j\| \\ &\leq \max_{u \in X} \|u a_j - u a_j\| (4R_0 + 1)\delta_0. \end{aligned}$$

Therefore

$$\begin{aligned}\|b_j - \hat{e}_j\| &\le \|b_j - a_j\| + \|a_j - \hat{e}_j\| \\ &< (4R_0 + 1)\delta_0 + \delta_0 \\ &= (4R_0 + 2)\delta_0 \\ &\le \delta.\end{aligned}$$

Now, since δ is taken to be small enough, we can assume that

$$\mathrm{sp}\,(b_j) \subseteq [-r, r] \cup [1 - r, 1 + r].$$

Using continuous functional calculus and Lemmas 1.1.38 and **??**, we can thus set

$$e_0 \equiv f(b_0),$$

and obtain

$$\|e_0 - b_0\| \le r,$$

so that

$$\|e_0 - \hat{e}_0\| < r + \delta \le 2r.$$

Finally, set $e_1 = 1 - e_0$ to obtain

$$\begin{aligned}\|e_1 - \hat{e}_1\| &= \|(1 - e_0) - (1 - \hat{e}_0)\| \\ &< 2r.\end{aligned}$$

Now, for all $x \in F$,

$$\begin{aligned}\|e_j x - x e_j\| &< 2R_0\|e_j - \hat{e}_j\| + \|\hat{e}_j x - x\hat{e}_j\| \\ &\le 2R_0(2r) + \delta_0 \\ &\le 2R_0(2r) + r \\ &= (4R_0 + 1)r \\ &\le \epsilon.\end{aligned}$$

Also,

$$\begin{aligned}\|\alpha(e_0) - e_1\| &\le \|e_0 - \hat{e}_0\| + \|e_1 - \hat{e}_1\| + \|\alpha(\hat{e}_0) - \hat{e}_1\| \\ &< 2r + 2r + \delta_0 \\ &\le 5r \\ &< \epsilon,\end{aligned}$$

so e_0 and e_1 are Rokhlin projections in $\bigotimes_{n=1}^{N-1} \mathcal{M}_{k(n)}$.

We now use these Rokhlin projections to prove the statement. Use Lemma 3.1.7 to produce two projections $p, q \in \mathcal{M}_{t(N-1)}$ from p_n and q_n. Now

$$\|\mathrm{Ad}\,(p - q)(e_0) - e_1\| < \epsilon < 1,$$

so $\mathrm{Ad}\,(p-q)(e_0) \overset{u}{\sim} e_1$ by Proposition 1.1.11. Therefore, there exists a unitary W such that

$$We_0W^* = e_1,$$

and we have

$$\mathrm{rank}\,(e_0) = \mathrm{rank}\,(e_1).$$

Without loss of generality, suppose e_0 and e_1 are rank-one projections onto the subspaces $\mathbb{C}\xi_0$ and $\mathbb{C}\xi_1$ respectively, for some $\xi_0, \xi_1 \in \mathbb{C}^2$ (otherwise write e_0 and e_1 as a sum of j rank-one projections for a fixed j). Then $\mathrm{span}\{\xi_0, \xi_1\}$ is W-invariant, and so W is the regular representation of $\mathbb{Z}_2$ on $\mathbb{C}^2$. Consequently, p and q are projections onto the two subspaces corresponding to the two irreducible unitary representations of $\mathbb{Z}_2$, and hence

$$\mathrm{rank}\,(p) = \mathrm{rank}\,(q) = 1.$$

($2 \Rightarrow 3$) Without loss of generality (see Lemma 3.1.7), let v_n once again be a partial isometry such that p_n and q_n are, respectively, its initial and final projections for all $n \in \mathbb{N}$. Lemma 3.1.6 immediately implies that $v_n + v_n^*$ is unitary and

$$(v_n + v_n^*)^2 = 1_{\mathcal{M}_{k_n}}.$$

Furthermore

$$\begin{aligned} \mathrm{Ad}\,(v_n + v_n^*)(p_n) &= v_n p_n v_n + v_n^* p_n v_n + v_n p_n v_n^* + v_n^* p_n v_n^* \\ &= (v_n)^2 + v_n^* v_n + (v_n)^2 (v_n^*)^2 + (v_n^*)^2 \\ &= q_n \end{aligned}$$

and similarly,

$$\mathrm{Ad}\,(v_n + v_n^*)(q_n) = p_n.$$

The map ψ_n in Lemma 3.1.8 can thus be written as the composite of the maps

$$\rho_n : \mathcal{M}_{t(n)} \oplus \mathcal{M}_{t(n)} \to \mathcal{M}_{t(n+1)} \quad \text{and} \quad \sigma_n : \mathcal{M}_{t(n)} \to \mathcal{M}_{t(n)} \oplus \mathcal{M}_{t(n)}$$

given by

$$\rho_n(a, b) = a \otimes p_n + b \otimes q_n \quad \text{and} \quad \sigma_n(a) = (a, \mathrm{Ad}\,(1 \otimes (v_n + v_n^*))(a)).$$

As such, $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ is the direct limit of the inductice system

$$\mathcal{M}_{t(1)} \overset{\rho_1 \circ \sigma_1}{\to} \mathcal{M}_{t(2)} \overset{\rho_2 \circ \sigma_2}{\to} \cdots$$

which is a UHF algebra by definition.

($3 \Rightarrow 4$) This is Theorem 1.3.23.

($4 \Rightarrow 2$) Suppose, by contradiction, that there exists n_0 such that for all $n > n_0$ we have $\mathrm{rank}\,(p_n) > \mathrm{rank}\,(q_n)$. Using Corollary 3.1.9, let $\eta_{n_0} \equiv (1, -1) \in K_0(\mathcal{M}_{t(n_0)} \rtimes_{\alpha_n} \mathbb{Z}_2)$. Fix $N > n_0$, and use Lemma 3.1.7 to get projections

$e_N, f_N \in \bigotimes_{n=n_0+1}^{N} \mathcal{M}_n$ such that $e_N + f_N = 1$ and $\text{rank}\,(e_N) > \text{rank}\,(f_N)$. By Lemma 3.1.8, the resulting image η_N of η_{n_0} in $K_0(\mathcal{M}_{t(N)} \rtimes_{\alpha_{t(N)}} \mathbb{Z}_2)$ is

$$(\text{rank}\,(e_N) - \text{rank}\,(f_N), \text{rank}\,(f_N) - \text{rank}\,(e_N)).$$

Thus, since $\text{rank}\,(e_N) - \text{rank}\,(f_N) \neq 0$, neither $\eta_N > 0$, nor $-\eta_N > 0$, and consequently, the image β of η_{n_0} in $K_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ is such that neither $\beta > 0$ nor $-\beta > 0$. □

Recall Example 2.3.4 given in the previous Chapter: the action of $\mathbb{Z}_2$ generated by

$$\bigotimes_{i=1}^{\infty} \text{Ad}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right)$$

on

$$\bigotimes_{i=1}^{\infty} \mathcal{M}_3.$$

We'd mentioned that this action is an example of an action with the tracial but not the strict Rokhlin property, which we did not prove. With the above result, it is trivial to check it indeed does not have the latter: the projections involved have ranks 2 and 1 respectively.

3.1.3 The tracial Rokhlin property for product type actions of order 2

We now proceed towards a similar characterization of, this time, the tracial Rokhlin property for the action α in terms of the ranks of the projections p_n and q_n, the tracial states on the crossed product and the dual action. To do so we need some extra machinery.

Definition 3.1.11. A simple separable unital C*-algebra $\mathcal{A}$ is said to be of **tracial rank zero** if for every finite set $f \subseteq \mathcal{A}$, every $\epsilon > 0$ and every nonzero positive element $x \in \mathcal{A}$ there exists a projection $p \in \mathcal{A}$ and a finite dimensional C*-subalgebra $\mathcal{B} \subseteq p\mathcal{A}p$ such that

1. $\mathcal{B}$ is unital and its unit is p.

2. $\text{dist}(pap, \mathcal{B}) < \epsilon$ for all $a \in F$.

3. $\|pa - ap\| < \epsilon$ for all $a \in F$.

4. $1 - p$ is murray-von Neumann equivalent to a projection in $\overline{x^*\mathcal{A}x}$.

Theorem 3.1.12 (cf. [17], Theorem 1.9)**.** *Let $\mathcal{A}$ be a simple separable infinite dimensional unital C*-algebra of tracial rank zero. Let $\alpha : G \to \text{Aut}\,(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. Suppose that for every finite set $F \subseteq \mathcal{A}$ and every $\epsilon > 0$ there exist positive elements $a_g \in \mathcal{A}$ for each $g \in G$, such that:*

1. $0 \leq x_g \leq 1$ *for all* $g \in G$.

2. $\|a_g a_h\| < \epsilon$ *for all* $g, h \in G$ *with* $g \neq h$.

3. $\|\alpha_g(a_h) - a_{gh}\| < \epsilon$ *for all* $g, h \in G$.

4. $\|a_g c - c a_g\| < \epsilon$ *for all* $g \in G$ *and* $c \in F$.

5. $\tau\left(1 - \sum_{g \in G} a_g\right) < \epsilon$ *for every* $\tau \in T(\mathcal{A})$.

Then α *has the tracial Rokhlin property.*

Theorem 3.1.13 (cf. [17], Remark 1.7, [20], Theorem 3.11 and [20], Proposition 6.1)**.** *Let* $\mathcal{A}$ *be an infinite dimensional simple separable unital C*-algebra, and let* $\alpha : G \to \mathrm{Aut}\,(\mathcal{A})$ *be an action of a finite abelian group* G *on* $\mathcal{A}$. *If* α *has the tracial Rokhlin property, then*

$$\tau \circ \hat{\alpha} = \tau$$

for every tracial state τ *on* $\mathcal{A} \rtimes_\alpha G$.

Theorem 3.1.14. *The following conditions are equivalent:*

1. α *has the tracial Rokhlin property.*

2. $\hat{\alpha}$ *is trivial on* $T(\mathcal{A} \rtimes_\alpha \mathbb{Z}_2)$.

3. $\Lambda(m, \infty) = 0$ *for all* $m \in \mathbb{N}$.

4. $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ *has a unique tracial state*

Proof. $(1 \Rightarrow 3)$ This is Theorem 3.1.13.

$(2 \Rightarrow 3)$ Recall that we denote by $\mathrm{tr}_{\,m}$ the unique tracial state on the matrix algebra $\mathcal{M}_m$. We know that a tracial state on $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ corresponds to a sequence $(\tau_n)_{n \in \mathbb{N}}$ of tracial states τ_n on $\mathcal{M}_{t(n)} \oplus \mathcal{M}_{t(n)}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{t(n-1)} \oplus \mathcal{M}_{t(n-1)} & \xrightarrow{\psi_n} & \mathcal{M}_{t(n)} \oplus \mathcal{M}_{t(n)} \\ & {}_{\tau_{n-1}} \searrow & \downarrow \tau_n \\ & & \mathbb{C}. \end{array}$$

Furthermore, by Proposition 1.2.6, there exist $r_n, s_n \in [0, 1]$ such that $r_n + s_n = 1$ and

$$\tau_n = r_n \mathrm{tr}_{\,t(n)} + s_n \mathrm{tr}_{\,t(n)}.$$

Fix $r \in [0, 1]$, and define

$$r_n \equiv \frac{1}{2}(1 - \Lambda(n, \infty)) + r\Lambda(n, \infty) \quad \text{and} \quad s_n \equiv \frac{1}{2}(1 + \Lambda(n, \infty)) - r\Lambda(n, \infty).$$

One then checks that the sequence $(r_n, s_n)_{n \in \mathbb{N}}$ verifies the above conditions and as thus defines a sequence of tracial states τ_n on the inductive system, which in turn defines a tracial state on the crossed product. It is then clear that if $\Lambda(j,\infty) \neq 0$ for some $j \in \mathbb{N}$, $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$ has at least 2 tracial states.

($3 \Rightarrow 1$) Let F be a finite subset of the dense subalgebra $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. Then there exists $N \in \mathbb{N}$ such that $F \subseteq \mathcal{A}_N$. Furthermore, by hypothesis, there exists $T > N$ such that $\Lambda(N,T) < \epsilon$. By Lemma 3.1.7 there exist projections $p, q \in \bigotimes_{n=N+1}^{T} \mathcal{M}_{k(n)}$ such that $p + q = 1$, $\mathrm{rank}\,(p) \geq \mathrm{rank}\,(q)$,

$$\bigotimes_{n=N+1}^{T} \mathrm{Ad}\,(p_n - q_n) = \mathrm{Ad}\,(p - q),$$

and

$$\mathrm{rank}\,(p) - \mathrm{rank}\,(q) = \Lambda(N,T)(\mathrm{rank}\,(p) + \mathrm{rank}\,(q)).$$

Choose a partial isometry $v \in \bigotimes_{n=N+1}^{T} \mathcal{M}_{k(n)}$ such that $vv^* \leq p$ and $v^*v = q$ and define $f_0, f_1 \in \bigotimes_{n=N+1}^{T} \mathcal{M}_{k(n)}$ by

$$f_0 \equiv \frac{1}{2}(vv^* + q + v + v^*) \quad \text{and} \quad f_1 \equiv \frac{1}{2}(vv^* + q - v - v^*).$$

Calculations show that both f_0 and f_1 are projections and obey the following properties:

- $f_0 \perp f_1$
- $f_0 + f_1 = vv^* + q$
- $(p - q)f_0(p - q)^* = f_1$
- $(p - q)f_1(p - q)^* = f_0$.

The required projections in $\mathcal{A}_T$ for the tracial Rokhlin property are then

$$e_0 \equiv 1_{\mathcal{A}_N} \otimes f_0 \quad \text{and} \quad e_1 = 1_{\mathcal{A}_N} \otimes f_1$$

since both commute with every element of F and $\alpha(e_0) = e_1$ and $\alpha(e_1) = e_0$. Finally, the unique tracial state τ satisfies

$$\tau(1 - e_0 - e_1) = 1 - 2\frac{\mathrm{rank}\,(q)}{\mathrm{rank}\,(p) + \mathrm{rank}\,(q)} = \Lambda(N,T) < \epsilon$$

which finishes the proof.

($2 \Rightarrow 4$) The description of the dual action given by Lemma 3.1.8 shows that there is only one $\hat{\alpha}$-invariant tracial state on $\mathcal{A} \rtimes_\alpha \mathbb{Z}_2$.

($4 \Rightarrow 2$) Obvious. □

Back in Example 2.3.4, we left out the proof that the action had the tracial Rokhlin property. It is now trivial to see it does. Indeed, the ranks of the projections p_n and q_n are 2 and 1 respectively, for all $n \in \mathbb{N}$. So, for all $m \in \mathbb{N}$

$$\Lambda(m,\infty) = \lim_{n\to\infty} \frac{1}{3^n} = 0.$$

3.2 Product type actions of arbitrary finite order

We now give a small step further and consider actions of arbitrary finite cyclic groups $\mathbb{Z}_\gamma$. We again start by considering a UHF algebra $\mathcal{A}$ of type $\prod k_n$, and view it as the inductive limit of

$$\mathcal{A}_1 \to \mathcal{A}_2 \to \cdots ,$$

where

$$\mathcal{A}_N \equiv \bigotimes_{n=1}^{N} \mathcal{M}_{k(n)}.$$

To produce an action α of $\mathbb{Z}_\gamma$ on $\mathcal{A}$, we choose, for each $n \in \mathbb{N}$, a family of pairwise orthogonal projections $\{p_1^{(n)}, \dots, p_\gamma^{(n)}\} \subseteq \mathcal{M}_{k(n)}$ such that

$$\sum_{j=1}^{\gamma} p_j^{(n)} = 1.$$

Then the element

$$\sum_{j=1}^{\gamma} e^{\frac{2\pi i j}{\gamma}} p_j^{(n)}$$

is readily seen to be unitary, and we accordingly set

$$\alpha \equiv \bigotimes_{n=1}^{\infty} \mathrm{Ad} \left(\sum_{j=1}^{\gamma} e^{\frac{2\pi i j}{\gamma}} p_j^{(n)} \right).$$

We can immediately state the following, whose proof is analogous to the corresponding results for $\mathbb{Z}_2$.

Theorem 3.2.1. *With $\mathcal{A}$ and α as above,*

1. *If α has the Rokhlin property, then for infinitely many $n \in \mathbb{N}$, $\mathrm{rank}\,(p_i^{(n)}) = \mathrm{rank}\,(p_j^{(n)})$ for all $i, j \in \{1, \dots, \gamma\}$.*
2. *If α has the tracial Rokhlin property, then $\hat{\alpha}$ is trivial on $T(\mathcal{A} \rtimes_\alpha \mathbb{Z}_\gamma)$.*

Generalizing the other implications is not as straightforward. The methods used in the previous section should work, but they need to be carefully analysed. We here only make a partial observation.

We first make a statement similar to Lemma 3.1.6.

Lemma 3.2.2. *Let i_1, i_2, i_3 be different indices. Suppose there are two partial isometries v and w such that, setting $p \equiv p_{i_1}$, $q \equiv p_{i_2}$ and $r \equiv p_{i_3}$ for some different indices i_1, i_2 and i_3, we have*

- *$p = vv^* = ww^*$.*

- $q = v^*v$.
- $r = w^*w$.

Then

$$vw = v^*w = vw^* = v^*w^* = 0.$$

Proof. As expected, the proof relies almost exclusively on orthogonality, and we here only prove two of the claims.

$$\begin{aligned} & pr = 0 \\ \Rightarrow \quad & vv^*w^*w = 0 \\ \Rightarrow \quad & v^*vv^*w^*ww^* = 0 \\ \Rightarrow \quad & vw = 0. \end{aligned}$$

All other equalities are proved analogously, with the exception of $vw^* = 0$, whose proof goes as follows.

$$\begin{aligned} & p^2 = p \\ \Rightarrow \quad & vv^*ww^* = vv^* \\ \Rightarrow \quad & v^*vv^*ww^*w = v^*vv^*w^* \\ \Rightarrow \quad & vw^* = v^*w^* = 0. \end{aligned}$$

□

Recall now that for proving $(2 \Rightarrow 1)$ in Theorem 3.1.10, which states that, for an action of order 2, if the ranks of the projections are equal for infinitely many terms in the tensor product, then the action has the strict Rokhlin property, we explicitly constructed Rokhlin projections in terms of a partial isometry generating both projections.

The search for such an explicit formula for the general case is a matter requiring further investigation, but the previous lemma greatly limits the possible choices. It turns out though, that no natural formula works. For instance, in the case $\gamma = 3$, one might be tempted to define, just as in the above mentioned proof,

$$f = \frac{1}{3}(1 + v_{1,2} + v_{1,2}^* + v_{2,3} + v_{2,3}^* + v_{1,3} + v_{1,3}^*)$$

where $v_{i,j}$ is a partial isometry such that $v_{i,j}v_{i,j}^* = p_i$ and $v_{i,j}^*v_{i,j} = p_j$. Unfortunately, f does not define a projection in this way.

Another attempt is to define, for γ even,

$$f \equiv \frac{1}{2}\left(1 + \sum_{i=1}^{\frac{\gamma}{2}}(v_{2i,2i-1} + v_{2i,2i-1}^*)\right).$$

This formula can be used to build γ projections, defined by

$$\mathrm{Ad}^{\,i}(f) \text{ for } i = 0, \ldots, \gamma - 1.$$

Ultimately though, f is not necessarily orthogonal to $\mathrm{Ad}\,(f)$ and the attempt also fails.

The construction of the Rokhlin projections on the general case is thus not so easy. One cannot expect such a “simple” formula as in the case $\gamma = 2$, and maybe one shouldn't expect a constructive proof at all.

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