Abstract

Mathematics is an essential tool in engineering, therefore, its simplicity is the key element for its better understanding and consequently to its advance. The excessive employment of Descartes’ coordinate system, caused by the scientists’ misconception of vectors, has led developers to an unnecessary complexity in calculations. In this context, the aim of this work is to help to improve the language of physics, through a mathematic formalism that integrates all the common algebraic systems such as complex numbers, matrix and vectors algebra and which is directly conceived from the physical world. Geometric algebra offers powerful new capabilities, such as the spinor theory, an invariant (coordinate-free) formulation for boosts and rotations providing a simple and intuitive understanding of special relativity in two-dimensions. The subsequently rising of hyperbolic numbers is comprehended with the extension of the general numeral system, with the revision of the hypercomplex numbers. Consequently, the general problem of relativistic non-collinear addition of velocities in the four-dimensional space-time is addressed with the introduction of a new encoding algebraic structure, gyrogroups. Furthermore, for moving media, the coordinate-free approach provides the comprehension and reduction of Maxwell equations in space-time. The results obtained avoid some rather cumbersome algebraic manipulations from the tensor (or dyadics) approach, thus, attaining simpler and more elegant solutions.

Keywords: Geometric algebra; Bivector; Geometric product; Boost; Relativistic optic; Minkowski; Lorentz; Moving media; Spinor; Vacuum Form Reduction; Hypercomplex numbers; Gyrogroups; Gyrocommutativity

1. Introduction

The main purpose of this dissertation is to introduce geometric algebra as a mathematical language with powerful capabilities, which brings new insights to the treatment of many engineering topics and offer new simplifications. Therefore in this dissertation, with this language, one intends to explore all the relevant features within the theory of relativity in order to attain more efficient calculations and consequently to offer a better understanding, impelling new advanced matters.

2. Geometric Algebra

Mathematics is a logical language which has advanced hand in hand with the development of physics, in order to be able to serve it. As Einstein stated “as far as the laws of mathematics refer to reality, they are not certain and as far as they are certain, they do not refer to reality”. Thus, it is important to conceive mathematics as an open language which must be free to evolve in order to attain its maximum purpose.

The vector is born from the mathematical perception of geometry. It is a directed line segment endowed with a direction and a length. The vector’s direction is given by the rotation needed to obtain a vector from another vector. Therefore, there is no apparent reason to insert a vector on an axis as Descartes developed for analytic geometry. To relate vectors, in a simpler way, avoiding unnecessary auxiliary methods, a new mathematic language is born, the geometric algebra, achieving, thus, a coordinate-free method. Geometric algebra is conceived to answer a natural question: How to multiply vectors. To answer the previous question one considers the already developed inner product, the purpose of which is to achieve a new step, a symbolic expression presenting vectors with magnitude and direction in algebraic system. This offers an opportunity to better understand vectors and distinguish them from scalars.

Inner Product

To meet the desired specifications the inner product uses the already established concept of perpendicular projection, when two vectors are presented, the projection of one is only dependent on the angle between them and its magnitude.

\[ \mathbf{a} \cdot \mathbf{b} = ||\mathbf{b}|| \cos \alpha \]  

(1)

Thus, this product is commutative, associative and distributive.
Outer Product

The Outer product is intended to complement the inner product, with the ability to describe perpendicular vectors and with a new notion, the oriented k-blade.

\[ |a \wedge b| = |a||b| \sin \theta \]  

(2)

Two non-collinear vectors, with the outer product produce a bivector, an oriented parallelogram with a magnitude of the corresponding area. The k-blade orientation is given by the order of the displayed vectors. Thus, this product is anti-commutative, distributive and associative. The outer product of three orthogonal vectors originates 3-blade, which is an oriented parallelepiped, a trivector, with the magnitude of its correspondent volume.

Geometric product

Both the inner and outer products complement one another. Thus, to fully obtain the relations between two vectors it is desirable to relate them. Therefore, the geometric product is given by

\[ C = ab = a \cdot b + a \wedge b \]  

(3)

As a sum of a commutative product and an anti-commutative product, this product is non-commutative and, obviously, is distributive and associative. In order to fully understand and apply the geometric algebra it is greatly useful to become conscious of the establishment of the convention of precedence. The outer product has priority when facing the inner product, being at the “top of preferences”.

\[ A \cdot B \wedge C = A \cdot (B \wedge C) \neq A \cdot B \wedge C \]  

(4)

Right next to the outer product is the inner product

\[ A \cdot BC = (A \cdot B)C \neq A \cdot (BC) \]  

(5)

\[ A \wedge BC = (A \wedge B)C \neq A \wedge (BC) \]  

(6)

Inversion operation is a useful tool provided by the geometric algebra, defined by

\[ A^{-1} = A^T AA^T \]  

(7)

The traditional approach to analytic geometry is with the so called coordinate system. This system uses a set of scales called coordinates to identify a point in space. Although, with the correct use of geometric algebra it may become obsolete, due to superfluous information that is carried through, entailing unnecessary complications. For a proper use of the geometric algebra, it is important to describe a point as geometric figure, because a point is not theoretically defined as a mathematical entity. It is possible to represent a point geometrically since any of the geometric spaces and figures are in fact nothing more than a set of points. Thus, to describe the tree-dimensional space, also called tree-dimensional physical world, in order to guarantee that this precise space is inserted in Euclidean space, geometric figures will be used.

\[ r^2 = |r|^2. \]  

(8)

Hence this vector can be projected in N-orthogonal vectors by the expression \( r = \sum r_i e_i \). In this figure the vector \( r \) is signed by \( r = r_1 e_1 + r_2 e_2 + r_3 e_3 \in \mathbb{R}^3 \). This system defines a tree-dimensional Euclidean space, although this same space could have also been conceived with the use of two non-collinear bivectors or a single trivector.

In the next table, we introduce a multivector structure in geometric algebra for each dimensional space

<table>
<thead>
<tr>
<th>Multivector</th>
<th>Multivector Structure</th>
<th>u = α</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Cl_0 )</td>
<td>( \Lambda^0 \mathbb{R} )</td>
<td>( u = α )</td>
</tr>
<tr>
<td>( Cl_1 )</td>
<td>( \Lambda^0 \mathbb{R} \oplus \Lambda^1 \mathbb{R} )</td>
<td>( u = α + a )</td>
</tr>
<tr>
<td>( Cl_2 )</td>
<td>( \mathbb{R} \oplus \mathbb{R}^2 \oplus \Lambda^2 \mathbb{R} )</td>
<td>( u = α + a + F + V )</td>
</tr>
<tr>
<td>( Cl_3 )</td>
<td>( \mathbb{R} \oplus \mathbb{R}^3 \oplus \Lambda^3 \mathbb{R} )</td>
<td>( u = α + a + F + V )</td>
</tr>
</tbody>
</table>

Where \( u \) stands for a multivector presented as a direct sum the different k-vectors. Each k-vector can be designated by

| Scalars      | \( \left< a \right> \) | \( \mathbb{R} \) | \( α \) |
| Vectors      | \( \left< a \right> \) | \( \mathbb{R}^2 \), \( \mathbb{R}^3 \) | \( a \) |
| Bivectors    | \( \left< a \right> \) | \( \mathbb{R}^3 \), \( \mathbb{R}^2 \) | \( F \) |
| Trivectors   | \( \left< a \right> \) | \( \Lambda^3 \mathbb{R} \) | \( V \) |

Similarly to the complex conjugation, the geometric algebra has three involutions, for \( u = α + a + F + V \), we have

- Grade involution: \( \hat{u} = α - a - F - V \)
- Reversion: \( \tilde{u} = α + a - F - V \)
- Clifford – conjugation: \( u^* = α - a - F + V \)

Geometric algebra is a direct sum of two of its subspaces, which form two different sub-algebras

- The even part \( Cl^e = \mathbb{R} \oplus \Lambda^2 \mathbb{R}^3 \)
- The odd part \( Cl^o = \mathbb{R} \oplus \Lambda^3 \mathbb{R}^3 \)
The even sub-algebra is not commutative and is isomorphic to the quaternion algebra from Hamilton. Note that the odd part is not a sub-algebra. These subspaces satisfy the inclusion relations

\[
C_{\ell}^+, C_{\ell}^+ \subset C_{\ell}^+, C_{\ell}^+, C_{\ell}^+, C_{\ell}^+ \subset C_{\ell}^+
\]

The elements which commute with all elements of the algebra define the center of the algebra. Therefore,

\[
\text{Cent}(C_{\ell}) = \mathbb{R} \oplus \wedge^2 \mathbb{R}^3
\]

Note that the center of this algebra is isomorphic to \( \mathbb{C} \). A complex number is a sum of two different parts, the real and the imaginary numbers. Equivalently, a multivector \( vu \) also known as spinor is in fact the sum of two diverse blades, the scalars and the bivectors. In the Euclidean space, a bivector like an imaginary number when squared is negative, since \( i j = -e e \)\( (9) \)

Rotations and contractions

The geometric algebra offers a new algebraic notion of direction, without the need to use auxiliary axes

\[
ab = |a||b|\exp\theta
\]

Therefore,

\[
ab = |a|\cos(\theta)\hat{a}_a b + |a|\sin(\theta)\hat{a}_b b
\] (11)

Given a vector \( b \) and a vector \( a \), it is possible to deduce the collinear and two orthogonal components of one in the other.

\[\text{The parallel projection corresponds to } \hat{a}_a = a \cdot b b^{-1} \text{ and the perpendicular projection to } \hat{a}_b = a \wedge b b^{-1} . \text{ The reflection of the vector } a \text{ along } b \text{ is } a' = b a b^{-1} = a' - a_\perp . \text{ A double reflection is actually considered to be a rotation.} \]

\[
r'' = (ba)r(a^{-1}b^{-1})
\] (12)

The geometric product in a two dimensional space between a vector \( r \) and the multivector \( ab \), results in a rotation of the vector \( r \). Where the rotated angle is twice the given angle defined by \( a \) and \( b \),

\[
r^* = r \exp\left(2(\alpha + \beta)B\right) \quad \text{for } \quad ab = |a||b|\exp\left(\alpha + \beta)B\right)
\]

In the presence of a bivector, in a three-dimensional space, a vector can be decomposed in a parallel and a perpendicular form

\[\text{The product between these two blades origins a multivector, which is defined by the sum between a vector and a trivector } \]

\[
aB = aB + a_\perp B
\] (13)

The orthogonal is \( a_\perp = a \wedge BB^{-1} \) and the parallel projection

\[
a_\parallel = (a \cdot B)B^{-1}
\] (14)

The operation presented is called left contraction

\[
a_\parallel(b \wedge c) = (a \cdot b)c - (a \cdot c)b.
\] (15)

By symmetry the right contraction is

\[
a_\parallel B = -B_\parallel a
\] (16)

Thus,

\[
aB = -Ba_\perp
\] (17)

\[
a_\parallel B = Ba_\parallel
\] (18)

Hence, we are now able to understand a three-dimensional rotation in space \( a' = R a \bar{R} \), where \( R \) is a multivector called rotor, \( R = mn = \exp\left(\frac{\theta}{2}B\right) \) the geometric product of two unitary vectors. This rotation is represented in the following figure
Analogously, it is easily deduced from equations (13) that

\[ a'_1 = Ra_1\tilde{R} = a_1 \tilde{R} = a_1 \text{ since } R \tilde{R} = 1 \]  

\[ a'_1 = Ra_1\tilde{R} = R^3a_1 \]  

(19)

(20)

3. Introduction to relativity

Einstein’s relativity theory is originated based on this inconsistence between the electromagnetic theory and Newton’s mechanics which conceives the interaction between any two objects independent of the inertial frame so that all inertial systems are equivalent in what the laws of physics are concerned. When considering the second postulate from Einstein “The speed of light (in a vacuum) is the same in all inertial frames of reference, regardless of the motion of the light source”, any real event occurs at a particular time and space, described by

\[ r = cte_0 + \vec{r} \]  

(21)

For this expression to be valid when describing a photon (the particle of the light), it has to be a straight line with the expression form

\[ c^2 (te_0)^2 - (re_0)^2 = 0 \]  

(22)

Or for a different inertial frame

\[ c^2 (\Delta t e_0)^2 - (\Delta r e_0)^2 = 0 \]  

(23)

Where \( c \) is the velocity of the light a constant that transforms units of time into units of length. Since any point in spacetime can belong in a light signal history, where the choice of the origin is arbitrary, it is possible to find an invariant relation between points in different inertial frames,

\[ r^2 = \tilde{r}^2 \]  

(24)

We are now facing a non-Euclidean space

\[ r^2 \neq |r|^2 \]  

(25)

4. Hypercomplex numbers

The admittance of a non-Euclidean system proves that the general number system so far considered is insufficient. The complex numbers are distinguished as an extension of real numbers, as the square of negative numbers and reveal to be an emblematic discovery for mathematics and sciences, allowing the formalizing of a two-component Euclidean space, although unequipped for non-Euclidean systems, such as in the hyperbolic numbers. The analysis of this issue led to the redefinition of the group where the complex numbers were inserted and to the adoption of a more generalized form called the hypercomplex numbers. The group of the hyperbolic number provides the convergence of the Euclidean and the Minkowski space algebra, spreading to a N-dimensional space once combined geometric algebra.

Now, the hypercomplex numbers can be given by

\[ \mathbf{h} = e_0 \vec{r} \]  

(26)

The element \( \mathbf{h} \) is now treated as spinor \( e_0 \vec{r} \)

Where \( \mathbf{h}^2 = \eta + \xi \mathbf{h} \), therefore \( (e_0 \vec{r})^2 = \eta + \xi (e_0 \vec{r}) \)  

(27)

The spinor \( e_0 \vec{r} = e_0 \vec{r} + e_0 \vec{r} \) therefore one can now obtain the following equations

\[ \left( \vec{e}_0 \vec{r} \right)^2 = \eta + \xi (\vec{e}_0 \vec{r}) \]  

(28)

Solving this system one obtains

\[ \xi = 2(\vec{e}_0 \vec{r}) \]  

(29)

and

\[ \eta = -e_0 \vec{r} \]  

(30)

The assumption \( e_0^2 = 1 \) entails \( e_0^2 = -\eta \)

\[ \begin{cases} e_0^2 = \vec{r}^2, & \eta = 1 \\ \vec{r}^2 = 0, & \eta = 0 \\ \vec{r}^2 = -e_0^2, & \eta = 1 \end{cases} \]  

(31)

Hence, in order to understand the non-Euclidean space-time from Einstein’s special theory of relativity one must consider \( \eta = 1 \). In a two-dimensional space an event can be represented by

\[ r^2 = (c^2re_0 + xe_0)^2 = (ct)^2 e_0^2 + x^2 e_1^2 = (ct)^2 - \eta x^2 \]  

(32)

Note that for \( \eta = -1 \) we are still in a Euclidean algebra. For \( \eta = 0 \), we are in fact facing the classical Galilean relativity, which is proved here also to be a non Euclidean relativity. Therefore, from the previous equation we were able to describe, three different relativistic theories.

If we now explore the spinor \( e_0 \), from the Euclidean relativity and from Einstein’s relativity varying the signal of \( \eta \)

\[ e_{00} = \eta \]  

(33)

\[ e_{00}^{34} = \eta e_{00} \]  

(34)

One obtains, applying the Taylor series expansion for the Euclidean relativity
exp(\(\theta e_n\)) = \sum_{k=0}^{n} (-1)^k \frac{\theta^{2k}}{(2k)!} + e_n \sum_{k=0}^{n} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}

= \cos \theta + e_n \sinh \theta.

And for Einstein's hyperbolic functions

\[ \exp(\zeta e_n) = \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{(2k)!} + e_n \sum_{k=0}^{\infty} \frac{\zeta^{2k+1}}{(2k+1)!} \]

= \cosh \zeta + e_n \sinh \zeta.

Therefore, the inner and outer product are characterized in the special theory of relativity by the following hyperbolic functions

\[ \cosh \zeta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}, \quad \sinh \zeta = \frac{||\mathbf{u} \wedge \mathbf{v}||}{||\mathbf{u}|| ||\mathbf{v}||} \quad (35) \]

5. Special theory of relativity

In space-time, where the primed system has the velocity of \(v\) to the unprimed system, a rest particle is in the primed fame represented in the world line by

\[ \mathbf{r}' = c t \mathbf{f}_0 \quad (36) \]

In the unprimed frame the same particle is observed with this trajectory

\[ \mathbf{r}_0 = c t \mathbf{e}_0 + x \mathbf{e}_1, \quad x = v t \quad (37) \]

Since \(\mathbf{r}' = (\mathbf{r}_0)\), in order to maintain space-time distance invariance between both frames while relating them, it is used a linear transformation, a boost, called in the two-dimensional space the Lorentz transformation.

\[ \mathbf{r} = \mathbf{r}_0 L \]

Alike, in the Euclidean space \(L\) is a spinor with \(L r_0 = \mathbf{r}_1\) and \(L \mathbf{r}_0 = \mathbf{r}'\).

Consequently,

\[ \mathbf{r} = \mathbf{r}_0 L \]

Therefore one can induce that

\[ \frac{d\mathbf{r}_0}{d\tau} = \frac{d\mathbf{r}}{d\tau} L \]

Deriving in order of \(\tau\), the invariance is unchanged. Now four trajectories can be related by

\[ \frac{\gamma}{\gamma} \left(c \mathbf{e}_0 + v \mathbf{e}_1\right) = c \mathbf{f}_1 \left(\frac{\gamma}{\gamma} L\right) \]

This transformation is also known as the passive transformation. Where

\[ L^2 = \frac{d\mathbf{f}_1}{d\tau} \left| L \right|^2 \quad (42) \]

As a spinor, \(L^2 = \exp(2\zeta \mathbf{B})\) thereby \(r^2 = (\mathbf{r} L)^2 = \mathbf{r}(L^2) L^2\)

Hence \(L^2 \cdot L^2 = 1 \left(\frac{dt}{d\tau}^2 \right) \left(1 - \frac{v^2}{c^2}\right)\) for \(\gamma = (1 - \beta)^{1/2}\)

Therefore, \(\frac{\gamma}{\gamma} \geq \frac{\gamma}{\gamma} t = \gamma t / \tau \)

Note that given the linear trajectories equations \(\frac{d\gamma}{d\tau} = \gamma / \tau\) and \(c \mathbf{e}_0 + \mathbf{e}_1 = c \mathbf{f}_1 L^2\)

Finally, after understanding the Lorentz transformation as an ordinary space-time boost in two-dimensional analogously to a space rotation, we are now able relate different moving frames.

\[ (c \mathbf{f}_0 + \mathbf{f}_1) L^2 = c t \mathbf{e}_0 + x \mathbf{e}_1 \quad (43) \]

Or to the correspondent inverse operation

\[ c \mathbf{f}_0 + \mathbf{f}_1 = L^2 (c t \mathbf{e}_0 + x \mathbf{e}_1) \quad (44) \]

One can deduce in the moving frame the time and the space trajectory with equations (42) and (44)

The time is expressed by \(\mathbf{f}_0 = \gamma (\tau - (v x / c^2)) \mathbf{f}_0\)

And the space by \(\mathbf{f}_1 = \gamma (v t - x) \mathbf{f}_0\).

After the disposal of all necessary tools to derive the Lorentz transformations and carefully analyzing the geometric algebra in space-time, we are able to write

\[ \mathbf{r} = \mathbf{r}_0 L \quad \mathbf{r}_0 = \mathbf{r}_0 L \]

Therefore, \((c \mathbf{f}_0 + \mathbf{f}_1) L^2 = c \mathbf{f}_0 (c t \mathbf{e}_0 + x \mathbf{e}_1)\)

We can now represent the following matrix

\[ \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} \]

From the matrix one can deduce

\[ f_0 e_0 = \gamma (1 + \beta e_{10}) = \exp(\zeta e_{10}) \]

From the spinor \(L\), one can now deduce, the hyperbolic functions

\[ L^2 = \exp(2\zeta \mathbf{e}_0) = \cosh \zeta + \mathbf{e}_0 \sinh \zeta \]

Comparing with the equation (42) one obtains

\[ L^2 = \exp(2\zeta \mathbf{e}_0) + \mathbf{e}_0 \sinh \zeta \]

Where

\[ f_0 e_0 = \gamma = \cosh \zeta \quad f_0 \wedge e_0 = \gamma \beta e_{10} = \sinh \zeta \]

Obtaining the following figure

\[ \phi = \tan^{-1} \left(\frac{v}{c}\right) \]

We are able to distinguish three different types of categories with the scalars resultant from squaring space-time intervals.
Which are
\[ r^2 = 0, \text{lightlike} \quad r^2 < 0, \text{spacelike} \quad r^2 > 0, \text{timelike} \]

In the figure, the light-like is represented by all the lines that define the cones area. The time-like lines are presented as the innumerous possible lines passing through the origin, defined in the volume limited by the light-like lines, which is the space inside the cone. Finally, the space-like lines belonging to the remaining volume, defined in a volume, symmetric to the time-like lines have the light-lines as axis of symmetry.

After reviewing the equation (43), one extracts the following expression
\[ \tau = 0 \quad cT \tau L^2 = cT \tau (e_0 + \beta e_i) \] (48)

In this case the particle in the primed frame is in its proper frame (at rest) measuring the time \( \tau = \tau \). In the unprimed frame the particle describes the following trajectory \( cT e_0 + \nu T e_i \). Thus we can right the following expression
\[ cT f_0 = cT e_0 + \nu T e_i \] (49)

Where according to the expression (46)
\[ cT f_0 = cT e_0 + \nu T e_i = cT \gamma (e_0 + \beta e_i) = cT (\cosh \zeta e_0 + \sinh \zeta e_i) \]

This equation represented in a blue curve, is presented in the next figure

For \( f_0 \) is squared it is obtained
\[ 1 = (\cosh \zeta)^2 - (\sinh \zeta)^2 \] (50)

From the blue line to the origin, there are innumerous possibilities for the normalized vector \( f_0 \) to be inserted, depending on the angle \( \zeta \). In this simple example the decision for the angle \( \zeta \) for the figure was purely arbitrary
\[ f_0 = \exp (\zeta e_i) e_0 \] (51)

Observing this figure, even though \( f_0 \) is by definition normalized, it appears to be larger than the vectors \( e_i \) in the rest frame. This fact can be helpful once more to infer the time dilatation. In order to insert this vector in the frame at rest it must be projected in the correspondent time vector \( e_0 \). To find this projection one uses an inner product, this projection actually corresponds to the temporal interval from the unprimed frame
\[ \cosh \zeta = \gamma \] (52)

Where
\[ \cosh \zeta > 1 \] (53)

Thus, the time vector in a moving frame observed from the frame at rest is actually larger than the proper time, as time dilates. Analogously, one can induce the space contraction.

Special relativity in \( \mathbb{C}_{1,3} \)

The kinematic relativity can be spatially described in three dimensions. The addition of time within a space-time relativistic approach enables us to obtain a four-dimensional field. Geometric algebra is a mathematical tool believed to be the ideal for this mathematical framework able to generalize Einstein’s relativity. The only relativity theory which apparently does not present unphysical properties, also intended as Minkowski’s space-time approach. This theory conceives basing on the division ring of Hamilton’s quaternion \( \mathbb{H} \) with the use of geometric algebra. \( \mathbb{C}_{1,3} = \text{Mat}(2, \mathbb{H}) \) [16]

The union between time and tridimensional space results in the sum \( r = (ct)e_0 + \vec{r} \in \mathbb{R}^{1,3} \) with the quadratic form
\[ \mathbf{Q}(\mathbf{r}) = e^{-\mathbf{r} \cdot \mathbf{r}} = e^{-x^2 - y^2 - z^2} \text{.} \]

Where the spatial vector \( \mathbf{r} \) is anti-Euclidean \( \langle \mathbf{r} \rangle = -|\mathbf{r}|^2 \) where \( \langle \mathbf{r} \rangle = -x^2 - y^2 - z^2 \).

Therefore, \( C_{1,3} \) is the sum of the following subspaces
\[ C_{1,3} = \mathbb{R} \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3} \]

A multivector is now represented in each subspace by
\[ u = \alpha + \mathbf{a} + \mathbf{F} + \mathbf{T} + \mathbf{V} \]

where \( \langle \mu \rangle_0 \in \mathbb{R}^{1,3}, \langle \mu \rangle_1 \in \mathbb{R}^{1,3}, \langle \mu \rangle_2 \in \mathbb{R}^{1,3}, \langle \mu \rangle_3 \in \mathbb{R}^{1,3} \).

The dimension of \( C_{1,3} \) can be given by the Pascal triangle
\[ \dim(C_{1,3}) = 1 + 4 + 6 + 4 + 1 = 2^4 = 16 \]

A multivector \( I = e_x \wedge e_y \wedge e_z \) is a quadrivector in \( C_{1,3} \) that aggregates all orthogonal unit vectors within and is also known as a pseudoscalar. This object offers the opportunity to rewrite a multivector in the following form
\[ u = \alpha + \mathbf{a} + \mathbf{F} + \mathbf{T} + \mathbf{V} \]

When \( u = \alpha + \mathbf{a} + \mathbf{bI} + \mathbf{F} + \mathbf{\beta I} \) is conjugated it results in
\[ u' = \alpha - \mathbf{a} - \mathbf{F} - \mathbf{bI} + \mathbf{\beta I} \]

Note that the pseudoscalar is unchanged in its conjugation \( \mathbf{1} = \mathbf{1} \) and when squared it is negative \( \mathbf{1}^2 = \mathbf{1} \), thus,
\[ u^2 = -\mathbf{b} + \mathbf{F} + \alpha \mathbf{I} + \alpha \mathbf{I} \]

The following equations describe how spinors perform in four-dimensional non-Euclidean space. In \( C_{1,3} \) hyperbolic bivectors result from the geometric product formed by time and space vectors. The Elliptical bivectors, consequently, are the outcome of the geometric product of two non-collinear space vectors or two different time vectors. Finally, the null bivectors are the geometric product of any collinear or null vectors.
\[ \mathbf{F}^2 = 0 \quad \mathbf{I}^2 = 0 \quad \mathbf{F}^2 = 0 \quad \exp(\alpha \mathbf{F}) = 1 + \mathbf{F} \alpha \]
\[ \mathbf{F}^2 > 0 \quad (\mathbf{IF})^2 < 0 \quad L = \exp(\alpha \mathbf{F}) = \cos \alpha + \mathbf{F} \sin \alpha \]
\[ \mathbf{F}^2 < 0 \quad (\mathbf{IF})^2 > 0 \quad U = \exp(\alpha \mathbf{F}) = \cos \alpha + \mathbf{F} \sin \alpha \]

**Grade involution**
\[ \hat{u} = \alpha - \mathbf{a} + \mathbf{b} - \mathbf{I} \beta \]

**Reversion**
\[ \tilde{u} = \alpha + \mathbf{a} - \mathbf{b} - \mathbf{I} \beta \]

**Clifford conjugation**
\[ u' = \alpha - \mathbf{a} + \mathbf{b} + \mathbf{I} \beta \]

[16] The grade involution unlike the anti-automorphisms in \( \alpha \wedge \mathbf{a} \) and Clifford-conjugation is an automorphism \( \hat{u}v = \hat{v}u \). The grade involution induces the even-odd grading, \( C_{1,3} \) Al.ike a Euclidean space, it can be divided into substructures, the odd and the even part. Both of these aggregated elements once geometrically multiplied change or maintain their subspace, analogously to the conventional "scalars parity" concept, as demonstrated in the following figure
\[ C_{1,3} = \mathbb{R} \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3} \]

Hence, the even part is composed by
\[ C_{1,3}^e = \mathbb{R} \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3} \quad u = \alpha + \mathbf{F} + \beta \mathbf{I} \]

And for the remaining objects, the odd part
\[ C_{1,3}^o = \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3} \quad u = \alpha + \mathbf{bI} \]

In this case the Lorentz groups are given by \( L(x) = sxs^{-1} \) for \( L \in O(1,3) \) and \( L(x) = sxs^{-1} \) for \( L \in SO(1,3) \) and \( L \in SO(1,3) \), where they are doubly covered by
\[ \text{Spin}(1,3) = \{ s \in C_{1,3}^o \mid \forall x \in \mathbb{R}^{1,3}, sxs^{-1} \in \mathbb{R}^{1,3} \} \]
\[ \text{Spin}_+(1,3) = \{ s \in C_{1,3}^o \mid s \hat{s} = \pm 1 \} \]
\[ \text{Spin}_-(1,3) = \{ s \in C_{1,3}^o \mid s \hat{s} = 1 \} \]

**Relative velocity**

The assumption that the Lorentz boosts do not form a group in \( C_{1,3} \), by Thomas and Wigner leads to the composition of two non-parallel boost equivalent to the composition of a boost and a rotation. Therefore one derivates the composition of velocities for non-collinear velocities in the four-dimensional space, though, \( L_{S x} = R = LU \).

Consider now three different observers, an observer in frame \( S \), other in \( S' \) and another in \( S \) with the relative velocities \( v_1, v_2 \) and \( v \).
The breakdown with commutability gives rise to the gyrocomutativity, where
\[ \tilde{\beta} \oplus \tilde{\beta}_1 = \text{gyr} [\tilde{\beta}_1, \tilde{\beta}] (\tilde{\beta} \oplus \tilde{\beta}_1) \] (62)

Analogously, one can understand as well gyroassociativity [15]

Left Gyroassociative
\[ \tilde{\beta}_1 \oplus (\tilde{\beta}_2 \oplus \tilde{\beta}_3) = (\tilde{\beta}_1 \oplus \tilde{\beta}_2) \oplus \text{gyr} \left[ \tilde{\beta}_1, \tilde{\beta}_3 \right] \tilde{\beta}_3 \] (63)

Right Gyroassociative
\[ (\tilde{\beta}_1 \oplus \tilde{\beta}_2) \oplus \tilde{\beta}_3 = \tilde{\beta}_1 \oplus (\tilde{\beta}_2 \oplus \text{gyr} \left[ \tilde{\beta}_1, \tilde{\beta}_3 \right] \tilde{\beta}_3) \] (64)

Through analogy, with the emergence of gyroassociative and gyrocommutative binary operations, the gyrogroup’s approach to Einstein’s relativity theory narrows the gap between Einstein’s velocity composition and Newton’s velocity addition.

Vacuum form reduction

There are two homogeneous Maxwell equations expressing the magnetic flux conservation
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \] (65)
\[ \nabla \cdot \vec{B} = 0 \] (66)

Note that the electromagnetic fields \( \vec{E} \) and \( \vec{B} \) are each here represented in a space three-dimensional \( \mathbb{R}^{3,1} \). Nevertheless they are still homogeneous in \( C_{1,3} \).

The bivector \( \vec{F} \) is the invariant form of the space-time constitutive relation for vacuum.
\[ \vec{F}_v = -\frac{1}{c} \vec{E} + \vec{IB} \] (67)

For plane wave propagation we have \( \vec{F} = \mathcal{R} [F_v \exp \{-i (\vec{k} \cdot \vec{r})\}] \) (68)
\( \vec{k} \) stands for wave vector, given by
\[ \vec{k} = \frac{-i}{c} (\vec{e}_0 + \vec{k}) \] (69)
a four-vector corresponding to the time-harmonic electromagnetic wave with a temporal component given by \( \partial_t / c \) and the spatial component \( \vec{k} \).

Therefore,
\[ \vec{k} \wedge \vec{F}_v = 0 \] (70)

The bivector \( \vec{F} \) is the invariant form of the space-time constitutive relation for vacuum. Therefore, the electric and the magnetic fields in vacuum are covariant, and alike space and time, their measures are dependent of the frame of reference. Therefore for \( \vec{v} = \vec{v}_0 \), \( \vec{F}_v = \vec{v}^2 \vec{F}_v \), the velocity in the frame where the media is at rest, one obtains

Hyperbolic bivector
\[ c^{-1} \vec{E} = (\vec{F} - c^2 \vec{e}_0 \vec{F} \vec{e}_0) / 2 \] (71)

Elliptic bivector
\[ \vec{I} \vec{B} = (\vec{F} + c^2 \vec{e}_0 \vec{F} \vec{e}_0) / 2 \] (72)
For
\[ F_i = -\frac{1}{c} E + iB \]  
(73)

Where \( F_i \) is a boost transformation of \( F \) with the covariants conserving the same magnitude whereas the electric field is in an opposite direction.

For \( \tilde{D}(\tilde{H}) \) two inhomogeneous Maxwell equations can be resulted
\[
V \times \tilde{H} = j + \frac{\partial \tilde{D}}{\partial t} \tag{74}
\]
\[
V \cdot \tilde{D} = -\rho \tag{75}
\]

For a current density \( J = \rho \mathbf{e}_a + \frac{1}{c} \mathbf{j} \in \mathbb{R}^3 \).

\[ \partial_a \mathbf{G} = \mathbf{J} \]  
(76)

Thus, for source-free regions
\[ k_a \mathbf{G}_0 = 0 \]

Analogously, the Maxwell bivector is
\[ \mathbf{G} = \mathbf{D} + \frac{1}{c} \mathbf{I} \mathbf{H} \]  
(77)

The relation between the invariant bivectors is given in the vacuum by the following impedance
\[ \mathbf{G}_0 = \frac{1}{\eta_0} \mathbf{F}_0 \]  
(78)

Note that \( \eta_0 = \mu_0 / \varepsilon_0 \)

Therefore, in vacuum we have
\[ k_a \mathbf{F}_0 = k \wedge \mathbf{F}_0 = 0 \]  
(79)

Since,
\[ k \mathbf{F}_0 = k_a \mathbf{F}_0 + k \wedge \mathbf{F}_0 \]  
then
\[ k \mathbf{F}_0 = 0 \]  
(80)

Comparing both invariant bivectors
\[
\begin{align*}
\alpha_i &= \frac{E_i}{\eta_0} \Rightarrow \mathbf{G} = \mathbf{D} + \frac{1}{c} \mathbf{I} \mathbf{H} = \frac{\alpha_1}{c} \mathbf{E} + \alpha_2 \mathbf{IB}. \\
\alpha_i &= \frac{1}{\eta_0 \mu} 
\end{align*}
\]  
(81)

due to the derivation of the electric and magnetic field by the simple subtraction or sum between \( \mathbf{F}_0 \) and \( \mathbf{F} \).

Then,
\[
\begin{align*}
\beta_1 \mathbf{F} + \beta_2 \mathbf{F}_i &= \beta_1 \left( \frac{1}{c} \mathbf{E} + \mathbf{IB} \right) + \beta_2 \left( -\frac{1}{c} \mathbf{E} + \mathbf{IB} \right) \\
&= \frac{1}{c} \left( (\beta_1 - \beta_2) \mathbf{E} + (\beta_1 + \beta_2) \mathbf{IB} \right) \\
&\Rightarrow \\
\beta_1 - \beta_2 &= \alpha_1, \quad \beta_1 + \beta_2 = \alpha_2 \\
\alpha_1 &= \frac{1}{2} (\alpha_1 + \alpha_2) \left( \frac{e + 1}{\mu} \right), \\
\alpha_2 &= \frac{1}{2} (\alpha_1 - \alpha_2) \left( \frac{e - 1}{\mu} \right)
\end{align*}
\]

Finally, it is obtained
\[ \mathbf{G} = \frac{1}{2 \eta_0} \left( e + \frac{1}{\mu} \right) \mathbf{F} - \frac{1}{2 \eta_0} \left( e - \frac{1}{\mu} \right) \mathbf{F}_i \]  
(82)

Although, this is an interesting result it is possible to achieve a more simplified expression, once more with the hyperbolic functions.

Thus, for \( \exp(\xi) = \sqrt{\mu} = n_\xi \)

\[
\begin{align*}
\cosh \xi &= \frac{1}{2 \sqrt{\mu}} \left( e + \frac{1}{\mu} \right) = \frac{n_\xi + 1}{2 n_\xi} \\
\sinh \xi &= \frac{1}{2 \sqrt{\mu}} \left( e - \frac{1}{\mu} \right) = \frac{n_\xi - 1}{2 n_\xi}
\end{align*}
\]  
(83)

Considering
\[ r_v(F) = F_i = v^a F v \]  
(84)

And
\[ \eta = \eta_0 \sqrt{\mu / \varepsilon} \]  
(85)

Thus,
\[ G = \frac{1}{2 \eta_0} \left( e + \frac{1}{\mu} \right) F - \frac{1}{2 \eta_0} \left( e - \frac{1}{\mu} \right) F_i \]

\[ = \frac{1}{\eta} \left( \cosh \xi F - \sinh \xi F_i \right) \]

\[ = \frac{1}{\eta} \left( \cosh \xi - r_v \sinh \xi \right) F \]

\[ = \frac{1}{\eta} \exp(-\xi r_v) F \]  
(86)

Formally, the space-time constitutive relation in an isotropic medium is more complex than in vacuum, given by
\[ G = \frac{1}{\eta} \exp(-\xi r_v) F. \]  
(87)

This equation enables us to understand the constitutive relation \( (\mathbf{D}, \mathbf{H}) \) with \( (\mathbf{E}, \mathbf{H}) \) for all inertial frames, given the invariance form of Maxwell bivector space-time constitutive relation.

6. Conclusion

The geometric algebra is introduced in the Euclidean space based on the developed geometric product. Afterwards, this algebra is inserted in a non-Euclidean space with the introduction of the Minkowski space-time, originating the spacetime algebra. Hypercomplex numbers are revised to obtain a bridge between geometric algebra and space-time algebra. The former were able to provide an extension of our intuitive understanding from Euclidean to hyperbolic geometry, and similarly, from classical mechanics to relativistic mechanics. With the spinor theory (from geometric algebra), the Lorentz transformation is seen as a linear transformation, a boost in a two-dimensional field that leaves the space-time distance invariant, alike a rotation which is also a linear transformation, leaving the distance between spatial points invariant. This is a coordinate-free approach, a powerful tool from the geometric algebra.

From a four-dimensional spacetime, where the kinematic relativity can be fully conveyed, we are able to conclude that the boosts do not form a group by the introduction of the Thomas rotation. From the gyrogroup’s approach to Einstein’s relativity theory we are able to narrow the gap between Einstein’s velocity composition and Newton’s...
velocity addition. The study of the kinematic’s relativity allows us to conclude that the geometric algebra is an ideal language to embrace space and space-time geometry, providing new tools to treat vectors, offering the opportunity to attain more efficiency and new aspirations, due to its simplicity and clearness. From the invariant Lorentz force, in this algebra, it is possible to extract the faraday’s bivector, also invariant, which is interpreted as the reduction form of the two Maxwell homogenous equations. From this equation, the electric and the magnetic fields in vacuum are comprehended as covariant fields, which means that their values are dependent of the observer’s referential frame. This is analogous to an isotropic field with the electrical displacement and the magnetic field strength. Finally, also with the application of the invariant Faraday bivector in a nonreciprocal bianisotropic media, interpreted as a moving isotropic media in its proper frame, a novel simplification form is achieved, with a fictitious space-time, this media is now explored in a similar way as in vacuum. The results presented in this dissertation are in full agreement with the conventional tensor and dyadic analyses. Nevertheless we circumvent some complex algebraic manipulations within, achieving, because of geometric algebra, more elegance and more sophisticated results.


