Modal logics for reasoning about distance spaces

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Extended Abstract

The goal of this dissertation was to study modal logics, particularly those which allow us to reason about distance spaces.

We started by thoroughly presenting modal logics, basing our discussion of syntax, semantics, calculus and decidability on the *basic modal language*. This language is the most simple modal language, allowing for only one modality, the possibility modality $\Diamond$, not counting its obvious converse, the necessity modality $\Box$. Semantics for the *basic modal language* were then defined, based on the concept of Kripke structures, more specifically Kripke frames and Kripke models.

We interpret formulas of the *basic modal language* on these structures, through the concepts of satisfiability and validity. Satisfiability is a concept used with models in a much more local environment than validity, which deals with frames. Local and global Entailment, a semantic notion of logical consequence between formulas was introduced to finish our section about the semantics involving modal logics.

For our discussion of axiomatization we introduced *normal modal logics*, defined as a set of formulas, closed for modus ponens and necessitation, which includes all tautologies and at least two axioms (one of them being the K axiom). We then prove soundness and completeness of the most basic normal modal logics, the **K** logic, on the class of all frames. That is, we prove that the set of valid formulas in every frame equals the set of formulas in **K**. As is usual, the completeness proof was the more complex proof of the two, where notions of consistency of sets of formulas and canonical models had to be introduced. Given the range of the arguments used, expansion of the completeness and soundness results to other *normal modal logics* becomes most of the time a very simple intelectual task.

We end the chapter on modal logics with a more general and less technical discussion over logic decidability. We address several proof methods for decidability, including decidability via quasi-models, which is the method used for the main result of this dissertation. With that in mind, we introduce a thourough yet generic proof sketch of decidability via quasi-models.

We now turn our attention to spatial modal logics. Firstly distance spaces are presented, a more general class of spaces than metric spaces, for symmetry and the triangle inequality are not required. Upon those we interpret the logic which will occupy us for the rest of the dissertation, the *Comparative Similarity Logic* ($\mathcal{CSL}$). This logic uses two spatial modalities, the closer ($\rightrightarrows$) and the realized (ⓡ) operators, which translates the closer property between sets and whether distances to a set are realized, respectively.

Afterwards we exemplify how the logic and its semantics function by proving a statement about local entailment and a statement about satisfiability over a formula representing the notion of topological closure.

Now comes the main result of this dissertation. Expanding the decidability discussion already presented to much more technically driven arguments, we prove that the $\mathcal{CSL}$ is decidable on the class of symmetric distance models. That is, we prove that we can decide, in a finite amount of time, whether a $\mathcal{CSL}$ formulas is satisfiable or not in a symmetric distance model. This proof is made using Hintikka sets and quasi-models, finite structures which are meant to resemble actual models, over which we can construct a model where some formula is going to be satisfied. Throughout this extensive proof several intuitive insights are usually given, in order to ease the technical complexity involved.

To end this dissertation we applied a graph-theoretical approach to our logic. Our goal using this approach is to create a framework where logics will be defined in order to facilitate their comparing and fibring. Using multi-graphs, graphs which can have any finite amount of sources but only one target, we create a multi-graph based signature and then a multi-graph based model. This two multi-graphs are related through a graph morphism to guarantee that the relations between the construction of formulas and their interpretation remain intact.