Odd Khovanov Homology for Virtual Knots

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1 Abstract

Khovanov constructed a Homology theory that categorifies the Jones Polynomial. The category in which he constructs his complex was for some time considered as unique since the associated (1+1)-TQFT was determined by a specific Frobenius system. Later Ozsvath, Rasmussen and Szabo (ORS) constructed a new categorification for the Jones Polynomial using the Odd Khovanov Homology theory. The algebraic structure behind this category cannot be described in terms of a Frobenius Algebra and thus does not conflict with the previous uniqueness.

Beliakova and Wagner argued that the appropriate algebraic structure for this new category is given by an EQFT - Extended Quantum Field Theory.

Our objective was to extend, with the structure defined by Beliakova and Wagner, the Odd Khovanov Homology to Virtual knots following the work done by Turaev and Turner who showed how to extend the Khovanov Homology.

We did not accomplish this task but we can show that the way ORS classified faces and cubes in Odd Khovanov Homology does not extend to virtual knots in a straightforward fashion, at least not in a way that will enable us to define the complex as they did.

This document will, hopefully, provide the reader with some basic notions about the subject, pointing out some of the difficulties which arise and are solved in each of these theories.

Keywords:
Khovanov homology, Odd Khovanov homology, Khovanov homology for virtual knots, Odd Khovanov homology for virtual knots

2 Odd Khovanov Homology

2.1 Ozsvath, Rasmussen and Szabo approach

Ozsvath, Rasmussen and Szabo (ORS) found a modified version of Khovanov homology - the Odd Khovanov Homology - which associates to a collection of embedded planar circles the exterior algebra of the vector spaces generated by them instead of a tensor product of Frobenius Algebras as before.

They build a category \( \mathcal{C} \) where the objects are disjoint unions of circles and the morphisms are generated by \textit{births} (zero-handle additions), \textit{merges} (one-handle additions where the feet of the one-handle lie on two different components), \textit{splits} (one-handle additions where the feet of the one-handle lie on the same component) and \textit{deaths} (two-handle additions).

If \( L \) is a oriented link diagram we can associate to it, as you can check in [7], a cube of oriented resolutions. Where each crossing can be thought as giving a cobordism from a 0-smoothing resolution to a 1-smoothing resolution, consisting of a single one-handle. Each vertex in the cube corresponds to an object in the category above and each edge to a morphism.

We will call each square in the cube, a \textit{face}. There are 4 types of faces:
Definition 1 (Edge assignments) An edge assignment for a diagram $L$ is a map that assigns a positive or a negative sign to each edge of the cube. Given an edge assignment, we say that a face is even or odd, depending of the number of negative edges it contains.

A type $X$ edge assignment is one where all faces of types $A$ and $X$ are even and all faces of type $C$ and $Y$ are odd.

A type $Y$ edge assignment is one where all faces of types $A$ and $Y$ are even and all faces of type $C$ and $X$ are odd.

Theorem 2 (ORS) Any diagram $D$ has an edge assignment of type $X$, and one of type $Y$.

Lemma 3 Each cube in the cube of resolutions contains an even number of squares of type $A + X$ and an even number of squares of type $A + Y$.

All possible cubes and their possible classifications in terms of faces of type $A, C, X$ and $Y$. 2
With just these few rules ORS construct the edge assignments needed to build the chain complex and show that the complex is independent of the edge assignment chosen and the orientations. The Odd Khovanov homology derived from this complex will also prove to be independent of the link projection.

2.2 Beliakova and Wagner idea

Beliakova and Wagner in [9] argued that the appropriate algebraic structure is given by an EQFT - Extended Quantum Field Theory. This EQFT is a 2-functor from a semistrict monoidal 2-category of cobordisms, called an extension, to an abelian category. It is called an extension because after we replace the original category by a group we get a normal extension of that group. In [9] they construct extensions of the category $\text{Cob}^3(\mathcal{O})$ and recover the Odd Khovanov homology as a semistrict extension.

The reader can check [9] to understand how this extension is constructed, here we will simply post the results.

2.2.1 The OddCob extension

Definition 4 (OddCob) The extension OddCob of $\text{Cob}^3(\mathcal{O})$ has finite ordered sets of circles for objects and the morphisms are generated by

- Merge
- Birth
- Split
- Death
- Permutation

and are subject to the following relations:

- Commutativity relation
- Anti-commutativity relation
- Associativity relation
- Co-associativity relation
- Frobenius relations
- Unit relation
- Co-unit relation
- 1st Permutation relation
- 2nd Permutation relation
- Unit permutation relation
- Co-unit permutation relation
- Merge permutation relation
All other commutation relations hold with a plus sign.

If we think of the classifications given by ORS to the faces of the cubes, we can check that the 1st Commutation relation and the Co-associativity relation fall under A-type classification. Since faces of type A are anti-commutative, both perspectives seem to be consistent. On the other hand the Associativity relation, the Frobenius relation, and the other Commutations that hold with plus sign should be classified as type C faces and since type C faces are commutative this reinforces the consistency.

This EQFT gives rise to the Odd Khovanov homology when applied to the Khovanov cube (when we mod out the cobordisms by the S relation, the T relation and the 4Tu relation). The reader should check [5] to know more about it.

3 Our Conjecture

Our initial motivation was to pick up on the work of Wagner and Beliakova in [8] and take the Odd Khovanov Homology theory, derived from the constructed EQTF extension $\text{OddCob}$, apply the work done by Turaev and Turner in [9] and try to extend this homology to virtual links.

To do this we started by building the khovanov cubes associated with the virtual knots with up to 3 classical crossings, we consulted [10] to obtain this knots, and try and understand if the face classifications and results introduced by ORS in [5] would still apply on these new cubes. We’ll leave you the cubes:
Virtual 3.2 inverse cube

Virtual 3.3 cube

Virtual 3.3 inverse cube

Virtual 3.4 cube

Virtual 3.4 inverse cube

Virtual 3.5 cube

Virtual 3.5 inverse cube

Virtual 3.7 cube

Virtual 3.7 inverse cube

On a first analysis we came up with the following classification:
### Classification results - 1st table

#### Virtual Knots

<table>
<thead>
<tr>
<th>3.1</th>
<th>3.1 INV</th>
<th>3.2</th>
<th>3.2 INV</th>
<th>3.3</th>
<th>3.3 INV</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Front</td>
<td>3.1 INV Protection</td>
<td>3.2 Base</td>
<td>3.2 INV Left Lat</td>
<td>3.2 INV Right Lat</td>
<td>3.3 Top</td>
</tr>
<tr>
<td>A</td>
<td>C or A</td>
<td>A</td>
<td>A</td>
<td>A</td>
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</tr>
<tr>
<td>3.1 Left Lat</td>
<td>3.1 Top</td>
<td>3.1 INV Base</td>
<td>3.1 INV Right</td>
<td>3.3 Front</td>
<td>3.3 INV Right Lat</td>
</tr>
<tr>
<td>X/Y</td>
<td>X/Y</td>
<td>C or A</td>
<td>C or A</td>
<td>A</td>
<td>A</td>
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<tr>
<td>3.1 Right Lat</td>
<td>3.1 INV Front</td>
<td>C or A</td>
<td>A</td>
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</tr>
<tr>
<td>3.2 Top</td>
<td>3.2 INV Base</td>
<td>3.3 Front</td>
<td>3.3 INV Right</td>
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<td>C or A</td>
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</tbody>
</table>

How to understand these tables?

The labels on the top (2nd classifying row) represent the virtual knots with 3 classical crossings. The two columns on the left represent the cobordisms in a given square of the cube. So, for example, the first square inside the table, the one that says "3.1 Front" and "A", means that the front square of the Khovanov cube we built for the 3.1 virtual knot has, as edges, the cobordisms given in the respective column on the left. The "A" means that, following the classification given by ORS, shown in Chapter 2.1, of face types; this face should be of type A.
The global results are presented in the table as follows:

A knot in a green square, means that this knot satisfies the ORS lemma 3, under the underlying classification of faces for it’s cube. A knot in a red square means the opposite. Yellow squares are faces in which there are no Möbius cobordisms (so we conjecture that the Ozsvath classification of faces is always correct in these squares).

Analysing this first table we immediately see that there are 3 knots for which the classifications given don’t work, that is, the number of faces $A + X$ and $A + Y$ in their cubes are not even. So we will try a different approach, some faces might be badly classified because we didn’t consider gauss codes on them. From this approach, the table will become:

<table>
<thead>
<tr>
<th>VirtualKnots</th>
<th>3.1</th>
<th>3.1 INV</th>
<th>3.2</th>
<th>3.2 INV</th>
<th>3.3</th>
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<th>3.4</th>
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</table>

Classification results - 2nd table

In this new table you can see the faces that changed classification in the orange squares. A knot in a blue square, means a knot that under this classification can still obey the lemma 3 under an appropriate choice of, yet undetermined, face classifications.

So, from the first table to the second one we made things a little worse. Now it seems that only the knots 3.5, 3.5 inverse, 3.7 and 3.7 inverse satisfy the lemma. There is still hope for knots 3.1 and 3.2 inverse but all others seem not to obey the lemma at all.

We have not yet been able to determine a classification that makes the cubes of all these knots satisfy the lemma but we came across some curious facts. For example, the reason why there are still faces in this last table for which we can choose which specific classification they should have under an underlying orientation is because of the classification of faces of type $X$ and $Y$.

If you think of a face of type $X$ and reverse the inside with the outside, this face will become a face of type $Y$ (and vice-versa). More importantly there are faces whose Gauss-code suggest they are of $X/Y$ ($X$ or $Y$) type, but under specific orientations we can’t make out if they’re $X$, $Y$, both or neither. For example, take the classification of the oriented faces of the 3.3 cube.
Face classifications for the 3.3 Virtual knot cube

Here, green squares are the faces in which we can’t really make out which classification to attribute.

Another interesting fact is that the Gauss-code diagram assigned to the 3.3 cube cannot have a planar realization, whereas for all other examples it can.

Returning to the objective of our purposes, all we can say for now is that the face classifications as given in [5] do not fully extended to the faces obtained in virtual knots cubes or, at least, they don’t satisfy the lemma 3 from which ORS derives their theorem 2, and with it constructs the complex.

4 Bibliography

References