Modal logics for reasoning about distance spaces

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Abstract

We start by providing an overview of modal logic including the language, Kripke semantics and axiomatization. General soundness, completeness and decidability results are provided. We concentrate our attention on the modal logic $\mathcal{CSL}$ (Comparative Similarity Logic) capable of reasoning about comparative distances in distance spaces. After introducing the language and the semantics, we prove its decidability over a specific class of distance models. Finally, we produce a graph-theoretic account of $\mathcal{CSL}$ having in mind the possibility of combining such a logic with other logics either by fibring or by "spatial" modalization.

Keywords: Modal logic, distance spaces, spatial logics, decidability, graph-theoretical account of logics

Resumo

Começamos por dar uma visão global da lógica modal incluindo a sua linguagem, semântica de Kripke e axiomatização. São apresentados resultados gerais de correcção, completude e decidibilidade. Depois, concentramos a nossa atenção na lógica modal $\mathcal{CSL}$ (Comparative Similarity Logic) capaz de raciocinar sobre distâncias comparativas em espaços métricos. Depois de introduzirmos a linguagem e a semântica, provamos que a lógica é decidível sobre uma classe específica de espaços métricos. Finalmente, apresentamos uma definição de $\mathcal{CSL}$ usando a abordagem de multigrafos à definição de lógicas tendo em vista a possibilidade de combinar esta lógica com outras, seja por fibrilação seja por modalização "espacial".

Palavras-chave: Lógica modal, espaços de distância, lógicas espaciais, decidibilidade, abordagem de multigrafos à definição de lógicas

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Chapter 1

Introduction

Modal logic is a kind of logic originally conceived as the logic of possibility and necessity. In fact, as early as Ancient Greece, Aristotle started reasoning with notions of possibility and necessity.

Despite such an early interest in these notions, only recently, in 1918, modal logic as we know it was presented, by C.I. Lewis, where he introduced notions of impossibility and strict implication [9].

Later, in 1932, in another breakthrough book, Lewis, along with C.H. Langford [10], presented a more detailed exposition of Lewis' ideas, where the possibility symbol, $\Diamond$, was introduced as primitive. The five well known axiomatic systems **S1**-**S5** were also defined.

Until the 1950's, most modal logic discussions were essentially syntactic. Interpretation was made based on boolean algebras, where modal connectors were seen as operators on them. Using this techniques, the first results regarding spatial logics appeared. The most interesting for this dissertation being the proof by Tarski and McKinsey [11] that **S4** can be used for reasoning about topological spaces. In fact, they showed that **S4** is complete when interpreted topologically, with $\Diamond$ being the topological closure of a set and $\Box$ the interior.

Only in the late 1950's, an intuitive semantics for modal logics was created. Through the notions of relational structures, frames and models, the 19-year old Saul Kripke revolutionized the way modal logics were handled and utilised. By relating the syntactic approach of the first half of the century with the new semantic approach, problems such as distinguishing logics or proving that the set of valid formulas equals the set of generated formulas were suddenly reduced to a straightforward semantic argument. For example, it was now clear that **S4** is complete in the set of all reflexive and transitive frames [2, 1].

This relational semantics, often called Kripke semantics, gave birth to a whole new set of possibilities. During the next few decades, modal logics evolved like never before, into several new concepts unforeseen until then. It became clear its many uses and importance in several fields of study.

Despite the many breakthroughs, one branch of modal logic that didn't receive the same attention as, for example, temporal and normal logics, was spatial modal logics, which garnered only some scattered attention in the literature.

However, later in the century, more focus was given to spatial logics. An important advance to this dissertation pertains to the introduction done by van Benthem to comparative distances in modal languages [18].

Since then, some attempts have been made at creating a cohesive study of this discipline, the most noble attempt resulting in a rather recent and detailed handbook [1].

For a more thorough historical perspective on modal logics and the specific branch of spatial logics, the reader should consult [8, 2, 7, 12].

The structure of this dissertation is as follows. In Chapter 2, we introduce modal logics, along with Kripke semantics and notions of satisfiability, validity, soundness, completeness and decidability. The discussion is particularized with help from the *basic modal language*. Chapter 3 is dedicated to the introduction of distance spaces and a spatial logic which reasons about comparative distances and topology. A major decidability result is proved. At last, Chapter 4 defines the spatial logic in graph-theoretical terms, as introduced in [13].

Chapter 2

An overview of modal logic

In our first chapter we present the main components behind modal logics. As with any logic it consists on the language, semantics and calculus. The modal language is an extension of a propositional language. Satisfaction and validity are defined over Kripke structures. The calculus is presented for normal modal logics. We proceed by introducing soundness and completeness of the calculus and include, in the end of the chapter, a discussion on the decidability of modal logics.

2.1 Language

We will start this study of modal logic by, without further unnecessary explanation, presenting the syntax for the most general of modal languages, the *basic modal language*, after which we will define the semantics of this logic via Kripke structures. The signature (or alphabet) of the *basic modal language* is, as in the case of the propositional logic, a set of propositional symbols. The formal grammar of the *basic modal language* has a new modality operator, not included in the propositional case. This operator is in many cases, the necessity operator $\Box$. Herein, we preferred to use the $\Diamond$ modality as primitive.

Definition 1 (Basic modal language) The formulas of the *basic modal language* are constructed using Φ according to the following formation rules:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond\varphi$$

where Φ is a set of proposition symbols and p $\in \Phi$.

We say that Φ is the signature for the *basic modal language*. By changing Φ we get a new modal language. As usual, we also use the abbreviations for conjunction, implication, equivalence, constant truth value ('top') and the converse of the $\Diamond$ symbol, the $\Box$ symbol:

$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$$

$$\varphi \to \psi := \neg\varphi \vee \psi$$

$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$$

$$\top := \neg\bot$$

$$\Box\varphi := \neg\Diamond\neg\varphi$$

2.2 Semantics

In this section we present the semantic framework that allow us to interpret modal formulas. In our case we adopted Kripke structures and not modal algebras. By introducing different Kripke structures, we can define, constructively, a local and a global interpretation of modal formulas. The notion of semantic entailment between sets of formulas and formulas is also defined.

2.2.1 Kripke structures

Now that we have defined the basic of modal languages, we are ready to introduce Kripke semantics, by firstly defining Kripke frames.

Definition 2 (Kripke frame) A *Kripke frame* $\mathcal{F}$ (or simply a frame) consists of a pair (W, R) where:

1. W is a non-empty set;

2. $R : W \times W \to \{0, 1\}$ is a binary relation on W.

The elements of W are called worlds, states or points, while the binary relation R is usually called the accessibility relation.

The fact that the Kripke frame is a very basic structure that has knowledge only about the logic's universe and doesn't take into account the signature of the language on which it is going to be used, means that frames will only be used to reason globally. To provide interpretation of the propositional symbols and the constructors we need the notion of Kripke model.

Definition 3 (Kripke model) A *Kripke model* (simply called model from now on) for the basic modal language is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F} = (W, R)$ is a frame, and $V : \Phi \to \wp W$ is a map.

Such map V assigns to each $p \in \Phi$ the worlds in W where p is perceived to be true.

We may also use the expression *interpretation structure* when referring to Kripke models.

So we can see that, contrary to a frame, a model provides us with a bridge, through the valuation function V, between the modal language on which it is based and the universe of the frame associated with the model.

2.2.2 Interpretation

With the definitions of frames and models, we are now ready to semantically interpret the *basic modal language* introduced, by defining with precision various notions of formula evaluation, including satisfaction (pertaining to models) and validity (pertaining to frames) in modal logics.

Definition 4 (Satisfaction in a model) Given a model $\mathcal{M} = (\mathcal{F}, V)$, a world $w \in W$ and a formula φ of the basic modal logic, we say that the formula φ is *true* (or *satisfied*) in $\mathcal{M}$ in the world w if $\mathcal{M}, w \Vdash \varphi$, where the relation $\Vdash$ is inductively defined as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p), \text{ where } p \in \Phi \\
\mathcal{M}, w \Vdash \bot & & \text{never} \\
\mathcal{M}, w \Vdash \neg\varphi & \text{iff} & \text{not } \mathcal{M}, w \Vdash \varphi \\
\mathcal{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Diamond\varphi & \text{iff} & \mathcal{M}, v \Vdash \varphi \text{ for some } v \in W \text{ with } R(w, v)
\end{array}$$

If, on the other hand, $\mathcal{M}$ does not satisfy φ in world w, we write $\mathcal{M}, w \nVdash \varphi$.

By definition of implication, constant truth ($\top$) and the box symbol ($\Box$), we get:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \rightarrow \psi & \text{iff} & \mathcal{M}, w \nVdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \leftrightarrow \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \rightarrow \psi \text{ and } \mathcal{M}, w \Vdash \psi \rightarrow \varphi \\
\mathcal{M}, w \Vdash \top & & \text{always} \\
\mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \mathcal{M}, v \Vdash \varphi \ \forall v \in W \text{ with } R(w, v)
\end{array}$$

This notion of satisfaction is a very internal and local one, hence being usually called *local satisfaction*. Formula interpretation is done inside the models, according to some specific current state w. Moreover, the interpretation of $\Diamond$ is also done according to local relations, for only the states that are accessible from our current state (according to the accessibility relation R) are going to be scanned for satisfaction.

We can extend this definition of satisfaction to sets of formulas. We say that a model $\mathcal{M}$ satisfies the set of formulas Γ in the world $w \in W$, written

$$\mathcal{M}, w \Vdash \Gamma$$

iff $\mathcal{M}, w \Vdash \gamma$ for every $\gamma \in \Gamma$.

Other definitions of satisfaction will be used. For example, we say that a formula φ is satisfiable in a model $\mathcal{M}$ if there exists a world

$w \in W$ where $\mathcal{M}, w \Vdash \varphi$. If $\mathcal{M}, w \Vdash \varphi$ happens for all worlds $w \in W$ then we say that the formula is globally true in $\mathcal{M}$.

Now we will present another type of formula evaluation within Kripke semantics, formula validity.

Definition 5 (Validity in a frame) Given a frame $\mathcal{F} = (W, R)$, a world $w \in W$ and a formula φ, we say that the formula φ is *valid in the frame $\mathcal{F}$ in w*, denoted by

$$\mathcal{F}, w \Vdash \varphi,$$

if $\mathcal{M}, w \Vdash \varphi$ for every model $\mathcal{M}$ based on the frame $\mathcal{F}$.

Also, the formula is valid in the frame $\mathcal{F}$, denoted by

$$\mathcal{F} \Vdash \varphi,$$

if the formula is valid in $\mathcal{F}$ for all worlds $w \in W$. The formula is simply valid, denoted by

$$\Vdash \varphi$$

if it is valid for all possible frames.

As we can see, the definitions of satisfiability and validity, although presented in a similar way, have a big difference in the way they work. With satisfiability, we are interested in the valuations given by the models, so satisfaction is a more local concept. Formulas are satisfied within a model depending on the different valuations given to each propositional symbol. On the other hand, validity, by being applied to every model, discards the valuations of each, and is left only with the frame defined, and so makes evaluations concerning our perceived universe, not the different truth-values included in the models.

In essence, while satisfiability gives us a local idea of our universe by accepting specific truth-values assigned to the different worlds, validity gives us a global idea of our universe, by taking into account only the relations between each world, not their specific valuation.

2.2.3 Entailment

Sometimes we may not be so interested in how each formulas relate to our universe by means of frames and models, but more in the sense of how formulas relate to each other. We may want to understand some formulas as being a basic logical consequence to another set of formulas in some local or global environment. For that purpose, two notions of semantic entailment are introduced, a local definition and a global one.

Definition 6 (Local semantic entailment) Let Φ be a logical signature, $\mathcal{M}$ be a model, and $\Gamma \cup \{\varphi\}$ a set of formulas of some language over Φ. We say that Γ *locally entails* φ in $\mathcal{M}$ (or φ is a local semantic consequence of Γ in $\mathcal{M}$), written

$$\Gamma \models^{l}_{\mathcal{M}} \varphi$$

if for every point $w \in \mathcal{M}$, we have $\mathcal{M}, w \Vdash \varphi$ whenever $\mathcal{M}, w \Vdash \Gamma$.

This definition of semantic entailment demands that if some set of formulas Γ is true at some point in some model, then φ must also be true at the exact same point in the same model. Hence, this is a definition that treats semantic entailment point to point. Sometimes we may be interested in a notion where instead of forcing the truthness of a set of formulas to be preserved in each and every point, we want to preserve the truthness in the model as a whole. That is the essence of the global semantic entailment.

Definition 7 (Global semantic entailment) As in the previous definition, let Φ be a logical signature, $\mathcal{M}$ be a model, and $\Gamma \cup \{\varphi\}$ a set of formulas of some language over Φ. We say that Γ *globally entails* φ in $\mathcal{M}$ (or φ is a global semantic consequence of Γ in $\mathcal{M}$), written

$$\Gamma \models^{g}_{\mathcal{M}} \varphi$$

if we have $\mathcal{M} \Vdash \varphi$ whenever $\mathcal{M} \Vdash \Gamma$.

We also use the definition of global semantic entailment over frames. If F is a set of frames, then $\Gamma \vDash^g_F \varphi$ if $\Gamma \vDash^g_{\mathcal{M}} \varphi$ for every model $\mathcal{M}$ based on some frame $\mathcal{F} \in F$.

As we can see, this definition makes the assumption that Γ is globally true in a model, and is based on the preservation of that global truth. The difference between these notions can be seen through the classic example of the formulas p and $\Box p$. If we take the local semantic entailment definition, we can see that just because p is true in some point w in some model $\mathcal{M}$, doesn't mean that p will also be true at every point accessible from w. Hence, there are models such that $p \nvDash^l_{\mathcal{M}} \Box p$. But, if it is true in every point in some model, then it is also true at every point accessible from some other point, and so, $p \vDash^g_{\mathcal{M}} \Box p$. And so clearly global entailment doesn't imply local entailment. But we can easily show that local entailment implies global entailment.

Theorem 8 Let Φ, $\mathcal{M}$, and $\Gamma \cup \varphi$ be defined as in **Definition 7**. Then, if $\Gamma \vDash^l_{\mathcal{M}} \varphi$, it is also the case that $\Gamma \vDash^g_{\mathcal{M}} \varphi$.

Proof: Suppose $\Gamma \vDash^l_{\mathcal{M}} \varphi$. We want to show that $\Gamma \vDash^g_{\mathcal{M}} \varphi$, which is the same as saying that if $\mathcal{M} \Vdash \Gamma$ then $\mathcal{M} \Vdash \varphi$. So suppose that $\mathcal{M} \Vdash \Gamma$. Then:

$\mathcal{M} \Vdash \Gamma$	(global truth definition)
$\mathcal{M}, w \Vdash \Gamma$ *for every point* $w \in \mathcal{M}$	(local entailment hypothesis)
$\mathcal{M}, w \Vdash \varphi$ *for every point* $w \in \mathcal{M}$	(global truth definition)
$\mathcal{M} \Vdash \varphi$	(global entailment definition)
$\Gamma \vDash^g_{\mathcal{M}} \varphi$	

QED

2.3 Axiomatization

We have discussed Kripke structures that allow us to reason semantically about modal logic. From now on we concentrate on normal modal logics to provide calculi.

2.3.1 Normal modal logics

We will start this introduction of normal modal logics by presenting it and its various nuances, while leaving the specific usage of these modal logics in soundness and completeness results to the next sections.

Before actually defining normal modal logics, we need the simple definition of tautological formula.

Definition 9 (Tautological formula) A *tautological formula* φ over Φ is a modal formula such that there is a set Π of propositional symbols, a propositional formula γ over Π and a map μ from Π to the modal language over Φ such that

- $\bar{\mu}(\gamma) = \varphi$;
- γ is a propositional tautology;

where $\bar{\mu}$ is the obvious extension of μ to formulas over Π.

Simple and obvious examples of propositional tautologies include: $(\Box p) \rightarrow (\Box p)$ and $(\Box p) \rightarrow (q \rightarrow (\Box p))$. The last tautological formula comes from the first axiom of the usual Hilbert calculus for propositional logic.

To prove that the last formula is in fact a tautological formula, let us take into account **Definition 9** and assume the following:

- $\Pi = \{p1, p2\}$,
- $\gamma = (p1 \rightarrow (p2 \rightarrow p1))$,
- $\Phi = \{p, q\}$
- μ a map from Π to the modal language over Φ such that:
  - $\mu(p1) = \Box p$,
  - $\mu(p2) = q$.

Then, it is clear that $\bar{\mu}(\gamma) = (\Box p) \to (q \to (\Box p))$, and since γ is a propositional tautology, then $(\Box p) \to (q \to (\Box p))$ is a tautological formula over Φ.

Now comes the definition of *normal modal logics*.

Definition 10 (Normal modal logics) A *normal modal logics* is a set Λ of formulas that must contain (at least) all tautological formulas and the following axioms:

$$\begin{array}{ll} (K\ axiom) & \Box(p \to q) \to (\Box p \to \Box q) \\ (Dual\ axiom) & \Diamond p \leftrightarrow \neg\Box\neg p \end{array}$$

Λ must also be closed under the following rules of inference:

- *Modus ponens*: given φ and $\varphi \to \psi$, prove ψ.
- *Generalization*: given φ, obtain $\Box\varphi$.

So a normal modal logic is simply a set of formulas that satisfy some deduction rules. There is a connection, however, between the formulas in a normal modal logic, and the valid formulas in some set of frames.

We say that a rule of inference preserves a specific type of semantic interpretation, for example, satisfaction, if it is true that the satisfaction of the premises in some model and some point guarantees the satisfaction of the conclusion in the same model and same point.

Given that definition, note that although *Modus ponens* preserves every semantic interpretation, whether it being satisfiability, global truth or validity, *generalization*, on the other hand, does not preserve satisfiability. Just because a formula is true in some point in some model, doesn't mean it is true in every accessible point in the same model. Global truth and validity though, are preserved, for if it is true in every point, then it is true also in every accessible point. This statement will be proved carefully later. Because of this notion, we are more interested in proving the connection in terms of frames, since we don't need to be careful using generalization around formulas semantically and syntactically valid. All our completeness and soundness results are to be expressed and examplified recurring back to frames, never models.

The smallest of normal modal logics is obtained by simply starting with exactly the axioms proposed: a minimal set of tautologies, the *K axiom* and the *dual axiom*, and using the previously stated deduction rules. As we will see, it is a fact that the formulas that can be deduced simply from this axioms and rules of proof are exactly the formulas from the basic modal language that are valid in the set of all frames. We refer to this modal logic simply by **K**, taking its name from the axiom that it includes.

Other, more specific and more expressive normal modal logics can be created by enriching them with some other axioms. These axioms usually reflect the type of frames that we want to connect with. For example, if we add to the **K** logic the axiom that reflects transitivity, $\Diamond\Diamond p \to \Diamond p$ (this axiom is also referred to as the (4) axiom), we get a normal modal logic, called the **K4** system (since it includes the K and 4 axiom), whose formulas are exactly the formulas valid on all transitive frames.

There are two interesting remarks that can be made about normal modal logics and lead to other ways of thinking about normal modal logics.

Remark 11

- The set of every possible formula of the basic modal language is a normal modal logic (called the inconsistent logic).
- If $\{\Lambda_i \mid i \in I\}$ is a collection of normal modal logics, then $\bigcap_{i\in I} \Lambda_i$ is also a normal modal logic.

By the previous remark, we realize that, given a set of formulas Γ, there is a minimal normal modal logic containing Γ. We call this modal logic $\mathbf{K}\Gamma$ and say that $\mathbf{K}\Gamma$ is axiomatized by Γ. Note that that implies that if Γ is the empty set, or the (4) axiom, then $\mathbf{K}\Gamma$ equals **K** or **K4**, respectively.

These normal modal logics also allow for a Hilbert-type deductive system. We will present it for the **K** system.

Definition 12 (K-proof) A ***K**-proof* is a finite sequence of formulas, where each formula is either an axiom or a formula that follows from previous ones by a deduction rule. The axioms and rules of proof for this formula deduction system is the same used to first define normal modal logics.

We say that a formula φ is **K**-provable if it appears in a **K**-proof. If that is the case, then we write $\vdash_{\mathbf{K}} \varphi$. If the normal modal logics system used in the proof (in this case the **K** system) is clear from the context, we can simply write $\vdash \varphi$.

As a final definition concerning **K**-proofs, we have the definition of local syntactical consequence. Let $\Sigma \cup \varphi$ be a set of formulas and Λ a normal modal logic. We say that φ is a local syntactic consequence of Σ (or φ is deducible from Σ) in Λ written $\Sigma \vdash_{\Lambda} \varphi$ if there is a **K**-proof of φ which may use Σ as hypothesis, but the generalization rule is never used on formulas which originate from any of the hypothesis $\sigma \in \Sigma$.

2.3.2 Soundness

We have now the right ingredients to start connecting our syntactic and semantic analysis of modal logic. The simplest notion to prove is the soundness of modal logics, which will ultimately lead to the more specific proof that every formula in the **K** system, meaning every formula that has a **K**-proof, is also valid on the class of all frames.

Definition 13 (Soundness) Let F be a class of frames. A normal modal logic Λ is *sound* with respect to F if for every $\varphi \in \Lambda$, and for every frame $\mathcal{F} \in F$, we have that $\vdash_{\Lambda} \varphi$ implies $\mathcal{F} \Vdash \varphi$.

This is the same as saying that a normal modal logic is sound with respect to a class of structures if every formula in that normal modal logic is valid in every frame from the class.

So this gives us one way of the bridge we want to create. From a syntactically defined logic, we try to get the same logic defined semantically, by defining the frames on which the logic is valid.

Since every normal modal logic is defined only by their axioms and the rules of proof, to prove that a normal modal logic is sound with respect to some set of frames, all we need to prove is that the axioms are valid in that class and that the rules of proof preserve validity. Proving that a rule of proof preserves validity is the same as proving that if the assumptions Γ are valid in a class of frames then so is the conclusion γ. That is, if $F \Vdash \Gamma$ then $F \Vdash \gamma$.

As we said before, the formulas in **K** are exactly the same as the modal formulas valid in the set of all frames. Let's now try to prove one side of the equation. All we need to prove is that the 2 axioms from **K**, the *K axiom* and the *dual axiom*, are valid in all frames, and that modus ponens and generalization preserve validity. These proofs are generally straight forward, so we'll just provide a proof for the K axiom and the generalization.

Lemma 14 (K axiom is valid in every frame) The K axiom is valid in every possible frame, that is,

$$\Vdash \Box(p \to q) \to (\Box p \to \Box q).$$

Proof: To prove that the K axiom is valid in every frame, we must have, for every model $\mathcal{M}$ and every point w,

$$\mathcal{M}, w \Vdash \Box(p \to q) \to (\Box p \to \Box q).$$

So, suppose $\mathcal{M}, w \Vdash \Box(p \to q)$. By our definition of satisfaction we get:

$\mathcal{M}, w \Vdash \Box(p \to q)$	$(\Box -$ satisfaction$)$
$\mathcal{M}, v \Vdash p \to q,$ for every v such that $R(w, v)$	$(\to -$satisfaction$)$
$\mathcal{M}, v \nVdash p$ or $\mathcal{M}, v \Vdash q$	$(\Box -$ satisfaction$)$
$\mathcal{M}, w \nVdash \Box p$ or $\mathcal{M}, w \Vdash \Box q$	$(\to -$satisfaction$)$
$\mathcal{M}, w \Vdash (\Box p \to \Box q)$	

Which is exactly what we wanted to prove. So we see that the K axiom is valid in all frames. QED

The generalization rule proof does not take as much work. Suppose that p is valid in a frame, ($\Vdash p$). Then it is satisfied in every model and every point. So if we take a simple model $\mathcal{M}$ and point w it is also true that p is valid in every point accessible to w, and so $\Box p$ is also true at w. Because the choice of $\mathcal{M}$ and w was arbitrary, then $\Box p$ is true in every point and every model, and so it is valid in every frame, $\Vdash \Box p$, as we wanted to prove.

The rest of the proofs for the **K** system and any other subsequent system are usually as trivial as these, so we omit them. To end this soundness section we simply state the primary theorem:

Theorem 15 (Soundness of the K system) The **K** system is *sound* with respect to the formulas valid in the class of all frames. That is:

$$\vdash_{\mathbf{K}} \varphi \text{ implies } \Vdash \varphi.$$

2.3.3 Canonical models and completeness

After proving that every formula deducible in the **K** system is also valid in the set of all frames, we are going to present a proof for the other side of the equation, that is, we are going to prove that every formula valid in the set of all frames is deducible in the **K** system. The proof technique we will present can be used for many other logics and classes of frames, as will be announced in due time. After the proof, we can finally state that the set of formulas deducible in **K** and the set of formulas valid in all frames coincide, they are the exact same set.

The other implication side of soundness is called completeness, we shall now define it precisely.

Definition 16 (Completeness) Let F be a class of frames. A normal modal logic Λ is *strongly complete* with respect to F if for every set $\Gamma \cup \varphi$ of formulas, if $\Gamma \vDash^g_F \varphi$ then $\Gamma \vdash_\Lambda \varphi$.

The logic Λ is weakly complete with respect to F if for every formula φ, $F \Vdash \varphi$ implies $\vdash_\Lambda \varphi$.

In this section, strong completeness is the concept that will be the main focus of our proofs, all results that will be proven will be about strong completeness. Also note that if a logic is strongly complete with respect to some class of frames, then it is weakly complete with respect to the same class as well.

Here's a small definition that will be of use for the completeness proof, later in this section.

Definition 17 (Λ-consistency) Let Λ be a modal logic. A set of formulas Γ is called Λ-*consistent* if $\Lambda \nvdash \bot$, Λ-*inconsistent* otherwise.

It is also easy to check that Γ is Λ-inconsistent iff there exists a formula φ such that $\Gamma \vdash_\Lambda \varphi \wedge \neg\varphi$.

The basic idea behind the basic modal logic completeness proof is beset on the following proposition regarding other ways of proving strong completeness. This proposition will be introduced without proof.

Proposition 18 Λ is strongly complete with respect to some class of frames F iff every Λ-consistent set of formulas is satisfiable in some model based on $\mathcal{F} \in F$. Λ is weakly complete if the same happens only for every Λ-consistent formula.

By this proposition we get that to prove that some logic Λ is strongly complete with respect to some class of structures, it suffices to find a model where every Λ-consistent set is satisfiable on.

So we are left with the task of finding and proving that such a model exists. This model will be based on the notion of *maximal* Λ-*consistent sets* of formulas and the *canonical models* construction derived from it.

Definition 19 (Λ-MCS) A set Γ of formulas from the logic Λ is said to be *maximal* Λ-*consistent* if Γ is a Λ-consistent set of formulas, and every set Γ^+ of formulas such that $\Gamma \subsetneq \Gamma^+$ is Λ-inconsistent. We call this Γ set a Λ-*MCS*.

These so called Λ-MCS have some interesting properties, one of which we will prove next and that will give some idea about the mechanism behind the completeness proof.

Proposition 20 Let Λ be a normal modal logic, $\mathcal{M}$ be a model for Λ and w a point of $\mathcal{M}$. Then the set $\Gamma = \{\varphi \mid \mathcal{M}, w \Vdash \varphi\}$ is actually a Λ-MCS.

Proof: By the definition of Λ-MCS, we need to prove that:

1. Γ is a Λ-consistent set.

2. Every Γ^+ set of formulas such that $\Gamma \subsetneq \Gamma^+$ is Λ-inconsistent.

Let us first start with the proof for the Λ-consistency.

1. We need to prove that $\Gamma \nvdash_\Lambda \bot$, which is equal to proving that there exists no φ such that $\Gamma \vdash_\Lambda \varphi \wedge \neg\varphi$.

   We will prove this by contradiction. Let us suppose that such φ in fact exists. By the abbreviation of the implication symbol, that is the same as saying that $\Gamma \vdash_\Lambda \neg(\varphi \to \varphi)$.

   Firstly note that since Λ is a normal modal logic, and as we know all axioms from Λ are valid and *Modus Ponens* preserves satisfiability, then from the definition of Γ, we get that $\Gamma \vdash_\Lambda \neg(\varphi \to \varphi)$ iff $\neg(\varphi \to \varphi) \in \Gamma$.

   So we can understand that $\Gamma \vdash_\Lambda \neg(\varphi \to \varphi)$ iff $\mathcal{M}, w \Vdash \neg(\varphi \to \varphi)$. This becomes then a simple semantic interpretation:

$\mathcal{M}, w \Vdash \neg(\varphi \to \varphi)$	(By negation)	iff
$\mathcal{M}, w \nVdash \varphi \to \varphi$	(By implication)	iff
$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \nVdash \varphi$		

   That is obviously a contradiction so no such φ exists and Γ is a Λ-consistent set.

2. Suppose Γ^+ is a set of formulas such that $\Gamma \subsetneq \Gamma^+$.

   Let φ be a formula such that $\varphi \in \Gamma^+$ but $\varphi \notin \Gamma$ which obviously exists. Because $\varphi \notin \Gamma$, then $\mathcal{M}, w \nVdash \varphi$ and so $\neg\varphi \in \Gamma$.

   The proof of the Λ-inconsistency can be made by finding a formula φ such that $\Gamma^+ \vdash_\Lambda \varphi \wedge \neg\varphi$. We shall prove that the formula φ defined above satisfies this condition.

   Since $\varphi \in \Gamma^+$ and $\vdash_\Lambda$ is closed under *Modus Ponens*, then if we can prove that $\varphi \to (\varphi \wedge \neg\varphi) \in \Gamma^+$ we automatically get $\Gamma^+ \vdash_\Lambda \varphi \wedge \neg\varphi$.

   So the proof can be simplified to proving that $\varphi \to (\varphi \wedge \neg\varphi) \in \Gamma$. By the definition of Γ, this proof turns into a yet again simple semantic interpretation problem:

$$\begin{aligned} &\mathcal{M}, w \Vdash \varphi \to (\varphi \wedge \neg\varphi) && \text{iff} \\ &\mathcal{M}, w \nVdash \varphi \text{ or } \mathcal{M}, w \Vdash (\varphi \wedge \neg\varphi) && \text{iff} \\ &\mathcal{M}, w \Vdash \neg\varphi \text{ or } \mathcal{M}, w \Vdash (\varphi \wedge \neg\varphi). && \end{aligned}$$

   We already know that $\mathcal{M}, w \Vdash \neg\varphi$, so we get $\Gamma^+ \vdash_\Lambda \varphi \wedge \neg\varphi$ and by definition Γ^+ is Λ-inconsistent as we wanted to prove.

QED

From this proposition, we see that every model and every point in that model is directly and uniquely associated to a Λ-MCS. And since points in one model are related, this Λ-MCSs are also expected to be related, as we will see going forward.

The idea behind our proof then is, if we can construct a model (the so called *canonical model*) where each point is in fact a Λ-MCS, and so is related to one and only one point, we will be able to prove that every Λ-consistent set is satisfiable on such model, by providing a Λ-MCS that extends this Λ-consistent set. That is, by providing one "point" in the model where the Λ-consistent set is satisfied.

We need to specify more precisely what we mean by a Λ-MCS that extends some Λ-consistent set. That can be done by presenting the Lindenbaum's Lemma.

Lemma 21 (Lindenbaum's lemma) Let Γ be a Λ-consistent set of formulas. Then there exists Γ^+ such that and $\Gamma \subseteq \Gamma^+$ and Γ^+ is a Λ-MCS.

Although we don't give an actual proof of Γ^+ being a Λ-MCS, by enumerating the formulas of our language by $(\varphi_0, \varphi_1, \varphi_2, \ldots)$, we can define Γ^+ as a union of countable sets:

$$\begin{aligned} \Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if this set is } \Lambda - \text{consistent} \\ \Gamma_n \cup \{\neg\varphi_n\}, & \text{otherwise} \end{cases} \\ \Gamma^+ &= \bigcup_{n \geq 0} \Gamma_n \end{aligned}$$

We are now ready to present the very important definition of *canonical models* that will ultimately lead to the desired completeness proof.

Definition 22 (Canonical model) Let Λ be a normal modal logic. The *canonical model* for Λ is $\mathcal{M}^\Lambda = (W^\Lambda, R^\Lambda, V^\Lambda)$, where:

1. W^Λ is the set of all Λ-MCSs.

2. R^Λ is the binary relation on W^Λ defined by $R^\Lambda(w, u)$ if for every formula φ, if $\varphi \in u$ then $\Diamond\varphi \in w$.

3. V^Λ is the valuation on W^Λ defined by

$$V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}.$$

There are some comments that have to be made regarding this definition.

Since W^Λ is the set of all Λ-MCSs, by the Lindenbaum's lemma, every Λ-consistent set is in fact a subset of a Λ-MCS and this will be the exact point in the model where such Λ-consistent set will be satisfied.

The valuation V^Λ definition tells us that every propositional letter will be true at some "point" (read Λ-MCS) if the same propositional letter is a member of the Λ-MCS. This membership quality will later be defined not only for propositional letters, but for every formula.

The binary relation announces that for every formula in an accessible state, there must be information in the present state that such accessibility exists. As will be announced next without proof, the contrary is also true. If there is some information in the present state that some formula is true in some other state, then that state must exist, along with the desired accessibility relation. That is called the existence lemma.

Lemma 23 (Existence lemma) If Λ is a normal modal logic and $w \in W^\Lambda$ some state where $\Diamond\varphi \in w$, then there exists a state v such that $R^\Lambda(w, v)$ and $\varphi \in v$.

This lemma's proof consists in finding a Λ-consistent set v^- that contains φ and mantains a coherent basis of relations between its formulas and the formulas of w. Then, through Lindenbaum's lemma, find the Λ-MCS, denominated v, that extends it.

Now we need to address the extension of the valuation of propositional letters to formulas, so that we can assure that every formula is satisfiable at some point in the canonical model if and only if it is included in that same point.

Lemma 24 (Truth lemma) If Λ is a normal modal logic and φ is any formula, then $\mathcal{M}^\Lambda, w \Vdash \varphi$ iff $\varphi \in w$.

Proof: This lemma is not going to be presented with its full proof, but the idea behind it being a simple induction on the degree of φ, with the basis (proposition letters) being proved by the valuation of the canonical models, and the induction step having to be proven for the negation, implication and the modalities.

We shall prove the modality part, for the others are quite trivial.

Assume that $\mathcal{M}^\Lambda, w \Vdash \Diamond\varphi$. Then:

$\mathcal{M}^\Lambda, w \Vdash \Diamond\varphi$	iff	($\Diamond$ definition)
$\exists v(\mathcal{R}^\Lambda(w,v)$ and $\mathcal{M}^\Lambda, v \Vdash \varphi)$	iff	(Induction hypothesis)
$\exists v(\mathcal{R}^\Lambda(w,v)$ and $\varphi \in v)$	iff	($\rightarrow$: $\mathcal{R}^\Lambda$ definition $\leftarrow$: By the existence lemma.
$\Diamond\varphi \in w$.		

QED

The final abstract theorem, before concluding the completeness proof for the **K**-system states the following:

Theorem 25 (Canonical model theorem) Every normal modal logic is strongly complete with respect to its canonical model.

Proof: Suppose Γ is a Λ-consistent set of formulas based on the normal modal logic Λ. Then, by the Lindenbaum's lemma, there exists Γ^+ extending Γ, where Γ^+ is in fact and Λ-MCS, so it is a point of the canonical model $\mathcal{M}^\Lambda$. Finally, since $\Gamma \in \Gamma^+$ and by the Truth lemma, we get:

$$\mathcal{M}^\Lambda, \Gamma^+ \Vdash \Gamma$$

Which is the same as saying that the canonical model satisfies Γ in the point (Λ-MCS) that extends Γ. QED

And now, instead of focusing on canonical models, we can extend this result to classes of frames, and through that we can give the main result for the **K**-system.

Theorem 26 (K-strong completeness) **K** is strongly complete with respect to the class of all frames.

Proof: Let Γ be a **K**-consistent set of formulas. We need to find a model based on a frame of that class (so any frame whatsoever) and a state in the model where Γ is satisfied. Well, if we take the model to be the canonical model $\mathcal{M}^{\mathbf{K}}$ and Γ^{+} to be a **K**-MCS extending Γ, then by the *canonical model theorem*, we have:

$$\mathcal{M}^{\mathbf{K}}, \Gamma^{+} \Vdash \Gamma$$

QED

To finalize this chapter on modal logic, we can note that our proof for the **K**-system was quite simple, for we didn't need to worry about which frame the model had to be based on, so the canonical model could be used without worries. If we wanted to show some other interesting results about other logics, for example, that the **K4**-system is strongly complete with regards to the class of transitive frames, then we would have to find a model that is based on a transitive frame. Here, though, comes into play the beauty of canonical models. In this case, and many more, the canonical model of the **K4**-system is itself a model based on a transitive frame, and so only by proving that fact, the proof of strong completeness would be reduced to our proof for the **K**-system.

In fact, many other normal modal logics have the property that their canonical models are included in the class of frames with respect to which we want to prove strong completeness. But these results lie outside the scope of this these.

2.4 Decidability

In the last two sections, we introduced mechanisms to interpret formulas in a semantic environment and to generate sets of formulas (called theories), from a specified set of axioms and rules of proof. These two mechanisms were then related, through soundness and completeness.

In this section, we introduce decidability of modal logics, mainly through the problems of satisfiability and validity. These problems are based in a much more computational environment then the environments of the last two sections. With decidability, we are more interested in whether or not it can be computed if a specific formula is satisfiable or valid in a class of models. For other discussion on this topic consult [3, 2].

Before probing further, we shall start by precisely defining the problems that we will tackle along this section.

Definition 27 (Satisfiability and validity problems) Let M and φ be a set of models and a formula over the same modal language.

The *satisfiability* problem corresponds to determining whether there exists a model $\mathcal{M} \in M$ such that φ is satisfiable in $\mathcal{M}$ ($\mathcal{M}, w \Vdash \varphi$, for some world w in $\mathcal{M}$).

Conversely, the *validity problem* corresponds to determining whether φ is valid in every model $\mathcal{M} \in M$ ($M \Vdash \varphi$).

There are some remarks that have to be made regarding these definitions.

Firstly, we can clearly see that the satisfiability and the validity problems are duals of each other. Because a formula φ is satisfiable in a set of models iff $\neg\varphi$ is not valid in those sets of models, and φ is valid iff $\neg\varphi$ is not satisfiable in that same set of models, we can state the following proposition:

Proposition 28 The satisfiability (or validity) problem for a formula φ and a class of models M has a positive answer iff the validity (or satisfiability) problem for $\neg\varphi$ has a negative answer.

Given this proposition, we will from now on say that a modal logic Λ is decidable for some class of models M if the Λ-satisfiability problem is decidable for every formula based on the same language as Λ and the class M.

Because of this duality, we are moving forward thinking mainly about the satisfiability problem, that is, about trying to find a model that satisfies some formula, or prove that such a model doesn't exist.

These problems obviously have a very important computational aspect, for we are trying to create an efficient procedure or algorithm capable of determining whether φ is in fact satisfiable. Despite the importance of the computational side of the problem, we will treat it mostly theoretically, we are not interested in how such computation of models is performed. Despite not going into any detail, it is worth to note that to create such procedures and algorithms, one usually resorts to Turing machines, mainly because of its simplicity and expressiveness.

Given that we want to decide whether such problems can be computed, it is also of great importance to give bounds on how much resources (of time and space) are needed to answer the problem. But, as is the case for the technical definition of computation, we are not interested in this side of the decidability discussion, both these aspects fall outside of the scope of this paper.

As a final remark, note that our discussion so far has been mainly semantic, in terms of satisfiability and validity. But, going back to the section on the axiomatization of modal logics, we can see that, for example, for the most basic normal modal logic, **K**, its formulas correspond to the set of valid formulas in the set of all frames (and so of all models). It is clear then that φ, based on the *basic modal language*, is valid in every frame iff $\mathbf{K} \vdash \varphi$, and the **K**-validity problem equals the **K**-provability. Because of the duality between validity and satisfiability, and between provability and consistency, it should also be clear that a formula φ is satisfiable in some model iff it is **K**-consistent.

In fact, this proposition holds for all sets of formulas and classes of frames such that those sets of formulas are sound and complete with respect to the class of frames. Since all logic systems here introduced are in fact sound and complete, then our decidability discussion (through satisfiability), can easily be translated to a discussion about consistency and provability.

Because every logic system has very specific properties, there isn't one general method to prove whether such a logic system is decidable or not with regard to a class of models. Here are some of the most common techniques for proving decidability:

- **Decidability Proof via the Finite Model Property**

  This is the proof method most commonly associated with logic decidability, and the main method presented in literature ([3, 17]).

In this proof method, we try to prove that some logic has a strong variant of the finite model property, which, grossly stated, says that every theorem is satisfied in a finite model. Every logic that has that strong variant of the finite model property is in fact decidable. With this method we are able to prove that logics such as **K**, **K4**, **T** (**K** with reflexive axiom) are decidable. This is the most common method, but there are some limitations, for sometimes the strong finite model property is difficult to prove, or is not even true.

- **Decidability via Interpretations**

  This method is based on reducing the satisfiability problem of some logic to another problem, which is known to be decidable. The problem to which we want to reduce has to be chosen carefully, for it should share some properties and ideas with the initial logic, to make the reduction as smooth as possible. For modal logics, such a decidable problem is, for example, the SnS, a set of theories on which we can easily interpret modal logics (hence the name, interpretations), even if they don't have the finite model property. One example of a logic without the finite model property that can be shown to be decidable using interpretations is the logic **KvB**. A thorough definition of this method, of the SnS theories or of the logic **KvB** are not included in this book. For more information consult [2].

- **Decidability via Quasi-models**

  When a logic doesn't have the finite model property, we can still construct some finite structures that resemble actual models, called quasi-models, through which we equal membership in a quasi-model to satisfaction in a uniquely constructed model of the logic.

  This is the method we will use to prove that the logic $\mathcal{CSL}$, which will be introduced in the next chapter is decidable.

The properties of the logic influence profoundly the properties of our quasi-model, and so influence as well the path our decidability proof takes. Because of that, we will next present a detailed scheme of the decidability proof steps for this method, without particularizing the proof for any specific logic.

- **Decidability via mosaics**

  This method consists of a generalization of the quasi-model method, where the mosaics, instead of being very similar to the model like the quasi-model, are simply finite structures that contain the instructions needed to create the model that satisfies certain formula. In fact, a mosaic can be seen as a finite sequence of domains of quasi-models, where in each step we get closer to the model that we want to create. This method is used for proving, for example, decidability for the $\mathbf{K}_t\mathbf{N}$ logic, the tense logic of the naturals.[2]

As was stated before, we will concentrate our discussion in the quasi-model method. Here is a general proof scheme for a decidability proof using quasi-models.

The proof of decidability via the usage of quasi-models is a useful method when the finite model property is either untrue or difficult to prove.

The main points of each decidability proof via quasi-models are very similar, with minor differences in the construction of the structures, which leads to some more complex differences in the development of the actual decidability theorem.

For proving that some term φ is satisfiable in some class of models, we start by taking a finite closure of the term φ and its subformulas. Then, from that closure, we try to extract another set (an *Hintikka set*) that satisfies some usual model-satisfaction-like properties, and finally construct a finite model-like structure (the *quasi-model*), where its points are basically the so-called Hintikka sets. This quasi-model construction varies markedly accordingly to the logic being used. With this construction we will be ready to prove that such a quasi-model exists for the term φ iff there exists a model in that class of models that satisfies φ.

At last the decidability proof would be ready to be proved. Given that our goal is to construct a real model that satisfies the formula the quasi-model is based on, the diversity of quasi-model definitions create different arguments for the construction of such models. Examples for possible proof techniques include:

- The creation of a homomorphism between the quasi-model and a model of the logic, where the valuation function for the formula would be carefully constructed.

- The creation of trees on which each node shall be related to an Hintikka set, where the creation of edges between the different nodes would also have to be carefully planned.[1]

Once our model is created, by a simple induction argument on the complexity of φ, we prove that, for every formula of the closed set, satisfaction in that model equals membership in the quasi-model.

Now that we have given a rather informal approach to a decidability proof via quasi-models, we shall define precisely and formally the full general argument of the proof. To give a more complete grasp on the structures defined for the proof, we will present them, when particularization is necessary, for formulas in the *basic modal language*. As was stated before, the notion of a closed set and the closure of a set is where our proof starts taking shape.

Definition 29 (Closed sets) Let Γ be a set of formulas in the basic modal language. We say that Γ is *closed* if it is closed for subformulas, single negation and disjunction. That is:

- If $\varphi \in \Gamma$ and ψ is a subformula of φ, then $\psi \in \Gamma$.

[1]This technique will be used for proving decidability for the $\mathcal{CSL}$ logic, introduced in the next chapter.

- If $\varphi \in \Gamma$ and φ is not of the form $\neg\psi$ for some formula ψ, then $\neg\varphi \in \Gamma$.
- If $\varphi, \psi \in \Gamma$, then $\varphi \vee \psi \in \Gamma$.

Definition 30 (Closure of a set) Let Γ be a set of formulas. The *closure* of the set Γ, $Cl(\Gamma)$, is the smallest closed set of formulas containing Γ. That is, $Cl(\Gamma)$ is a closed set, $\Gamma \subseteq Cl(\Gamma)$, and if $\Phi \supseteq \Gamma$ is closed, then $Cl(\Gamma) \subseteq \Phi$.

The closure of a set is in itself not a very interesting structure, since for every formula it contains, there is another formula that denies its statement. But from it we can construct some maximal subsets that have some important properties. We call such subsets Hintikka sets.

Definition 31 (Hintikka sets) Let Γ be a closed set of formulas in the basic modal language and H a subset of Γ. H is said to be a *Hintikka set* if H is maximal and the following conditions hold:

- $\bot \notin H$.
- $\neg\varphi \in H$ iff $\varphi \notin H$.
- $\varphi \vee \psi \in H$ iff $\varphi \in H$ or $\psi \in H$.

If a set H satisfies the 3 conditions above, we say that H is a *boolean closed set.*

Suppose that $H \subset \Gamma$ is a Hintikka set. Since H is a maximal subset of Γ that satisfies the conditions given, then every set J such that $H \subsetneq J \subseteq \Gamma$ is not a Hintikka set.

Even more, given our definitions of $\wedge$, $\rightarrow$, $\top$, we also have, in H:

- $\top \in H$.
- $\varphi \rightarrow \psi \in H$ iff $\varphi \notin H$ or $\psi \in H$, for $\varphi \rightarrow \psi \in \Gamma$.
- $\varphi \wedge \psi \in H$ iff $\varphi \in H$ and $\psi \in H$, for $\varphi \wedge \psi \in \Gamma$.

These Hintikka sets can be seen as the points of our quasi-model. Because of its construction, we are preventing the set to have clear contradictions when it comes to model satisfaction. In that sense, if a formula belongs to the Hintikka set (or, in other words, is satisfied in some point of the quasi-model), then every boolean-related formula will also be satisfied or not accordingly.

Since modalities are used not to talk about a single point in a model, but to relate various points in the model, they are not a part of the creation of the Hintikka sets (for, again, they represent a single point in our quasi-model). Our conditions on modalities will be made in our next definition, that of the quasi-model. More precisely, these conditions will be imposed on the accessibility relation of the quasi-model.

Definition 32 (Quasi-model) Let φ be some formula over a language. A *quasi-model* for φ is a structure $\mathcal{Q}$=($\mathcal{F}$, λ) where:

- $\mathcal{F} = (W, R)$ is a finite frame.
- $\lambda : W \to \wp Cl(\{\varphi\})$, called a labeling, is a function mapping states of $\mathcal{F}$ to subsets of $Cl(\{\varphi\})$ that satisfies the following conditions:

    1. $\varphi \in \lambda(w)$ for some $w \in W$,
    2. $\lambda(w)$ is a Hintikka set, for each $w \in W$.

    As was stated before, conditions for the preservation of the modalities properties must also be imposed. As an example, for the case of a set of formulas over the *basic modal language*, we must also have:

    3. $\Diamond\varphi \in \lambda(w)$ iff $\varphi \in \lambda(v)$ for some v with $R(w, v)$, for all $\Diamond\varphi \in Cl(\{\varphi\})$.
    4. $\varphi \in \lambda(v)$ for some v with $R(w, v)$ but not $R(v, w)$, for all $\Diamond\varphi \in \lambda(w)$.

These are all the logic structures needed for the final proof of decidability. We are going to present the basic lemma that will lead us to the decidability via quasi-models. Since the creation of the real model varies significantly from logic to logic and quasi-model to quasi-model, we are not going to present any general proof of that argument, for such does not exist.

Lemma 33 Let φ be a formula over a language. Then φ is satisfiable in a class of models over that language iff there is a quasi-model for φ.

Proof sketch: The left to right direction is normally very simple, for from satisfiability of φ it is usually easy to construct a quasi-model for φ.

The right to left direction involves a higher degree of complexity. The techniques for creating the model from the quasi-model, although not very complicated, usually incorporate some steps that demonstrate a smart, careful and interesting analysis on the properties of the class of models and the logic associated with the formula.

Suppose that a model $\mathcal{M} = (W_{\mathcal{M}}, R_{\mathcal{M}}, V_{\mathcal{M}})$ with the necessary properties is created. The proof that, for all $\psi \in Cl_{\varphi}$, we have $\mathcal{M}, w_{\mathcal{M}} \Vdash \psi$ iff $\psi \in \lambda(w)$, where w and $w_{\mathcal{M}}$ are somehow related is made through induction on the complexity of ψ.

The induction basis is proved according to the construction and properties of the valuation function $V_{\mathcal{M}}$, while the boolean operators step is the same in every proof for it deals only with the definition of Hintikka sets and the λ function conditions on boolean connectives. In fact, here is the proof for the boolean connective $\neg$:

$\mathcal{M}, w_{\mathcal{M}} \Vdash \neg\psi$	($\neg$ satisfaction)
$\mathcal{M}, w_{\mathcal{M}} \nVdash \psi$	(Induction hypothesis)
$\psi \notin \lambda(w)$	(Because $\lambda(w)$ is a Hintikka set)
$\neg\psi \in \lambda(w)$	

All other boolean connectives would have a trivially similar proof.

On the other hand, the proof for the modalities step is always the more complex one, for it involves every structure defined so far in the decidability subsection, and so we can not generalize it for every proof. The induction proof ends the proof of our lemma. □

Before stating and proving at last the decidability theorem, we shall make a small remark. Since we are not interested in the computational complexity associated with finding such quasi-models and models, we didn't mention that the quasi-model for φ has in fact some size constraints. That fact will be a part of our final proof, the decidability proof.

Suppose that $\mathcal{L}$ is a logic on which we proved the previous lemma, for the class of models M.

Theorem 34 $\mathcal{L}$ is decidable on M.

Proof: Let φ be a formula based on the same language as $\mathcal{L}$.

Then, by the previous lemma, φ is satisfiable iff there exists a quasi-model for φ, where the quasi-model has some size constraints. Because the number of different quasi-models is finite, then the size constraints can be defined a priori. We can, therefore, create every possible quasi-model candidate up to that specified size constraint, and check, for each candidate, if it is a quasi-model for φ. In case we find such a quasi-model, we terminate the program and claim that φ is satisfiable on M. If we never find such a quasi-model (which would only take a finite pre-determined amount of time and space to compute), we claim that φ is not satisfiable on M. QED

The computational techniques for encoding and computing quasi-models, as was said before, fall outside of the scope of this book.

And so, without the assertion of the finite model property, we realize that there is still the possibility of creating finite structures that encode some properties of the model, through which we can construct the necessary models that satisfy formulas in our logic of interest.

Now that we have presented an as thorough as needed introduction to modal logic, we will particularize every general discussion about modal logics to a logic, called *Comparative Similarity Logic* ($\mathcal{CSL}$), that will be used for reasoning qualititatively about several types of distance spaces.

Chapter 3

Spatial modal logic

The basic idea behind this chapter will be to introduce a logic which can be used to reason about comparative distances using distance spaces.[16, 15]

We start by introducing the notion of distance space, after which the introduction of the logic will be made, syntactically and semantically, with a final proof of decidability of the logic according to a specific class of models.

3.1 Distances spaces

The logic we will introduce in this chapter is going to be based upon the notion of distance spaces, so a small introduction to distance spaces and their properties is needed.

Definition 1 (Distance spaces) A *distance space* is a tuple (Δ, d), where Δ is a non empty set, and $d : \Delta \times \Delta \to \mathbb{R}_0^+$ is a map such that:

- $d(u, v) \geq 0$
- $d(u, v) = 0$ iff $u = v$.

The distance of a point to a set is defined as follows. Given a point $u \in \Delta$ and a set $X \subseteq \Delta$, the distance $d(u, X)$ from u to X is defined by

$$d(u, X) = \begin{cases} \inf_{v \in X} \ d(u, v), & if \ X \neq \emptyset \\ \infty, & if \ X = \emptyset \end{cases}$$

As was said before, our discussion is going to be focused mainly on qualitative terms. So, given that topology abstracts away from the quantitative aspects of geometry, we shall introduce two topological concepts that will service that exact purpose, the closer and the realized operator.

The closer operator ($\leftleftharpoons$) is a binary operator, that given an ordered pair of sets, returns the points in our space which are closer to the first set, in relation to the second. So, given $X, Y \subseteq \Delta$, we have

$$X \leftleftharpoons Y = \{u \in \Delta \mid d(u, X) < d(u, Y)\}.$$

The realized operator (Ⓡ) is a unary operator, that given a set, returns the points in our space where the distance between the set and those points are realized. We say a distance from u to X is realized if there is a point $x \in X$ such that $d(u, X) = d(u, x)$. The definition comes now easily. Given a set $X \subset \Delta$, we have

$$Ⓡ X = \{u \in \Delta \mid \exists x \in X \ d(u, X) = d(u, x)\}.$$

As was our goal, by introducing the closer and the realized operator, we are suddenly not interested in the numerical quantification of the distance between points and sets in our space, but rather in which points of our space some sets are realized or closer to other sets, disregarding the actual numerical distances. Hence, the discussion is now qualitative instead of quantitative.

Now we proceed to the definition of the syntax and semantics of the *comparative similarity logic*.

3.2 Comparative similarity logic - $\mathcal{CSL}$

As was the case with the *basic modal logic*, to introduce the $\mathcal{CSL}$ we will first present its syntax, followed by the semantics of the logic, using the previously defined Kripke semantics.

The signature of the $\mathcal{CSL}$'s language is a countably infinite set $P = \{p_1, p_2, \ldots\}$, called *atomic terms* or *spatial variables.*

The formal grammar of the $\mathcal{CSL}$'s language has two added modalities when compared to the grammar of the *propositional logic*, the realized and the closer operator.

Definition 2 ($\mathcal{CSL}$ formulas) The *formulas of the $\mathcal{CSL}$* are constructed using the signature and modalities introduced above according to the following formation rules:

$$\tau ::= p_i \mid \bot \mid \neg\tau \mid \tau_1 \wedge \tau_2 \mid ⓡ\tau \mid \tau_1 \leftleftarrows \tau_2$$

The usual abbreviations for $\top, \vee$ and $\rightarrow$ are used. Note that in this logic, there is no converse for the two modalities operators.

The semantic interpretation of the $\mathcal{CSL}$ is made with resort to distance spaces. In fact, a frame $\mathcal{F}$ for the $\mathcal{CSL}$ is nothing more then a distance space (Δ, d). Let's first of all note that this frame is a little more complex than the one used for the *basic modal language.* The relation between the worlds in Δ is now not simply binary, there is an actual positive real number assigned to each relation, that as we know, reveals the distance between each point.

A model for the $\mathcal{CSL}$ is simply the pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F} = (\Delta, d)$ is a frame as defined in the last paragraph, and $V : P \rightarrow \wp\Delta$ is a map.

A model for the $\mathcal{CSL}$ is also called a distance model.

Let τ be a $\mathcal{CSL}$ formula. We define $\tau^{\mathcal{M}}$ to be the interpretation of τ according to the model $\mathcal{M}$. Such definition is made according to the following inductive process:

$$\begin{aligned}
p_i^{\mathcal{M}} &= V(p_i),\\
\bot^{\mathcal{M}} &= \emptyset,\\
(\neg\tau)^{\mathcal{M}} &= \Delta \setminus \tau^{\mathcal{M}},\\
(\tau_1 \wedge \tau_2)^{\mathcal{M}} &= \tau_1^{\mathcal{M}} \cap \tau_2^{\mathcal{M}},\\
(ⓡ\tau)^{\mathcal{M}} &= \{u \in \Delta \mid \exists x \in \tau^{M}\ d(u, \tau^{M}) = d(u, x)\},\\
(\tau_1 \leftleftarrows \tau_2)^{\mathcal{M}} &= \{u \in \Delta \mid d(u, \tau_1^{\mathcal{M}}) < d(u, \tau_2^{\mathcal{M}})\}.
\end{aligned}$$

Given this definition, the satisfaction in a distance model is easily defined as well.

Definition 3 (Satisfaction in a distance model) Let $\mathcal{M}$ be a distance model, $u \in \Delta$ be a point and τ a $\mathcal{CSL}$ formula. The formula τ is *satisfied* in the distance model $\mathcal{M}$ in the point u (designated as we know by $\mathcal{M}, u \Vdash \tau$) if and only if $u \in \tau^{\mathcal{M}}$.

Obvious extensions of this definitions include: τ is called *satisfiable* if there is a distance model $\mathcal{M}$ such that $\tau^{\mathcal{M}} \neq \emptyset$. It is called *satisfiable in a model* $\mathcal{M}$ if $\tau^{\mathcal{M}} \neq \emptyset$. If $\tau^{\mathcal{M}} = \Delta$, then τ is *globally satisfied*. It is called *valid* if, for every distance model $\mathcal{M}$, we have $\tau^{\mathcal{M}} = \Delta$.

All others extensions that were presented in the last chapter, related with satisfaction, validity and semantic entailment are easily translated to this logic.

Also, two formulas τ_1 and τ_2 are *equivalent*, written $\tau_1 \equiv \tau_2$ if $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$ for every distance model $\mathcal{M}$.

Now that the syntax and semantics of the $\mathcal{CSL}$ were presented, we are going to illustrate the type of reasoning being made within this logic with help from two examples.

The first being the illustration of the satisfiability of a $\mathcal{CSL}$ formula and how models can be constructed to satisfy specific formulas.

Example 4 (Satisfiability of a $\mathcal{CSL}$ formula) Our goal in this example is to prove the satisfiability of the following formula:

$$(\neg(\top \leftleftarrows p_1)) \leftleftarrows p_1.$$

Proof: Intuitively, this formula represents the property of the closure of a set being closer than the set itself.

With that in mind, before trying to construct the satisfying model, we will first interpret the formula in a general model, specifically the first part of the formula. Let $\mathcal{M}$ be any model and u a point in that model. By the definition of $\mathcal{CSL}$ formula satisfaction,

$$(\neg(\top \leftleftarrows p_1))^{\mathcal{M}} = \{u \in \Delta : d(u, p_1^{\mathcal{M}}) \leq d(u, \Delta)\}.$$

Since $d(u, \Delta) = 0$, $(\neg(\top \leftleftarrows p_1))^{\mathcal{M}}$ returns the points that are exactly at distance 0 from $p_1^{\mathcal{M}}$, that is, it returns the topological closure of $p_1^{\mathcal{M}}$.

So our goal is to find a model such that the distance between one specific point and the closure of a set is less than the distance between that point and the actual set.

Commonly, models are created with resort to the real numbers and an associated metric. So, let $\mathcal{F}_{\mathbb{R}^2} = (\mathbb{R}^2, d_e)$ be a frame, where d_e is the usual euclidean metric, and let $\mathcal{M}_{\mathbb{R}^2} = (\mathcal{F}_{\mathbb{R}^2}, V)$ be any model based on $\mathcal{F}_{\mathbb{R}^2}$.

In this case, though, such a model will never satisfy the intended formula, as we are going to show.

The first remark to be made is that the euclidean metric satisfies the triangle inequality, which states that $\forall x, y, z \in \mathbb{R}^2$, $d(x, z) \leq d(x, y) + d(y, z)$.

Now, consider points $x \in \mathbb{R}^2$ and $y \in (\neg(\top \leftleftarrows p_1))^{\mathcal{M}_{\mathbb{R}^2}}$. Using the triangle inequality, and the fact that $\forall u \in \mathbb{R}^2$,

$$u \in (\neg(\top \leftleftarrows p_1))^{\mathcal{M}_{\mathbb{R}^2}} \quad \text{iff} \quad d(u, p_1^{\mathcal{M}_{\mathbb{R}^2}}) = 0,$$

we get

$$d(x, p_1^{\mathcal{M}_{\mathbb{R}^2}}) \leq d(x, y) + d(y, p_1^{\mathcal{M}_{\mathbb{R}^2}}) = d(x, y).$$

Finally, since y is an arbitrary point of $(\neg(\top \leftleftarrows p_1))^{\mathcal{M}_{\mathbb{R}^2}}$, we have that $d(x, p_1^{\mathcal{M}_{\mathbb{R}^2}}) \leq d(x, (\neg(\top \leftleftarrows p_1))^{\mathcal{M}_{\mathbb{R}^2}})$, and so, as we wanted to show, for any point $x \in \mathbb{R}^2$,

$$\mathcal{M}_{\mathbb{R}^2}, x \nVdash (\neg(\top \leftleftarrows p_1)) \leftleftarrows p_1.$$

To finalize our example, we are going to construct a model that indeed satisfies the formula. The main point will be that this model will not abide by the triangle inequality.

So, let $\mathcal{M} = (\Delta, d, V)$ be a model with symmetry ($\forall x, y \in \Delta,\ d(x, y) = d(y, x)$), with the following characteristics:

$$\Delta = \{u, v, x_i | i \in \mathbb{N}\}, \quad V(p_1) = \{x_i | i \in \mathbb{N}\}$$
$$d(u, v) = 1, \qquad d(u, x_i) = 2,\ \forall i \in \mathbb{N}, \quad d(v, x_i) = 2^{-i},\ \forall i \in \mathbb{N}$$

Given this attributes, we clearly have $v \in (\neg(\top \leftleftarrows p_1))^{\mathcal{M}}$, and $1 = d(u,v) < d(u, p_1^{\mathcal{M}}) = 2$, and so $d(u, (\neg(\top \leftleftarrows p_1))^{\mathcal{M}}) < d(u, p_1^{\mathcal{M}})$ as we wanted, which allow us to conclude

$$\mathcal{M}, u \Vdash (\neg(\top \leftleftarrows p_1)) \leftleftarrows p_1.$$

QED

In this next example we relate two $\mathcal{CSL}$ formulas by the notion of entailment.

Example 5 (Entailment between $\mathcal{CSL}$ formulas) We are going to prove the following proposition:

$$[(p_1 \vee p_2) \leftleftarrows p_3] \wedge [\neg(p_2 \leftleftarrows p_3)] \models^l (p_1 \leftleftarrows p_3).$$

Proof: According to the definition of local entailment ($\models^l$), such an assertion is true iff for each model $\mathcal{M} = (\Delta, d, V)$ and each point $u \in \Delta$, we have $\mathcal{M}, u \Vdash [(p_1 \vee p_2) \leftleftarrows p_3] \wedge [\neg(p_2 \leftleftarrows p_3)]$ implies $\mathcal{M}, u \Vdash (p_1 \leftleftarrows p_3)$.

So we start by supposing that for some model and point as describe, the first condition is true. Then,

$$\mathcal{M}, u \Vdash [(p_1 \vee p_2) \leftleftarrows p_3] \wedge [\neg(p_2 \leftleftarrows p_3)] \quad \text{iff} \quad (\wedge - \text{satisfaction})$$

$$\begin{cases} \mathcal{M}, u \Vdash [(p_1 \vee p_2) \leftleftarrows p_3] & \text{iff} \quad (\leftleftarrows \text{ satisfaction}) \\ \mathcal{M}, u \Vdash [\neg(p_2 \leftleftarrows p_3)] & \text{iff} \quad (\neg \text{ satisfaction}) \end{cases}$$

$$\begin{cases} d(u, (p_1 \vee p_2)^{\mathcal{M}}) \; < \; d(u, p_3^{\mathcal{M}}) & \text{iff} \quad (\tau^{\mathcal{M}} \text{ definition}) \\ \text{not } \mathcal{M}, u \Vdash (p_2 \leftleftarrows p_3) & \text{iff} \quad (\leftleftarrows \text{ satisfaction}) \end{cases}$$

$$\begin{cases} d(u, p_1^{\mathcal{M}} \cup p_2^{\mathcal{M}}) \; < \; d(u, p_3^{\mathcal{M}}) \\ d(u, p_2^{\mathcal{M}}) \; \geq \; d(u, p_3^{\mathcal{M}}). \end{cases}$$

Now, from the definition of distance, we have that

$$inf_{v \in p_1^{\mathcal{M}} \cup p_2^{\mathcal{M}}} \; d(u,v) < d(u, p_3^{\mathcal{M}}) \text{ and } inf_{v \in p_2^{\mathcal{M}}} \; d(u,v) \geq d(u, p_3^{\mathcal{M}}).$$

So, we can infer that the point belonging to $p_1^{\mathcal{M}} \cup p_2^{\mathcal{M}}$ which minimizes the distance between u and $p_1^{\mathcal{M}} \cup p_2^{\mathcal{M}}$ is in fact a part of $p_1^{\mathcal{M}}$. So, $d(u, p_1^{\mathcal{M}}) = d(u, p_1^{\mathcal{M}} \cup p_2^{\mathcal{M}}) < d(u, p_3^{\mathcal{M}})$.
Hence, $d(u, p_1^{\mathcal{M}}) < d(u, p_3^{\mathcal{M}})$, and, from the definition of $\leftleftarrows$-satisfaction, we finally get, as we wanted,

$$\mathcal{M}, u \Vdash (p_1 \leftleftarrows p_3).$$

QED

After the illustration of the possibilities of the $\mathcal{CSL}$, we can start to prove important results for this logic, more specifically, in the next section we are going to present a proof of the decidability of the $\mathcal{CSL}$.

After the illustration of the possibilities of the $\mathcal{CSL}$, we can start to prove important results for this logic. Despite lying outside the scope of this thesis, it is noteworthy to state that $\mathcal{CSL}$ is in fact axiomatizable, sound and complete with respect to several classes of distance models (including symmetric, symmetric with triangle inequality and metric) as is proven in [15]. Continuing our study of the $\mathcal{CSL}$, in the next section we are going to present a proof of its decidability.

3.3 Decidability of $\mathcal{CSL}$

Extending the concepts presented in the decidability section of the last chapter, we are going to present a proof that the $\mathcal{CSL}$ is decidable on the class of symmetric distance models.

Since this logic doesn't have the finite model property, for clearly distance spaces can not be resumed to a finite space, this proof will be made via quasi-models, by particularizing the steps introduced in our decidability discussion to the $\mathcal{CSL}$.

So our plan is, given a formula φ from the $\mathcal{CSL}$, to define the structures necessary to prove if such a formula is satisfiable in any model on the class of symmetric distance models. Those structures include Hintikka sets and quasi-models, as were discussed in the last chapter. Afterwards, we will create the bridge between the existence of quasi-models and the satisfaction in a $\mathcal{CSL}$-model.

Although it is not shown in this thesis, the decidability problem for $\mathcal{CSL}$ on the class of symmetric distance models has been proven to be ExpTime-complete

3.3.1 Quasi-model and substructures

The first structure to be introduced is, as was the case in **Section 2.4** on page 30, the closure of an appropriate set, created from our formula φ.

Before the definition of the closure of φ, we need some additional notation. We denote by **sub** φ to be the set of subformulas of φ and **com** φ to be the set of comparisons made in φ, that is

$$\mathbf{com}\ \varphi = \{\psi_1, \psi_2 \mid \psi_1 \leftleftarrows \psi_2 \in \mathbf{sub}\varphi\} \cup \{\psi \mid ⓡ\psi \in \mathbf{sub}\varphi\} \cup \{\bot, \top\}.$$

Definition 6 (Closure of φ) The *closure* of φ, denoted by **cl** φ, is the closure of the set

$$\mathbf{sub}\ \varphi \cup \{\psi_1 \leftleftarrows \psi_2 \mid \psi_1, \psi_2 \in \mathbf{com}\ \varphi\} \cup \{ⓡ\psi \mid \psi \in \mathbf{com}\ \varphi\}.$$

The extraction of the Hintikka sets from the closure of the formula φ is a somewhat more complicated process than the Hintikka sets created for the *basic modal logic*. Before defining syntactically our Hintikka sets, we are going to provide some semantic overview into the reasons that lead to their way of construction.

Through this semantic overview, we are going to try to make clear the type of syntactic constraints we will have to introduce to our syntactic definition of Hintikka sets, in order to mirror some necessary semantic properties, such as the transitivity of $\leftleftarrows$ or the non-realization of $\bot$.

We know, from **Definition 31** in **Chapter 2**, that the construction of an Hintikka set prevents the set to have clear contradictions when it comes to model satisfaction. With that in mind, intuitively, an Hintikka set for φ will be a subset of **cl** φ which is satisfiable in some points in the distance space. Each Hintikka set is then directly connected with the distance space.

So, consider $\mathcal{M} = (\Delta, d, V)$ to be a distance model. An Hintikka set for φ in the model $\mathcal{M}$ is a set of formulas in $\mathbf{cl}\ \varphi$ that is satisfied in some subset of Δ. Because an Hintikka set will be uniquely defined by the atoms, comparisons and realizations (and related negations) it contains, since $\mathbf{at}\ \varphi$ and $\mathbf{com}\ \varphi$ are finite, then the number of Hintikka sets is also finite. So, if the distance space is infinite, there will have to be at least one Hintikka set related to multiple, infinite in fact, points.

So, given a point $u \in \Delta$, the function

$$h^{\mathcal{M}} : \Delta \to \wp\mathbf{cl}\ \varphi$$

returns a semantic definition of what an Hintikka set is, where:

$$h^{\mathcal{M}}(u) = \{\psi \in \mathbf{cl}\ \varphi \mid u \in \psi^{\mathcal{M}}\}$$

This definition satisfies the conditions proposed in **Definition 31** for the *basic modal language*, for clearly

- $\bot \notin h^{\mathcal{M}}(u)$
- $\neg\psi \in h^{\mathcal{M}}(u)$ iff $\psi \notin h^{\mathcal{M}}(u)$, since $\neg\psi \in \mathbf{cl}\ \varphi$
- $\psi_1 \wedge \psi_2 \in h^{\mathcal{M}}(u)$ iff $\psi_1, \psi_2 \in h^{\mathcal{M}}(u)$, since $\psi_1 \wedge \psi_2 \in \mathbf{cl}\ \varphi$.

Another important remark concerning this definition is that, according to the formulas present in the Hintikka set $h^{\mathcal{M}}(u)$, we have the information about which of the formulas in $\mathbf{com}\ \varphi$ are closer to u in our interpretation and whether the distances between those subformulas and u are realized.

This information will be given by our creation of the binary relations $\leq_{h^{\mathcal{M}}(u)}$ and $<_{h^{\mathcal{M}}(u)}$, and the subset $\varrho_{h^{\mathcal{M}}(u)}$ of $\mathbf{com}\ \varphi$, which will be defined by,

$$\psi_1 \leq_{h^{\mathcal{M}}(u)} \psi_2 \ \mathbf{iff}\ d(u, \psi_1^{\mathcal{M}}) \leq d(u, \psi_2^{\mathcal{M}}) \ \mathbf{iff}\ \neg(\psi_2 \leftleftarrows \psi_1) \in h^{\mathcal{M}}(u),$$

$$\psi_1 <_{h^{\mathcal{M}}(u)} \psi_2 \ \mathbf{iff}\ d(u, \psi_1^{\mathcal{M}}) < d(u, \psi_2^{\mathcal{M}}) \ \mathbf{iff}\ (\psi_1 \leftleftarrows \psi_2) \in h^{\mathcal{M}}(u),$$

$$\psi \in \varrho_{h^{\mathcal{M}}(u)} \ \mathbf{iff}\ \exists v \in \psi^{\mathcal{M}} d(u, v) = d(u, \psi^{\mathcal{M}}) \ \mathbf{iff}\ \textcircled{r}\psi \in h^{\mathcal{M}}(u).$$

We will also say that $\psi \in \mathbf{com}\ \varphi$ is a $\leq_{h^{\mathcal{M}}(u)}$-minimal element (or $\leq_{h^{\mathcal{M}}(u)}$-maximal element) if $d(u, \psi^{\mathcal{M}}) = 0$ (or $d(u, \psi^{\mathcal{M}}) = \infty$).

One final semantic observation concerning this last definition needs to be made, before pursuing a more syntactical approach.

Remark 7

- ψ is a $\leq_{h^{\mathcal{M}}(u)}$-minimal element only if u is in the closure of the set $\psi^{\mathcal{M}}$. In that case, $\psi \in \varrho_{h^{\mathcal{M}}(u)}$ iff $u \in \psi^{\mathcal{M}}$ iff $\psi \in h^{\mathcal{M}}(u)$;
- ψ is a $\leq_{h^{\mathcal{M}}(u)}$-maximal element only if $\psi^{\mathcal{M}} = \emptyset$, and so $\psi \notin \varrho_{h^{\mathcal{M}}(u)}$.

So now that we understand the type of conclusions we can draw, semantically, from a boolean closed set based on some distance model, we can start our syntactical discussion.

Such discussion will obviously start with a boolean closed subset $h \subset \mathbf{cl}\ \varphi$. Given this subset, we are going to present the syntactic analogue to the relations $\leq_{h^{\mathcal{M}}(u)}, <_{h^{\mathcal{M}}(u)}, \varrho_{h^{\mathcal{M}}(u)}$. So, for all $\psi_1, \psi_2 \in \mathbf{com}\ \varphi$, we define the binary relations $\leq_h, <_h$ and the subset ϱ_h as follows:

$$\psi_1 \leq_h \psi_2 \ \textbf{iff}\ \neg(\psi_2 \leftleftarrows \psi_1) \in h, \quad \psi_1 <_h \psi_2 \ \textbf{iff}\ (\psi_1 \leftleftarrows \psi_2) \in h.$$

$$\psi_1 \in \varrho_h \ \textbf{iff}\ \textcircled{r}\psi_1 \in h.$$

The definition of $\leq_h$-minimal element and $\leq_h$-maximal element must also be reintroduced, syntactically.

We say that $\psi \in \mathbf{com}\ \varphi$ is a $\leq_h$-minimal element iff $\neg(\top \leftleftarrows \psi) \in h$. Also, ψ is a $\leq_h$-maximal element iff $\neg(\psi \leftleftarrows \bot) \in h$.

Note that these definitions, when applied to any distance model, are exactly the same as the ones made for the semantic approach.

We also call min h and max h to the sets of $\leq_h$-minimal and $\leq_h$-maximal elements, respectively. Also $\psi_1 \simeq_h \psi_2$ if $\psi_1 \leq_h \psi_2$ and $\psi_2 \leq_h \psi_1$.

Finally we have all the tools to present the definition of an Hintikka set for φ.

Definition 8 (Hintikka set for $\mathcal{CSL}$) Let φ be a $\mathcal{CSL}$-formula. A boolean closed subset h of $\mathbf{cl}\ \varphi$ is an *Hintikka set* if it is maximal and the following conditions are satisfied:

- $\leq_h$ is a total quasi-order (Reflexive, transitive and total binary relation) on $\mathbf{com}\ \varphi$,
- $h \cap \mathbf{com}\ \varphi$ is the set of the $\leq_h$-minimal elements that belong to ϱ_h,
- $\bot$ is a $\leq_h$-maximal element, and no $\leq_h$-maximal element belongs to ϱ_h.

Since an Hintikka set is by definition a boolean closed subset, then no boolean contradictions may occur during the semantic interpretation.

The first condition for the Hintikka set guarantees that the relation $\leq_h$ satisfies the same constraints as its counterpart, $\leq_{h^{\mathcal{M}}(u)}$, again preventing blatant contradictions.

The second and third conditions impose the correct behavior of the formulas on both sides of the distance spectrum, as was noticed in the **Remark 7**.

Now that the definition of Hintikka sets was made, we can start investigating the notion of the quasi-model. The quasi-model, according to the definition already introduced in the last chapter, will resemble a finite model, where each point will be an Hintikka set, and the relation between Hintikka sets will have to be carefully created.

As we defined above, the Hintikka sets are related to the points in our distance space according to a one-to-many relationship. So each point is related to only one Hintikka set.

Moreover, going back to a semantic discussion, let $\mathcal{M}$ be a distance model. Given a point $u \in \Delta$, and a satisfiable formula $\psi \in \mathbf{com}\ \varphi$, not satisfied in u, there is another point satisfying ψ, $v \in \Delta$, that is closer to u than any other such point. This is the basis of the quasi-model's accessibility relation, two Hintikka sets will be related according to points they are related to and a formula in **com** φ satisfied in one of the points but not the other.

To make this clearer, we are going to introduce a semantic lemma, without proof, that illustrates the idea of the relation between Hintikka sets and the formulas in **com** φ.

First, a small definition. Let h be an Hintikka set. Define $h^{\mathcal{M}} = \{u \in \Delta \mid h^{\mathcal{M}}(u) = h\}$. So, $h^{\mathcal{M}}$ returns the points where the formulas that are satisfied in that point in the distance model are exactly the formulas of the Hintikka set.

Lemma 9 Let $\mathcal{M}$ be a distance model, $u, v \in \Delta$, $\psi_1, \psi_2 \in$ **com** φ. Then,

1. $\psi_1 <_{h^{\mathcal{M}}(u)} \bot$ iff $\psi_1 <_{h^{\mathcal{M}}(v)} \bot$ iff $\psi_1^{\mathcal{M}} \neq \emptyset$.

2. Suppose that $u \notin \psi_1^{\mathcal{M}}$ but $\psi_1^{\mathcal{M}} \neq \emptyset$. Then there is an Hintikka set h such that $\psi_1 \in h$, and $d(u, \psi_1^{\mathcal{M}}) = d(u, h^{\mathcal{M}})$, with each being realized or not simultaneously. We also have:

   (a) If $\psi_2 \in h$, then $\psi_2 \leq_{h^{\mathcal{M}}(u)} \psi_1$.

   (b) If $\psi_1 \simeq_{h^{\mathcal{M}}(u)} \psi_2$ and $\psi_2 \in h$, then $\psi_1 \in \varrho_{h^{\mathcal{M}}(u)}$ iff $\psi_2 \in \varrho_{h^{\mathcal{M}}(u)}$.

Item 1. of the lemma is obvious, so we will focus our attention on item 2.

Item 2.(a) stems from the fact that h was chosen to minimize the distance between u and $\psi_1^{\mathcal{M}}$. That is, the point which minimizes such distance is in $h^{\mathcal{M}}$. Since $\psi_2 \in h$, then $h^{\mathcal{M}} \subset \psi_2^{\mathcal{M}}$, so the minimizing point is also in $\psi_2^{\mathcal{M}}$ and item 2.(a) of the lemma comes easily.

Item 2.(b) states that any other formula $\psi_2 \in h$ where h also minimizes the distance between that formula and u is realized iff ψ_1 is realized.

From this lemma we can declare that a bridge indeed exists between one point and its related Hintikka set (in this case, $h^{\mathcal{M}}(u)$), and the point and Hintikka set where ψ_1 is not only satisfied, but also closest to u.

Finally, much the same way that we treated the definition of Hintikka sets, we are going to present the relations between two Hintikka sets in a syntactic way, but always taking into account the semantic observations.

Then, let h_1, h_2 be Hintikka sets, $\psi_1 \in \mathbf{com}\ \varphi$ and $\psi_1 \notin h_1$. There is a relation between h_1 and h_2 (we call such relation a ψ_1-link) if, for every $\psi_2 \in \mathbf{com}\ \varphi$, we have:

- $\psi_2 <_{h_1} \bot$ iff $\psi_2 <_{h_2} \bot$.
- If $\psi_2 \in h_2$, then $\psi_2 \leq_{h_1} \psi_1$.
- If $\psi_1 \in \varrho_{h_1}$, $\psi_1 \simeq_{h_1} \psi_2$, and $\psi_2 \in h_2$, then $\psi_2 \in \varrho_{h_1}$.

Thus, a ψ_1-link (h_1,h_2) simply provides h_1 with the closest Hintikka set containing ψ_1, according to $\leq_{h_1}$.

A quasi-model is no more then a set of Hintikka sets with ψ-links for every satisfiable formula ψ. A precise definition follows.

Definition 10 (Quasi-model for $\mathcal{CSL}$) A *quasi-model* for φ (which can also be called a *φ-diagram*) is a frame $Q = (D, R)$ where the following conditions are satisfied:

- There exists $h_* \in D$ with $\varphi \in h_*$,
- $\forall h_1, h_2 \in D$ and $\psi \in \mathbf{com}\ \varphi$, we have $\psi <_{h_1} \bot$ iff $\psi <_{h_2} \bot$,
- for every $h_1 \in D$ and $\psi \notin h_1$ with $\psi <_{h_1} \bot$, there exists $h_2 \in D$ such that (h_1, h_2) is a ψ-link. That is, we have $R(h_1, h_2)$.

This definition in fact agrees with the one made in the last chapter, with clear differences, created by the different logics associated with each.

Now that we have presented all the necessary structures for the decidability proof, we are ready to state the theorem and provide the details of its proof.

3.4 Decidability proof

Theorem 11 (Decidability of $\mathcal{CSL}$)
Let φ be a $\mathcal{CSL}$ formula. Then the following statements are equivalent:

1. φ is satisfied in a symmetric distance model.

2. There exists a quasi-model for φ.

Proof:
(1. $\Longrightarrow$ 2.)
Let $\mathcal{M} = (\Delta, d, V)$ be a distance model with $\varphi^{\mathcal{M}} \neq \emptyset$. Then, $D = \{h^{\mathcal{M}}(u) | u \in \Delta\}$ is a quasi-model for φ.

The proof of this argument comes from the careful relation between the syntactic definition of the quasi-model and the semantic observations for any distance model. We will not present the proof, for most of it was already mentioned during the discussion of the last chapter.

(2. $\Longleftarrow$ 1.)

This proof is going to be made, as was said in **Section 2.4**, with resort to a tree argument. The proof will be made in two different parts.

1. Construction of the model from the quasi-model structure.

2. Proof that the model constructed satisfies φ.

So, suppose that $Q = (D, R)$ is a quasi-model for φ.

1. Let us start by defining the satisfiable formulas in **com** φ. Let

$$\{\psi_0, \ldots, \psi_{k-1}\} = \{\psi \in \textbf{com}\ \varphi \mid \psi <_h \bot, \forall h \in D\},$$

   where ψ_i are all distinct.

   Let T be a tree where λ is the root node, and for every node α, the children of α will be called $\alpha + i$, with $i \in \{0, \ldots, k-1\}$. This will be the tree over which we will unfold the quasi-model.

Our plan will be to construct the nodes of the tree inductively, along with a labeling process ($hs : \Delta \rightarrow D$), where each node will be associated with a Hintikka set. Once the tree is constructed, we will define a distance function, d, that will assign the distances between nodes, taking the necessary steps to assure that the relations between hintikka sets are respected. Finally, this structure (T, d, hs) will be the model over which we will prove that φ is satisfiable.

We start the construction of our tree by labeling the root node with the Hintikka set to which φ belongs to. So, $hs(\lambda) = h_*$.

Now, at every step, find some node α where $hs(\alpha)$ hasn't been developed yet. Now, take the Hintikka set $hs(\alpha)$. By the definition of the quasi-model Q, for every $i < k$ with $\psi_i \notin hs(\alpha)$, there exists $h_i \in D$ such that $(hs(\alpha), h_i)$ is a ψ_i-link. Given this fact, the necessary information over the children of the node α will be encoded in the following definition:

- if $\psi_i \in hs(\alpha)$, then $\alpha(i, j) \notin T, \forall j \in \mathbb{N}$ (for no link is needed to materialize the distance, since ψ_i is already in $hs(\alpha)$).
- if $\psi_i \in \varrho_{hs(\alpha)} \setminus hs(\alpha)$, then $\alpha(i, 0) \in T$, $hs(\alpha(i, 0)) = h_i$, and $\alpha(i, j) \notin T, \forall j > 0$ (for a single link and ψ_i-witness, is needed, since the distance from $hs(\alpha)$ to ψ_i is realized).
- if $\psi_i \notin \varrho_{hs(\alpha)}$, then $\alpha(i, j) \in T, hs(\alpha(i, j)) = h_i, \forall j \in \mathbb{N}$ (because infinite ψ_i-witnesses of the same type are needed, for the distance from $hs(\alpha)$ to ψ_i is not realized).

finally, for $\alpha \in T$ and $i < k$, we set the i_{th} children of α as

$$\alpha + i = \begin{cases} \{\alpha\} & \text{if } \psi_i \in hs(\alpha), \\ \{\alpha(i, 0)\} & \text{if } \psi_i \in \varrho_{hs(\alpha)} \setminus hs(\alpha), \\ \{\alpha(i, j) | j \in \mathbb{N}\} & \text{if } \psi_i \notin \varrho_{hs(\alpha)}. \end{cases}$$

Finally, $\alpha+ = \cup_{i<k}(\alpha + i)$ consists of all of α's children.

Until every Hintikka set $hs(\alpha)$ is developed, find a new α that hasn't been yet and repeat the children process. Since the number of Hintikka sets is finite, then the tree must also be finite.

Now that the tree and the labeling process has been constructed, all that is left is the distance function. In order to simplify the notation, we write d^α for $d(\alpha', \alpha)$, where α' is the parent of α. so for every node $\alpha \in T$, the values d^α are inductively defined the following way.

Set $d^\lambda = 1$. Suppose that the distance d^α, is already defined. We shall now define the distances of its children. Since $hs(\alpha)$ is a type, we can choose, for each $i < k$, any numbers $d^{\alpha+i} \in [0, d^\alpha)$, satisfying the following conditions, for $l < k$:

$$d^{\alpha+i} \leq d^{\alpha+l} \text{ iff } \psi_i \leq_{hs(\alpha)} \psi_l,$$
$$d^{\alpha+i} = 0 \text{ iff } \psi_i \in min\ hs(\alpha).$$

Note that these numbers can be completely random, since the *Comparative Similarity Logic* deals only with comparative distances, the only important relation to maintain is the distance order, the actual numbers have no meaning in the logic, since the logic itself is not capable of understanding any distance number other than 0.

So we set, for each children of α, the following distances:

$$\begin{array}{ll} d^{\alpha(i,0)} = d^{\alpha+i}, & \text{if } \psi_i \in \varrho_{hs(\alpha)} \setminus hs(\alpha), \\ d^{\alpha(i,j)} = d^{\alpha+i} + (d^\alpha - d^{\alpha+i})/(2+j), & \text{if } \psi_i \notin \varrho_{hs(\alpha)}. \end{array}$$

Note that the definition of $d^{\alpha(i,j)}$ was simply created so that the distance would be as close to $d^{\alpha+i}$ as needed, the actual expression is not important.

Since $\lim_{j\to\infty} d^{\alpha(i,j)} = d^{\alpha+i}$, we get, for every possibility, $d(\alpha, \alpha+i) = d^{\alpha+i}$.

Finally, let $d : T \to \mathbb{R}_0^+$ be a function defined by, for any $\alpha, \beta \in T$,

$$d(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ d^\alpha & \text{if } \alpha \text{ is a child of } \beta, \\ d^\beta & \text{if } \beta \text{ is a child of } \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

So we see that d is a symmetric distance function on T.

Now take, for $\psi \in \mathbf{cl}\ \varphi$, $\psi^T = \{\alpha \in T \mid \psi \in hs(\alpha)\}$. So ψ^T is simply the nodes of the tree whose Hintikka sets include ψ.

At last, the structure $\mathcal{M} = (T, d, V)$, where $V(p_i) = p_i^T$, is exactly the symmetric model where φ is going to be proved to be satisfied. Note that obviously $p_i \in \mathbf{cl}\ \varphi$.

Now that we have designed our model, we are able to prove the last part of the theorem, that φ is satisfied in $\mathcal{M}$.

2. We know, from the construction of the quasi-model, that $\varphi \in hs(\lambda)$. So, if we can prove that if for each formula ψ, we have $\alpha \in \psi^{\mathcal{M}}$ if $\alpha \in \psi^T$, then we have that φ is satisfied in $\mathcal{M}$.

   Before proving that result, though, we are in need of an auxiliary result.

   Lemma 12 Let $\alpha \in T$ and $i < k$. Then $d(\alpha, \psi_i^T) = d^{\alpha+i}$.

   Proof: As we know $d(\alpha, \alpha + i) = d^{\alpha+i}$ and $\alpha + i \subseteq \psi_i^T$. So, since by construction $d(\alpha, \alpha(i, j_1)) > d(\alpha, \alpha(i, j_2))$ when $j_1 < j_2$, all we need to prove is that for each possibility of the children nodes $\alpha + i$, we have:

$$\forall \beta \in \psi_i^T\ \exists \beta_i \in \alpha + i,\ d(\alpha, \beta_i) \leq d(\alpha, \beta).$$

   Also, if $\beta \in \psi_i^T \setminus \alpha+$, then we have $d(\alpha, \beta) = 1$ or $d(\alpha, \beta) = d^\alpha$. In any case, $d(\alpha, \beta_i) < d(\alpha, \beta)$.

So we only need to be concerned about the case $\beta \in \psi_i^T \cap \alpha+$. Consider an arbitrary $\beta \in \psi_i^T \cap \alpha+$. We have 3 possibilities:

- $\psi_i \in hs(\alpha)$.
  Then $\alpha + i = \{\alpha\}$, $\beta_i = \alpha$ and so $d(\alpha, \beta_i) = 0 \leq d(\alpha, \beta)$.
- $\psi_i \in \varrho_{hs(\alpha)}$.
  Then $\alpha + i = \{\alpha(i, 0)\}, \beta_i = \alpha(i, 0), \alpha(i, 0) \in \psi_i^T$.
  On the other hand, we have $\psi_i \in hs(\beta)$ and $(hs(\alpha), hs(\beta))$ is a $\psi_l - link$, for some l<k. By the second property in the definition of a link, because $\psi_i \in hs(\beta)$, we have $\psi_i \leq_{hs(\alpha)} \psi_l$. Finally, by the conditions the numbers $d^{\alpha+j}$ satisfy, we get $d(\alpha, \beta_i) = d^{\alpha(i,0)} = d^{\alpha+i} \leq d^{\alpha+l} \leq d^{\beta} = d(\alpha, \beta)$.
- $\psi_i \notin \varrho_{hs(\alpha)}$.
  Then $\alpha + i = \{\alpha(i, j) \mid j \in \mathbb{N}\}$, with $\alpha(i, j) \in \psi_i^T$.
  On the other hand, we have $\psi_i \in hs(\beta)$ and $(hs(\alpha), hs(\beta))$ is a $\psi_l - link$, for some l<k. Again, we have $\psi_i \leq_{hs(\alpha)} \psi_l$.
  Now, by the last property in the definition of a link, since we have $\psi_i \notin \varrho_{hs(\alpha)}$ and $\psi_i \in hs(\beta)$, we must have either
  (a) $\psi_l \notin \varrho_{hs(\alpha)}$.
  In this case we get $d^{\alpha+i} \leq d^{\alpha+l} < d^{\beta}$.
  (b) $\psi_l \in \varrho_{hs(\alpha)}$ and $\psi_i \neq_{hs(\alpha)} \psi_l$.
  In this case we get $\psi_i <_{hs(\alpha)} \psi_l$ and so $d^{\alpha+i} < d^{\alpha+l} = d^{\beta}$.

  In either case, we have $d^{\alpha+i} < d^{\beta}$, so it is possible to get $\beta_i \in \alpha$ such that $d(\alpha, \beta_i) \leq d(\alpha, \beta)$.

QED

We are now ready to prove the final result, which allow us to conclude that φ is satisfied in our constructed distance model.

Lemma 13 For each $\psi \in \mathbf{cl}\,\varphi$, $\alpha \in \psi^T$ iff $\alpha \in \psi^{\mathcal{M}}$.

Proof: We are going to present the proof in full detail by induction on the construction of ψ.

- Basis: (ψ *is* p_i).
  By the definition of the distance model, we have $p_i^T = p_i^{\mathcal{M}}$.
- Step:
  - (ψ *is* $\neg\varphi_0$).
    Suppose $\alpha \in (\neg\varphi_0)^T$. Then:

$$\begin{array}{lll} \neg\varphi_0 \in hs(\alpha) & \text{iff} & \alpha \text{ is boolean closed} \\ \varphi_0 \notin hs(\alpha) & \text{iff} & \text{Induction hypothesis} \\ \alpha \notin \varphi_0^{\mathcal{M}} & \text{iff} & \text{By } \neg\text{-satisfaction} \\ \alpha \in (\neg\varphi_0)^{\mathcal{M}}. & & \end{array}$$

  - (ψ *is* $\varphi_1 \cap \varphi_2$).
    Suppose $\alpha \in (\varphi_1 \cap \varphi_2)^T$. Then:

$$\begin{array}{lll} \varphi_1 \cap \varphi_2 \in hs(\alpha) & \text{iff} & \alpha \text{ is boolean closed} \\ \varphi_1 \in hs(\alpha) \text{ and } \varphi_2 \in hs(\alpha) & \text{iff} & \text{Induction hypothesis} \\ \alpha \in \varphi_1^{\mathcal{M}} \text{ and } \alpha \in \varphi_2^{\mathcal{M}} & \text{iff} & \text{By } \cap\text{-satisfaction} \\ \alpha \in (\varphi_1 \cap \varphi_2)^{\mathcal{M}}. & & \end{array}$$

  - (ψ *is* $\varphi_1 \leftleftarrows \varphi_2$).

    ($\Longrightarrow$)
    Suppose $\alpha \in (\varphi_1 \leftleftarrows \varphi_2)^T$.
    Then $\varphi_1 <_{hs(\alpha)} \varphi_2$. So $\varphi_1 \notin max\ hs(\alpha)$, and also $\varphi_1 = \psi_i$, for some i<k. By induction hypothesis, $\varphi_1^T = \varphi_1^{\mathcal{M}}$, which is non empty because $\lambda + i \in \psi_i^T$.
    Now, if $\varphi_2 \notin \{\psi_0, \ldots, \psi_{k-1}\}$, then $\varphi_2 \in max\ hs(\alpha)$ and so $\varphi_2^{\mathcal{M}} = \varphi_2^T = \emptyset$, so $\alpha \in (\varphi_1 \leftleftarrows \varphi_2)$, because $d(\alpha, \varphi_1^{\mathcal{M}}) < d(\alpha, \varphi_2^{\mathcal{M}}) = \infty$.
    Suppose now, on the other hand, that $\varphi_2 = \psi_l$ for some l<k. Then $d^{\alpha+i} < d^{\alpha+l}$ by definition of the distances d^j. So, using first the induction hypothesis and then lemma 12, we get

$$d(\alpha, \varphi_1^{\mathcal{M}}) = d(\alpha, \varphi_1^T) = d^{\alpha+i} < d^{\alpha+l} = d(\alpha, \varphi_1^T) = d(\alpha, \varphi_1^{\mathcal{M}}).$$

From that, $\alpha \in (\varphi_1 \leftrightarrows \varphi_2)^{\mathcal{M}}$.

$(\Longleftarrow)$
Suppose $\alpha \in (\varphi_1 \leftrightarrows \varphi_2)^{\mathcal{M}}$.
Then $\varphi_1^{\mathcal{M}} \neq \emptyset$, so $\varphi_1 = \psi_i$ for some $i < k$. So, by induction hypothesis, $\varphi_1^T = \psi_i^{\mathcal{M}}$.
Now, if $\varphi_2 \notin \{\psi_0, \ldots, \psi_{k-1}\}$, then $\varphi_2 \in max\ hs(\alpha)$ and so $(\varphi_1 \leftrightarrows \varphi_2) \in hs(\alpha)$, or $\alpha \in (\varphi_1 \leftrightarrows \varphi_2)^T$.
If $\varphi_2 = \psi_l$, for some l<k, then by the induction hypothesis, we have $\varphi_2^T = \psi_l^{\mathcal{M}}$. Also, $d(\alpha, \varphi_1^T) < d(\alpha, \varphi_2^T)$. By lemma 12, we get $d^{\alpha+i} < d^{\alpha+l}$, and $(\varphi_1 \leftrightarrows \varphi_2) \in hs(\alpha)$, or $\alpha \in (\varphi_1 \leftrightarrows \varphi_2)^T$.

– $(\psi\ is\ \text{ⓡ}\varphi_0)$.

$(\Longrightarrow)$
Suppose $\alpha \in (\text{ⓡ})^T$. Then $\varphi_0 \in \varrho_{hs(\alpha)}$ and by lemma 12, $d(\alpha, \varphi_0^T) = d^{\alpha+i}$, for some i<k where $\varphi_0 = \psi_i$. By induction hypothesis, $d(\alpha, \varphi_0^{\mathcal{M}}) = d^{\alpha+i}$ and since $d(\alpha, \psi_i) = d^{\alpha+i}$, the distance is realized, so $\alpha \in (\text{ⓡ}\varphi_0)^{\mathcal{M}}$.

$(\Longleftarrow)$
Suppose $\alpha \in (\text{ⓡ})^{\mathcal{M}}$, i.e., $d(\alpha, \varphi_0^{\mathcal{M}})$ is realized. By induction hypothesis, $\varphi_0^{\mathcal{M}} = \varphi_0^T$, so $\varphi_0 \in \varrho_{hs(\alpha)}$ and by lemma 12, $d(\alpha, \varphi_0) = d^{\alpha+i}$, where $\varphi_0 = \psi_i$ for some i<k. Since $d(\alpha, \psi_i) = d^{\alpha+i}$, $d(\alpha, \varphi_0^T)$ is realized, so $\alpha \in (\text{ⓡ}\varphi_0)^T$.

QED

Conclusion of the proof of the Theorem.
Since $\varphi \in hs(\lambda)$, we get $\varphi^{\mathcal{M}} = \varphi^T \neq \emptyset$. That is, φ is satisfied in a symmetric distance model.

QED

Chapter 4

A graph-theoretic semantics of $\mathcal{CSL}$

In this chapter, we define the graph-theoretical approach to logics and present the $\mathcal{CSL}$ in graph-theoretical terms. In the graph-theoretical approach, we create a set of guidelines where each logic can be defined. Using this approach, all logics will be based under the same foundation and so the comparing, relating and fibring of logics become simplified. Instead of fibring logics that are presented in two very different ways, we may now unite logics that are defined in the same way, no matter how different their core may be.

Aside from the fibring itself, it is of great importance to understand whether properties are retained under the fibring of logics. Some properties have already been studied and proved to be retained, while others have yet to be studied. For example, it is known that the fibring of two complete logics may in fact be complete, depending on some specific conditions of the logics (see [14]).

The graph-theoretical approach is based on the notion of the multigraph (or m-graph for short), a graph where each edge can have a finite number of sources but only one target.

Signatures, models and deductive systems are defined as m-graphs, while concepts such as formulas, their satisfaction and deduction can be seen as paths over those m-graphs. In this dissertation, we will simply introduce the notions of signature and interpretation. For more information on the subject, consult [13].

An m-graph is then a tuple

$$G = (V, E, src, trg)$$

where V is a set of vertices, E is a set of m-edges and $src : E \to V^+$ and $trg : E \to V$ are maps that return the sources and target of an edge, respectively. V^+ is a finite collection of elements of V.

A signature will be an m-graph, one where V are the sorts, indicating the kind of notions available in the signature, and E represent the formation rules available in the construction of our language.

So, a signature is a tuple $\Sigma = (G, \Diamond, \pi)$, where G is an m-graph as defined before, $\Diamond$ and π are sorts, representing the concrete sort (the propositional symbols) and the propositions sort (the schema formulas), respectively. Others sorts may need to be included, but we will refrain from using any example, for it is not needed in most of the logics mentioned in this dissertation: the propositional logic, basic modal logic and $\mathcal{CSL}$.

Now we will define precisely the graph-theoretic signature for the $\mathcal{CSL}$.

Definition 1 ($\mathcal{CSL}$'s graph-theoretic signature) Let Φ be a set of propositional symbols.

The *signature for the* $\mathcal{CSL}$ is a tuple $\Sigma_{\mathcal{CSL}} = (G, \Diamond, \pi)$, where $G = (V, E, src, trg)$ is an m-graph, and:

- $V = \{\Diamond, \pi\}$, and
- $E = \{p_i, \neg, \wedge, ⓡ, \leftleftarrows\}$ is such that:
  - $p_i : \Diamond \to \pi, \ \forall p \in \Phi$,
  - $\neg : \pi \to \pi$,
  - $\wedge : \pi\pi \to \pi$,
  - $ⓡ : \pi \to \pi$,

– $\leftleftarrows: \pi\pi \to \pi$.

The graph-theoretical semantics of a logic also relies upon an m-graph, called the operations graph. The operations graph has the possible truth values as its nodes and the edges represent the effect of the formation rules on the truth values.

In order to construct a model for a logic, we also need to know how the operations m-graph and the signature m-graph are related. We need to know which of the possible truth values can be assigned to each sort, and which operations are related to each formation rules. For that purpose we introduce the notion of an m-graph morphism.

Definition 2 (M-graph morphism) Let $G_1 = (V_1, E_1, src_1, trg_1)$, $G_2 = (V_2, E_2, src_2, trg_2)$ be two m-graphs. $h : G_1 \to G_2$ is called an *m-graph morphism* if $h = (h^v, h^e)$, and $h^v : V_1 \to V_2, h^e : E_1 \to V_2$ are two maps such that:

- $src_2 \circ h^e = h^v \circ src_1$,
- $trg_2 \circ h^e = h^v \circ trg_1$.

Also needed for the model is a set D called the distinguished values, meant to represent the values where a schema formula would be true, and a possible truth value, db, called the concrete value, which is related to the concrete sort in the signature m-graph.

So, the model for the $\mathcal{CSL}$ can now be defined.

Definition 3 ($\mathcal{CSL}$'s graph theoretic model) Let $\mathcal{F} = (\Delta, d)$ be a distance frame and $P = \{p_1, p_2, ...\}$ be a set of propositional symbols.

A *model $\mathcal{M}$ for the $\mathcal{CSL}$* over the frame $\mathcal{F}$ is a tuple (G', α, D, db), where $G' = (V', E', src', trg')$ is an operations graph, $\alpha : G' \to G$ is an m-graph morphism, $D \subset (\alpha^v)^{-1}(\pi)$ is non-empty and $db \in (\alpha^v)^{-1}(\pi)$.

Moreover, we have

- G' is such that:
  - $V' = \wp\Delta \cup db$;

- $E^{'} = \{p_1^{'}, p_2^{'}, \ldots\} \cup \{\neg_A : A \in \wp\Delta\} \cup \{\wedge_{AB} : A, B \in \wp\Delta\} \cup \{ⓡ_A : A \in \wp\Delta\} \cup \{\leftleftarrows_{AB} : A, B \in \wp\Delta\}$;
- $src^{'}$ and $trg^{'}$ are such that:
  - $p_1^{'} : db \to A_1,\ A_1 \in \wp\Delta$;
  - $p_2^{'} : db \to A_2,\ A_2 \in \wp\Delta$;
  - $\ldots$
  - $\neg_A : A \to \Delta \backslash A$;
  - $\wedge_{AB} : A \times B \to A \cap B$;
  - $ⓡ_A : A \to \{b \in \Delta : \exists a \in A,\ d(b, A) = d(b, a)\}$;
  - $\leftleftarrows_{AB} : A \times B \to \{c \in \Delta : d(c, A) < d(c, B)$.

- $\alpha : G^{'} \to G$ is such that:
  - $\alpha^v(A) = \pi,\ A \in \wp\Delta$;
  - $\alpha^v(db) = \Diamond$;
  - $\alpha^e(p_i^{'}) = p_i,\ i \in \mathbb{N}$;
  - $\alpha^e(\neg_A) = \neg$;
  - $\alpha^e(\wedge_{AB}) = \wedge$;
  - $\alpha^e(ⓡ_A) = ⓡ$;
  - $\alpha^e(\leftleftarrows_{AB}) = \leftleftarrows$.
- $D = \Delta$.

The development of our graph-theoretical model is pretty straightforward, everything comes easily from our previous definitions of distance models and interpretation.

Given this representation of $\mathcal{CSL}$, we could now fibre it with any other graph-theoretically defined logic, using the framework present in [14].

The properties of the resulting logic is a discussion that far outgrows the intent of this dissertation. In fact, many problems regarding fibring are still open, as is expressed in [14, 13, 4, 19].

Chapter 5

Conclusion

We introduced modal logics, starting with the definition of the *basic modal language*. Semantics and axiomatization over that language provided us with the necessary mechanisms to prove soundness and completeness of the logic.

A more general discussion about decidability followed, where various methods of proving logic decidability were introduced.

We then presented a modal logic for reasoning about topological notions and relative distances. The topological component of the logic is accomplished by the introduction of two special operators: $\leftleftarrows$ and $\textcircled{r}$. The operator $\leftleftarrows$ is capable of reasoning over relative closeness while the operator $\textcircled{r}$ is capable of determining realization of a distance. Our main result was proving decidability of the $\mathcal{CSL}$ over the class of symmetric distance models.

The logic $\mathcal{CSL}$ was firstly introduced as a sublanguage of a more general language, $\mathcal{QML}$ (Qualitative Metric Logic) ([15, 16]). Regarding this logic, many interesting problems are still unresolved. For information about the outlook of this branch of spatial logics consult [15].

A diagrammatic representation of the spatial language was displayed. This way of representing logics, firstly introduced in [13], adds nothing to the notion and expressive power of the logic itself. That is, despite a completely different presentation of the logic, it still maintains all its characteristics and fundamentals. The graph-theoretical approach's main purpose is then, not on the representation of single logics, but on the creation of a framework which facilitates the combination of logics, whichever is their background. In fact, logics as different as modal logics and linear logics were combined in [4]. This type of combination, known as fibring (firstly introduced in [6]), has already been studied in several articles, where problems such as the preservation of soundness, completeness and the finite model property were studied ([14, 4]).

Given that the main result of this dissertation is the decidability of the $\mathcal{CSL}$, it is noteworthy to investigate in which conditions preservation of decidability through fibring is accomplished, and whether the $\mathcal{CSL}$ satisfies such conditions. This problem is, however, still open, along with many other problems, concerning, for example, preservation of cut elimination, quantifier elimination and interpolation. Moreover, it is also interesting to investigate modalization, in the sense of [5], via $\mathcal{CSL}$.

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