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Étale Cohomology

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Júri

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Something smart

Abstract

This work is a brief exposition of étale cohomology and its main properties. We start by introducing étale morphisms and proceed to introduce the concept of the étale site, which generalizes the notion of topological space. Several concepts related to the study topological spaces (such as sheaves) are then adapted to this new framework. After studying in detail sheaves over the étale site, we define étale cohomology and present some concrete examples of the main techniques used to study it, specially in the case of curves. We finish by focusing on the version of Poincaré duality for the étale cohomology of curves, which is one of the good properties that the theory shares with other cohomology theories.

Resumo

Esta dissertação consiste numa breve exposição sobre a cohomologia étale e muitas das suas propriedades úteis. Começamos por introduzir o conceito de morfismos étale e de seguida o de sítio étale, uma topologia generalizada. Depois disto introduzem-se os conceitos usualmente usados para estudar espaços topológicos, agora adaptados à nossa nova estrutura topológica, em particular técnicas cohomológicas. Segue-se então um estudo mais detalhado de feixes sobre o sítio étale e finalmente introduzimos a cohomologia de feixes sobre o sítio étale. Por fim focamos-nos em alguns exemplos concretos de cohomologia étale sobre curvas e algumas das suas propriedades como é o caso da dualidade de Poincaré. Esta é uma das boas propriedades que falha quando consideramos a cohomologia com a topologia de Zariski.

Keywords

- Étale Morphisms;
- Sites;
- Fundamental Group;
- Étale site;
- Étale sheafs;
- Étale cohomology;
- Poincaré Duality.

Notations and Conventions

An injection is denoted by $\hookrightarrow$ (we will not attribute any symbol to a surjection).

The symbol $X \approx Y$ means X is isomorphic to Y, $X \simeq Y$ denotes that X and Y are canonically isomorphic (or there is a given or unique isomorphism) and $X \stackrel{\text{def}}{=} Y$ means that X is defined to be Y, or that X equals Y by definition.

For a ring A, $A^{\times}$ denotes the group of units of A, A_p denotes the localization on the prime ideal p, $Frac(A)$ denotes the field of fractions of A, i.e., $\mathrm{Frac}(A) \stackrel{\text{def}}{=} A_{(0)}$. When A is local, $\mathfrak{m}_A$ will denote the unique maximal ideal of A and $k(\mathfrak{m}_A) \stackrel{\text{def}}{=} A/\mathfrak{m}_A$, its residue field.

Our terminology concerning schemes is standard, except that we assume that our rings are Noetherian and that our schemes are locally Noetherian.

The word "variety" means a separated, reduced scheme of finite type over a field, and a curve is a variety of dimension one. This is motivated by [Har77, page 105].

The symbols μ_n, G_m, GL_n denote certain group schemes (2.3.7).

We will denote a algebraically closure of the field k by k^{al} and the separable closure by k^{sep}, recall that both are unique up to isomorphism.

Contents

Introduction

By the mid XXth century, André Weil stated his famous conjectures [Wei49] concerning the number of solutions of polynomial equations over finite fields. These conjectures suggested a deep connection between the arithmetics of algebraic varieties over finite fields and the topology of algebraic varieties over the complex numbers.

Grothendieck was the first to suggest étale cohomology (1960) as an attempt to solve the Weil Conjectures. By that time, it was already known by Serre that if one had a suitable cohomology theory for abstract varieties defined over the complex numbers, then one could deduce the conjectures using the various standard properties of a cohomology theory.

The Weil conjectures are not the only motivation for the study of étale cohomology. The usual invariants from algebraic topology such as the fundamental groups and the cohomology groups, are very useful and one would like to have their analogues in the context of general algebraic varieties.

The problem was that the Zariski topology was too coarse to allow a detailed study of these topological invariants. Serre had already proved that it provided satisfactory results for the study of coherent sheaves. Moreover, he proved that for complex varieties these invariants would agree with the ones given by the usual topological techniques. However, the whole framework failed when studying other types of sheaves, for instance, when considering constant sheaves, all the cohomology groups are trivial, i.e. the rth cohomology groups for $r > 0$, are zero. The Zariski topology had other problems: the inverse mapping theorem fails and the usual fiber bundles are not locally trivial. There does not seem to exist a good way to fix these problems using a finer topology on general algebraic varieties.

Grothendieck's key insight was a clear understanding of the role played by the open sets of a topological space. He noticed that the important feature was not the open sets themselves but the category of all of them. In this setting, he introduced a generalization of the concept of a topological space, by replacing the category of open sets by a general category whose objects play the role of the open subsets, together with a notion of covering families for each object - a site.

He foresaw étale sites as a particular type of this generalized topology, by considering the open subsets of an algebraic variety as a certain class of maps over the variety, required to satisfy certain basic properties already known in other theories, like Galois Cohomology.

In Chapter 1, we start by presenting the concept of an étale morphism, that is, a morphism that at each point induces an isomorphism between the "tangent" spaces. We start by studying the case of étale morphisms between nonsingular varieties and prove that they satisfy the conditions of the implicit function theorem. Then, we introduce the general definition of an étale morphism between schemes.

In the beginning of Chapter 2 we introduce a new topological perspective: we define the notion of a site and generalize the usual structures used to study topological spaces (sheaves and continuous maps) to this new framework. After this brief introduction to the general theory of sites, we will focus on the étale site: the category of étale morphisms over a scheme X. We present give a brief description of the étale fundamental group of a scheme, followed by some properties about sheaves over the étale site. All of this is done with the Zariski topology always in mind, since one should understand that most of the results are straightforward generalizations, but there are some differences. For instance, the structure of the local ring when considering the étale topology (stalk) is richer than in the Zariski case. These extra structures appearing will allow further results, like the exactness of the Kummer sequence

$$0 \to \mu_n \to G_m \xrightarrow{t \mapsto t^n} G_m \to 0,$$

since it will be exact on the stalks, and in general this is not true in the Zariski topology. The Kummer sequence will turn out to be very important when computing the cohomology groups of μ_n,.

In our last chapter we aim to study some concrete examples of the étale cohomology of curves over an algebraically closed field, and compare it with Zariski cohomology. Firstly, we briefly overview some of the usual definitions and cohomological constructions, this time using the étale topology on a scheme. Following this review we start studying our concrete examples of the usage of cohomological methods on curves. We prove that the étale cohomology with values on the sheaf G_m, the analog of $\mathcal{O}_X^\times$, gives exactly the same values for the étale cohomology as for the Zariski cohomology.Under the same conditions we get

$$H^r\left(X_{et}, G_m\right)\begin{cases} = \Gamma(X, \mathcal{O}_X^\times), & r = 0 \\ \simeq \operatorname{Pic}(X), & r = 1 \\ = 0, & r > 1. \end{cases}$$

On the other hand, if we consider instead values on the sheaf $\mu_n \approx \mathbb{Z}/n\mathbb{Z}$, we find non-trivial cohomology groups, contrarily to what happens using the Zariski topology. Again, with some conditions, the main result is

$$H^r\left(X_{et}, \mu_n\right)\begin{cases} = \mu_n(k), & r = 0 \\ \approx \left(\mathbb{Z}/n\mathbb{Z}\right)^{2g}, & r = 1 \\ \simeq \mathbb{Z}/n\mathbb{Z}, & r = 2 \\ = 0, & r > 2. \end{cases}$$

More interesting is that the cohomology groups with coefficients in μ_n are analog to the results obtained when considering the sheaf $\mathbb{Z}$, with the complex topology. At last we focus on Poincaré duality for curves. Again, this result fails when considering the Zariski topology. In all these three examples we strongly use the fact that curves are defined over an algebraically closed field.

As main references we used Milne's lecture notes on Étale Cohomology [Mil08], and as supporting references also his book Étale Cohomology [Mil80] and Hartshorne's book on Algebraic Geometry [Har77].

Chapter 1

Étale Morphisms

An étale morphism is the algebraic analogue of the notion of a local isomorphism between manifolds in differential geometry, a covering of Riemann surfaces without branch points, etc. They satisfy the hypotheses of the implicit function theorem, but because open sets in the Zariski topology are so large, they are not necessarily local isomorphisms. Whenever we consider the special case of nonsingular algebraic varieties we require the induced map on tangent spaces be an isomorphism. In what follows, we try to give some intuition about what are étale morphisms on algebraic varieties. We will skip most of the proofs and focus on examples and results.

1.1 Étale morphisms of algebraic varieties

For the sake of simplicity, throughout this section, all the varieties will be defined over an algebraically closed field k, and all points are closed.

Definition 1.1.1. *Let V and W be two nonsingular algebraic varieties over k. We say that a morphism $\varphi : V \to W$ is **étale at a point** $v \in V$, if $d\varphi : T_v V \to T_{\varphi(v)} W$, the map induced on the tangent spaces, is an isomorphism. We say that φ is **étale** if it is étale at every point.*

Proposition 1.1.2. *Let $V \overset{def}{=} \operatorname{Spec}(A)$ be a nonsingular affine variety over k, and let W be the subvariety of $V \times \mathbb{A}^n$ defined by the equations*

$$g_i\left(Y_1, \ldots, Y_n\right) = 0, \quad g_i \in A\left[Y_1, \ldots, Y_n\right], \quad i = 1, \ldots, n.$$

The projection map $W \to V$ is étale at a point $(p; b_1, \ldots, b_n)$ if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is nonsingular at that point.

Proof: See [Mil08, 2.1 page 16]. □

Remark 1.1.3. *Notice that a nonsingular Jacobian matrix is a necessary condition for the inverse function theorem to hold.*

Example 1.1.4. *Let $V \overset{def}{=} \operatorname{Spec}(A)$ be a nonsingular variety over k, and let $\sum_{i=1}^{N} a_i T^i \overset{def}{=} f(T) \in A\left[T\right]$. We regard each coefficient of f, a_i, as a global regular function and therefore $f(p;T)$ as a continuous family of polynomials parametrized by $p \in V$. Let $W = \operatorname{Spec}(A[T]/f(T))$ (assuming the ring to be reduced), we regard W as a subvariety of $V \times \mathbb{A}^1$ defined by the zero set of the polynomial f as in Proposition 1.1.2, where the projection $\pi : V \times \mathbb{A}^1 \to V$ corresponds to the inclusion $A \hookrightarrow A[T]/f(T)$. Observe that $\pi^{-1}(p_0)$ corresponds to the zero set of the polynomial $f(p_0, T) \in k[T]$. Moreover:*

*(i) $\pi^{-1}(p_0)$ is finite if and only if p_0 is not a common zero of all the a_i; hence π is **quasi-finite**[1] if and only if $(a_1, \ldots, a_n) = A$, since the a_i having a common zero is equivalent to the ideal $(a_1, \ldots, a_n)$ being included in the maximal ideal associated to that point.*

[1] We say that a morphism f is quasi-finite if the inverse image of a point is always finite.

(ii) *π is **finite*** [2] *if and only if a_N is a unit in A.*

This comes from the fact that $A[T]/f(T)$ is finitely generated as an A-module if and only if f is monic. One direction is clear: if f is monic the quotient can be finitely generated by $1, T, \ldots, T^{N-1}$, since for all $k \geq N$ we have $T^k = T^N T^{k-N} = (T^N - a_N^{-1} f(T))T^{k-N} \in A[T]/f(T)$ that now has at most degree $k-1$ and we continue this process until all the powers of T have degree less than N. For the other direction, suppose that the quotient is finitely generated as a module by $\tilde{e_1}, \ldots, \tilde{e_n}$, take e_i to be the representatives of each class with minimal degree, and M to be the maximal degree of e_i. We know that we can write $T^{M+1} = \sum b_i \tilde{e_i}$ in $A[T]/f[T]$, so $T^{M+1} - \sum b_i e_i = g(T)f(T)$, but the left hand side is monic, hence f and g are monic.

(iii) *π is étale at $(p_0; c)$ if and only if c is a simple root of $f(p_0, T)$. This follows directly from the Proposition 1.1.2.*

Example 1.1.5. *Consider the map $x \mapsto x^n : \mathbb{A}^1 \to \mathbb{A}^1$. Since $\frac{dX^n}{dX} = nX^{n-1}$ from the Proposition 1.1.2 follows that the map is étale at no point if the characteristic of the field is n and is étale at all $x \neq 0$ otherwise.*

The tangent space works perfectly for nonsingular points of a variety, but when we try to apply this concept to singular points the results are not the expected ones, see the below Example 1.1.7. To get the expected results we need to generalize the concept of tangent space, and look at the tangent cone.

Definition 1.1.6. *Take $V = \mathrm{Spec}(k[X_1, \ldots, X_n]/I)$, where k is a field. Let $I_* = \{f_* : f \in I\}$, where f_* is the homogeneous part of f of lowest degree. With this we define the **tangent cone** at the origin p as*

$$C_p(V) \stackrel{def}{=} \mathrm{Spec}(k[X_1, \ldots, X_n]/I_*).$$

The tangent cone for a general point q is obtained by the natural coordinate transformation mapping q to the origin.

Example 1.1.7.

(i) *For the variety defined by the equation $Y^2 = X^3$, the tangent cone at $(0,0)$ is defined by $Y^2 = 0$, i.e., the $X - axis$ "doubled".*

(ii) *In the case that the variety is defined by $Y^2 = X^2(X+1)$, the tangent cone at the origin is defined by $Y^2 = X^2$, i.e., the union of the lines $Y = \pm X$.*

In both cases if we had used the usual definition of tangent space, we would get that the tangent spaces would be $\mathbb{A}^2$, which clearly fails to represent our intuitive idea of tangent space.

Now we can expand our definition of étale morphism for general varieties.

Definition 1.1.8. *We say that $\varphi : W \to V$, a morphism of varieties over an algebraically closed field k, is **étale at a point** $p \in W$ if it induces an isomorphism $C_p(W) \to C_{\varphi(p)}(V)$ of tangent cones.*

There are two different natural problems emerging from the previous considerations:

- **Question 1**: How to compute the tangent cone?
- **Question 2**: How to compute the induced map?

The computation of the tangent cone is not as trivial as it may seem by the previous examples. Clearly if $I = (f)$ then $I_* = (f_*)$, but the same does not hold when we have more generators, i.e., if $I = (f_1, \ldots, f_n)$ when $n > 1$ then generally $I_* \neq (f_{1*}, \ldots, f_{N*})$. Consider the following example.

Example 1.1.9. *Let $\sqrt{I} = I \stackrel{def}{=} (XY, XZ + Z(Y^2 - Z^2))$. We have that*

$$f \stackrel{def}{=} YZ(Y^2 - Z^2) = Y(XZ + Z(Y^2 - Z^2)) - Z(XY) \in I,$$

and since f is homogeneous, $f \in I_*$. Clearly $f \notin (XY, XZ)$. In fact

$$I_* = (XY, YZ, YZ(Y^2 - Z^2))$$

[2] We say that a morphism $f : W \to V$ is finite if there exists a covering of V by open affine subsets $V_i = \mathrm{Spec}(B_i)$ such that $f^{-1}(V_i) = \mathrm{Spec}(A_i)$ where A_i is a B_i algebra which is finitely generated as a B_i-module.

This observation raises the following question: how to find the generators for I_* when generators of I are given? Fortunately there is an algorithm to solve this problem, see [CLO07, page 467].

The map induced on the rings $k[X_1, \dots, X_n]/I_*$ by the map of algebraic varieties is not the obvious one, i.e., it is not necessarily induced by the same map on polynomial rings as the original map. To solve this problem we need to introduce more concepts, so we can get a more intrinsic definition of the tangent cone. Let us start by introducing some notation.

Definition 1.1.10. *Let A be a local ring with maximal ideal m. The* ***associated graded ring*** *is*

$$gr(A) \stackrel{def}{=} \oplus_{i \geq 0} m^i/m^{i+1}.$$

For more details [AM69, Chapter 10].

Remark 1.1.11. *Observe that if $A = B_n$ for some ring B and $m = nA$, then $gr(A) = \oplus n^i/n^{i+1}$. This comes from the fact $n^i/n^j \cong m^i/m^j$.*

Proposition 1.1.12. *Let V be an algebraic variety and $\mathcal{O}_V$ the structure sheaf associated to it. The map $k[X_1, \dots, X_n]/I_* \to gr(\mathcal{O}_{V,p})$ sending the class of X_i in $k[X_1, \dots, X_n]/I_*$ to the class of X_i in $gr(\mathcal{O}_{V,p})$ is an isomorphism.*

Proof: Let m be the maximal ideal in $k[X_1, \dots, X_n]/I$ corresponding to the closed point $p \stackrel{\text{def}}{=}$origin. Now we have

$$\begin{aligned} gr(\mathcal{O}_{V,p}) &= \sum m^i/m^{i+1} \\ &= \sum (X_1, \dots, X_n)^i/(X_1, \dots, X_n)^{i+1} + I \cap (X_1, \dots, X_n)^i \\ &= \sum (X_1, \dots, X_n)^i/(X_1, \dots, X_n)^{i+1} + I_i \end{aligned}$$

where I_i is the homogeneous piece of I_* of degree i, i.e., the subspace of I_* consisting of homogeneous polynomials of degree i). But now it is clear that

$$(X_1, \dots, X_n)^i/(X_1, \dots, X_n)^{i+1} + I_i = i\text{th homogeneous piece of } k[X_1, \dots, X_n]/I_*$$

For any other point q it follows in the same way after we do the natural coordinate transformation mapping q to the origin. □

A **local homomorphism** [3] $\varphi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ clearly induces a homomorphism on the associated graded rings $gr(A) \to gr(B)$, in this case it is really a **graded homomorphism** [4] since $\varphi(\mathfrak{m}_A^i) \subset \mathfrak{m}_B^i$. It can be proved that if $gr(A) \to gr(B)$ is an isomorphism, so is the map induced on the completions $\hat{A} \to \hat{B}$ [AM69, 10.23 page 112]. The converse is also true, because $gr(A) = gr(\hat{A})$ [AM69, 10.22 page 111].

With this we get the following criterion:

> A regular map $\varphi : W \to V$ of varieties over an algebraically closed field k is étale at a point $p \in W$ if and only if the map $\widehat{\mathcal{O}}_{V,\varphi(p)} \to \widehat{\mathcal{O}}_{W,p}$ induced by φ is an isomorphism.

Observe that the tangent cone holds information about the singularity type of the point, so if p maps to q, then a necessary condition for φ to be étale at p, is that V must have the same type of singularity at q as W has at p.

Remark 1.1.13. *Since a morphism between V and W induces a morphism between the local k-algebras, it also induces a local homomorphism between the stalks $\mathcal{O}_{V,\varphi(p)} \to \mathcal{O}_{W,p}$. Hence we get the induced map on tangent cone.*

Example 1.1.14. *(i) Take again the variety V defined by the equation $Y^2 = X^3 + X^2$. Consider the map*

$$\begin{aligned} \varphi : \mathbb{A}^1 &\longrightarrow V \\ t &\longmapsto (t^2 - 1, t(t^2 - 1)). \end{aligned}$$

[3] We say that a morphism between two local rings $\varphi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a local homomorphism if it is a homomorphism and $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

[4] Let $A = \oplus A_i$ and $B = \oplus B_i$ be graded rings, we say that a homomorphism $\varphi : A \to B$ is graded if $\varphi(A_i) \subset B_i$.

As k-algebras this corresponds to the map:

$$\begin{aligned}\tilde{\varphi} : k[X,Y](Y^2 - X^2 - X^3) &\longrightarrow k[T]\\ X &\longmapsto T^2 - 1\\ Y &\longmapsto T(T^2-1).\end{aligned}$$

Now we are going to check if this map is étale at $p = 1 \in \mathbb{A}^1$. The tangent cone of $\mathbb{A}^1$ at p is $k[s]$ where s denotes the class of $T-1$ in $\mathfrak{m}_p/\mathfrak{m}_p^2 = (T-1)/(T-1)^2$. In the case of V we get that the tangent cone at the origin, $\varphi(1) = (0,0) \overset{def}{=} q$, is $k[x,y]/(x^2 - y^2)$ where x and y denote the classes of X and Y in $\mathfrak{m}_q/\mathfrak{m}_q^2$.
Now the induced morphism on the graded rings is the one that takes x to the class T^2-1 in $\mathfrak{m}_p/\mathfrak{m}_p^2$ and y to the class $T(T^2-1)$. Recall that $\mathfrak{m}_p/\mathfrak{m}_p^2 = (s)/(s)^2$, so

$$\begin{aligned}x &\longmapsto T^2 - 1 \equiv (T-1)(T+1) \equiv s(s+2) \equiv s^2 + 2s \equiv 2s \in \mathfrak{m}_p/\mathfrak{m}_p^2\\ y &\longmapsto T(T^2-1) \equiv T(T-1)(T+1) \equiv (s+1)s(s+2) \equiv s^3 + 3s^2 + 2s \equiv 2s \in \mathfrak{m}_p/\mathfrak{m}_p^2.\end{aligned}$$

So $gr(\tilde{\varphi}) : k[x,y]/(x^2-y^2) \to k[s]$ on degree zero is the identity and on degree 1 is

$$\begin{aligned}gr(\tilde{\varphi})_1 : (x,y)/(x,y)^2 \simeq (X,Y)/(X,Y)^2 + (X,Y) \cap (X^2 - Y^2) &\longrightarrow (s)/(s)^2 \simeq (T-1)/(T-1)^2\\ x &\longmapsto 2s\\ y &\longmapsto 2s.\end{aligned}$$

Hence this map is not an isomorphism, and so the map is not étale at p.

(ii) *Let us now consider the map*

$$\begin{aligned}\varphi : \mathbb{A}^1 &\longrightarrow V = Spec\big(k[X,Y]/(Y^2 - X^3)\big)\\ t &\longmapsto (t^2, t^3)\end{aligned}$$

Consider $p = 0$, the tangent cone at the origin of the affine line is clearly $k[t]$ where t denotes the class of T in $\mathfrak{m}_p/\mathfrak{m}_p^2 = (T)/(T)^2$. The tangent cone of V at $\varphi(0) = 0 \overset{def}{=} q$ is $k[x,y]/y^2$ where x and y denote the class of X and Y on $\mathfrak{m}_q/\mathfrak{m}_q^2$ respectively.
The map induced on the algebras is

$$\begin{aligned}\tilde{\varphi} : k[X,Y]/(Y^2 - X^3) &\longrightarrow k[T]\\ X &\longmapsto T^2\\ Y &\longmapsto T^3.\end{aligned}$$

Similarly to the previous example,

$$\begin{aligned}x &\longmapsto t^2 \equiv 0 \in \mathfrak{m}_p/\mathfrak{m}_p^2\\ y &\longmapsto t^3 \equiv 0 \in \mathfrak{m}_p/\mathfrak{m}_p^2,\end{aligned}$$

and so the map induced on the graded rings $gr(\tilde{\varphi}) : k[x,y]/y^2 \to k[t]$ is the identity on degree zero and

$$\begin{aligned}gr(\tilde{\varphi})_1 : (x,y)/(x,y)^2 \simeq (X,Y)/(X,Y)^2 + (X,Y) \cap (Y^2) &\longrightarrow (t)/(t)^2 \simeq (T)/(T)^2\\ x &\longmapsto 0\\ y &\longmapsto 0\end{aligned}$$

on degree one, Clearly it is not an isomorphism.

Remark 1.1.15. *In the case that k is not algebraically closed, it is easy to generalize the idea of étale morphism. We say that a morphism $\varphi : W \to V$ is étale at p if for some algebraic closure of k^{al} of k, $\varphi_{k^{al}} : W_{k^{al}} \to V_{k^{al}}$ is étale at the points of $W_{k^{al}}$ mapping to p.*

1.2 Étale morphisms of schemes

To define étale morphisms of schemes we need to recall some concepts of commutative algebra.

Definition 1.2.1. *A homomorphism of rings $A \to B$ is said to be **flat** if the functor $M \to B \otimes_A M$ from A-modules to B-modules is exact. Sometimes we will say say that B is a **flat** A**-algebra**.*

Remark 1.2.2. *It is a well known result of commutative algebra that the functor $M \to B \otimes_A M$ is always right exact. So we only need to check the left exactness [AM69, 2.18 page 28]. Another useful result is the fact that $f : A \to B$ is flat if and only if $A_{f^{-1}(\mathfrak{m})} \to B_{\mathfrak{m}}$ is flat for every maximal ideal of B [AM69, 3.10 page 41].*

Definition 1.2.3. *A morphism of schemes $\varphi : Y \to X$ is **flat** if for any pair of affine open sets V and U such that $\varphi(V) \subset U$, the induced map $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is flat, or equivalently (by the previous remark) if the local homomorphisms $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ are flat for all closed points $y \in Y$.*

Definition 1.2.4. *A morphism of schemes $\varphi : Y \to X$ that is locally of finite type is said to be **unramified** at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_{\varphi(y)}\mathcal{O}_{Y,y}$ is a finite separable field extension of $k(\varphi(y)) = \mathcal{O}_{X,\varphi(y)}/\mathfrak{m}_{\varphi(y)}$.*

*In terms of rings, this says that a homomorphism $f : A \to B$ of finite type is **unramified** at $\mathfrak{q} \in \operatorname{Spec}(B)$ if and only if $\mathfrak{p} \stackrel{def}{=} f^{-1}(\mathfrak{q})$ generates the maximal ideal in $B_{\mathfrak{q}}$ and $k(\mathfrak{q}) = B_{\mathfrak{q}}/\mathfrak{q}$ is a finite separable field extension of $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$.*

*A morphism of schemes $\varphi : Y \to X$ is **unramified** if it is unramified at all $y \in Y$.*

Remark 1.2.5. *A map is unramified at $x \in X$ if and only if $\Omega^1 Y/X_x$ is zero, [Sta10, Lemma O2GF], and $\Omega^1 Y/X$ is a coherent sheaf, hence to verify that the map is unramified it is sufficient to check that it is unramified for the closed points.*

Definition 1.2.6. *A morphism $\varphi : Y \to X$ of schemes is **étale** if it is flat and unramified, hence locally of finite type.*

*A homomorphism of rings $f : A \to B$ is **étale** if the induced map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is étale, which means:*

(i) *B is a finitely generated A-algebra;*

(ii) *B is a flat A-algebra;*

(iii) *for all maximal ideals m of B, $B_m/f(n)B_m$ is a finite separable field extension of A_n/n where $n = f^{-1}(m)$.*

Example 1.2.7. *For any proper ideal $I \subset A$ the map $A \to A/I$ is unramified, but not generally flat.*

Take $B \stackrel{def}{=} A \times A$, and let $I \stackrel{def}{=} (1, 0)$. The quotient map $B \to B/I \simeq A$ is flat.

In fact, a flat morphism $\varphi : Y \to X$ of varieties is the analogue in algebraic geometry of a continuous family of manifolds $Y_x \stackrel{\text{def}}{=} \varphi^{-1}(x)$ parametrized by the points of X. If φ is flat morphism of schemes over a field k then

$$\dim(\mathcal{O}_{Y_x,y}) = \dim(\mathcal{O}_{Y,y}) - \dim(\mathcal{O}_{X,x})$$

[Har77, III.9.5] and the converse is true if X and Y are regular varieties [Har77, Ex III.10.9]. The next example follows from this observation:

Example 1.2.8. *A inclusion of a closed subscheme is flat if and only if the subscheme is a connected component. This is a generalization of the Example 1.2.7.*

Proposition 1.2.9. *For a morphism $\varphi : Y \to X$ between varieties over an algebraically closed field, the definition of "étale" in this subsection agrees with the one in the previous section.*

PROOF: See [Mil08, 2.9 page 20]. □

Now some examples.

Example 1.2.10.

(i) **The Jacobian criterion**: *Proposition 1.1.2 still holds with V an regular affine scheme.*

(ii) **Fields**: *Let k be a field and $f : k \to A$ an étale homomorphism. We will call A an **étale k-algebra**. Since A is unramified over k, for every maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is finite separable extension of k, hence A has dimension 0 and therefore is an Artin ring [AM69, 8.5 page 90]. Thus, A is a finite product of Artinian local rings [AM69, 8.7 page 90], in our case this means that A is a finite product of finite separable extensions of k. Hence, A is a étale k-algebra if and only if it is a finite product of finite separable extensions of k.*

(iii) **Standard étale morphisms**: *Let A be a ring and $f(T) \in A[T]$ a monic polynomial, thus $A[T]/f(T)$ is a free A-module of finite rank, hence flat. For any $b \in A[T]/f[T]$ such that $f'(T)$ is invertible in $(A[T]/f(T))_b$, the homomorphism $A \to (A[T]/f[T])_b$ is étale, (1.1.4) and (1.1.2).*

*An étale homomorphism $\varphi : V \to U$ is said to be **standard** if it is isomorphic to the* Spec *of such homomorphism, i.e., if the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\approx} & \operatorname{Spec}\left((A[T]/f(T))_b\right) \\ \downarrow{\scriptstyle \varphi} & & \downarrow \\ U & \xrightarrow{\approx} & \operatorname{Spec}(A). \end{array}$$

In fact, every étale morphism is locally standard: for any étale morphism $\varphi : Y \to X$ and any $y \in Y$, there exists open affine neighborhoods V of y and U of $\varphi(y)$ such that $\varphi(V) \subset U$ and $\varphi|V : V \to U$ is standard [Mil80, I 3.14 page 26].

As we already said, étale morphism are the analog of local isomorphisms. We will now just present some very well known properties of étale morphisms, that justify this analogy.

Proposition 1.2.11.

(i) Any open immersion is étale.

(ii) The composite of two étale morphisms is étale.

(iii) Any base change of an étale morphism is étale.

(iv) If $\varphi \circ \psi$ and φ are étale, then so also is ψ.

(v) Let $\varphi : Y \to X$ be a morphism of finite type. The set of points $U \subset Y$ where φ is étale is open in Y.

Now let $\varphi : Y \to X$ be an étale morphism.

(vi) For all $y \in Y$, $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,\varphi(y)}$ have the same Krull dimension.

(vii) Whenever X is normal, Y is normal.

(viii) Whenever X is regular, Y is regular.

(ix) The morphism φ is quasi-finite.

(x) The morphism φ is open.

PROOF:

(i) – *(iii)* See [Mil80, I 3.3 page 22].

(iv) It is a particular case of [Mil80, I 3.6 page 24].

(v) See [Mil80, I 3.8 page 24].

(vi) – *(viii)* See [Mil80, I 3.17 page 27].

(ix) Consequence of locally being a standard étale morphism, [Mil80, I 3.14 page 26].

(x) Consequence of [Mil80, Proposition I.2.12 page 14].

□

There is a special type of étale maps that will be very useful throughout the rest of the text. As we will see in the next chapter they will be a special case of the analogue of finite covering spaces.

Definition 1.2.12. *Let $\pi : Y \to X$ be a* ***faithfully flat***[5] *morphism and let G be a finite group acting on Y over X on the right. Then π is called a* ***Galois covering of X with group*** *G if the morphism $Y \times G \to Y \times_X Y : (y, g) \mapsto (y, y \cdot g)$ is an isomorphism.*

Remark 1.2.13. *In other words we are saying that two points are in the same fiber if and only if we can take one to the other by the action of G, and the action is transitive.*

Here $Y \times G$ denotes $\coprod_g Y_g$, where $Y_g = Y$. If π is a Galois covering, then π is also surjective, finite and étale of degree equal to the order of G. Conversely, it is possible to prove that if $Y \to X$ is surjective, finite and étale of degree equal to the order of $\mathrm{Aut}_X(Y)$, then it is Galois with group $\mathrm{Aut}_X(Y)$.

Remark 1.2.14. *For any finite étale morphism $Y \to X$, there exists a Galois covering $Y' \to X$ that factors through $Y \to X$ [Mur67, 4.4.1.8].*

Remark 1.2.15. *If $Y \to X$ is a Galois covering with group G and Y is connected then, $G = \mathrm{Aut}_X(Y)$. Since Y is connected $Aut_X(Y)$ acts faithfully on the fiber, which is G. On the other hand, G acts transitively on the fiber, hence we have a bijection [Mil80, page 40].*

Example 1.2.16. *Let $\mathbb{A}^1$ be the affine line over an algebraically closed field k of characteristic zero. Consider the finite étale maps*

$$\begin{aligned} \varphi_n : \mathbb{A}^1\backslash\{0\} &\to \mathbb{A}^1\backslash\{0\} \\ t &\mapsto t^n. \end{aligned} \qquad (\textit{Example } 1.1.5)$$

Let $X_n \to \mathbb{A}^1\backslash\{0\}$ denote the covering φ_n, then

$$Aut_{\mathbb{A}^1\backslash\{0\}}(X_n) = \mu_n(k). \qquad (\textit{the group of } n\textit{th roots of 1 in } k)$$

The action of $\mu_n(k)$ on X_n over $\mathbb{A}^1\backslash\{0\}$ is defined by $x \mapsto \zeta x$, where $\zeta \in \mu_n(k)$. Therefore, φ_n is a Galois covering of $\mathbb{A}^1\backslash\{0\}$ with group $\mu_n(k)$.

[5]A morphism $f : Y \to X$ is **faithfully flat** if it is flat and surjective.

Chapter 2

The étale site

2.1 Sites

In order to extend the usage of sheaves to any category we need to generalize the role played by the families of open coverings in the category of open sets of a topological space. We try to do this in the following way:

Definition 2.1.1. *Let* $\boldsymbol{C}$ *be a category such that for each* $U \in Obj(\boldsymbol{C})$ *we have a distinguished set of families maps* $\{U_i \to U\}_{i\in\mathcal{I}}$*, called* ***coverings*** *of U, satisfying the following axioms:*

(i) for any covering family $\{U_i \to U\}_{i\in\mathcal{I}}$ *and any morphism* $V \to U$ *in* $\boldsymbol{C}$*, the fibre products* $U_i \times_U V$ *exists, and* $\{U_i \times_U V \to V\}_{i\in\mathcal{I}}$ *is a covering family of* V*;*

(ii) if $\{U_i \to U\}_{i\in\mathcal{I}}$ *is a covering family of* U*, and if for each* $i \in \mathcal{I}$*,* $\{V_{ij} \to U_i\}_{j\in\mathcal{J}}$ *is a covering family of* U_i*, then the family* $\{V_{ij} \to U\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ *is a covering family of U;*

(iii) for any U in $\boldsymbol{C}$*, the family* $\{U \xrightarrow{id} U\}$ *consisting of a single map is a covering family of U.*

The system of covering families is called a ***(Grothendieck) topology****, and* $\boldsymbol{C}$ *together with the topology is called a* ***site****.*

Example 2.1.2. *Let* X *be a topological space, we define* $O(X)$ *to be the category where the objects are the open sets and whose morphisms are the inclusions. Then the families* $\{U_i \to U\}_{i\in\mathcal{I}}$ *such that* $U = \cup_{i\in\mathcal{I}} U_i$ *are coverings for the Grothendieck Topology. In this case, for open subsets* U *and* U' *of* V *we have* $U \times_V U' = U \cap U'$

Definition 2.1.3. *Let* $\boldsymbol{C}$ *be a site, we define a* ***presheaf*** $\mathcal{F}$ *over* $\boldsymbol{C}$ *with values in a category* $\boldsymbol{A}$*, as a contravariant functor from* $\boldsymbol{C}$ *to* $\boldsymbol{A}$*.*

Remark 2.1.4. *Observe that this definition does not depend on the topology of* $\boldsymbol{C}$*.*

Definition 2.1.5. *A presheaf* $\mathcal{F}$ *is called a* ***sheaf*** *if for every covering* $\{U_i \to U\}_{i\in\mathcal{I}}$ *of an open subset U the following sequence is exact in* $\boldsymbol{A}$*:*

$$\mathcal{F}(U) \to \prod_{i\in\mathcal{I}} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in\mathcal{I}} \mathcal{F}(U_i \times_U U_j).$$

(This definition makes sense whenever the category $\boldsymbol{C}$ *has all equalizers, which includes* $\boldsymbol{C} =$ ***Sets*** *or any abelian category.)*

We define a ***morphism of presheaves*** *as natural transformation and a* ***morphism of sheaves*** *is a morphism of presheaves.*

In the cases that we are interested, the exactness of the previous sequence by definition is equivalent to say that the first arrow maps $F(U)$ injectively into the subset of $\prod_{i\in\mathcal{I}} \mathcal{F}(U_i)$, where the next two arrows agree.

When $\mathbf{A} = \mathbf{Ab}$, the category of abelian groups, this is equivalent to the exactness of the following sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \prod_{i\in\mathcal{I}} \mathcal{F}(U_i) & \longrightarrow & \prod_{i,j\in\mathcal{I}} \mathcal{F}(U_i \times_U U_j) \\ & & f & \longmapsto & (f|U_i) & & \\ & & & & \{f_i\} & \longmapsto & (f_i|U_i \times_U U_j - f_j|U_i \times_U U_j) \end{array}$$

Remark 2.1.6. *When $\boldsymbol{C}$ arises from a topological space like in Example 2.1.2, the definitions of presheaves and sheaves coincide with usual ones.*

Definition 2.1.7. *A family of regular maps or a morphisms schemes $\{\varphi_i : U_i \to U\}_{i\in I}$ is said to be* ***surjective*** *if $U = \cup_{i\in I}\varphi_i(U_i)$.*

Example 2.1.8. *Let X be a scheme.*

(i) **The Zariski site on X***: The site X_{zar} is the one associated with X regarded as topological space for the Zariski topology as in Example 2.1.2, i.e., the coverings are families of Zariski open immersions that are (as a family) surjective.*

(ii) **The étale site on X***: The site X_{et} is the category Et/X, whose objects are the étale morphism $U \to X$ and whose arrows are the commutative diagrams*

$$\begin{array}{ccc} U & \longrightarrow & U' \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

The coverings of X_{et} are the surjective families of étale morphisms $\{U_i \to U\}$ in Et/X.

(iii) **The flat site on X***: The site X_{Fl} is the category Sch/X, of all X-schemes. The coverings are the surjective families of X-morphisms $\{U_i \xrightarrow{\varphi_i} U\}$ with each φ_i flat and of finite-type.*

(iv) **The big étale site on X***: The site X_{Et} is the category Sch/X. The coverings are the surjective families of étale X-morphisms $\{U_i \xrightarrow{\varphi_i} U\}$.*

Example 2.1.9. *In the case that $x = \mathrm{Spec}(k)$ and k an algebraically closed field, the site x_{et} is trivial, in the sense that a point is trivial as a topological space. Recall that the site is equivalent to the category of étale k-algebras, and in this case an étale k-algebra is a finite product of k, (1.2.10). Therefore, any covering is just a finite disjoint union of copies of x, that project to x.*

Definition 2.1.10. *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two sites. A functor $\boldsymbol{T}_2 \to \boldsymbol{T}_1$ transforming coverings into coverings is called a continuous map from $\boldsymbol{T}_1$ to $\boldsymbol{T}_2$.*

Example 2.1.11.

A map of topological spaces $Y \to X$ defines a continuous map between corresponding sites if and only if it is continuous in the usual sense.

There are obvious continuous maps of sites

$$X_{Fl} \to X_{Et} \to X_{et} \to X_{zar}.$$

Which correspond to the natural inclusions

$$X_{zar} \hookrightarrow X_{et} \hookrightarrow X_{Et} \hookrightarrow X_{Fl}.$$

2.2 The étale fundamental group

Many notions commonly present in general topology have analogs on the étale site. For example one can wonder what plays the role of a point. Example 2.1.9 implies that next definition makes sense. This generalizes the idea of point on the étale site.

Definition 2.2.1. *Let $x \in X$ and Ω be a* ***separably algebraically closed field***[1] *that contains the residue field at x, i.e., $k(x) \stackrel{def}{=} \mathcal{O}_{X,x}/m_x \subset \Omega$. A* ***geometric point*** *$\bar{x}$ of X over x is a map $\bar{x} : \mathrm{Spec}(\Omega) \to X$ factoring through x.*

Remark 2.2.2. *Our use of the concept of a geometric point will be very convenient, according to the situation, i.e., we defined it as a map, sometimes we will identify it only as the "point"* Spec(Ω) *and more often we will just say "Let $\bar{x} \to X$ be a geometric point of X".*

Example 2.2.3. *When x is a closed point and X is defined over an algebraically closed field k then $x = \overline{x}$[2], i.e., all the closed points are geometric points, since every residue field of a closed point is naturally isomorphic k.*

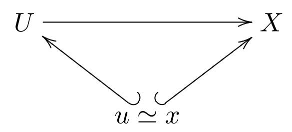
This is not the case when we are not over a separably algebraically closed field. First we do not have the Hilbert's Nullstellensatz theorem to tell us that the residue fields of the closed points of the variety will be our base field. Even if that happened, since our base field is not separably algebraically closed, the closed points might not be geometric points.

A more concrete example: when our base field is $\mathbb{R}$, there are two types of closed points of a variety over $\mathbb{R}$. They are classified by their residue field which is either $\mathbb{R}$ or $\mathbb{C}$. In both cases the geometric points will always come from Spec($\mathbb{C}$).

Now we will define the concept of neighborhood in the étale topology.

Definition 2.2.4. *An* ***étale neighborhood*** *of $x \in X$ is a pair (U, u) where U is an étale X-scheme and u is a point of U mapping to x such that $k(u) \simeq k(x)$.*

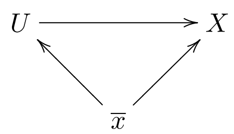
Remark 2.2.5. *Clearly this is equivalent to say that the diagram:*



is commutative.

Hence, the next definition makes sense.

Definition 2.2.6. *Let $\overline{x} \to X$ a geometric point of X. An* ***étale neighborhood*** *of $\overline{x}$ is a commutative diagram:*



where $U \to X$ is étale.

There are two ways to define the fundamental group $\pi_1(X, x_0)$ of the topological space X (satisfying the usual requirements, see [Mun75]) at the base point x_0. The first one is to notice the natural group structure on the set of closed paths through x_0 modulo homotopy equivalence; the second strategy is to define it as the group of automorphisms of the universal covering space of X. Since there are too few algebraically defined closed paths, the first does not generalize well to schemes, but the second does. Now we will sketch the construction of the étale fundamental group.

For proofs of the statements in this subsection and much more details about the following construction see [Mur67]. An 8 page summary of properties of the étale fundamental group is available in [Mil80, I.§5].

Let X be a connected scheme, and recall that a finite map $\pi : Y \to X$ is closed [Mil09, 8.7 page 135]. By (1.2.11) we conclude that a finite étale map is closed and open, hence surjective. Moreover, for each $x \in X$ there is an étale neighborhood of $(U, u) \to (X, x)$ such that $Y \times_X U$ is a disjoint union of open subschemes U_i each of which is mapped isomorphically onto U [Sta10, Lemma 04HN]. Thus, a finite étale map is the natural analogue of a finite covering space.

[1]We say that a field k is **separably algebraically closed** if any separable element of k^{al}, the algebraic closure of k, over k belongs to k.

[2]Strictly speaking this is not correct since $\overline{x}$ can be given by a transcendental extension of $k(x)$.

Let **FEt/X** be the category whose objects are the finite étale maps $\pi : Y \to X$, sometimes referred to as finite étale coverings, and whose arrows are the X-morphisms. We define the functor $F : \mathrm{FEt}/X \to \mathrm{Sets}$ to be the functor sending (Y, π) to the set of $\overline{x}$-valued points of Y lying over x, so $F(Y) = \mathrm{Hom}_X(\overline{x}, Y)$. In the case that X is a variety over an algebraically closed field and $\overline{x} = x$ is a closed point (2.2.3), then $F(Y) = \pi^{-1}(x)$.

If X is a topological space with a universal covering space, it can be proved that the fiber functor F is representable by the universal covering. Not every space has a universal cover. Even though, in general, it may be proved that F is strictly prorepresentable, that is, there exists a projective system $\tilde{X} = \{X_i\}_{i\in I}$ of finite étale-coverings of X indexed by a directed set I such that

$$F(Y) = \mathrm{Hom}(\tilde{X}, Y) \stackrel{\text{def}}{=} \varinjlim \mathrm{Hom}(X_i, Y) \quad \text{functorial in Y.}$$

This projective system $\tilde{X}$ plays the role of the universal covering space of X. It is possible to choose $\tilde{X}$ such that each X_i is Galois over X, with X_i connected, i.e., $X_i \to X$ has degree over X equal to the order of $\mathrm{Aut}_X(X_i)$ (1.2.15). Choosing this path $\{\mathrm{Aut}_X(X_i)\}_{i\in I}$ becomes a projective system, hence we can define

$$\pi_1(X, \overline{x}) = \mathrm{Aut}_X(\tilde{X}) \stackrel{\text{def}}{=} \varprojlim \mathrm{Aut}_X(X_i).$$

Thus, $\pi_1(X, \overline{x})$ generalizes the deck transformation group.

Example 2.2.7. *Let $\mathbb{A}^1$ be the affine line over an algebraically closed field k of characteristic zero. In Example 1.2.16 we computed some of the Galois coverings of $\mathbb{A}^1\backslash\{0\}$. In fact these coverings are sufficient to calculate the fundamental group of the punctured affine line [Mil80, I.5.1c]. Hence,*

$$\pi_1(\mathbb{A}^1\backslash\{0\}, \overline{x}) = \varprojlim(\mu_n(k) \approx \mathbb{Z}/n\mathbb{Z}) \approx \hat{\mathbb{Z}},$$

where the ˆ denotes the profinite completion, i.e., $\hat{\mathbb{Z}} \simeq \prod \mathbb{Z}_p$. The limit is over the primes and $\mathbb{Z}_p$ denotes the p-adic integers.

Example 2.2.8. *Let $\mathbb{P}^1$ be the projective line over an algebraically closed field. The differential $\omega = dz$ on the projective line has a double pole at ∞ and no other poles or zeros. For any étale covering $\pi : Y \to \mathbb{P}^1$ with Y connected and degree n, the differential $\pi^*(w)$ has $2n$-poles and no zeros. Thus, the degree of the divisor of $\pi^*(w)$ is -2n. By the Riemann–Hurwitz formula this equals $2\,\mathrm{genus}(Y) - 2$, hence the degree of π must be one, therefore π is an isomorphism. Hence, $\pi_1(\mathbb{P}^1, \overline{x}) = \{1\}$.*

The same argument shows that $\pi_1(\mathbb{A}^1, \overline{x}) = \{1\}$ when the base field has characteristic zero. In general when the characteristic is different from zero we don't have $\pi_1(\mathbb{A}^1, \overline{x}) = 0$.

The action of $\pi_1(X, \overline{x})$ on $\tilde{X}$ (on the right), defines a left action of $\pi_1(X, \overline{x})$ on $F(Y)$ for each finite étale covering Y of X, since $\mathrm{Hom}(-, Y)$ is a contravariant functor. This action is continuous when $F(Y)$ is given the discrete topology and $\pi_1(X, \overline{x})$ has the usual topology of profinite groups [Sha72]. In this case, it can be proved that the action factors through some finite quotient $\mathrm{Aut}_X(X_i)$ for some $i \in I$ [Sha72]. Observe that this is the analogue of the case when X is a topological space with a universal covering space.

Remark 2.2.9. *When $Y \to X$ is a Galois covering with Y connected , it follows immediately (1.2.15) that*

$$F(Y) \approx \mathrm{Aut}_X(Y) \simeq \mathrm{Aut}_X(\tilde{X})/\,\mathrm{Aut}_Y(\tilde{X}) \stackrel{def}{=} \pi_1(X, \overline{x})/\pi_1(Y, \overline{x}),$$

and the action of $\pi_1(X, \overline{x})$ on $F(Y)$ is just the group multiplication on the left.

Theorem 2.2.10. *The functor F defines an equivalence between the category of finite étale coverings of X and the category of finite sets with a continuous $\pi_1(X, \overline{x})$-action.*

Thus, $\pi_1(X, \overline{x})$ classifies the finite étale coverings of X in the same way that the topological fundamental group classifies the covering spaces of a topological space.

Corollary 2.2.11. *The Galois coverings of X with group G are classified by the continuous homomorphisms $\pi_1(X, \overline{x}) \to G$.*

Proof:(Sketch) Restricting the previous equivalence to the Galois coverings with group G, we get a bijection between the Galois coverings of X with group G and the isomorphism classes of continuous actions of $\pi_1(X, \overline{x})$ in G on the left, and each of these classes is represented up to isomorphism by a continuous homomorphism $\pi_1(X, \overline{x}) \to G$.

We will sketch the other direction. Given $\varphi \in \mathrm{Hom}_{\mathrm{conts}}(\pi_1(X, \overline{x}), G)$, we know that φ will factor through some $G_i \overset{\mathrm{def}}{=} \mathrm{Aut}_X(X_i)$, i.e., we have a commutative diagram:

$$\begin{array}{ccc} \varprojlim G_i & \xrightarrow{\ \varphi\ } & G \\ & {\scriptstyle \pi_i} \searrow \quad \nearrow {\scriptstyle \varphi_i} & \\ & G_i. & \end{array}$$

Let $Y \overset{\mathrm{def}}{=} X_i \times G/\sim$ where $(x, g) \sim (x', g')$ if there exists $h \in G_i$ such that $(x', g') = (x \cdot h, \varphi_i(h^{-1})g)$. Y will be a Galois covering with group G. For example, in the case that $G_i = G$ and $\varphi = \pi_i$ we get $Y = X_i$ and if φ is the trivial homomorphism, $\varphi = \mathrm{id}_G$, then $Y = X \times G$.

Now we need to confirm that the action of $\pi_1(X, \overline{x})$ on the fiber of Y induces φ as the continuous homomorphism associated to the action. Let ψ be the continuous homomorphism associated to the action, clearly it will factor through $\mathrm{Aut}_X(X_i)$, so we just need to verify that $\varphi_i = \psi_i$. Let $h \in G_i$ and $(x, g) \overset{\mathrm{def}}{=} y \in Y$. Then $y \cdot h = (x \cdot h, g)$, but $(x \cdot h, g) \sim (x, \varphi_i(h)g)$, hence the homomorphisms to G coincide.

□

Corollary 2.2.12. *The connected Galois coverings of X with group G are classified by the continuous epimorphisms $\pi_1(X, \overline{x}) \to G$.*

Proof: Follows exactly as the previous Corollary. If we add the restriction that the Galois coverings must be connected, the action of $\pi_1(X, \overline{x})$ on the fiber will be transitive (2.2.9), and this gives a rise to a continuous epimorphism of $\pi_1(X, \overline{x})$ to $Aut_X(Y) = G$.

Given an continuous epimorphism, φ, we do as above. Let $Y \overset{\mathrm{def}}{=} X_i \times G/\sim$. Since each X_i is connected it is sufficient to show that all elements can represented in the same copy of X_i, i.e., for a given $g \in G$ and any $(x, h) \in Y$, $(x, h) \sim (x', g)$. Let $l = gh^{-1} \in G$ and $f \in G_i$ such that $\varphi_i(f^{-1}) = l$, hence $(x, h) \sim (x \cdot f, \varphi_i(f^{-1})h = g)$. □

Corollary 2.2.13. *For each finite quotient of $\pi_1(X, \overline{x})$, $\pi_1(X, \overline{x})/H$, there is a Galois covering of $Y \to X$ with group equal to the quotient group, and $\pi_1(Y, \overline{x}) = H$. Conversely, for each connected Galois covering of $Y \to X$ with group G, $G = \pi_1(X, \overline{x})/\pi_1(Y, \overline{x})$.*

Proof: Given a finite quotient $\pi_1(X, \overline{x})/H$ for some normal subgroup H of the étale fundamental group of X, we have the projection map $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x})/H$ which is continuous. Thus, by the previous corollary, there will be a connected Galois covering $Y \to X$ with group $\pi_1(X, \overline{x})/H \simeq \pi_1(X, \overline{x})/\pi_1(Y, \overline{x})$ (2.2.9). Since we have the same projection map associated to $\pi_1(X, \overline{x})/H$ and $\pi_1(X, \overline{x})/\pi_1(Y, \overline{x})$, we conclude that $H = \pi_1(Y, \overline{x})$.

The rest of the corollary follows from Remark 2.2.9. □

2.2.1 Nonsingular varieties over $\mathbb{C}$

In this section X is a nonsingular variety over $\mathbb{C}$, and for any variety Y over $\mathbb{C}$, $Y(\mathbb{C})$ denotes Y endowed with the complex topology. Recall that if $Y \to X$ is an étale morphism then Y is also a nonsingular variety (1.2.11).

The main result of this subsection is the following.

Theorem 2.2.14. *(Riemann Existence Theorem) Let X be a nonsingular variety over $\mathbb{C}$. There is an equivalence of categories between the FEt/X and the finite covering spaces of $X(\mathbb{C})$.*

Proof: See [GR58]. □

Putting this together with Corollary 2.2.13 and the analogue of Theorem 2.2.10 for topological spaces with a universal covering we conclude that $\pi_1(X(\mathbb{C}), x)$ and $\pi_1(X_{et}, \overline{x})$ have the same finite quotients (taking $x = \overline{x}$). Therefore, their profinite completion must be isomorphic, but the étale fundamental group is already complete by definition, hence $\pi_1(X_{et}, \overline{x}) = \pi_1(\widehat{X(\mathbb{C})}, x)$.

This confirms our previous examples when we take the base field to be the complex numbers.

2.3 Sheaves on the Étale Site

Recall that a sheaf $\mathcal{F}$ on X_{et} is a contravariant functor from Et/X with values in a category **C** such that

$$\mathcal{F}(U) \to \prod_{i \in \mathcal{I}} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in \mathcal{I}} \mathcal{F}(U_i \times_U U_j)$$

is exact for every $U \to X$ étale and every étale covering family $\{U_i \to U\}_{i \in I}$.

Notice that every Zariski covering family is an étale covering family, and in this case, this condition is the usual one, so by restriction we have a sheaf on U_{zar} for every $U \to X$ étale, hence when $U = \coprod_{i \in \mathcal{I}} U_i$ we get that $\mathcal{F}(U) \simeq \prod_{i \in \mathcal{I}} \mathcal{F}(U_i)$, therefore a sheaf takes disjoints unions to products. In the case that $\mathcal{I} = \emptyset$, i.e., $U = \emptyset$, we get that $\mathcal{F}(\emptyset)$ is the zero group if we are considering **C**=**Ab** or a set with one element if **C**=**Sets**.

Now, if we consider a Galois covering $Y \to X$ with group G (1.2.12), since G acts on the right on Y, and a presheaf $\mathcal{P}$ is a contravariant functor G acts on $\mathcal{P}(Y)$ on the left.

Proposition 2.3.1. *Let $\pi : Y \to X$ be a Galois covering of schemes with group G. Let $\mathcal{F}$ be a presheaf on X_{et} that takes disjoint unions to products. Then $\mathcal{F}$ satisfies the sheaf condition for the covering family consisting of a simple map $\{\pi : Y \to X\}$ if and only if the image of the map $\mathcal{F}(X) \to \mathcal{F}(Y)$ is $\mathcal{F}(Y)^G$, the elements of $\mathcal{F}(Y)$ fixed by G.*

Proof: There is a commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{\pi} & Y & \leftleftarrows & Y \times_X Y \\
\uparrow{\scriptstyle \text{id}} & & \uparrow{\scriptstyle \text{id}} & & \uparrow{\scriptstyle \simeq} \\
X & \xleftarrow{\pi} & Y & \leftleftarrows & Y \times G \simeq \coprod_{g \in G} .Y
\end{array}$$

The vertical map is the one given by the definition of a Galois covering (1.2.12), so the horizontal projection maps on the top, $Y \times Y \rightrightarrows Y$, are the obvious ones, and the horizontal maps on the bottom row $Y \times G \rightrightarrows Y$ are $(y, g) \mapsto y$ and $(y, g) \mapsto y \cdot g$. By applying $\mathcal{F}$ to it we get the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y) & \rightrightarrows & \mathcal{F}(Y \times_X Y) \\
\downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \simeq} \\
\mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y) & \rightrightarrows & \prod_{g \in G} \mathcal{F}(Y).
\end{array}$$

Now the maps on the bottom right row are $s \mapsto (s, \dots, s, \dots)$ and $(1s, \dots, gs, \dots)$, hence they agree on $\mathcal{F}(Y)$ if and only if $gs = s$ for all $g \in G$. □

One can wonder if there is an easy criterion for checking whether a presheaf on X_{et} is a sheaf. In fact there is one, which it will make our lives simpler.

Proposition 2.3.2. *To verify that a presheaf $\mathcal{F}$ on X_{et} is a sheaf, X a scheme, it suffices to check that:*

(i) For each $U \to X$ étale, $\mathcal{F}$ is a sheaf when restricted to U_{zar};

(ii) $\mathcal{F}$ satisfies the sheaf condition for the étale covering families $\{V \to U\}$ of a single map, with V and U both affine.

Proof:(Sketch) Both conditions are clearly necessary. The idea of the proof is reducing everything to the second condition. We will take a étale covering family $\{U_i \to U\}_{i \in I}$ and change it in a series of single coverings, similar to $\coprod U_i \to U$.

When both U and U_i are affine, and I is finite we simply take $\coprod U_i \to U$. In the general case, we just need to work with the decomposition of U and U_j as a union of affine schemes and decompose the covering into a covering of the same type as above. See [Mil80, II.1.5]. □

More interesting is the fact that when checking the second condition in (2.3.2) we only use the fact that $V \to U$ is surjective and flat (i.e., we shall not need to use that is unramified). The reason for this is the following result.

Lemma 2.3.3. *For any **faithfully flat**[3] homomorphism of rings $A \to B$, the next sequence is exact*

$$0 \to A \to B \xrightarrow{b \mapsto (1 \otimes b - b \otimes 1)} B \otimes_A B$$

PROOF: See [Mil08, 6.8 page 43]. □

2.3.1 Examples of sheaves on X_{et}

Constant sheaves

For any set Λ and $U \to X$ étale, define

$$\mathcal{F}_\Lambda(U) \stackrel{\text{def}}{=} \Lambda^{\pi_0(U)}$$

where $\pi_0(U)$ is the set of connected components of U, hence $\Lambda^{\pi_0(U)}$ is the product of copies of Λ indexed by $\pi_0(U)$. When Λ is a group, then $\mathcal{F}_\Lambda$ is a sheaf of groups. Sometimes we make an abuse of notation and denote $\mathcal{F}_\Lambda$ just by Λ.

Remark 2.3.4. *It is also common to call $\mathcal{F}_\Lambda$ the sheaf of locally constant Λ-valued functions.*

Coherent sheaves

Before talking about coherent sheaves we need to define what is our structure sheaf. Take the presheaf defined by

$$\mathcal{O}_{X_{et}}(U) = \Gamma(U, \mathcal{O}_U).$$

Clearly its restriction to U_{zar} is a sheaf for any U étale over X. When we consider an étale covering family of a single map between affine spaces, $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, like in Proposition 2.3.2, the sequence associated to that cover is exactly the sequence in the Lemma 2.3.3, thus $\mathcal{O}_{X_{et}}$ is a sheaf on X_{et}.

Definition 2.3.5. *Let X be a scheme. The sheaf $\mathcal{O}_{X_{et}}$ is the **structure sheaf** associated to the site X_{et}.*

Let $\mathcal{M}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X_{zar} in the usual sense of algebraic geometry [Har77, II.5 page 109]. For any $\varphi : U \to X$ étale, we obtain a coherent $\mathcal{O}_U$-module $\varphi^*\mathcal{M}$ on U_{zar}. Recall that when φ is affine, i.e., $\varphi : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, if $\mathcal{M}(\operatorname{Spec}(A)) = M$, then $\varphi^*\mathcal{M} = B \otimes_A M$.

Clearly there is a presheaf $U \mapsto \Gamma(U, \varphi^*\mathcal{M})$ on X_{et}, which we denote by $\mathcal{M}^{et}$.

Example 2.3.6. *For the case that $\mathcal{M} = \mathcal{O}_{X_{zar}}$ note that $(\mathcal{O}_{X_{zar}})^{et} = \mathcal{O}_{X_{et}}$.*

To verify that $\mathcal{M}^{et}$ is a sheaf it suffices, by (2.3.2), to show that the sequence

$$0 \to M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M$$

is exact whenever $A \to B$ is faithfully flat and M is of finite rank. The exactness of this sequence follows from the argument in the proof of the Lemma 2.3.3.

[3] We say that a homomorphism $A \to B$ is faithfully flat if $Spec(B) \to Spec(A)$ is surjective.

The sheaf defined by a scheme Z

An X-scheme Z defines a contravariant functor:

$$\mathcal{F}_Z : \mathrm{Et}\,/X \to \mathrm{Sets}, \mathcal{F}_Z(U) = \mathrm{Hom}_X(U, Z)$$

Sometimes we will just denote $\mathcal{F}_Z(U)$ by $Z(U)$. Clearly it satisfies the sheaf criterion for Zariski coverings. By applying Proposition 2.3.2 it is sufficient to show that we have the exactness of

$$\mathcal{F}_Z(\mathrm{Spec}(A)) \to \mathcal{F}_Z(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}_Z(\mathrm{Spec}(B \otimes_A B)).$$

When Z is affine it follows directly from (2.3.3). Let $Z = \mathrm{Spec}(C)$, for C a ring. The exactness of the previous sequence is equivalent to the exactness of

$$0 \to \mathrm{Hom}_{A-\mathrm{alg}}(C, A) \to \mathrm{Hom}_{A-\mathrm{alg}}(C, B) \to \mathrm{Hom}_{A-\mathrm{alg}}(C, B \otimes_A B),$$

which is exact by (2.3.3) and the fact that the functor $\mathrm{Hom}_{A-\mathrm{alg}}(C, -)$ is left exact.

In the nonaffine case we just need to decompose Z in a union of affine open sets and use arguments similar to those used to prof that it is a sheaf in the Zariski site.

If Z has a group structure, then $\mathcal{F}_Z$ is a sheaf of groups.

Example 2.3.7.

(i) Let μ_n be the scheme defined by the equation $T^n = 1$. Then $\mu_n(U)$ is the group of nth roots of 1 in $\Gamma(U, \mathcal{O}_U)$.

(ii) Let G_m be the affine line with the origin omitted, regarded as a group under multiplication. Then $G_m(U) = \Gamma(U, \mathcal{O}_U)^\times$.

(iii) Let GL_n be the group scheme defined by $T \cdot det(T_{ij}) = 1$, equivalently $det(T_{ij}) \neq 0$. Then $GL_n(U) = GL_n(\Gamma(U, \mathcal{O}_U))$, the group of nonsingular matrices with coefficients in $\Gamma(U, \mathcal{O}_U)$.

Sheaves on $\mathrm{Spec}(k)$

From Example 1.2.10 we get that the category $Et/\,\mathrm{Spec}(k)$ is equivalent to Et/k, the category of étale k-algebras. Hence a presheaf on $\mathrm{Spec}(k)_{et}$ can be regarded as a covariant functor on Et/k. Applying Proposition 2.3.1 we conclude that such functor will be a sheaf if $\mathcal{F}(\prod A_i) = \oplus \mathcal{F}(A_i)$ for every finite family $\{A_i\}$ of k-algebras and $\mathcal{F}(k') \xrightarrow{\simeq} \mathcal{F}(K)^{\mathrm{Gal}(K/k')}$ whenever K/k' is a finite Galois extension of fields with k'/k a finite extension. Picking a separable closure k^{sep} of k, and letting $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$, for a sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)_{et}$ we define

$$M_{\mathcal{F}} = \varinjlim \mathcal{F}\,(k')$$

where k' runs through the subfields k' of k^{sep} that are finite and Galois over k. The idea is that $M_{\mathcal{F}}$ would be the value of the sheaf on k^{sep} if it could be defined there. It happens that $M_{\mathcal{F}}$ is a **discrete G-module**[4].

Conversely, if M is a discrete G-module we can associate a sheaf to it by defining

$$\mathcal{F}_M(A) = \mathrm{Hom}_G\left(\mathrm{Hom}_{k-\mathrm{alg}}(\,A, k^{\mathrm{sep}}\,), M\right),$$

where A is an étale k-algebra. This will be a sheaf on $\mathrm{Spec}(k)_{et}$.

The functors $\mathcal{F} \mapsto M_{\mathcal{F}}$ and $M \mapsto \mathcal{F}_M$ define an equivalence between the category of sheaves on $\mathrm{Spec}(k)_{et}$ and the category of discrete G-modules [Mil80, II.1.9 page 53].

Example 2.3.8. *If $\mathcal{F} = G_m$ on $\mathrm{Spec}(k)_{et}$, it follows by definition, that $M_{\mathcal{F}} = (k^{sep})^\times$, where k^{sep} is a separable closure of k.*

[4]Let G be a profinite group. A **discrete G-module** is a G-module M such that when M is given the discrete topology, the multiplication $G \times M \to M$ is continuous.

2.3.2 Stalks and the local ring for the étale topology

Definition 2.3.9. *Let X be a scheme and $\mathcal{F}$ be a presheaf on X_{et}. The **stalk** of $\mathcal{F}$ at a geometric point $\overline{x} \to X$ is*

$$\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U),$$

where the limit is over the étale neighborhoods of $\overline{x}$.

The local ring for the étale topology

The structure of the local ring when considering the étale topology is richer than in the Zariski case. One consequence of this extra structure is for example the exactness of the Kummer sequence 2.4.9. The proofs for this section results can be found in [Mil08, Chapter 4 page 31], for the approach via varieties, or in [Mil80, I.§4 page 33] for the general approach.

Definition 2.3.10. *We say that a local ring A is **Henselian** if it satisfies the following condition: given any $f(T)$, a monic polynomial in $A[T]$, and $\overline{f}(T)$ its image in $k[T]$, where $k = A/\mathfrak{m}_A$, then any factorization of $\overline{f}$ into a product of two monic relatively prime polynomials lifts to a factorization of f into the product of two monic polynomials.*

In other words, if $\overline{f} = g_0h_0$ where g_0 and h_0 are monic and relatively prime in $k[T]$, then $f = gh$ with g and h monic, $\overline{g} = g_0$ and $\overline{h} = h_0$.

Example 2.3.11. *(i) Any field is a Henselian ring;*

(ii) Every quotient of a Henselian ring is Henselian;

(iii) Rings of algebraic power series over a field are Henselian, i.e., the ring

$$k\,[[X_1, \ldots, X_n]] \cap k\,(X_1, \ldots, X_n)^{al}$$

is Henselian. You can see the proof in [Mil08, 4.17 page 36], but it uses concepts that were not defined here, yet.

Remark 2.3.12. *The elements g and h in the above factorization are strictly coprime, i.e., $A[T] = (g, h)$. To see this, note that $M \stackrel{\text{def}}{=} A[T]/(g, h)$ is finitely generated as an A-module, and we also have $M = \mathfrak{m}_A M$, since $\overline{g}$ and $\overline{f}$ are coprime in $k[T]$, hence by Nakayama's lemma $M = 0$.*

The polynomials g and h in above factorization are unique. Suppose that $f = gh = g'h'$. The previous argument also shows that g and h' are strictly coprime, hence there exist $r, s \in A[T]$ such that $gr + h's = 1$. Therefore

$$g' = g'gr + g'h's = g'gr + ghs,$$

and so g divides g'. Since g and g' have the same degree and are monic, they must be equal.

Lemma 2.3.13. *A ring is Henselian if and only if given $f_1, \ldots, f_n \in A[T_1, \ldots, T_n]$ (with the same notation as above); every common zero a on k^n of the $\overline{f_i}$ for which $\det(\partial\overline{f_i}/\partial T_j)(a)$ is nonzero, lifts to a common zero of the f_i in A^n.*

Using this lemma we can prove the next result.

Theorem 2.3.14. *For any geometric point $\overline{x}$ of a scheme X, $\mathcal{O}_{X_{et},\overline{x}}$ is Henselian.*

Proof: (Sketch) The idea is to show that the ring satisfies the condition mentioned in the Lemma 2.3.13. Assuming that we have $f_1, \ldots, f_n \in \mathcal{O}_{X_{et},\overline{x}}[T_1, \ldots, T_n]$, we can take representatives $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_U)[T_1, \ldots, T_n]$ for some sufficiently small étale neighborhood U, since the coefficients that define the polynomials are finite.

Let $k \stackrel{\text{def}}{=} \mathcal{O}_{X_{et},\overline{x}}/\mathfrak{m}_{\overline{x}}$, $B \stackrel{\text{def}}{=} \Gamma(U, \mathcal{O}_U)$ and $C \stackrel{\text{def}}{=} B[T_1, \ldots, T_n]/(f_1, \ldots f_n)$, the projection $B \to C$ defines a map on the associated schemes. A common zero in k^n of the polynomials corresponds to a closed point in $\mathrm{Spec}(C)$ over x, and if at that common zero $\det(\partial\overline{f_i}/\partial T_j) \neq 0$, the map between schemes is étale at that point (1.2.10).

Since the étale property is open whenever the map is finite, (1.2.11), we conclude that there exists $g \in C$ such that $\mathrm{Spec}(C_g) \to U$ is étale. Hence $\mathrm{Spec}(C_g)$ is an étale neighborhood of $\overline{x}$. The canonical map $C_g \to \mathcal{O}_{X_{et},\overline{x}}$, inclusion in the limit, gives a common zero of the polynomials in $\mathcal{O}_{X_{et},\overline{x}}$ (such map corresponds to picking images of each T_j such that the homomorphism is well defined, i.e., the image of each f_i is zero). □

Proposition 2.3.15. *Let $\varphi : Y \to X$ be a morphism of schemes such that it is étale at y, then the map $\mathcal{O}_{X_{et},\overline{\varphi(y)}} \to \mathcal{O}_{Y,\overline{y}}$ induced by φ is an isomorphism.*

PROOF:We can assume that φ is étale (if not we may always restrict it to an open set such that it becomes étale, which is possible because the étale property is open (1.2.11). Hence the map is the inclusion of the limit along a cofinal set (formed by those étale neighborhoods that factor through the map φ). □

This last proposition, gives us another confirmation that étaleness is the right concept that one should use. It says that if the map is étale at a point, the étale local rings on the point and its image are isomorphic. This is the same as saying that Y and X around those points are very similar to each other, i.e., they are locally isomorphic at y in the étale topology.

Proposition 2.3.16. *Let x be a nonsingular point on a variety X over an algebraically closed field, and let $d = dim(X)$. Then there is a regular map $\varphi : U \subset X \to \mathbb{A}^d$ étale at x.*

The last proposition strengthens what we said before. It says that varieties around nonsingular points are locally isomorphic in the étale topology to the affine space of same dimension as the variety, something that could not be true in the Zariski topology.

Definition 2.3.17. *Let A be a local ring. A local homomorphism $A \to A^h$ of local rings with A^h Henselian is called the* ***Henselization*** *of A if any other local homomorphism $A \to B$ with B Henselian factors uniquely into $A \to A^h \to B$.*

Clearly the Henselization of a local ring is unique up to isomorphism (if it exists). In fact the Henselization always exists and can be obtained by a direct limit. In the case of $A = \mathcal{O}_{X_{zar},x}$ we have the following result.

Proposition 2.3.18. *Let X be a scheme and $x \in X$, then $\mathcal{O}^h_{X_{zar},x} \simeq \varinjlim \Gamma(U, \mathcal{O}_U)$, where the limit runs over all connected étale neighborhoods of x.*

Remark 2.3.19. *The above proposition says that when X is a scheme over an algebraically closed field, then for a closed point $x \in X$ we have $\mathcal{O}_{X_{et},x} = \mathcal{O}^h_{X_{zar},x}$.*

Example 2.3.20. *The Henselization of $k[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$ is $k[[X_1, \dots, X_n]] \cap k(X_1, \dots, X_n)^{al}$ [Mil08, 4.17 page 36].*

Hence, in the case that we have a variety X over an algebraically closed that is nonsingular at a closed point $x \in X$, then $\mathcal{O}_{X_{et},\overline{x}} \simeq k[[X_1, \dots, X_d]] \cap k(X_1, \dots, X_d)^{al}$, where d is the dimension of the variety at the point x.

More generally $\mathcal{O}_{X_{et},\overline{x}}$ is always related to $\mathcal{O}_{X_{zar},x}$.

Definition 2.3.21. *A local ring A is said to be* ***strictly Henselian*** *if it is Henselian and its residue field is separably algebraically closed, or equivalently, if every monic polynomial $f(T) \in A[T]$ such that $\overline{f}(T)$ is separable splits into factors of degree 1.*

Let A be a local ring. A local homomorphism $A \to A^{sh}$ from A into a strict Henselian ring A^{sh} is a ***strict Henselization*** *of A if any other local homomorphism from A into a strictly Henselian ring H extends to A^{sh}, and moreover, the extension is uniquely determined once the map $A^{sh}/\mathfrak{m}^{sh} \to H/\mathfrak{m}_H$ on the residue fields has been specified.*

Proposition 2.3.22. *Let X be a scheme and $\overline{x}$ be a geometric point of X, then $\mathcal{O}_{X_{et},\overline{x}}$ is a strict Henselization of $\mathcal{O}_{X_{zar},x}$.*

Therefore, $\mathcal{O}_{X_{et},\overline{x}}$ is characterized by being the "smallest" Henselian ring containing $\mathcal{O}_{X_{zar},x}$, such that its residue field is separably algebraically closed.

Example 2.3.23.

(i) The stalk of $\mathcal{F}_Z$ *at* $\overline{x}$*,* Z *a a scheme of finite type over* X*, is* $Z(\mathcal{O}_{X_{et},\overline{x}})$*, i.e, the set of* $\mathcal{O}_{X_{et},\overline{x}}$*-valued points of* Z*. Hence the stalks of* μ_n, G_m GL_n *at* $\overline{x}$ *are* $\mu_n(\mathcal{O}_{X_{et},\overline{x}}), \mathcal{O}^{\times}_{X_{et},\overline{x}}$ *and* $\mathrm{GL}_n(\mathcal{O}_{X_{et},\overline{x}})$ *respectively.*

(ii) The stalk of $\mathcal{M}^{et}$*,* $\mathcal{M}$ *a coherent* $\mathcal{O}_X$*-module, at* $\overline{x}$ *is* $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_{et},\overline{x}}$ *where* $\mathcal{M}_x$ *is the stalk of* $\mathcal{M}$ *at* x *as a sheaf for the Zariski topology.*

(iii) For a sheaf $\mathcal{F}$ *on* $\mathrm{Spec}(k)$*,* k *a field, the stalk at* $\overline{x} = \mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)$ *is* $M_{\mathcal{F}}$ *regarded as an abelian group by forgetting the module structure (see Section 2.3.1).*

2.3.3 Skyscraper sheaves

On a topological space, a sheaf $\mathcal{F}$ is said to be a **skyscraper** sheaf if the stalks are zero with exception of a finite number of points (whence the name skyscraper). On a Hausdorff topological space X, given a point $x \in X$, and an abelian group Λ, we define

$$\Lambda^x(U) \begin{cases} \Lambda & x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Then Λ^x is a skyscraper sheaf on X, with stalk Λ at x and zero otherwise. It easily follows from the definitions that for any sheaf $\mathcal{F}$ on X we have

$$\mathrm{Hom}(\mathcal{F}, \Lambda^x) \simeq \mathrm{Hom}(\mathcal{F}_x, \Lambda).$$

This intrinsic property will be used to define skyscraper sheaves on a general site.

When X is a scheme over a algebraically closed field, it is very easy to generalize this concept, since the closed points are geometric points. Thus, for $x \in X$ a closed point and $\varphi : U \to X$ an étale map, we define

$$\Lambda^x(U) = \oplus_{u \in \varphi^{-1}(x)} \Lambda.$$

This is a sheaf on X_{et} and all the properties from the topological case are still valid, so we can continue our generalization process.

Let X be a scheme, and $i : \bar{x} \to X$ a geometric point of X. For any étale map $\varphi : U \to X$, define

$$\Lambda^{\bar{x}}(U) = \oplus_{\mathrm{Hom}_X(\bar{x}, U)} \Lambda.$$

Observe that $Hom_X(\bar{x}, U)$ is a straightforward generalization of $\varphi^{-1}(x)$. Again this gives a sheaf on X_{et}, and for every sheaf $\mathcal{F}$ we have $\mathrm{Hom}(\mathcal{F}, \Lambda^{\bar{x}}) \simeq \mathrm{Hom}(\mathcal{F}_{\bar{x}}, \Lambda)$. However, if $x \stackrel{\text{def}}{=} i(\bar{x})$ is not closed, then it need not be true that $(\Lambda^{\bar{x}})_{\bar{y}} = 0$ when the image of $\bar{y}$ is X is different from x.

Example 2.3.24. *Let* X *be a connected normal scheme. Let* $\overline{x}$ *be a geometric point over the generic point of* X*, then* $\Lambda^{\bar{x}}$ *is the constant sheaf* Λ*.*

In fact, let $\varphi : U \to X$ *be an étale morphism with* U *connected. From Proposition 1.2.11 we know that* U *is also normal, hence it only has one generic point, which can only be the preimage of the generic point of* X*. Therefore* $\mathrm{Hom}_X(\bar{x}, U)$ *has just one element, thus* $\Lambda^{\bar{x}}(U) = \Lambda$*.*

Therefore, unless x is closed, $\Lambda^{\bar{x}}$ is not necessarily a skyscraper sheaf. In the case that x is closed we say that $\Lambda^{\bar{x}}$ is the skyscraper sheaf over x with group Λ.

2.3.4 Locally constant sheaves

Definition 2.3.25. *A sheaf* $\mathcal{F}$ *on* X_{et} *is* ***locally constant*** *if there is a covering family* $\{U_i \to X\}$ *such that the restriction of the sheaf* $\mathcal{F}$ *to each* $U_{i\,et}$ *is constant.*

Example 2.3.26. *Let* X *be variety over* $\mathbb{R}$*. Clearly* μ_n *is not necessarily constant, but it will be a locally constant sheaf: take for instance the covering* $X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \to X$*, which is étale by (1.1.15).*

If $X = \mathbb{A}^1$ *is the affine live over* $\mathbb{R}$*, then* $\mu_3(\mathbb{A}^1) = \{1\}$*, but* $\mu_3(\mathbb{A}^1 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})) = \mathbb{Z}/3\mathbb{Z}$*.*

As in general topology, we have the following result.

Proposition 2.3.27. *Let X be connected and $\overline{x}$ a geometric point of X. The map $\mathcal{F} \to \mathcal{F}_{\overline{x}}$ defines an equivalence between the category of locally constant sheaves of sets (respectively abelian groups) with finite stalks on X and the category of finite discrete $\pi_1(X, \overline{x})$-sets (respectively modules).*

Remark 2.3.28. *In fact, one shows that, if $Z \to X$ is a finite étale map, then the sheaf $\mathcal{F}_Z$ is a locally constant sheaf with finite stalks, and every such sheaf is of this form for some Z, i.e., $Z \to \mathcal{F}_Z$ defines an equivalence between FEt/X and the category of locally constant sheaves on X_{et} with finite stalks.*

PROOF: See [Mil80, V.1.1 page 155]. □

2.4 The category of Sheaves

2.4.1 Generalities on categories

Now we will recall some basic concepts of category theory, needed to start our study of the category of sheaves on X_{et}.

Throughout the rest of the text we will heavily use abelian categories [Mil08, Chapter 7 page 49].

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be abelian categories. We say that the functors $L : \mathbf{C}_1 \to \mathbf{C}_2$ and $R : \mathbf{C}_2 \to \mathbf{C}_1$ are **adjoint** if

$$\mathrm{Hom}_{\mathbf{C}_2}(LX_1, X_2) \simeq \mathrm{Hom}_{\mathbf{C}_1}(X_1, RX_2).$$

We also say that L is the **left adjoint** of R, and that R is the **right adjoint** of L, this relation is unique up to a unique isomorphism.

When $\mathrm{Hom}(-, I)$ is an **exact functor**[5] we say that the object I is **injective**.

Proposition 2.4.1.

(i) A functor R that admits a left adjoint is left exact.

(ii) A functor L that admits a right adjoint is right exact.

(iii) A functor R that admits an exact left adjoint preserves injectives.

PROOF: We will prove (i), the others are *mutatis mutandis.* Let

$$0 \to A \to B \to C$$

be an exact sequence in $\mathbf{C}_2$, i.e., such that for any $T \in \mathbf{C}_2$

$$0 \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B) \to \mathrm{Hom}(T, C)$$

is exact. We want to see that if R is applied to the first sequence then for any $T' \in \mathbf{C}_1$, the sequence

$$0 \to \mathrm{Hom}(T', RA) \to \mathrm{Hom}(T', RB) \to \mathrm{Hom}(T', RC)$$

is exact. Since R admits a left adjoint, L, the previous sequence is isomorphic to

$$0 \to \mathrm{Hom}(LT', A) \to \mathrm{Hom}(LT', B) \to \mathrm{Hom}(LT', C)$$

which is exact since the first sequence is exact. □

2.4.2 The category of presheaves

Recall that the site X_{et} is the category Et/X, whose objects are the étale morphisms $U \to X$ and whose arrows are the commutative diagrams, and by definition the presheaves of abelian groups on X_{et} are the contravariant functors Et/$X \to$ Ab. Now there is a minor set-theoretical problem here, since Et/X is not a small category[6]. However since every étale map is locally standard Et/X is equivalent to small category, so it is harmless to pretend that it is a small category.

[5] We say that a functor is **exact** if it preserves exact sequences.

[6] A category is small if its objects form a set (rather than a class). In this case we can talk about the abelian category of the contravariant functors to the abelian groups.

By omitting this issue we define the category $\mathrm{PreSh}(X_{et})$ of presheaves of abelian groups on X_{et} as the category of all contravariant functors from Et/X to the abelian groups. It is an abelian category where

$$\mathcal{P}_1 \to \mathcal{P}_2 \to \mathcal{P}_3$$

is exact if and only if

$$\mathcal{P}_1(U) \to \mathcal{P}_2(U) \to \mathcal{P}_3(U)$$

is exact for all $U \to X$ étale. Moreover the kernels, cokernels, products, direct sum, inverse limits, direct limits, etc., are obtained objectwise.

2.4.3 The sheaf associated with a presheaf

Definition 2.4.2. *Let $i : \mathcal{P} \to a\mathcal{P}$ be a homomorphism from a presheaf $\mathcal{P}$ to a sheaf $a\mathcal{P}$; then $a\mathcal{P}$ is said to be the* ***sheaf associated with*** *$\mathcal{P}$ (or to be the sheafification of $\mathcal{P}$) if all other homomorphism from $\mathcal{P}$ to a sheaf factor uniquely through $i : \mathcal{P} \to a\mathcal{P}$, i.e., if i induces a canonical isomorphism $\mathrm{Hom}_{\mathrm{PreSh}}(\mathcal{P}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{Sh}}(a\mathcal{P}, \mathcal{F})$, for all sheaves $\mathcal{F}$.*

Clearly if the pair $(i,a\mathcal{P})$ exists, it will be unique up to a unique isomorphism. In fact it exists and the construction for the étale site is just a *verbatim* copy of the construction for a topological space [Har77, II.1.2 page 64]. Therefore we have the following result.

Proposition 2.4.3. *For any presheaf $\mathcal{P}$ on X_{et}, then there exists an associated sheaf $i : \mathcal{P} \to a\mathcal{P}$. The map i induces isomorphisms on $\mathcal{P}_{\overline{x}} \to (a\mathcal{P})_{\overline{x}}$ on the stalks.*

Proof: See [Mil80, II.2.11 page 61]. □

2.4.4 The category of sheaves

We define the category $\mathrm{Sh}(X_{et})$ to be the full subcategory of $\mathrm{PreSh}(X_{et})$ whose objects are sheaves of abelian groups. Now we check what exactness means in $\mathrm{Sh}(X_{et})$: it will be the complete analogue of the topological case.

Definition 2.4.4. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the* ***kernel*** *of φ, denoted by $\mathrm{Ker}\,\varphi$, to be the presheaf kernel of φ (which is a sheaf).*

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the ***image*** *of φ, denoted by $\mathrm{Im}\,\varphi$, to be the sheaf associated to the presheaf image of φ.*

A sequence of sheaves

$$\mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \xrightarrow{\psi} \mathcal{F}_3$$

is exact if $\mathrm{Im}\,\varphi = \mathrm{Ker}\,\psi$.

A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ of presheaves (or sheaves) is said to be ***locally surjective*** *if for every U and $s \in \mathcal{G}(U)$, there exists a covering $\{U_i \to U\}$ such that $s|U_i$ is in the image of $\alpha(U_i)$ for each i.*

Remark 2.4.5. *A morphism of sheaves is surjective, as a morphism of sheaves, if and only if it is locally surjective.*

Proposition 2.4.6. *Let*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a sequence of sheaves of abelian groups on X_{et}. The following are equivalent:

(i) the sequence is exact in the category of sheaves;

(ii) the map $\mathcal{F}_2 \to \mathcal{F}_3$ is locally surjective, and

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

is exact for all $U \to X$ étale;

(iii) the sequence

$$0 \to \mathcal{F}_{1\overline{x}} \to \mathcal{F}_{2\overline{x}} \to \mathcal{F}_{3\overline{x}} \to 0$$

is exact for each geometric point $\overline{x} \to X$.

PROOF: This is a very well known result when we are considering sheaves over a topological space (replacing geometric points by the usual points). The proof in the étale case is just a *verbatim* copy of the proof for the topological case. See [Mil08, 7.6 page 52]. □

Corollary 2.4.7. *The sheafification, the functor $a : \mathrm{PreSh}(X_{et}) \to \mathrm{Sh}(X_{et})$ (2.4.3), is exact.*

Corollary 2.4.8. *The inclusion functor $i : \mathrm{Sh}(X_{et}) \to \mathrm{PreSh}(X_{et})$ preserves injectives.*

Example 2.4.9. *(Kummer sequence) Let n be an integer that is not divisible by the characteristic of any residue field of X. For instance, if X is a scheme over a field k of characteristic $p \neq 0$, then we require that p does not divide n. Consider this sequence*

$$0 \to \mu_n \to G_m \xrightarrow{t \mapsto t^n} G_m \to 0.$$

In order to prove that this sequence is exact we have to check that it is exact on the stalks, i.e.,

$$0 \to \mu_n(\mathcal{O}_{X_{et},\overline{x}}) \to {\mathcal{O}_{X_{et},\overline{x}}}^{\times} \xrightarrow{t \mapsto t^n} {\mathcal{O}_{X_{et},\overline{x}}}^{\times} \to 0,$$

is exact for every geometric point of X, by (2.4.6) and (2.3.23).

It is obvious that the last sequence is always left exact. The problem is to show it is also right exact. We need to show that every element of ${\mathcal{O}_{X_{et},\overline{x}}}^{\times}$ is a nth power.

Recall that $\mathcal{O}_{X_{et},\overline{x}}$ is strictly Henselian (2.3.22), i.e., the ring is Henselian and the residue field is separably algebraically closed. Therefore, since n is coprime with the characteristic of the residue field, the polynomial $T^n - a$,with $a \in (\mathcal{O}_{X_{et},\overline{x}})^{\times}$, is separable ($\frac{d(T^n-a)}{dT} = nT^{n-1} \neq 0$), hence it has a root in the residue field. Again, since $\frac{d(T^n-a)}{dT} = nT^{n-1} \neq 0$ in the residue field, the root can be lifted to $\mathcal{O}_{X_{et},\overline{x}}$, (2.3.13). Hence, the sequence on the stalks is exact.

Remark 2.4.10. *The use of étale topology here is essential, the Kummer sequence usually is not exact in the Zariski topology.*

When n is divisible by the characteristic of some residue field of X, the sequence on stalks (for étale topology) at that point will not be exact.

The same sequence is exact in $\mathrm{Sh}(X_{fl})$ without any restriction on the characteristic of the residue fields [Mil80, II.§2 page 66] .

2.5 Images of sheaves

2.5.1 Direct images of sheaves

Let $\pi : Y \to X$ be a morphism of schemes, and let $\mathcal{P}$ be presheaf on Y_{et}. For $U \to X$ étale, define

$$\pi_*\mathcal{P}(U) = \mathcal{P}(U \times_X Y).$$

Since $U \times_X Y$ is the analogue of $\pi^{-1}(U)$ when $U \hookrightarrow X$ is just an open inclusion, this agrees with the usual definition of the direct image of a sheaf. It is also well defined since $U \times_X Y \to Y$ is étale (1.2.11). With the obvious restriction maps, $\pi_*\mathcal{P}$ becomes a presheaf on X_{et}.

Lemma 2.5.1. *If $\mathcal{F}$ is a sheaf, so is $\pi_*\mathcal{F}$.*

PROOF: Let $\{U_i \to U\}$ be a surjective family of étale maps in Et/X. Then $\{U_i \times_X Y\}$ is a covering family of $U \times_X Y$ (2.1.1), of étale maps in Et/Y, and so

$$\mathcal{F}(U \times_X Y) \to \prod \mathcal{F}(U_i \times_X Y) \rightrightarrows \mathcal{F}((U_i \times_X Y) \times_Y (U_j \times_X Y)) = \mathcal{F}((U_i \times_X U_j) \times_X Y)$$

is exact. But this is precisely the sequence

$$(\pi_*\mathcal{F})(U) \to \prod (\pi_*\mathcal{F})(U_i) \rightrightarrows (\pi_*\mathcal{F})(U_i \times_X U_j)$$

□

Remark 2.5.2. *Obviously, the functor* $\pi_* : \mathrm{PreSh}(Y_{et}) \to \mathrm{PreSh}(X_{et})$ *is exact. Therefore, its restriction* $\pi_* : \mathrm{Sh}(Y_{et}) \to \mathrm{Sh}(X_{et})$ *is left exact but in general it will not be exact.*

Example 2.5.3. *Let* X *be a scheme over a algebraically closed field* k*,* $\mathcal{F}$ *a sheaf on* X_{et}*,* $x \in X$ *a closed point, and* π *the constant morphism to* x*,* $\pi : X \to \{x\}$*.*

Since x *is also a geometric point, the site* x_{et} *is trivial (2.1.9), hence any sheaf here will be constant. In this case of* $\pi_*\mathcal{F} = \Gamma(X, \mathcal{F})$*.*

Since a exact sequence of sheaves on X_{et} *does not generally give rise to an exact sequence on the global sections, therefore, it is clear that in general this* π_* *is not exact.*

Studying the stalks of $\pi_*\mathcal{F}$ we will see some important cases in which π_* is exact.

Proposition 2.5.4. *The functor* $\pi_* : \mathrm{Sh}(Y_{et}) \to \mathrm{Sh}(X_{et})$ *is exact whenever* π *is a finite morphism, and in particular, if it is a closed immersion.*

PROOF: This follows by a simple analysis of the stalks. In the first case we get that $(\pi_*\mathcal{F})_{\overline{x}} = \oplus_{y \mapsto x} {\mathcal{F}_{\overline{y}}}^{d(y)}$ where $d(y)$ is the separable degree of $k(y)$ over $k(x)$.

In the second case, π is $Z \hookrightarrow X$, with Z a closed subscheme, and $(\pi_*\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}}$ if $x \in Z$, and zero otherwise. □

Example 2.5.5. *Let* X *be a variety over a algebraically closed field,* x *a closed point of* X*, and* π *its inclusion in* X*,* $\pi : x \hookrightarrow X$*. Then,* $\pi_*\mathcal{F}$ *is the skyscraper sheaf over* x *with group* $\mathcal{F}_x$*.*

Example 2.5.6. *Let* X *be the affine line,* $\mathbb{A}^1$*, over an algebraically closed field,* $U = \mathbb{A}^1 \backslash \{0\}$*,* $\mathcal{F} = \mathbb{Z}$ *the constant sheaf on* U_{et} *and* $\pi : U \hookrightarrow X$*.*

Obviously $\pi_*\mathcal{F} = \mathbb{Z}$ *in* X_{et}*, hence* $\pi_*\mathcal{F}_{\overline{0}} = \mathbb{Z}$*. So, whenever* π *is an open inclusion, the stalks outside the image may be non zero.*

2.5.2 Inverse images of sheaves

We will now define the left adjoint of π_*. Let $\mathcal{P}$ be a presheaf on X_{et}. For $V \to Y$ étale, define

$$\mathcal{P}'(V) = \varinjlim \mathcal{P}(U)$$

where the direct limit is over the commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & X \end{array}$$

with $U \to X$ étale. Unfortunately, $\mathcal{P}'$ may not be a sheaf even when $\mathcal{P}$ is. Thus, for a sheaf $\mathcal{F}$ on X_{et}, we define $\pi^*\mathcal{F} = a(\mathcal{F}')$.

Example 2.5.7. *In the case that* $\pi : U \to X$ *is an étale morphism one sees easily that* π^* *is just the restriction of* $\mathrm{Sh}(X_{et})$ *to* $\mathrm{Sh}(U_{et})$*.*

One can easily check that for any sheaf $\mathcal{G}$ on Y_{et} we have

$$\mathrm{Hom}_{\mathrm{Sh}(Y_{et})}(\pi^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{PreSh}(Y_{et})}(\mathcal{F}', \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Sh}(X_{et})}(\mathcal{F}, \pi_*\mathcal{G}).$$

Therefore, we have the following result.

Proposition 2.5.8. π^* *is a left adjoint to* $\pi_* : \mathrm{Sh}(Y_{et}) \to \mathrm{Sh}(X_{et})$*.*

Remark 2.5.9. *Let* $i : \overline{x} \to X$ *be a geometric point of* X*. For any sheaf* $\mathcal{F}$ *on* X_{et}*, it is clear from the definitions that* $(i^*\mathcal{F})(\overline{x}) = \mathcal{F}_{\overline{x}}$*. Therefore, for any morphism* $\pi : Y \to X$ *and geometric point* $i : \overline{y} \to Y$ *of* Y*, we have*

$$(\pi^*\mathcal{F})_{\overline{y}} = i^*(\pi^*\mathcal{F})(\overline{y}) = (\pi \circ i)^*\mathcal{F}(\overline{y}) = \mathcal{F}_{\overline{y}}.$$

Since this is true for every geometric point of Y *we have the next proposition.*

Proposition 2.5.10. π^* *is exact and therefore* π_* *preserves injectives, (2.4.1).*

2.5.3 Existence of enough injectives

Proposition 2.5.11. *Every sheaf $\mathcal{F}$ on $\mathrm{Sh}(X_{et})$ can embedded into an injective sheaf.*

PROOF: For each $x \in X$, choose a geometric point $i_x : \overline{x} \to X$ with image x and an embedding $\mathcal{F}_{\overline{x}} \hookrightarrow I(x)$ of the abelian group $\mathcal{F}_{\overline{x}}$ into an injective abelian group. Then $\mathcal{I}^x \stackrel{\text{def}}{=} i_{x*}(I(x))$ is an injective sheaf (2.5.10). In fact, it is just the skyscraper of $I(x)$ over x. Since a product of injective objects is injective, $\mathcal{I} \stackrel{\text{def}}{=} \prod \mathcal{I}^x$ will be an injective sheaf. The composite $\mathcal{F} \hookrightarrow \prod i_{x*}(\mathcal{F}_{\overline{x}}) \hookrightarrow \mathcal{I}$ is the embedding sought. □

2.5.4 Extension by zero

Let $j : U \hookrightarrow X$ be an open immersion. For a sheaf $\mathcal{F}$ on U_{et}, the stalks of $j_*\mathcal{F}$ need not be zero at points outside U, (2.5.6). We now define a functor $j_!$ "extension by zero" that will correct this flaw.

Let $\mathcal{P}$ be a presheaf on U_{et}. For any $\varphi : V \to X$ étale, define

$$\mathcal{P}_!(V) = \begin{cases} \mathcal{P}(V) & \text{if } \varphi(V) \subset U; \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, $\mathcal{P}_!$ need not be sheaf even when $\mathcal{P}$ is, the following example shows this:

Example 2.5.12. *Let $X = X_1 \sqcup X_2$, with X_1 and X_2 both nonempty schemes, and let j be the inclusion of X_1 in X. Let $\varphi_1 : U_1 \to X_1$ and $\varphi_2 : U_2 \to X_2$ be two étale maps, and $\varphi : U_1 \sqcup U_2 \to X$ the natural induced map (which is also étale).*

By definition we have

$$\mathcal{P}_!(\varphi_2 : U_2 \to X_2) = \mathcal{P}_!(\varphi : U_1 \sqcup U_2 \to X) = 0$$
$$\mathcal{P}_!(\varphi_1 : U_1 \to X_1) = \mathcal{P}(\varphi_1 : U_1 \to X),$$

so as long as $\mathcal{P}_!(U_1) \neq 0$ we have $\mathcal{P}_!(U_1 \sqcup U_2) \neq \mathcal{P}_!(U_1) \times \mathcal{P}_!(U_2)$, a necessary condition that $\mathcal{P}_!$ should satisfy in order to be a sheaf.

Thus, for a sheaf $\mathcal{F}$ on U_{et}, we define $j_!\mathcal{F}$ to be $a(\mathcal{F}_!)$.

Example 2.5.13. *If j is also a closed immersion we have $j_* = j_!$.*

Remark 2.5.14. *For any sheaf $\mathcal{G}$ on X_{et},*

$$\mathrm{Hom}_{\mathrm{Sh}(X_{et})}(j_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{PreSh}(X_{et})}(\mathcal{F}_!, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Sh}(U_{et})}(\mathcal{F}, \mathcal{G}|U) = \mathrm{Hom}_{\mathrm{Sh}(U_{et})}(\mathcal{F}, j^*\mathcal{G}).$$

Therefore, we have the following result.

Proposition 2.5.15. *Let $j : U \hookrightarrow X$ be an open immersion. $j_!$ is a left adjoint of $j^* : \mathrm{Sh}(X_{et}) \to \mathrm{Sh}(U_{et})$.*

Proposition 2.5.16. *Let $j : U \hookrightarrow X$ be an open immersion. For any sheaf $\mathcal{F}$ on U_{et} and geometric point $\overline{x} \to X$,*

$$(j_!\mathcal{F})_{\overline{x}} = \begin{cases} \mathcal{F}_{\overline{x}} & x \in U; \\ 0 & x \notin U. \end{cases}$$

PROOF: Clear from definition and Proposition 2.4.3. □

Corollary 2.5.17. *Let $j : U \hookrightarrow X$ be an open immersion. The functor $j_! : \mathrm{Sh}(U_{et}) \to \mathrm{Sh}(X_{et})$ is exact, thus j^* preserves injectives, (2.4.1).*

Let Z be the complement of U in X with the reduced scheme structure, $i : Z \hookrightarrow X$ be the natural inclusion, and $\mathcal{F}$ a sheaf on X_{et}. There is canonical morphism $j_!j^*\mathcal{F} \to \mathcal{F}$, which is adjoint to the identity map on $j^*\mathcal{F}$, and a canonical morphism $\mathcal{F} \to i_*i^*\mathcal{F}$, which is adjoint to identity map on $i^*\mathcal{F}$. In other words,

$$\mathrm{Hom}_{U_{et}}(j^*\mathcal{F}, j^*\mathcal{F}) \simeq \mathrm{Hom}_{X_{et}}(j_!j^*\mathcal{F}, \mathcal{F}) \quad \text{and} \quad \mathrm{Hom}_{Z_{et}}(i^*\mathcal{F}, i^*\mathcal{F}) \simeq \mathrm{Hom}_{X_{et}}(\mathcal{F}, i_*i^*\mathcal{F}),$$

and our morphisms, $j_!j^*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^*\mathcal{F}$, are the ones that on the left hand side correspond to the identity maps.

Proposition 2.5.18. *Let $j : U \hookrightarrow X$ be an open immersion, Z be the complement of U in X with the reduced scheme structure and $i : Z \hookrightarrow X$ be the natural inclusion. For any sheaf $\mathcal{F}$ on X, the sequence*

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

is exact.

PROOF: This can be checked on stalks. For $x \in U$, the sequence on the stalks is

$$0 \to \mathcal{F}_{\overline{x}} \xrightarrow{\text{id}} \mathcal{F}_{\overline{x}} \to 0 \to 0,$$

and for $x \notin U$, the sequence on the stalks is

$$0 \to 0 \to \mathcal{F}_{\overline{x}} \xrightarrow{\text{id}} \mathcal{F}_{\overline{x}} \to 0$$

□

Example 2.5.19. *Let X be the affine line over an algebraically closed field, $j : \mathbb{A}^1 \backslash \{0\} \hookrightarrow X$, $i : \{0\} \hookrightarrow X$ and $\mathcal{F} = \mathbb{Z}$. The sequence*

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

in this case is

$$0 \to \mathbb{Z}_! \to \mathbb{Z} \to \mathbb{Z}^{\bar{0}} \to 0.$$

Chapter 3

Cohomology

3.1 Definitions and basic properties

3.1.1 Definition of cohomology

As reference for this section we sugest [Wei94].

Definition 3.1.1. *We define* $H^r(X_{et}, -)$ *to be the* r*th derived functor of* $\Gamma(X, -) : \mathrm{Sh}(X_{et}) \to Ab$.

The theory of derived functors shows that, for any sheaf $\mathcal{F}$, $H^0(X_{et}, \mathcal{F}) = \Gamma(X, \mathcal{F})$; if $\mathcal{I}$ is injective, then $H^r(X_{et}, \mathcal{I}) = 0$ for $r > 0$; and a short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(X_{et}, \mathcal{F}_1) \to H^0(X_{et}, \mathcal{F}_2) \to H^0(X_{et}, \mathcal{F}_3) \to H^1(X_{et}, \mathcal{F}_1) \to H^1(X_{et}, \mathcal{F}_2) \to \cdots$$

and this assignment is functorial.

3.1.2 The dimension axiom

Let $x = \mathrm{Spec}(k)$ for some field k, and fix a geometric point $\overline{x} = \mathrm{Spec}(k^{\mathrm{sep}})$ for some separable closure k^{sep} of k. Recall from section 2.3.1 and Example 2.3.23, that the functor $\mathcal{F} \mapsto \mathcal{M}_{\mathcal{F}} \stackrel{\mathrm{def}}{=} \mathcal{F}_{\overline{x}}$ defines an equivalence between the category of sheaves on x_{et} and the category of discrete G-modules, where $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. We also saw that $\Gamma(x, \mathcal{F}) = (\mathcal{M}_{\mathcal{F}})^G$, thus exists a correspondence between the right derived functors of $M \mapsto M^G = H^0(G, M)$ and $\mathcal{F} \mapsto \Gamma(x, \mathcal{F}) = H^0(x_{et}, \mathcal{F})$. Therefore:

Proposition 3.1.2. *In the same setting we have,*

$$H^r(x_{et}, \mathcal{F}) \simeq H^r(G, \mathcal{M}_{\mathcal{F}}),$$

where $H^r(G, M)$ *is the* r*th right derived functor of* $M \to M^G$ *in the category of discrete* G*-modules.*

When we take x to be a geometric point, $x = \overline{x}$, (the analog of a point in the étale site) G is trivial and so we get $H^r(x_{et}, \mathcal{F}) = 0$, for $r > 0$ and all $\mathcal{F}$.

Corollary 3.1.3. *The geometric points satisfy the dimension axiom.*

Remark 3.1.4. *Let* k *be a non separably algebraically closed field and* Λ *the constant sheaf on* $\mathrm{Spec}(k)_{et}$*. The action of* $\mathrm{Gal}(k^{sep}/k)$ *on the sheaf is trivial, but the cohomology does not need to be trivial. See [Bro82, Example 2 page 58], where they treat the case with* G *cyclic and finite. For instance*

$$H^r(\mathrm{Spec}(\mathbb{R})_{et}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \quad \textit{for all } r \geq 0.$$

3.1.3 Cohomology and inverse limits of schemes

In general, the inverse limits of schemes do not commute with taking cohomology, but there is a special case when this happens. Let I be a directed set, and $\{X_i\}_{i\in I}$ be a projective system of schemes index by I. If the transition maps $\phi_i^j : X_j \to X_i$ when $j \geq i$, are **affine**[1], then the inverse limit scheme $X_\infty \stackrel{\text{def}}{=} \varprojlim X_i$ exists, see [Gro67, VII2.5].

Theorem 3.1.5. *Let I be a directed set, $\{X_i\}_{i\in I}$ be a projective system of X-schemes and $\mathcal{F}$ a sheaf on X. If all X_i are quasicompact and the transition maps are affine, then*

$$\varinjlim H^r(X_i, \mathcal{F}) \simeq H^r(X_\infty, \mathcal{F}).$$

Proof: See [AM72, VII5.8] □

Remark 3.1.6. *The condition of the transition maps being affine holds trivially in the case that X_i are affine schemes.*

3.1.4 Cohomology with support on closed subscheme Z

Let Z be a closed subscheme of X, and let $U = X\backslash Z$. For any sheaf $\mathcal{F}$ on X_{et}, we define

$$\Gamma_Z(X,\mathcal{F}) = \mathrm{Ker}(\Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F})).$$

Hence $\Gamma_Z(X,\mathcal{F})$ is the group of sections of $\mathcal{F}$ with support on Z. Clearly, the functor $\mathcal{F} \mapsto \Gamma_Z(X,\mathcal{F})$ is exact, and we denote its rth right derived by $H_Z^r(X_{et}, -)$, and we call it **the rth cohomology group with support on Z**.

Theorem 3.1.7. *For any sheaf $\mathcal{F}$ on X_{et} and closed $Z \subset X$, there is a long exact sequence*

$$\cdots \to H_Z^r(X_{et},\mathcal{F}) \to H^r(X_{et},\mathcal{F}) \to H^r(U_{et},\mathcal{F}) \to H_Z^{r+1}(X_{et},\mathcal{F}) \to \cdots.$$

The sequence is functorial in the pairs $(X, U = X\backslash Z)$ and $\mathcal{F}$.

Proof: See [Mil08, Chapter 10 page 65]. □

3.1.5 Excision

The excision property also holds for the étale topology.

Theorem 3.1.8. *(Excision) Let $\pi : X' \to X$ be an étale map and let $Z' \subset X'$ be a closed subscheme of X' such that $Z \stackrel{def}{=} \pi(Z')$ is closed in X, $\pi|Z'$ is an isomorphism onto Z, and $\pi(X'\backslash Z') \subset X\backslash Z$. Then for any sheaf $\mathcal{F}$ on X_{et}, the canonical map $H_Z^r(X_{et},\mathcal{F}) \to H_{Z'}^r({X'}_{et},\mathcal{F})$ is an isomorphism for all r.*

Proof: See [Mil08, 9.7 page 66]. □

Corollary 3.1.9. *Let x be a closed point of X. For any sheaf $\mathcal{F}$ on X,*

$$H_x^r(X_{et},\mathcal{F}) \simeq H_x^r\left(\mathrm{Spec}(\mathcal{O}_{X_{zar},x}^h)_{et},\mathcal{F}\right)$$

where $\mathcal{O}_{X_{zar},x}^h$ is the henselization of $\mathcal{O}_{X_{zar},x}$.

Proof. Recall that

$$\mathcal{O}_{X_{zar},x}^h \stackrel{\text{def}}{=} \varinjlim_{(U,u)} \Gamma(U,\mathcal{O}_U)$$

where the limit is taken over all the connected étale neighborhoods (U,u) of x, (2.3.18). According to the previous theorem, $H_x^r(X,\mathcal{F}) \simeq H_u^r(U,\mathcal{F})$ for any étale neighborhood (U,u) of x such that u is the only point of U mapping to x. Such étale neighborhoods are cofinal. Hence

$$\begin{aligned}
H_x^r\left(\mathrm{Spec}(\mathcal{O}_{X_{zar},x}^h)_{et},\mathcal{F}\right) &= H_x^r\left(\mathrm{Spec}\left(\varinjlim \Gamma(U,\mathcal{O}_U)\right)_{et},\mathcal{F}\right) && \text{(taking the limit over the cofinal set)}\\
&\simeq \varinjlim H_u^r(U_{et},\mathcal{F}) && \text{(Thm 3.1.5)}\\
&\simeq \varinjlim H_x^r(X_{et},\mathcal{F}) && \text{(Excision (3.1.8))}\\
&\simeq H_x^r(X_{et},\mathcal{F}).
\end{aligned}$$

□

[1] We say that a morphism is **affine** if the inverse image of a open affine is affine.

3.1.6 Cohomology with compact support

Definition 3.1.10. *Let X be a scheme. We say that sheaf $\mathcal{F}$ on X_{et} is a* ***torsion*** *sheaf if $\mathcal{F}(U)$ is a torsion abelian group for all $U \to X$ étale.*

Definition 3.1.11. *For any torsion sheaf $\mathcal{F}$ on a variety U, we define*

$$H_c^r(U, \mathcal{F}) = H^r(X, j_!\mathcal{F})$$

where X is any complete variety containing U as a dense open subvariety and j is the inclusion map. Following the terminology used in topology, we call $H_c^r(U, \mathcal{F})$ the ***cohomology groups of $\mathcal{F}$ with compact support****. For an explanation of this terminology see [Mil08, Chapter 18].*

An open immersion $j : U \hookrightarrow X$ from U into a complete variety X such that $j(U)$ is dense in X is called a ***completion*** *of U.*

The existence of a completion map is a consequence of Nagata's paper [Nag62], and when $\mathcal{F}$ is a torsion sheaf it can be shown that the groups $H^r(X, j_!\mathcal{F})$ are independent of the choice of the embedding $j : U \hookrightarrow X$ [Mil08, 18.2 page 116]. Hence, the cohomology groups with compact support are well defined.

Remark 3.1.12. *In the case of curves the completion is unique.*

Remark 3.1.13. *Since $j_!$ is exact, a short exact sequence of sheaves on U induces a long exact sequence of cohomology groups on the completion of U, X. However, since $j_!$ does not preserve injectives, $H_c^r(U, -)$ is not the rth derived functor of $H_c^0(U, -)$.*

3.1.7 Derived functors of the direct image

Proposition 3.1.14. *For any morphism of schemes $\pi : Y \to X$ and $\mathcal{F}$ a sheaf on Y_{et}, then, $R^r\pi_*\mathcal{F}$- the rth right derived functor of $\pi_*\mathcal{F}$ in $\mathrm{Sh}(Y_{et})$-is the sheaf on X_{et} associated with the presheaf $U \mapsto H^r(U \times_X Y, \mathcal{F})$.*

Proof: Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{PreSh}(Y_{et}) & \xrightarrow{(\pi_*)_p} & \mathrm{PreSh}(X_{et}) \\ \uparrow i & & \downarrow a \\ \mathrm{Sh}(Y_{et}) & \xrightarrow{\pi_*} & \mathrm{Sh}(X_{et}). \end{array}$$

Where i is the functor that regards a sheaf as a presheaf, a is the sheafification, $\pi_* : \mathrm{Sh}(Y_{et}) \to \mathrm{Sh}(X_{et})$ the direct image on sheaves and $(\pi_*)_p : \mathrm{PreSh}(Y_{et}) \to \mathrm{PreSh}(X_{et})$ the direct image on presheaves. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Hence

$$\begin{aligned} R^r\pi_*\mathcal{F} &\overset{\text{def}}{=} H^r(\pi_*(\mathcal{F} \to \mathcal{I}^\bullet)) && \text{(the rth cohomology group of the sequence $\pi_*\mathcal{F} \to \pi_*\mathcal{I}^\bullet$)} \\ &= H^r(a \circ (\pi_*)_p \circ i(\mathcal{F} \to \mathcal{I}^\bullet)) && (a \circ (\pi_*)_p \circ i = \pi_*) \\ &= a \circ (\pi_*)_p (H^r(i\mathcal{F} \to i\mathcal{I}^\bullet)). && \text{(exactness of a (2.4.7) and $(\pi_*)_p$ (2.5.2))} \end{aligned}$$

Since $H^r(i\mathcal{F} \to i\mathcal{I}^\bullet)$ is the presheaf defined by $V \mapsto H^r(V, \mathcal{F})$, $(\pi_*)_p(H^r(i\mathcal{F} \to i\mathcal{I}^\bullet))$ is the desired presheaf. □

Corollary 3.1.15. *The stalk of $R^r\pi_*\mathcal{F}$ at $\overline{x} \to X$ is $\varinjlim H^r(U \times_X Y, \mathcal{F})$ where the limit is taken over all the étale neighborhoods (U, u) of $\overline{x}$.*

Proof: By construction, (2.4.3), the stalks of $a \circ \pi_*(H^r(i\mathcal{I}^\bullet))$ and $\pi_*(H^r(i\mathcal{I}^\bullet))$ are isomorphic, and $\varinjlim H^r(U \times_X Y, \mathcal{F})$ is the stalk of $\pi_*(H^r(i\mathcal{I}^\bullet))$. □

Example 3.1.16. *Assume that X is connected and normal scheme, let $g : \eta \hookrightarrow X$ be the inclusion of the generic point of X and $\overline{x}$ be a geometric point of X. Let $\varphi : U \to X$ be an étale map with U connected and recall that since X is normal, U is also normal (1.2.11), so the generic point of U is mapped to the generic point of X. Observe also that the étale neighborhoods U, with U affine and connected are cofinal. Hence,*

$$\begin{aligned}
(R^r g_* \mathcal{F})_{\overline{x}} &= \varinjlim H^r\left(U \times_X \eta, \mathcal{F}\right) && \textit{(taking the limit over the cofinal set of neighborhoods)}\\
&\simeq \varinjlim H^r\left(\varphi^{-1}(\eta), \mathcal{F}\right) && \textit{(where } \varphi : U \to X\textit{)}\\
&\simeq \varinjlim H^r\left(\operatorname{Spec}\left(\Gamma\left(U, \mathcal{O}_U\right)_{(0)}\right), \mathcal{F}\right) &&\\
&\simeq H^r\left(\operatorname{Spec}\left(\left(\varinjlim \Gamma\left(U, \mathcal{O}_U\right)\right)_{(0)}\right), \mathcal{F}\right) && \textit{(Thm 3.1.5)}\\
&\overset{def}{=} H^r\left(\operatorname{Spec}\left(Frac\left(\mathcal{O}_{X_{et},\overline{x}}\right)\right), \mathcal{F}\right) && \textit{(Defn 2.3.9)}
\end{aligned}$$

3.1.8 Spectral Sequences

In this subsection we will present two standard results about spectral sequences. A three page explanation about this topic is available in [Mil80, page 307-309] and for a fourteen page explanation see [Sha72, page 39-52].

Theorem 3.1.17. *(Grothendieck's composite functor spectral sequence) Let A, B and C be abelian categories, such that A and B have enough injectives. Let $F : A \to B$ and $G : B \to C$ be left exact functors, such that $(R^r G)(FI) = 0$ for $r > 0$ if I is injective. Then for each object X of A there is a spectral sequence*

$$E_2^{rs} = (R^r G)(R^s F)(X) \Rightarrow R^{r+s}(GF)(X).$$

Proof. See [HS71, 9.3 page 299]. □

Remark 3.1.18. *If F takes injectives to injectives then $(R^r G)(FI) = 0$ for $r > 0$.*

Theorem 3.1.19. *(Leray Spectral Sequence) Let $\pi : Y \to X$ be a morphism of schemes. For any sheaf $\mathcal{F}$ on Y_{et}, there is a spectral sequence*

$$H^r\left(X_{et}, R^s \pi_* \mathcal{F}\right) \Rightarrow H^{r+s}\left(Y_{et}, \mathcal{F}\right).$$

Proof: This follows directly from (3.1.17) and the previous remark, since the functors $F = \pi_* : \operatorname{Sh}(Y_{et}) \to \operatorname{Sh}(X_{et})$ and $G = \Gamma(X, -) : \operatorname{Sh}(X_{et}) \to Ab$ are both left exact (2.5.2) and π_* preserves injectives (2.5.10) □

3.2 Čech Cohomology and its applications

3.2.1 Definition of the Čech groups

The computation of the cohomology groups arising from the derived functor approach is, in general, a very difficult task. To overcame this problem we compute the Čech cohomology groups. Which are, in general, more manageable. Under some non-restrictive hypothesis, the results are the same.

Let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be an étale covering family of X, and let $\mathcal{P}$ be a presheaf of abelian groups on X_{et}. We define the rth Čech Cohomology group of $\mathcal{P}$ relative to the covering $\mathcal{U}$, $\check{H}^r(\mathcal{U}, \mathcal{P})$, in the same way that we would define it for a topological space [Har77, III.4] but now fiber products will play the role of intersections.

Definition 3.2.1. *In the same setting, define*

$$C^r(\mathcal{U}, \mathcal{P}) \overset{def}{=} \prod_{(i_0, \ldots, i_r) \in I^{r+1}} \mathcal{P}(U_{i_0 \ldots i_r}), \quad \textit{where } U_{i_0 \ldots i_r} \overset{def}{=} U_{i_0} \times_X \cdots \times_X U_{i_r}.$$

For $s = (s_{i_0 \dots i_r}) \in C^r(\mathcal{U}, \mathcal{P})$, define $d^r s \in C^{r+1}(\mathcal{U}, \mathcal{P})$ by the rule

$$(d^r s)_{i_0 \dots i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j s_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}} | U_{i_0 \dots i_{r+1}}.$$

One verifies by straightforward calculation that

$$C^\bullet(\mathcal{U}, \mathcal{P}) \stackrel{def}{=} C^0(\mathcal{U}, \mathcal{P}) \to \cdots \to C^r(\mathcal{U}, \mathcal{P}) \xrightarrow{d^r} C^{r+1}(\mathcal{U}, \mathcal{P}) \to \cdots$$

is a complex. We define the rth ***Čech Cohomology groups of $\mathcal{P}$ relative to the covering $\mathcal{U}$*** *by*

$$\check{H}^r(\mathcal{U}, \mathcal{P}) \stackrel{def}{=} H^r(C^\bullet(\mathcal{U}, \mathcal{P})).$$

The ***Čech Cohomology groups*** *are defined as a limit over all covering families, i.e.,*

$$\check{H}^r(X_{et}, \mathcal{P}) \stackrel{def}{=} \varinjlim \check{H}^r(\mathcal{U}, \mathcal{P}).$$

For more details check [Mil08, Chapter 10].

Example 3.2.2. *To be added.*

Remark 3.2.3. *If $\mathcal{U}$ is a Zariski cover then we get the usual complex.*

Remark 3.2.4. *It is also possible to define the first Čech cohomology* group *for presheaves of groups (not necessarily abelian), later we will use this approach to interpret the first Čech cohomology* group *as the set of isomorphism classes of principal homogeneous spaces.*

Proposition 3.2.5. *Some standard properties:*

- $\check{H}^0(X_{et}, \mathcal{F}) = \Gamma(X, \mathcal{F})$ *for any sheaf $\mathcal{F}$ on X;*
- $\check{H}^r(X_{et}, \mathcal{I}) = 0$*, for all injective sheaves $\mathcal{I}$.*

Proof: The first one is obvious. For the second see [Mil80, III.2.4 page 98]. □

3.2.2 Comparison of the first cohomology groups

On any site, the first Čech cohomology group and the first derived functor coincide. In this subsection we aim to prove this. Since we have only been working with the étale site we only present a proof in this case, however the proofs also hold in a more general setting.

Proposition 3.2.6. *For any sheaf $\mathcal{F}$ on X_{et}, let $\mathcal{H}^r(\mathcal{F})$ denote the presheaf $U \mapsto H^r(U, \mathcal{F})$. For all $r > 0$, the sheaf associated with $\mathcal{H}^r(\mathcal{F})$ is 0.*

Proof: Consider the functors

$$\mathrm{Sh}(X_{et}) \xrightarrow{i} \mathrm{PreSh}(X_{et}) \xrightarrow{a} \mathrm{Sh}(X_{et}),$$

where i is the functor that regards a sheaf as a presheaf, and a is sheafification (2.4.2). Recall that i is left exact (2.4.6) and a is exact (2.4.7). Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Hence $\mathcal{H}^r(\mathcal{F}) \stackrel{\text{def}}{=} H^r(i\mathcal{F} \to i\mathcal{I}^\bullet)$ (the rth cohomology sheaf of the complex $i\mathcal{F} \to i\mathcal{I}^\bullet$), therefore,

$$\begin{aligned} a(\mathcal{H}^r(\mathcal{F})) &\stackrel{\text{def}}{=} a(H^r(i\mathcal{F} \to i\mathcal{I}^\bullet)) \\ &\simeq H^r(a(i\mathcal{F} \to i\mathcal{I}^\bullet)) && (a \text{ is exact}) \\ &= H^r(\mathcal{F} \to \mathcal{I}^\bullet) && (a \circ i = \mathrm{id}) \\ &= 0 \quad \text{for } r > 0 && (\mathcal{F} \to \mathcal{I}^\bullet \text{ is exact}) \end{aligned}$$

□

Remark 3.2.7. *$\mathcal{H}^r$ are the right derived functors of $i : \mathrm{Sh}(X_{et}) \to \mathrm{PreSh}(X_{et})$.*

Corollary 3.2.8. *Let $s \in H^r(X_{et}, \mathcal{F})$ for some $r > 0$. Then there exists an étale covering family $\{U_i \to X\}_{i\in I}$ such that the image of s in each group $H^r(U_i, \mathcal{F})$ is zero.*

These results allow us to construct an isomorphism between the groups $H^1(X_{et}, \mathcal{F})$ and $\check{H}^1(X_{et}, \mathcal{F})$: take an embedding of $\mathcal{F}$ into an injective sheaf, $\mathcal{F} \hookrightarrow \mathcal{I}$, and let $\mathcal{G}$ be the cokernel (as a sheaf) of the embedding. We then have an exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to \mathcal{H}^1(\mathcal{F}) \to 0.$$

Let $s \in H^1(X_{et}, \mathcal{F})$, and $t \in \mathcal{G}(X)$ a preimage of s. By Corollary 3.2.8, there is a covering $\{U_i \to X\}_{i\in I}$ such that $s|U_i = 0$, hence $t|U_i$ lifts to an a element $\tilde{t_i}|U_i \in \mathcal{I}(U_i)$. With this we can define $s_{ij} = \tilde{t_j}|U_{ij} - \tilde{t_i}|U_{ij}$ which is a 1-cocycle in $C^1(\{U_i \to X\}_{i\in\mathcal{I}}, \mathcal{F})$. Observe that if we make different choices, in the end they will differ by an element $f \in C^0(\{U_i \to X\}_{i\in\mathcal{I}}, \mathcal{F})$, and so will be cohomologous. Therefore the map $s \mapsto (s_{ij})$ to $\check{H}^1(X_{et}, \mathcal{F})$ is well defined.

Proposition 3.2.9. *The map $s \mapsto (s_{ij})$ defines an isomorphism $H^1(X_{et}, \mathcal{F}) \to \check{H}^1(X_{et}, \mathcal{F})$.*

Instead of giving the proof for the above result we will present an alternative method using spectral sequences.

Theorem 3.2.10. *There is a spectral sequence:*

$$\check{H}^r(X_{et}, \mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X_{et}, \mathcal{F})$$

Proof. First, one can show that $\mathrm{PreSh}(X_{et})$ has enough injectives. Moreover, we can generalize (3.2.5) and get that $\check{H}^r(X_{et}, \mathcal{I}) = 0$, for all injective presheaves $\mathcal{I}$ [Mil80, III.2.4 page 98].A short exact sequence of presheaves gives rise to a short exact sequence of complexes, hence, a long exact sequence on Čech cohomology groups.

Combining these observations we get that $\check{H}^r(X_{et}, -)$ is the rth right derived functor (on the category of presheaves) of $\check{H}^0(X_{et}, -)$. Consider the sequence of functors

$$\mathrm{Sh}(X_{et}) \xrightarrow{i} \mathrm{PreSh}(X_{et}) \xrightarrow{\check{H}^0(X_{et}, -)} \mathrm{Ab}\,.$$

As we saw before the rth derived functor of i is $\mathcal{H}^r$ (3.2.7). Recall that $\check{H}^0(X_{et}, -)$ is left exact, and we already saw that i is also left exact (2.4.6) and preserves injectives (2.4.8). Therefore, from Grothendieck's theorem on the existence of spectral sequences (3.1.17), we have the desired spectral sequence,

$$(R^r\check{H}^0(X_{et}, -))(R^s i)(\mathcal{F}) = \check{H}^r(X_{et}, \mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X_{et}, \mathcal{F}) = R^{r+s}(\check{H}^0(X_{et}, -) \circ i)(\mathcal{F}).$$

□

Corollary 3.2.11. *For a sheaf $\mathcal{F}$ on X we have*

$$\check{H}^r(X_{et}, \mathcal{F}) \simeq H^r(X_{et}, \mathcal{F}) \quad \textit{for } r = 0, 1.$$

PROOF: Follows directly from the previous theorem and the fact that $\check{H}^0(X_{et}, \mathcal{H}^s(\mathcal{F})) = 0$ for $s > 0$ (3.2.8). □

3.2.3 Principal Homogeneous Spaces and H^1

We have already seen that $\check{H}^1(X_{et}, \mathcal{F})$ and $H^1(X_{et}, \mathcal{F})$ coincide. Our aim in this subsection is to get a new interpretation for the first Čech group in terms of principal homogeneous spaces. For this it is convenient to consider $\check{H}^1(X_{et}, \mathcal{F})$ in a more general setting: we will only require that $\mathcal{F}$ is a sheaf of groups (not necessarily abelian).

Definition 3.2.12. *Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be an étale covering of X, and let $\mathcal{F}$ be a sheaf of groups on X_{et}.*

*We define $U_{ij\ldots}$ as $U_i \times_X U_j \times_X \cdots$. A **1-cocycle** for $\mathcal{U}$ with values in F is a family $g = \{g_{ij}\}_{(i,j)\in I\times I} \in \prod_{(i,j)\in I\times I} \mathcal{F}(U_{ij})$ such that $g_{ij} \cdot g_{jk} = g_{ik}$ in U_{ijk}.*

*We say that two cocycles g and g' are **cohomologous**, denoted $g \sim g'$, if there is $\{h_i\}_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $g'_{ij} = h_i \cdot g_{ij} \cdot h_j^{-1}$ in U_{ij}.*

The set of 1-cocycles modulo $\sim$ is denoted $\check{H}^1(X_{et}, \mathcal{F})$.

Remark 3.2.13. *In general, $\check{H}^1(X_{et}, \mathcal{F})$ is not a group.*

Definition 3.2.14. *Let G be a group and S be a set on which G acts on the right. The set S is said to be a* ***principal homogeneous space for*** *G if, there exists $s \in S$, such that the map $(g \mapsto sg) : G \to S$, is a bijection. This will be then true for each $s \in S$.*

Let $\mathcal{F}$ be a sheaf of groups on X_{et}, and let $\mathcal{S}$ be a sheaf of sets on which $\mathcal{F}$ acts on the right, i.e., for each $U \to X$ étale map, $\mathcal{S}(U)$ is a set on which $\mathcal{F}(U)$ acts on the right and the action is compatible with restriction maps. Then $\mathcal{S}$ is called a ***principal homogeneous space for*** *$\mathcal{F}$ if:*

- *there exists an étale covering family $\{U_i \to X\}$ of X such that, for all i, $S(U_i) \neq \emptyset$;*
- *for every $U \to X$ étale and $s \in \Gamma(U, \mathcal{S})$, the map $g \mapsto sg : \mathcal{F}|U \to \mathcal{S}|U$ is an isomorphism of sheaves.*

If $\mathcal{S}$ is a principal homogeneous space for $\mathcal{F}$ we say that a covering family $\{U_i \to X\}$ ***splits*** *$\mathcal{S}$ if $\mathcal{S}(U_i) \neq \emptyset$ for all i.*

Let $\mathcal{S}$ be a principal homogeneous space for $\mathcal{F}$, $\mathcal{U} = \{U_i \to X\}$ a étale covering family of X that splits $\mathcal{S}$, and choose $s_i \in \mathcal{S}(U_i)$ for each i. By the second condition there exists a unique $g_{ij} \in \mathcal{F}(U_{ij})$, such that $s_i \cdot g_{ij} = s_j$ in U_{ij}. Hence $\{g_{ij}\}_{(i,j)\in I\times I}$ is a cocycle, since

$$s_i \cdot g_{ij} \cdot g_{jk} = s_j \cdot g_{jk} = s_k = s_i \cdot g_{ik}.$$

Replacing s_i with $s_i' = s_i h_i$ for some $h_i \in \mathcal{F}(U_i)$ produces a cohomologous cocycle. Thus $\mathcal{S}$ defines a class $c(\mathcal{S})$ in $\check{H}^1(\mathcal{U}, \mathcal{F})$.

Example 3.2.15. *Let $\mathcal{G}$ be the constant sheaf on X_{et} defined by a finite group G. Let $\pi : Y \to X$ be a Galois covering with group G. Let $\mathcal{S}$ be the sheaf of sets defined by*

$$(\varphi : U \to X) \mapsto \mathcal{S}(\varphi : U \to X) = \{s : U \to Y \text{ morphisms such that } \pi \circ s = \varphi\}.$$

Recall that $\forall_{x\in X}\, \exists_{(U,u)\to(X,x)}$ such that $Y \times_X U \simeq U \times G$. For such a neighborhood we have $\mathcal{S}|U \simeq \mathcal{G}$, hence, $\mathcal{S}$ is a principal homogeneous space for $\mathcal{G}$.

Proposition 3.2.16. *The map $\mathcal{S} \mapsto c(\mathcal{S})$ defines a bijection from the set of isomorphism classes of principal homogeneous spaces for $\mathcal{F}$ split by $\mathcal{U}$ to $\check{H}^1(\mathcal{U}, \mathcal{F})$.*

Proof: The proof is straightforward. It can be read in [Mil80, III 4.6 page 123], or a more detailed version, see [Mil08, 11.1 page 75]. □

Proposition 3.2.17. *When $\mathcal{G}$ is defined by an affine scheme over X (Section 2.3.1), then $\mathcal{S}$ is representable by a scheme.*

Proof: See [Mil80, III 4.3 page 121]. □

Example 3.2.18. *Conversely to the previous example (3.2.15), every principal homogeneous space of $\mathcal{G}$ arises from a Galois covering with group G. Since G is finite, it can be considered as an affine group scheme over X, thus, by the previous proposition $\mathcal{S}$ can be represented by a scheme, that will be a Galois covering of X with group G.*

When X is connected, the Galois coverings of X with group G are classified canonically by the continuous homomorphisms from $\pi_1(X, \overline{x})$ to G, where $\overline{x}$ is a geometric point (2.2.11).

All this, together with (3.2.11) and (3.2.16), gives us that, for X connected and G abelian, there is a canonical isomorphism

$$H^1(X_{et}, \mathcal{G}) \simeq \check{H}^1(X_{et}, \mathcal{G}) \simeq \mathrm{Hom}_{conts}(\pi_1(X, \overline{x}), G).$$

Čech cohomology with coefficients in GL_n

The following result gives a geometric interpretation of the first cohomology group with coefficients in GL_n. It shows that in this very special case, the result is the same for th three different topologies: Zariski, étale and flat (2.1.8).

Definition 3.2.19. *Let $L_n(X_*)$ denote the set of isomorphism classes of locally free sheaves of $\mathcal{O}_{X_*}$-modules of rank n on the site X_*. Where $*$ stands for Zariski, étale or flat.*

Proposition 3.2.20. *(Hilbert's Theorem 90) There are natural bijections:*

$$\begin{array}{ccccc} \check{H}^1(X_{zar}, \mathrm{GL}_n) & \overset{\sim}{\longleftrightarrow} & \check{H}^1(X_{et}, \mathrm{GL}_n) & \overset{\sim}{\longleftrightarrow} & \check{H}^1(X_{fl}, \mathrm{GL}_n) \\ \updownarrow \simeq & & \updownarrow \simeq & & \updownarrow \simeq \\ L_n(X_{zar}) & \overset{\sim}{\longleftrightarrow} & L_n(X_{et}) & \overset{\sim}{\longleftrightarrow} & L_n(X_{fl}). \end{array}$$

Thus there is a canonical isomorphism

$$H^1(X_{et}, G_m) \simeq Pic(X) \overset{def}{=} L_1(X_{zar}).$$

PROOF: (Sketch) The first vertical map and the fact that it is a bijection are well known, (check [Har77, exc III.4.5 page 224] or [Mil08, 11.4 page 76]). For the other vertical maps the construction is the same and the proof of bijection follows *mutatis mutandis*.

Much more difficult is to prove that the horizontal maps are bijections induced by the canonical continuous morphisms of sites $X_{fl} \to X_{et} \to X_{zar}$ (2.1.11), the proof of this is sketched in [Mil08, 11.4 page 76].

For the case when $n = 1$ there is a much more elegant proof using the Leray Spectral Sequence (3.1.19) in a more general setting (where we consider π as the canonical morphism between the different sites). As an example, we get $H^r(X_{zar}, R^s\pi_*G_m) \Rightarrow H^{r+s}(X_{fl}, G_m)$. If $R^1\pi_*G_m = 0$ we will get the desired isomorphism. It is sufficient to prove $H^1(\mathrm{Spec}(A)_{fl}, G_m) = 0$, for A local, since this will imply that the stalks of $R^1\pi_*G_m$ are zero, by (3.1.15).

The full proof for $n = 1$, using spectral sequences can be found in [Mil80, III.4.9]. □

3.3 Cohomology of G_m

Now our aim is to compute the other cohomology groups of G_m.

3.3.1 Weil-divisor sequence

Definition 3.3.1. *Let $z \in X$, we define the codimension of z, codim(z), as the codimension of the Zariski closure of z in X.*

Proposition 3.3.2. *(Weil-divisor sequence) Let X be a connected normal variety, $g : \eta \to X$ be the inclusion of the generic point of X, η, into X and $i_z : z \to X$ denote the inclusion of a point z into X*

There is a sequence of sheaves on X_{zar} and X_{et}

$$0 \to G_m \to g_*G_m \to \oplus_{codim(z)=1} i_{z*}\mathbb{Z} \to 0.$$

It is always left exact, and it is exact if X is regular.

PROOF: For X_{zar}, the sequence is the well known sequence [Har77, II.6.11 page 141],

$$0 \to \mathcal{O}_X^\times \to K^\times \to \mathrm{Div} \to 0,$$

where K is the field of rational functions of X, i.e., the field of fractions of $\Gamma(U, \mathcal{O}_X)$, for any open affine $U \subset X$. The identification of these two sequences is given by the following remarks: it is clear that $G_m(U) \overset{\text{def}}{=} \Gamma(U, \mathcal{O}_X))^\times \overset{\text{def}}{=} \mathcal{O}_X^\times(U)$. Now take $U = \text{Spec}(A)$ to be an open affine set of X. Then $g_*G_m(U) \overset{\text{def}}{=} G_m(g^{-1}(U)) \simeq G_m(\text{Spec}(A_{(0)})) \overset{\text{def}}{=} A_{(0)}^\times \overset{\text{def}}{=} K^\times$, and finally

$$\text{Div}(U) \overset{\text{def}}{=} \oplus_{z \text{ a closed point}} \mathbb{Z} \overset{\text{def}}{=} \oplus_{\{z \in U : codim(z)=1\}} \mathbb{Z}.$$

By the same remarks we get that the sequence on $U = \text{Spec}(A)$ is

$$0 \to A^\times \to K^\times \xrightarrow{\text{ord}_\mathfrak{p}} \oplus_{\{\mathfrak{p} \in \text{Spec}(A)=1 : \text{height}(\mathfrak{p})=1\}} \mathbb{Z} \to 0.$$

On the stalks we get the same with A replaced by $\mathcal{O}_{X_{zar},x}$. A result from commutative algebra says that this sequence, with A integrally closed, is always left exact, and it is exact if and only if A is unique factorization domain. If X is regular, the stalks $\mathcal{O}_{X_{zar},x}$ are regular, and therefore unique factorization domains. This allow us to conclude that the above sequence is exact.

The result for the étale topology, follows directly from the Zariski case. For any étale map $U \to X$ with U connected, the restriction of the sequence to U_{zar} is exact by what we just proved. From this we automatically get left exactness.

Recall that whenever X is regular, U is regular (1.2.11), surjectivity and local surjectivity are equivalent for a morphism of sheaves (2.4.6), and any Zariski covering is also an étale covering. Hence, if a map is locally surjective for the Zariski topology it is also locally surjective for the étale topology. Therefore, when X regular

$$g_*G_m \to \oplus_{codim(z)=1} i_{z*}\mathbb{Z}$$

is locally surjective in the Zariski topology, hence locally surjective in the étale topology. □

Remark 3.3.3. *In the last two paragraphs of the proof we proved that if a sequence of sheafs on X_{et} is exact restricted to all U_{zar} with U étale over X then the sequence is also exact on X_{et}.*

Remark 3.3.4. *When X is over an algebraically closed field, X being regular is equivalent to X being a nonsingular variety.*

We will now use the exact sequence

$$0 \to G_m \to g_*G_m \to \oplus_{codim(z)=1} i_{z*}\mathbb{Z} \to 0$$

to compute the cohomology groups of G_m when X is a connected nonsingular curve over an algebraically closed field k. In order to do that we first need some results from number theory.

Definition 3.3.5. *A field k is said to be **quasi-algebraically closed** if every nonconstant homogeneous polynomial $f(T_1, \ldots, T_n) \in k[T_1, \ldots T_n]$ of degree $d < n$ has a nontrivial zero in k^n.*

Proposition 3.3.6. *The following fields are quasi-algebraically closed.*

(i) an algebraically closed field;

(ii) a function field of dimension one over an algebraically closed field;

(iii) the field of fractions of the strict henselization of $\mathcal{O}_{X_{zar},x}$, where X is a scheme over a field with characteristic zero.

Proof: The first is obvious. For the last two see [Sha72, Theorem 24 page 108 and Theorem 27 page 116] respectively. □

Proposition 3.3.7. *Let k be a quasi-algebraically closed field, $G = \text{Gal}(k^{sep}/k)$ and M a discrete G-module.*

(i) $H^r(G, M) = 0$ for $r > 2$ for any M.

*(ii) When $M = (k^{sep})^\times$, is also true for $r = 2$, i.e., $H^2(G, (k^{sep})^\times) = 0$. The cohomology group $H^2(G, (k^{sep})^\times)$ is called the **Brauer group** of k;*

(iii) If M is a torsion module then $H^r(G, M) = 0$ for $r > 1$;

PROOF: The last two are [Sha72, Corollary 1 page 107]. The first is a consequence of combining [Sha72, Proposition 13 page 53] and the last sentence. □

Lemma 3.3.8. *Let X be a connected normal curve over an algebraically closed field and $g : \eta \hookrightarrow X$ be the inclusion of the generic point of X. The cohomology groups $H^r(X_{et}, g_*G_m)$ and $H^r\left(X_{et}, \oplus_{codim(z)=1} i_{z*}\mathbb{Z}\right)$ are zero for all $r > 0$.*

PROOF: For a closed point $z \in X$, i_{z*} is exact (2.5.4), thus $H^r(X_{et}, i_{z*}\mathcal{F}) \simeq H^r(z_{et}, \mathcal{F})$. Since z is a geometric point (2.2.3), $H^r(z_{et}, \mathcal{F}) = 0$ for $r > 0$ (see Section 3.1.2). Therefore $H^r\left(X_{et}, \oplus_{codim(z)=1} i_{z*}\mathbb{Z}\right) = 0$ for $r > 0$.

Recall from (3.1.16) that,

$$(R^r g_*\mathcal{F})_{\overline{x}} = H^r\left(\text{Spec}\left(Frac\left(\mathcal{O}_{X_{et},\overline{x}}\right)\right), \mathcal{F}\right),$$

and when K is a field we have $H^r(\text{Spec}(K), \mathcal{F}) \simeq H^r(G, M_{\mathcal{F}})$ where $G = \text{Gal}(K^{sep}/K)$ and $M_{\mathcal{F}}$ is the discrete G-module associated to $\mathcal{F}$ (3.1.2).

In the case when $\mathcal{F} = G_m$ we have $M_{\mathcal{F}} = (K^{sep})^\times$ (2.3.8). Combining all this with Proposition 3.3.7 we get that

$$(R^r g_*G_m)_{\overline{x}} = H^r\left(\text{Spec}\left(Frac\left(\mathcal{O}_{X_{et},\overline{x}}\right)\right), G_m\right) = H^r\left(\text{Gal}(K^{sep}/K), (K^{sep})^\times\right) = 0 \text{ for } r > 1.$$

From Hilbert's Theorem 90 (3.2.20), it also follows that

$$(R^r g_*G_m)_{\overline{x}} = H^r\left(\text{Spec}\left(Frac\left(\mathcal{O}_{X_{et},\overline{x}}\right)\right), G_m\right) = 0 \text{ for } r = 1.$$

Since $R^r g_*G_m = 0$ for $r > 0$, the result follows from the Leray spectral sequence (3.1.19), and the fact that the field of rational functions on X is a function field of dimension one over k (3.3.2). Thus we have

$$\begin{aligned}
H^r(X_{et}, g_*G_m) &\simeq H^r(\eta_{et}, G_m) && \text{(Leray spectral sequence 3.1.19)}\\
&= H^r(\text{Spec}(K)_{et}, G_m) && \text{(where } K \text{ is the field of rational functions of } X)\\
&= H^r\left(\text{Gal}(K^{sep}/K), (K^{sep})^\times\right) && \text{(Prop 3.1.2)}\\
&= \begin{cases} 0 & \text{for } r > 1 \\ 0 & \text{for } r = 1 \end{cases} && \begin{matrix}\text{(Prop 3.3.7)}\\ \text{(Hilbert's Theorem 90 3.2.20)}\end{matrix}
\end{aligned}$$

□

Theorem 3.3.9. *For a connected nonsingular curve X over an algebraically closed field,*

$$H^r(X_{et}, G_m) \begin{cases} = \Gamma(X, \mathcal{O}_X^\times), & r = 0 \\ \simeq \text{Pic}(X) \simeq \text{Div}(X)/K^\times, & r = 1 \\ = 0, & r > 1, \end{cases}$$

where K is the field of rational functions on X.

PROOF: The Weil-divisor sequence, (3.3.2), gives rise to a long exact sequence,

$$\cdots \to H^r(X_{et}, G_m) \to H^r(X_{et}, g_*G_m) \to H^r\left(X_{et}, \oplus_{codim(z)=1} i_{z*}\mathbb{Z}\right) \to H^{r+1}(X_{et}, G_m) \to \cdots.$$

The previous lemma then implies that $H^r(X_{et}, G_m) = 0$ for $r > 1$. Looking at the initial terms of the sequence, the previous lemma, Hilbert's Theorem 90 (3.2.20) imply that

$$0 \to \Gamma(X, \mathcal{O}_X^\times) \to K^\times \to \text{Div}(X) \to H^1(X_{et}, G_m) \simeq \text{Pic}(X) \to 0.$$

The map $K^\times \to Div(X)$ is the map $f \mapsto \text{div}(f) = \sum \text{ord}_z.z$ where sum runs over the closed points of X. Hence, $\text{Pic}(X) \simeq \text{Div}(X)/K^\times$, (cf. [Har77, II.6.16]). □

Remark 3.3.10. *This theorem also holds in a more general setting when we consider* ***Dedekind Schemes***[2]*, also known as the "abstract nonsingular curves" [Har77, I.§6].*

[2] A **discrete valuation ring** or **DVR** is a Noetherian local domain that is a regular local ring.
A **Dedekind ring** is a Noetherian domain all of whose localizations at maximal ideals are discrete valuation rings.
A **Dedekind scheme** is a quasi-compact and irreducible scheme which has an open covering by spectra of Dedekind rings.

The Picard group of a Curve

Our aim is now to understand a little better the structure of the Picard group of a curve. As we saw before, (3.3.9), it fits in the exact sequence

$$K^\times \xrightarrow{\text{div}} \oplus_{\substack{x\in X\\ \text{closed}}} \mathbb{Z} \overset{\text{def}}{=} \text{Div}(X) \to \text{Pic}(X) \to 0.$$

Any divisor, $D \in \text{Div}(X)$, can be written as finite sum $D = \sum_{z\in X} n_x.x$, $n_x \in \mathbb{Z}$, and the **degree of** D is $\sum_{z\in X} n_x$. Observe that for $f \in K^\times$ the divisor $\text{div}(f)$ has degree zero, so we can define $\text{Div}^0(X)$ as the subgroup of divisors of degree 0, and $\text{Pic}^0(X) \overset{\text{def}}{=} \text{Div}^0(X)/K^\times$.

Proposition 3.3.11. *Let X be a complete connected nonsingular curve over an algebraically closed field k. The sequence*

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \mathbb{Z} \to 0$$

is exact. For any integer n relatively prime to the characteristic of k,

$$\text{Pic}^0(X) \to \text{Pic}^0(X) : z \mapsto nz$$

is surjective with kernel equal to a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2g$, where g is the genus of X.

PROOF: The first sequence is part of the **kernel-cokernel exact sequence**[3] of the pair of maps

$$K^\times \xrightarrow{\text{div}} \text{Div}^0(X) \hookrightarrow \text{Div}(X).$$

The second statement when $k = \mathbb{C}$ is Abel's Theorem [Ful95, 21.28 page 306]. The general case is analogous, but requires the algebraic theory of the Jacobian variety of X [Mil86]. □

3.3.2 Cohomology of G_m with support on x

Proposition 3.3.12. *Let U be a nonsingular curve over an algebraically closed field k. For any $x \in U$,*

$$H^1_x(U_{et}, G_m) = \mathbb{Z}, \quad H^r_x(U_{et}, G_m) = 0,\ r \neq 1.$$

PROOF: Let $R \overset{\text{def}}{=} \mathcal{O}_{U_{et},\overline{x}}$, $K \overset{\text{def}}{=} Frac(R)$ and $V \overset{\text{def}}{=} \text{Spec}(R)$. Combining 2.3.19) and (3.1.9), it follows that $H^r_x(U_{et}, G_m) \simeq H^r_x(V_{et}, G_m)$.

R is a regular local ring of dimension 1, and a domain, hence R is a discrete valuation ring and therefore V is a Dedekind Scheme. Thus, since V has only one closed point, $H^r(V_{et}, G_m) = 0$ for $r > 0$ (3.3.10).

We have that

$$V\backslash x = \text{Spec}(R)\backslash m_x = \text{Spec}\left(R_{(0)}\right) = \text{Spec}(K),$$

hence $H^r((V\backslash x)_{et}, G_m) = H^r(\text{Spec}(K)_{et}, G_m) = 0$ for $r > 0$, since K is separably algebraically closed, (see section 3.1.2). From the long exact sequence of the pair $(V, V\backslash x)$,

$$\cdots \to H^r(V_{et}, G_m) \to H^r((V\backslash x)_{et}, G_m) \to H^{r+1}_x(V_{et}, G_m) \to H^{r+1}(V_{et}, G_m) \to \cdots (3.1.7),$$

we get that $H^r_x(V_{et}, G_m) = 0$ for $r > 1$, since with the exception of $H^{r+1}_x(V_{et}, G_m)$, all others groups in the long exact sequence are known to be zero for $r > 0$. For $r = 0, 1$ we have

$$\begin{aligned} H^0_x(V_{et}, G_m) &= \text{Ker}\left(H^0(V_{et}, G_m) \to H^0((V\backslash x)_{et}, G_m)\right) \\ &= \text{Ker}\left(R^\times \hookrightarrow K^\times\right) = 0 \\ H^1_x(V_{et}, G_m) &= \text{Coker}\left(H^0(V_{et}, G_m) \to H^0((V\backslash x)_{et}, G_m)\right) \\ &\simeq \frac{H^0((V\backslash x)_{et}, G_m)}{H^0(V_{et}, G_m)} = K^\times/R^\times \\ &\simeq \mathbb{Z}. \qquad \text{(induced by the valuation on } K \text{ determined by } R) \end{aligned}$$

□

[3]The kernel-cokernel exact sequence of the pair $A \xrightarrow{f} B \xrightarrow{g} C$ is

$$0 \to \text{Ker}(f) \to \text{Ker}(g \circ f) \to \text{Ker}(g) \to \text{Coker}(f) \to \text{Coker}(g \circ f) \to \text{Coker}(g) \to 0.$$

3.4 Cohomology of μ_n

Now we are interested in the calculation of the cohomology groups of the sheaf μ_n on a curve, over an algebraically closed field in the étale topology.

There are two main reasons why we need to know these groups. The first one is that although μ_n is isomorphic to the constant sheaf being $\mathbb{Z}/n\mathbb{Z}$, when X is a $\mathbb{Z}[1/n][\sqrt[n]{1}]$-scheme, [Mil80, page 126], the cohomology groups are not necessarily zero for $r > 0$ as in the Zariski topology [Har77, III.2.5 page 208]. The other one is that these groups arise naturally in Poincaré's duality that we will study next.

Proposition 3.4.1. *Let X be a complete connected nonsingular curve over an algebraically closed field k with genus g. For any n prime relatively to char(k),*

$$H^0(X_{et},\mu_n) = \mu_n(k), \quad H^1(X_{et},\mu_n) \approx (\mathbb{Z}/n\mathbb{Z})^{2g}, \quad H^2(X_{et},\mu_n) \simeq \mathbb{Z}/n\mathbb{Z} \text{ and } H^r(X_{et},\mu_n) = 0,\ r > 2.$$

PROOF: From the Kummer sequence (2.4.9), we get the following long exact sequence

$$\cdots \to H^r(X_{et},\mu_n) \to H^r(X_{et},G_m) \xrightarrow{n} H^r(X_{et},G_m) \to H^{r+1}(X_{et},\mu_n) \to \cdots$$

Since X is complete and connected we have $\mathcal{O}_X(X) = k$. Therefore, $H^0(X_{et},\mu_n) = \mu_n(k)$ and $H^0(X_{et},G_m) = k^\times$. Thus we get the following exact sequences, where the first one comes from k being algebraically closed,

$$0 \to H^0(X_{et},\mu_n) = \mu_n(k) \to H^0(X_{et},G_m) = k^\times \xrightarrow{n} k^\times \to 0$$

$$0 \to H^1(X_{et},\mu_n) \to H^1(X_{et},G_m) = \operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X) \to H^2(X_{et},\mu_n) \to H^2(X_{et},G_m) = 0,$$
$$H^{r-1}(X_{et},G_m) = 0 \xrightarrow{n} 0 \to H^r(X_{et},\mu_n) \to H^r(X_{et},G_m) = 0, \quad r > 2.$$

Hence

$$\begin{aligned}
H^1(X_{et},\mu_n) &= Ker\left(\operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)\right) \\
&= Ker\left(\operatorname{Pic}^0(X) \xrightarrow{n} \operatorname{Pic}^0(X)\right) \\
&\approx (\mathbb{Z}/n\mathbb{Z})^{2g}, && \text{(Prop 3.3.11)} \\
H^2(X_{et},\mu_n) &= \operatorname{Coker}\left(\operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)\right) \\
&= \operatorname{Pic}(X)/n\operatorname{Pic}(X) \\
&\simeq \frac{\operatorname{Pic}(X)/\operatorname{Pic}^0(X)}{n\operatorname{Pic}(X)/\operatorname{Pic}^0(X)} && (\operatorname{Pic}(X) \neq 0 \text{and Prop 3.3.11}) \\
&\simeq \mathbb{Z}/n\mathbb{Z}, && (\operatorname{Pic}(X)/\operatorname{Pic}^0(X) \simeq \mathbb{Z}) \\
H^r(X_{et},\mu_n) &= 0, \quad r > 2.
\end{aligned}$$

□

Proposition 3.4.2. *Let U be a nonsingular curve over an algebraically closed field k. For any integer n prime relatively to char(k) and $x \in U$,*

$$H^2_x(U_{et},\mu_n) \simeq \mathbb{Z}/n\mathbb{Z}, \quad H^r_x(U_{et},\mu_n) = 0,\ r \neq 2.$$

PROOF: This follows from Proposition 3.3.12 and the long exact sequence of cohomology groups with support on x associated to the Kummer sequence (2.4.9),

$$\cdots \xrightarrow{n} H^{r-1}_x(U_{et},G_m) \to H^r_x(U_{et},\mu_n) \to H^r_x(U_{et},G_m) \xrightarrow{n} H^r_x(U_{et},G_m) \to H^{r+1}_x(U_{et},\mu_n) \to \cdots.$$

Recall from Proposition 3.3.12, $H^r_x(U_{et},G_m) = 0$ for $r \neq 1$ and $H^1_x(U_{et},G_m) = \mathbb{Z}$, therefore the map $H^r_x(U_{et},G_m) \xrightarrow{n} H^r_x(U_{et},G_m)$ is always injective, hence $H^r_x(U_{et},\mu_n) = 0$ for $r = 0, 1$ and $r > 3$. For $r = 2$ we have, $H^2_x(U_{et},\mu_n) \simeq \operatorname{Coker}\left(\mathbb{Z} \xrightarrow{n} \mathbb{Z}\right) = \mathbb{Z}/n\mathbb{Z}$. □

Proposition 3.4.3. *For any connected nonsingular curve U over an algebraically closed field k and integer n prime to char(k), there is a canonical isomorphism $H_c^2(U,\mu_n) \to \mathbb{Z}/n\mathbb{Z}$.*

PROOF: Let $j : U \hookrightarrow X$ be the completion of U, $Z \stackrel{\text{def}}{=} X \backslash U$ and $i : Z \hookrightarrow X$. By (2.5.18) we have the following short exact sequence

$$0 \to j_! j^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 0,$$

therefore we obtain the exact sequence

$$\cdots \to H^{r-1}(X, i_* i^* \mu_n) \to H^r(X, j_! j^* \mu_n) \to H^r(X, \mu_n) \to H^r(X, i_* i^* \mu_n) \to \cdots .$$

But

$$\begin{aligned} H^r(X, j_! j^* \mu_n) &\stackrel{\text{def}}{=} H_c^r(U, j^* \mu_n) & \\ &\simeq H_c^r(U, \mu_n) & (j^*\mu_n = \mu_n|U) \\ H^r(X, i_* i^* \mu_n) &\simeq H^r(Z, i^* \mu_n) & (Z \text{ closed } \Rightarrow i_* \text{ exact}) \\ &= \oplus_{i=1}^N H^r(z_i, i^*\mu_n|z_i) & (Z \text{ closed } \Rightarrow Z = \{z_1, \ldots, z_N\}) \\ &= 0 \quad \text{for } r > 0. & (\text{Section 3.1.2}) \end{aligned}$$

Hence $H_c^2(U,\mu_n) \to H^2(X,\mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$ is an isomorphism. □

Remark 3.4.4. *For any $x \in U$, torsion sheaf $\mathcal{F}$ on U, and $r \geq 0$, there is a canonical map $H_x^r(U,\mathcal{F}) \to H_c^r(U,\mathcal{F})$. For $\mathcal{F} = \mu_n$ and $r = 2$, this map is compatible with the isomorphisms in the two previous propositions.*

3.5 Poincaré Duality

Definition 3.5.1. *If $\mathcal{F}$ is a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules (notice that it is also a sheaf of abelian groups) with finite stalks, we will say that $\mathcal{F}$ is a* ***finite locally constant sheaf****, with n always implicit.*

A pairing $M \times N \to C$ of abelian groups is said to be ***perfect*** *if the induced maps $M \to \mathrm{Hom}(N,C)$ and $N \to \mathrm{Hom}(M,C)$ are isomorphisms.*

For a finite locally constant sheaf we define the sheaf $\check{\mathcal{F}}$ as $\check{\mathcal{F}}(V) = \mathrm{Hom}_V(\mathcal{F}|V, \mu_n|V)$.

Remark 3.5.2. *If $\mathcal{F}$ is a finite locally constant sheaf on U_{et} with U a connected scheme, it follows from Remark 2.3.28 that there exists some finite étale covering $V \to U$, such that $\mathcal{F}|V$ is the constant sheaf defined by a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module M.*

Remark 3.5.3. *Let U be a scheme over $\mathbb{Z}[1/n][\sqrt[n]{1}]$ and consider the finite locally constant sheaves $\mathbb{Z}/n\mathbb{Z}$ and μ_n on U_{et}, then we have $\mathbb{Z}/n\mathbb{Z} \approx \mu_n$, roughly the isomorphism $\mu_n \to \mathbb{Z}/n\mathbb{Z}$ is defined by choosing one nth root of 1 and mapping it to $1 \in \mathbb{Z}/n\mathbb{Z}$. Thus, it is clear that $\check{\mu}_n \approx \mathbb{Z}/n\mathbb{Z}$. In fact $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$, i.e, the isomorphism is independent of the choice of a generator of μ_n.*

Let α be a generator of μ_n. Any morphism is of the form $\alpha \mapsto \alpha^k$, where $k \in \mathbb{Z}/n\mathbb{Z}$. Picking another generator of μ_n, β , we have, for some l, $\beta = \alpha^l$, hence the morphism is given by $\beta = \alpha^l \mapsto (\alpha^k)^l = \beta^k$, so its representation in $\mathbb{Z}/n\mathbb{Z}$ is independent of the choice of a generator. This is also true for any other constant sheaf represented by a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 1.

Remark 3.5.4. *Recall that $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/(m,n)\mathbb{Z}$. If M is a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, $M \simeq \oplus \mathbb{Z}_{p_i^{\alpha_i}}$ where $p_i^{\alpha_i}|n$, therefore $\mathrm{Hom}(M, \mathbb{Z}/n\mathbb{Z}) \approx M$, so $M \simeq \mathrm{Hom}(\mathrm{Hom}(M,\mathbb{Z}/n\mathbb{Z}),\mathbb{Z}/n\mathbb{Z})$. Since $\mu_n \approx \mathbb{Z}/n\mathbb{Z}$ then $M \approx \check{M} \approx \check{\check{M}} \stackrel{def}{=} \mathrm{Hom}(\mathrm{Hom}(M,\mu_n),\mu_n)$. There is always a natural embedding of M into $\check{\check{M}}$, which is an isomorphism, since M and $\check{\check{M}}$ have the same rank. Hence $M \simeq \check{\check{M}}$.*

We conclude that, if $\mathcal{F}$ is a finite locally constant sheaf on U_{et} with U a connected $\mathbb{Z}[1/n][\sqrt[n]{1}]$-scheme, and $\mathcal{G} = \check{\mathcal{F}}$, then $\mathcal{F} \simeq \check{\mathcal{G}}$, just apply the previous reasoning to the covering where the sheaves are constant.

Since $M \approx \check{M}$, we have that whenever $\mathcal{F}$ is a finite locally constant sheaf, $\check{\mathcal{F}}$ is a finite locally constant sheaf.

Theorem 3.5.5. *(Poincaré Duality) Let U be a connected nonsingular curve over an algebraically closed field k and n be an integer which is relatively prime to char(k). For any finite locally constant sheaf $\mathcal{F}$ and $r \geq 0$, there is a* canonical *perfect pairing of (finite groups)*

$$H_c^{2-r}(U, \mathcal{F}) \times H^r\left(U, \check{\mathcal{F}}\right) \to H_c^2(U, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$$

First, we need to understand that this theorem does not just state that groups are dual to each other, but that they are dual with a *specific pairing*, the canonical one, i.e., independent of a choice. In this part of the text the difference between $\approx$ and $\simeq$ is critical, recall that we use $\approx$ to assert a existence of an isomorphism and $\simeq$ to state that there is an isomorphism that it is canonical or unique.

As an example, we know that since U is a k-scheme with k algebraically closed, it is also a $\mathbb{Z}[1/n][\sqrt[n]{1}]$-scheme, and the choice of a n^{th} root of 1 will give us $\mu_n \approx \mathbb{Z}/n\mathbb{Z}$, but this isomorphism is not canonical [Mil80, page 126]. On the other hand, we saw in the Example 3.5.3 that we have $\check{\mu}_n \simeq \mathbb{Z}/n\mathbb{Z}$, since the isomorphism is independent of a choice of a generator of μ_n.

The pairing in the theorem is the one induced by the following canonical pairing

$$\mathrm{Ext}^r_{\mathbf{A}}(A, B) \times \mathrm{Ext}^s_{\mathbf{A}}(B, C) \to \mathrm{Ext}^{r+s}_{\mathbf{A}}(A, C),$$

where $\mathbf{A}$ is an abelian category and $A, B, C \in \mathrm{Obj}(\mathbf{A})$. We choose to skip the definition of Ext and the details of how to get the pairing of the theorem. This is very well explained in [Mil80, page 167-168].

Remark 3.5.6. *For our sketch proof of Theorem 3.5.5 it is enough to observe that the pairing in the theorem is induced by the pairing*

$$\mathcal{F} \times \check{\mathcal{F}} \to \mu_n,$$

and in the sketch we will use the facts that the pairings for $r = 0, 2$ and $\mathcal{F} = \mu_n$, given by:

$$(\mathbb{Z}/n\mathbb{Z} \simeq H_c^2(U, \mu_n)) \times (H^0(U, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}) \to H_c^2(U, \mu_n); \qquad (r = 0)$$
$$(\mu_n(k) = H_c^0(X, \mu_n)) \times (H^2(X, \mathbb{Z}/n\mathbb{Z}) \approx \mu_n(k)) \to (H_c^2(X, \mu_n) \qquad (r = 2 \text{ and } X \text{ a complete curve})$$

are non degenerate, i.e., in these cases the theorem is verified. Observe that the first pairing is just the usual multiplication.

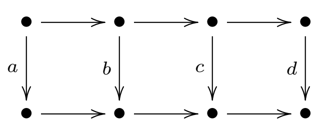
The proof of the Poincaré Duality can be read in [Mil80, V.2.1 page 175].

Remark 3.5.7. *In topology Poincaré Duality is usually stated using the rth homology group of $\mathcal{F}$ instead $H^r\left(U, \check{\mathcal{F}}\right)$. Here we do not have a version of homology, so $\mathcal{F}$ and $\check{F}$ are used to obtain a pairing with values in μ_n.*

3.5.1 Sketch of the proof of Poincaré duality

We will need some results that will be presented without proof.

Lemma 3.5.8. *(Five-lemma) Consider the following commutative diagram with exact rows,*



then

b,d injective, a surjective $\Rightarrow$ c injective;
a,c surjective, d injective $\Rightarrow$ b surjective.

Lemma 3.5.9. *The module $\mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}/n\mathbb{Z}$-module is injective.*

Proof: This follows from Baer's criterion, see [Wei94, 2.3.1 page 39]. □

Theorem 3.5.10. *(Cohomological Dimension) Let $\mathcal{F}$ be a torsion sheaf (3.1.10). For a variety X over an algebraically closed field we have that $H^r(X, \mathcal{F}) = 0$ for $r > 2\dim(X)$.*

PROOF: See [Mil80, page 221 1.1]. □

This theorem is the analog of the theorem about cohomological dimension found in algebraic topology. Observe that if $k = \mathbb{C}$ then $2\dim(X)$ is just the real dimension of our variety. The proof of this result uses very advanced tools that we have not introduced.

We will not present a proof of the fact that the diagrams presented in the course of this sketch commute.

We will divide the proof in various steps. We want to prove that the map

$$\phi^r(U,\mathcal{F}) : H^r(U,\mathcal{F}) \to H_c^{2-r}\left(U,\check{\mathcal{F}}\right)^*, \quad \text{for } r \geq 0,$$

induced by the pairing in the theorem (with $\mathcal{F}$ replaced by $\check{\mathcal{F}}$) is an isomorphism for all finite locally constant sheaves $\mathcal{F}$ on U, where * means $Hom(-,\mathbb{Z}/n\mathbb{Z})$.

Remark 3.5.11. *There is no need to replace $\mathcal{F}$ by $\check{\mathcal{F}}$, if we prove that $\phi^r(U,\mathcal{F})$ is an isomorphism for all finite locally constant sheaves, then it is also true for $\phi^r(U,\check{\mathcal{F}})$, since $\check{\mathcal{F}}$ is also a finite locally constant sheaf (3.5.4), and this map is just the replacement of $\mathcal{F}$ by $\check{\mathcal{F}}$ in the formula of $\phi^r(U,\mathcal{F})$.*

This follows from $j_!\check{F} = (j_!\check{\mathcal{F}})$ (see below), where $j : U \hookrightarrow X$ is the completion of U, and $\check{\check{\mathcal{F}}} \simeq \mathcal{F}$ (3.5.4).

The fact that $j_!\check{F} = (j_!\check{\mathcal{F}})$, follows straightforwardly from definition: Let $\varphi : V \to X$ be an étale map, if $\varphi(V) \subset U$ we have

$$\begin{aligned}
(j_!\check{\mathcal{F}})(V) &\overset{def}{=} \mathrm{Hom}_V(j_!\mathcal{F}|V, \mu_n|V) \\
&\overset{def}{=} \mathrm{Hom}_V(\mathcal{F}_!|V, \mu_n|V) \\
&\simeq \mathrm{Hom}_{V\times_X U}(\mathcal{F}|V \times_X U, \mu_n|V \times_X U) \qquad (\mathrm{Hom}_X(\mathcal{F}!,\mathcal{G}) = \mathrm{Hom}_U(\mathcal{F},\mathcal{G}|U),\ \textit{Rem } 2.5.14) \\
&\simeq \mathrm{Hom}_V(\mathcal{F}|V, \mu_n|V) \qquad (U \times_X V \simeq \varphi^{-1}(U) = V) \\
&\overset{def}{=} \check{F}(\varphi(V)) \\
&\overset{def}{=} (\check{F})_!(V),
\end{aligned}$$

otherwise

$$\begin{aligned}
(j_!\check{\mathcal{F}})(V) &\overset{def}{=} \mathrm{Hom}_V(j_!\mathcal{F}|V, \mu_n|V) \\
&\overset{def}{=} \mathrm{Hom}_V(\mathcal{F}_!|V, \mu_n|V) \\
&= 0 \\
&\overset{def}{=} (\check{F})_!(V).
\end{aligned}$$

Hence, $(\check{F})_! = (j_!\check{\mathcal{F}})$, thus $j_!\check{F} = (j_!\check{\mathcal{F}})$.

Step 1. *The theorem is true for $r \neq 0, 1, 2$.*

PROOF: For any finite locally constant sheaf $\mathcal{F}$, $H^r(U,\mathcal{F}) = 0$ for $r > 2$. Recall that $H_c^r(U,\mathcal{F}) = H^r(X, j_!\mathcal{F})$, so from the previous theorem we also have $H_c^r(U,\mathcal{F}) = 0$ for $r > 2$. Thus, the theorem holds for $r > 2$. □

Step 2. *A exact sequence of finite locally constant sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

gives rise to long exact sequence

$$\cdots \to H_c^{r-2}\left(U,\check{\mathcal{F}}_1\right)^* \to H_c^{r-2}\left(U,\check{\mathcal{F}}_2\right)^* \to H_c^{r-2}\left(U,\check{\mathcal{F}}_3\right)^* \to H_c^{r-1}\left(U,\check{\mathcal{F}}_1\right)^* \to \cdots.$$

PROOF: Since $\mathbb{Z}/n\mathbb{Z}$ is injective as $\mathbb{Z}/n\mathbb{Z}$-module, (3.5.9),the functor $\mathcal{F} \to \check{\mathcal{F}}$ is exact. Hence, we have the following short exact sequence

$$0 \to \check{\mathcal{F}}_3 \to \check{\mathcal{F}}_2 \to \check{\mathcal{F}}_1 \to 0,$$

that gives rise to exact sequence of cohomology groups

$$\cdots \to H_c^{r-1}\left(U,\check{\mathcal{F}}_1\right) \to H_c^{r-2}\left(U,\check{\mathcal{F}}_3\right) \to H_c^{r-2}\left(U,\check{\mathcal{F}}_2\right) \to H_c^{r-2}\left(U,\check{\mathcal{F}}_1\right) \to \cdots.$$

Finally, applying the functor $\mathrm{Hom}(-,\mathbb{Z}/n\mathbb{Z})$, which is exact by (3.5.9), we get the sequence sought. □

Step 3. *Let $\pi : U' \to U$ be a finite étale map. The theorem is true for $\mathcal{F}$ on U' if and only if it is true for $\pi_*\mathcal{F}$ on U.*

Proof: Since π_* is exact (2.5.4) and preserves injectives (2.5.10), we have $H^r(U, \pi_*\mathcal{F}) = H^r(U', \mathcal{F})$. In a similar way we have $H_c^{r-2}\left(U, \pi_*\check{\mathcal{F}}\right)^* = H_c^{r-2}\left(U', \check{\mathcal{F}}\right)^*$.

Usually the equality $\pi_*\check{\mathcal{F}} = (\pi_*\check{\mathcal{F}})$ does not hold. In this case it holds, and it follows straightforwardly from the definition:

$$\pi_*\check{\mathcal{F}}(V) \stackrel{\text{def}}{=} \operatorname{Hom}_{U'\times_U V}(\mathcal{F}|(U'\times_U V), \mu_n|(U'\times_U V));$$

$$(\pi_*\check{\mathcal{F}})(V) \stackrel{\text{def}}{=} \operatorname{Hom}_V(\pi_*F|V, \mu_n|V) \stackrel{\text{def}}{=} \operatorname{Hom}_V(\mathcal{F}|(U'\times_U V), \mu_n|V),$$

and as we already saw in Section 2.2, we can take V sufficiently "small" such that $U'\times_U V$ is a disjoint union of open subschemes V_i each of which is mapped isomorphically onto V [Sta10, Lemma 04HN]. Hence,

$$\begin{aligned}
\pi_*\check{\mathcal{F}}(V) &= \operatorname{Hom}_{\coprod V_i}\left(\mathcal{F}\left|\left(\coprod V_i\right), \mu_n\right|\left(\coprod V_i\right)\right) \\
&\simeq \prod_i \operatorname{Hom}_{V_i}(F|V_i, \mu_n|V_i) \\
(\pi_*\check{\mathcal{F}})(V) &= \operatorname{Hom}_V\left(\mathcal{F}\left|\left(\coprod V_i\right), \mu_n\right|V\right) \\
&\simeq \prod_i \operatorname{Hom}_{V_i}(F|V_i, \mu_n|V_i) \qquad (V \simeq V_i).
\end{aligned}$$

Therefore, $\phi^r(U, \pi_*\mathcal{F})$ can be identified with $\phi^r(U', \mathcal{F})$. □

Step 4. *The map $\phi^0(U, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism of $\mathbb{Z}/n\mathbb{Z}$-modules, i.e., the theorem holds for $r = 0$ and $\mathcal{F} = \mu_n$.*

Proof: The pairing for this case is

$$\mathbb{Z}/n\mathbb{Z} = H^0(U, \mathbb{Z}/n\mathbb{Z}) \times H_c^2(U, \mu_n) \to H_c^2(U, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$$

where the action of $\mathbb{Z}/n\mathbb{Z}$ on $H_c^2(U, \mu_n)$ is the usual action as $\mathbb{Z}/n\mathbb{Z}$-module, hence perfect, as we already observed previously (3.5.6). □

Step 5. *The theorem is true for r=0.*

Proof: Remark 3.5.2 says there is a finite étale covering $\pi : U' \to U$ where $\mathcal{F}|U'$ is constant, and is represented by a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module.

Recall that if $n = pq$, with p and q coprime, there is a natural embedding $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/n\mathbb{Z}$ given by the map $\alpha \mapsto q\alpha$, thus using the primary decomposition of the structure theorem for finitely generated modules over a principal ideal domain we can embed $\mathcal{F}|U'$ in $(\mathbb{Z}/n\mathbb{Z})^s$ for some s.

We also have a natural inclusion $\mathcal{F} \hookrightarrow \pi_*\pi^*\mathcal{F}$. This follows again from the fact that finite étale maps are the analogue of a finite covering space, i.e., for a sufficiently "small" étale map $V \to U$ we have $\mathcal{F}(V) \hookrightarrow \pi_*\pi^*\mathcal{F}(V) \stackrel{\text{def}}{=} \mathcal{F}(U'\times_U V) \simeq \prod_i \mathcal{F}(V_i) \simeq \prod_i \mathcal{F}(V)$, [Sta10, Lemma 04HN], where the inclusion is the diagonal map.

Combining the natural inclusion $\mathcal{F} \hookrightarrow \pi_*\pi^*\mathcal{F}$, $\pi^*\mathcal{F} = \mathcal{F}|U' \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^s$, and the fact that π_* is exact (2.5.4), we get an inclusion

$$\mathcal{F} \hookrightarrow \pi_*\pi^*\mathcal{F} = \pi_*\mathcal{F}|U' \hookrightarrow \pi_*(\mathbb{Z}/n\mathbb{Z})^s.$$

Let $\mathcal{G}$ be the cokernel of the previous map, which is also a finite locally constant sheaf. Yhe obvious short exact sequence gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \pi_*(\mathbb{Z}/n\mathbb{Z})^s) & \longrightarrow & H^0(U, \mathcal{G}) \\
\downarrow \simeq & & \downarrow \phi^0(U,\mathcal{F}) & & \downarrow \phi^0(U,\pi_*(\mathbb{Z}/n\mathbb{Z})^s) & & \downarrow \phi^0(U,\mathcal{G}) \\
H_c^3\left(U, \check{\mathcal{G}}\right)^* = 0 & \longrightarrow & H_c^2\left(U, \check{\mathcal{F}}\right)^* & \longrightarrow & H_c^2\left(U, \left(\pi_*(\mathbb{Z}/n\mathbb{Z})^s\right)^\vee\right)^* \simeq H_c^2(U, \pi_*(\mu_n)^s)^* & \longrightarrow & H_c^2\left(U, \check{\mathcal{G}}\right)^*
\end{array}$$

Applying steps 3 and 4 we see that $\phi^0\left(U,\pi_*\left(\mathbb{Z}/n\mathbb{Z}\right)^s\right)$ is an isomorphism. Applying the five lemma we get that $\phi^0(U,\mathcal{F})$ is injective (the two missing groups on the left hand side are zero).

We need is the vertical arrow to be an isomorphism. Since the injectivity of $\phi^0(U,\mathcal{F})$ is valid for every finite locally constant sheaf $\mathcal{F}$ we conclude that $\phi^0(U,\mathcal{G})$ is also injective. Applying again the five lemma we see that $\phi^0(U,\mathcal{F})$ is surjective. □

Step 6. *The map $\phi^1(U,\mathbb{Z}/n\mathbb{Z})$ is injective.*

PROOF: We know from Example (3.2.18), that $H^1\left(U,\mathbb{Z}/n\mathbb{Z}\right)\simeq \mathrm{Hom}_{\mathrm{conts}}(\pi_1(U,\bar{u}),\mathbb{Z}/n\mathbb{Z})$. Let $s\in H^1\left(U,\mathbb{Z}/n\mathbb{Z}\right)$ and $\pi: U'\to U$ be the Galois covering associated to the kernel of s. We have that

$$\pi_1(U,\bar{u})/\operatorname{Ker}(s)\simeq \operatorname{Im}(s)\subset \mathbb{Z}/n\mathbb{Z},$$

thus $\pi_1(U,\bar{u})/\operatorname{Ker}(s)$ is a finite quotient of the étale fundamental group, so by Proposition 2.2.13, we know that there is a Galois covering $\pi: U'\to U$ with group $\pi_1(U,\bar{u})/\operatorname{Ker}(s)$ and $\pi_1(U',\bar{u})=\operatorname{Ker}(s)$.

By construction s is mapped to zero in

$$\mathrm{Hom}_{\mathrm{conts}}(\operatorname{Ker}(s),\mathbb{Z}/n\mathbb{Z})=\mathrm{Hom}_{\mathrm{conts}}(\pi_1(U,\bar{u}),\mathbb{Z}/n\mathbb{Z})\simeq H^1\left(U',\mathbb{Z}/n\mathbb{Z}\right)\simeq H^1\left(U,\pi_*\mathbb{Z}/n\mathbb{Z}\right),$$

where the last isomorphism follows from step 3.

As in the previous proof, let $\mathcal{G}$ to be the cokernel of

$$\begin{aligned}\mathbb{Z}/n\mathbb{Z}\hookrightarrow \pi_*\pi^*\mathbb{Z}/n\mathbb{Z} &= \pi_*(\mathbb{Z}/n\mathbb{Z}|U') && \text{(Example 2.5.7)}\\ &= \pi_*\mathbb{Z}/n\mathbb{Z}, && (U' \text{ is connected by construction})\end{aligned}$$

and consider the following commutative diagram, where the rows are exact

$$\begin{array}{ccccccc} H^0\left(U,\pi_*(\mathbb{Z}/n\mathbb{Z})\right) & \longrightarrow & H^0\left(U,\mathcal{G}\right) & \longrightarrow & H^1\left(U,\mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & H^1\left(U,\pi_*\mathbb{Z}/n\mathbb{Z}\right)\\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\phi^1(U,\mathbb{Z}/n\mathbb{Z})} & & \downarrow{\phi^1(U,\pi_*\mathbb{Z}/n\mathbb{Z})}\\ H^2_c\left(U,\pi_*\mu_n\right)^* & \longrightarrow & H^2_c\left(U,\check{\mathcal{G}}\right)^* & \longrightarrow & H^1_c\left(U,\mu_n\right)^* & \longrightarrow & H^1_c\left(U,\pi_*\mu_n\right)^*. \end{array}$$

The vertical maps labeled as "$\simeq$" are isomorphism by the previous step. Since s is mapped to zero in $H^1\left(U,\pi_*\mathbb{Z}/n\mathbb{Z}\right)$, by diagram chasing we get that if $s\in\operatorname{Ker}(\phi^1(U,\mathbb{Z}/n\mathbb{Z}))$ then $s=0$. Therefore $\phi^1(U,\mathbb{Z}/n\mathbb{Z})$ is injective.

□

Step 7. *When X is a complete curve, the maps $\phi^r(X,\mathbb{Z}/n\mathbb{Z})$ are isomorphisms for $r=1,2$.*

PROOF: For $r=1$, since $\check{\mu_n}\simeq\mathbb{Z}/n\mathbb{Z}$, the map is $\phi^1(X,\mathbb{Z}/n\mathbb{Z}): H^1\left(X,\mathbb{Z}/n\mathbb{Z}\right)\to H^1\left(X,\mu_n\right)^*$. Recall that $\mu_n\approx\mathbb{Z}/n\mathbb{Z}$ (not canonically), hence $H^1\left(X,\mathbb{Z}/n\mathbb{Z}\right)\approx H^1\left(X,\mu_n\right)\approx(\mathbb{Z}/n\mathbb{Z})^{2g}$ (3.4.1), therefore $H^1\left(X,\mu_n\right)^*$ and $H^1\left(X,\mathbb{Z}/n\mathbb{Z}\right)$ have the same rank as $\mathbb{Z}/n\mathbb{Z}$-module. So by the previous step, we have that $\phi^1(X,\mathbb{Z}/n\mathbb{Z})$ is an isomorphism.

For $r=2$ the pairing is

$$H^2\left(X,\mathbb{Z}/n\mathbb{Z}\right)\times\left(H^0_c\left(X,\mu_n\right)=\mu_n(k)\right)\to\left(H^2_c\left(X,\mu_n\right)=H^2\left(X,\mu_n\right)\simeq\mathbb{Z}/n\mathbb{Z}\right).$$

So by Remark 3.5.6 we have that the pairing is perfect for $r=2$, $\mathcal{F}=\mu_n$ and X a complete curve.

□

Step 8. *If the maps $\phi^r(U,\mathbb{Z}/n\mathbb{Z})$ are isomorphisms for $r=1,2$ then the maps $\phi^r(U\backslash x,\mathbb{Z}/n\mathbb{Z})$ are also isomorphisms.*

PROOF: Let $x\in U$ and take $V=U\backslash x$. There is a commutative diagram, with exact rows:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^r_x\left(U,\mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & H^r\left(U,\mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & H^r\left(V,\mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & H^{r+1}_x\left(U,\mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & \cdots\\ & & \downarrow & & \downarrow{\phi^r(U,\mathbb{Z}/n\mathbb{Z})} & & \downarrow{\phi^r(V,\mathbb{Z}/n\mathbb{Z})} & & \downarrow & & \\ \cdots & \longrightarrow & H^{2-r}\left(x,\mu_n\right)^* & \longrightarrow & H^{2-r}_c\left(U,\mu_n\right)^* & \longrightarrow & H^{2-r}_c\left(V,\mu_n\right)^* & \longrightarrow & H^{1-r}\left(x,\mu_n\right)^* & \longrightarrow & \cdots. \end{array}$$

Where the upper sequence is the exact sequence of the pair $(U, V = U\backslash x)$, and the lower sequence is explained below.

Taking $h : V \hookrightarrow U$ and $i : x \hookrightarrow U$, from (2.5.18) we get a short exact sequence,

$$0 \to h_! h^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 0.$$

Thus, from step 2 we obtain an exact sequence

$$\cdots \to H_c^{2-r}\left(U, i_* i^* \mu_n\right)^* \to H_c^{2-r}\left(U, \mu_n\right)^* \to H_c^{2-r}\left(U, h_! h^* \mu_n\right)^* \to H_c^{1-r}\left(U, i_* i^* \mu_n\right)^* \to \cdots .$$

Let $j : U \hookrightarrow X$ be the completion of U, (3.1.11). From the previous sequence we get the lower exact sequence in the following way: first observe that $i_* i^* \mu_n$ is just the skyscraper of $\mu_n(k)$ on $x \in U$, (2.5.5), therefore

$$\begin{aligned} H_c^{2-r}\left(U, i_* i^* \mu_n\right) &\overset{\text{def}}{=} H^{2-r}\left(X, j_! i_* i^* \mu_n\right) \\ &= H^{2-r}\left(x, \mu_n\right) \qquad (j_! i_* i^* \mathbb{Z}/n\mathbb{Z} \text{ is the skyscraper of } \mu_n(k) \text{ at } x \in X). \end{aligned}$$

In the same way we get

$$\begin{aligned} H_c^{2-r}\left(U, h_! h^* \mu_n\right) &\overset{\text{def}}{=} H^{2-r}\left(U, j_! h_! h^* \mu_n\right) \\ &= H^{2-r}\left(U, (j \circ h)_! h^* \mu_n\right) \\ &\overset{\text{def}}{=} H_c^{2-r}\left(V, h^* \mu_n\right) \qquad (j \circ h \text{ is the completion of } V) \\ &= H_c^{2-r}\left(V, \mu_n\right) \qquad (h^* \mu_n = \mu_n | V \text{ Example 2.5.7}). \end{aligned}$$

Now we want to apply the five lemma to the diagram above. For that, we first need to have a better understanding of the unlabeled vertical maps in the diagram. From Proposition 3.4.2 and $\mathbb{Z}/n\mathbb{Z} \approx \mu_n$ we know that $H_x^r\left(U, \mathbb{Z}/n\mathbb{Z}\right) \approx H_x^r\left(U, \mu_n\right)$, hence

$$H_x^r\left(U, \mathbb{Z}/n\mathbb{Z}\right) \begin{cases} \approx \mathbb{Z}/n\mathbb{Z} = H^0\left(x, \mathbb{Z}/n\mathbb{Z}\right) & , r = 2 \\ = 0 & , r \neq 2. \end{cases}$$

Recall that since x is a geometric point then $H^r\left(x, \mu_n\right)$ is trivial, (see section 3.1.2), thus,

$$H^{2-r}\left(x, \mu_n\right) = \begin{cases} \mu_n(k) & , r = 2 \\ 0 & , r \neq 2. \end{cases}$$

So for $r \neq 2$, the unlabeled vertical arrows in the diagram above are the morphisms between zero groups, and for $r = 2$ the vertical map is induced by the pairing

$$H_x^2\left(U, \mathbb{Z}/n\mathbb{Z}\right) \times \mu_n(k) \to H_c^2\left(U, \mu_n\right),$$

i.e., the pairing of the theorem with $r = 2$ and $\check{\mathcal{F}} = \Gamma_x(-, \mathbb{Z}/n\mathbb{Z})$, see section 3.1.4.

Now let us focus on the exact sequence for $r = 1, 2$ for which we do not know yet that we have vertical isomorphisms.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^1\left(V, \mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & H_x^2\left(U, \mathbb{Z}/n\mathbb{Z}\right) \simeq \mathbb{Z}/n\mathbb{Z} & \longrightarrow & H^2\left(U, \mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow & H^2\left(V, \mathbb{Z}/n\mathbb{Z}\right) & \longrightarrow \cdots \\ & \Big\downarrow{\scriptstyle \phi^1(V, \mathbb{Z}/n\mathbb{Z})} & & \Big\downarrow & & {\scriptstyle \simeq}\Big\downarrow{\scriptstyle \phi^2(U, \mathbb{Z}/n\mathbb{Z})} & & \Big\downarrow{\scriptstyle \phi^2(V, \mathbb{Z}/n\mathbb{Z})} & \\ \cdots \longrightarrow & H_c^1\left(V, \mu_n\right)^* & \longrightarrow & H^0\left(x, \mathbb{Z}/n\mathbb{Z}\right)^* \simeq \mathbb{Z}/n\mathbb{Z} & \longrightarrow & H_c^0\left(U, \mu_n\right)^* & \longrightarrow & H_c^0\left(V, \mu_n\right)^* & \longrightarrow \cdots \end{array}$$

Applying the five lemma we get that the unlabeled map is a monomorphism between two free $\mathbb{Z}/n\mathbb{Z}$-modules of rank 1, hence an isomorphism. Applying the five-lemma several times it follows immediately that $\phi^r(V, \mathbb{Z}/n\mathbb{Z})$ are isomorphisms. □

Step 9. *The maps $\phi^r(U, \mathcal{F})$ are isomorphism for $r = 1, 2$.*

Proof: Recall that $U = X\backslash\{x_1, \ldots, x_n\}$ for X a complete curve. So by step 7 and 8 we get that $\phi^r(U, \mathbb{Z}/n\mathbb{Z})$ are isomorphisms for $r = 1, 2$. Applying the same argument of step 5 *mutatis mutandis* first to $r = 1$ and then to $r = 2$ we get the result. □

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