Étale Cohomology

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Abstract

This work is a brief exposition of étale cohomology and its main properties. We start by introducing étale morphisms and proceed to introduce the concept of the étale site, which generalizes the notion of topological space. Several concepts related to the study topological spaces (such as sheaves) are then adapted to this new framework. After studying in detail sheaves over the étale site, we define étale cohomology and present some concrete examples of the main techniques used to study it, specially in the case of curves. We finish by focusing on the version of Poincaré duality for the étale cohomology of curves, which is one of the good properties that the theory shares with other cohomology theories.

Extended Abstract

By the mid XXth century, André Weil stated his famous conjectures [Wei49] concerning the number of solutions of polynomial equations over finite fields. These conjectures suggested a deep connection between the arithmetics of algebraic varieties over finite fields and the topology of algebraic varieties over the complex numbers.

Grothendieck was the first to suggest étale cohomology (1960) as an attempt to solve the Weil Conjectures. By that time, it was already known by Serre that if one had a suitable cohomology theory for abstract varieties defined over the complex numbers, then one could deduce the conjectures using the various standard properties of a cohomology theory.

The Weil conjectures are not the only motivation for the study of étale cohomology. The usual invariants from algebraic topology such as the fundamental groups and the cohomology groups, are very useful and one would like to have their analogues in the context of general algebraic varieties.

The problem was that the Zariski topology was too coarse to allow a detailed study of these topological invariants. Serre had already proved that it provided satisfactory results for the study of coherent sheaves. Moreover, he proved that for complex varieties these invariants would agree with the ones given by the usual topological techniques. However, the whole framework failed when studying other types of sheaves, for instance, when considering constant sheaves, all the cohomology groups are trivial, i.e. the rth cohomology groups for $r > 0$, are zero. The Zariski topology had other problems: the inverse mapping theorem fails and the usual fiber bundles are not locally trivial. There does not seem to exist a good way to fix these problems using a finer topology on general algebraic varieties.

Grothendieck's key insight was a clear understanding of the role played by the open sets of a topological space. He noticed that the important feature was not the open sets themselves but the category of all of them. In this setting, he introduced a generalization of the concept of a topological space, by replacing the category of open sets by a general category whose objects play the role of the open subsets, together with a notion of covering families for each object - a site.

He foresaw étale sites as a particular type of this generalized topology, by considering the open subsets of an algebraic variety as a certain class of maps over the variety, required to satisfy certain basic properties already known in other theories, like Galois Cohomology.

In Chapter 1, we start by presenting the concept of an étale morphism, that is, a morphism that at each point induces an isomorphism between the "tangent" spaces. We start by studying the case of étale morphisms between nonsingular varieties and prove that they satisfy the conditions of the implicit function theorem. Then, we introduce the general definition of an étale morphism between schemes.

In the beginning of Chapter 2 we introduce a new topological perspective: we define the notion of a site and generalize the usual structures used to study topological spaces (sheaves and continuous maps) to this new framework. After this brief introduction to the general theory of sites, we will focus on the étale site: the category of étale morphisms over a scheme X. We present give a brief description of the étale fundamental group of a scheme, followed by some properties about sheaves over the étale site. All of this is done with the Zariski topology always in mind, since one should understand that most of the results are straightforward generalizations, but there are some differences. For instance, the structure of the local ring when considering the étale topology (stalk) is richer than in the Zariski case. These extra structures appearing will allow further results, like the exactness of the Kummer sequence

$$0 \to \mu_n \to G_m \xrightarrow{t \mapsto t^n} G_m \to 0,$$

since it will be exact on the stalks, and in general this is not true in the Zariski topology. The Kummer sequence will turn out to be very important when computing the cohomology groups of μ_n,.

In our last chapter we aim to study some concrete examples of the étale cohomology of curves over an algebraically closed field, and compare it with Zariski cohomology. Firstly, we briefly overview some of the usual definitions and cohomological constructions, this time using the étale topology on a scheme. Following this review we start studying our concrete examples of the usage of cohomological methods on curves. We prove that the étale cohomology with values on the sheaf G_m, the analog of $\mathcal{O}_X^\times$, gives exactly the same values for the étale cohomology as for the Zariski cohomology.Under the same conditions we get

$$H^r\left(X_{et}, G_m\right) \begin{cases} = \Gamma(X, \mathcal{O}_X^\times), & r = 0 \\ \simeq \operatorname{Pic}(X), & r = 1 \\ = 0, & r > 1. \end{cases}$$

On the other hand, if we consider instead values on the sheaf $\mu_n \approx \mathbb{Z}/n\mathbb{Z}$, we find non-trivial cohomology groups, contrarily to what happens using the Zariski topology. Again, with some conditions, the main result is

$$H^r\left(X_{et}, \mu_n\right) \begin{cases} = \mu_n(k), & r = 0 \\ \approx \left(\mathbb{Z}/n\mathbb{Z}\right)^{2g}, & r = 1 \\ \simeq \mathbb{Z}/n\mathbb{Z}, & r = 2 \\ = 0, & r > 2. \end{cases}$$

More interesting is that the cohomology groups with coefficients in μ_n are analog to the results obtained when considering the sheaf $\mathbb{Z}$, with the complex topology. At last we focus on Poincaré duality for curves. Again, this result fails when considering the Zariski topology. In all these three examples we strongly use the fact that curves are defined over an algebraically closed field.

As main references we used Milne's lecture notes on Étale Cohomology [Mil08], and as supporting references also his book Étale Cohomology [Mil80] and Hartshorne's book on Algebraic Geometry [Har77].

References

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