A NEW GEOMETRIC PERSPECTIVE ABOUT THE PROPAGATION OF ELECTROMAGNETIC WAVES IN ANISOTROPIC MEDIA

Luís Filipe da Silva Pragosa nº 55281

Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail: luispragosa@hotmail.com

ABSTRACT

This dissertation presents a new mathematics language, the geometric algebra or Clifford algebra. Initially, are discussed the key concepts of euclidean geometric algebra, and is introduced a new product between vectors, the product geometric, which represents the essential definition of this algebra. It also presents two new geometric objects, the bivector and trivector and concepts with extreme importance, as the rotors and contractions. Finally, is introduced the concept of anisotropy and is demonstrated the application of geometric algebra of euclidean space to anisotropic media, in particular, the uniaxial and biaxial crystals, where are studied their constitutive relations and their wave characteristics.

As is evident in this dissertation, the geometric algebra of euclidean space is a new mathematics tool very useful for the analysis of anisotropic media. This is due to its naturalness and simplicity, and currently has many applications in Classical Physics [1]. By itself, the theoretical formulation of geometric algebra proofs that this kind of algebra can become in future the unified language of modern Physics.

Index Terms – Geometric algebra, bivector, trivector, geometric product, anisotropic media, uniaxial crystals and biaxial crystals

1. INTRODUCTION

The history of geometric algebra got its start in Ancient Greece ([1] and [2]) with the writing of geometric relations in algebraic form, however, the formalism of geometric algebra as we know it today had its beginning only in nineteenth century. This algebra is often called the Clifford algebra due to include the geometric or Clifford product, which has been introduced by William Kingdon Clifford (1845-1879). In this new product are simultaneously present the usual dot (or inner) product and the outer (or exterior) product.

Quaternions were invented by Sir William Rowan Hamilton (1805-1865) in 1843, and this discovery was prompted by the hope of creating a type of hypercomplex numbers that were related to the three-dimensional space the same way that complex numbers are related to the plan. Today, quaternions are used as an important tool for modeling problems in most applied sciences: robotics, virtual reality, computer vision, navigation, etc. [3].

The exterior product was invented in 1844 by Hermann Günter Grassmann (1809-1877), proving that the relationship between algebra and geometry isn’t restricted to three dimensions, i.e., the exterior product is definable in n dimensions. This product doesn’t depend of a metric, and enjoys the property of anti-symmetry, however, isn’t invertible.

In 1878 Clifford unified the discoveries of Grassmann and Hamilton in one unique algebraic structure, which gave origin to geometric algebra.

Unlike the exterior product, the cross product of Gibbs introduced by Josiah Willard Gibbs (1839-1903) is definable only in three dimensions and, thus, is a limitation of this product. It should also be noted that the cross product of Gibbs requires a metric, instead of the exterior product that is independent of any metric.

Only in the beginning of the XX century when Einstein published the restrict-relativity theory, which was formulated in four dimensions, experts started to question the validity of Gibbs product when applied to more than three dimensions and began to give credit to Clifford’s work. So in 1920 geometric algebra was used in quantum mechanics to solve the problem of spin matrices.

Geometric algebra is a mathematics language which is relatively recent and was only properly formalized around 1960 due to the work of David Hestenes. Its applications are several and distinct, they go from object-oriented programming and computer vision to the electromagnetic theory. The geometric algebra also permits to unify several fields of mathematics such as Grassmann algebra, Berezin calculus, Lie group, Hamilton quaternions, dyadic calculus and several differential forms.

In last years, the geometric algebra has been investigated by an increasing number of people, highlighting the work of Pertti Lounesto, Chris Doran, Anthony Lasenby, Leo Dorst, Daniel Fontijne e Stephen Mann.

2. BASIC CONCEPTS OF GEOMETRIC ALGEBRA

\( C_{\ell} \)

The geometric product or Clifford’s product is the key definition of geometric algebra. To begin, consider that in \( C_{\ell} \) we have, in terms of unit vectors, \( |e_j| = |e_j| = |e_k| = 1 \) and a basis \( \mathcal{B} = \{e_1, e_2, e_3\} \). So that

\[
\begin{cases}
1, & j = k \\
0, & j \neq k
\end{cases}
\]

(1)
Considering a vector, \( \mathbf{r} = x \mathbf{e}_1 + y \mathbf{e}_2 + z \mathbf{e}_3 \in \mathbb{R}^3 \), so, we define the fundamental axiom of \( \mathbb{C}^3 \) as \( \mathbf{r}^2 = |\mathbf{r}|^2 \). In consequence of this axiom the geometric product is associative but non commutative, so that, \( \mathbf{e}_3 \mathbf{e}_2 = -\mathbf{e}_2 \mathbf{e}_3, \mathbf{e}_2 \mathbf{e}_3 = -\mathbf{e}_3 \mathbf{e}_2 \), and \( \mathbf{e}_3 \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_3 \). The non commutativity leads to an important result in geometric algebra:

\[
(e,e)^2 = (e,e)(e,e) = e_i(e,e)e_j = -e_i(e,e)e_j = -(e,e)(e,e) = -e_i^2e_j^2 = -1
\]

\[
\therefore (e,e)^2 = -1.
\]

This algebra has an object who square negatively. This object is called a bivector. A bivector is a directed plane segment. \( \mathbf{F} = e_{12} = e_1 e_2 \) represents a unit bivector as shown in Figure 1.

Figure 1 – Representation of the unit bivector \( \mathbf{F} \).

Given two generic vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) the geometric product, \( \mathbf{a} \mathbf{b} \), is the graduate sum of a scalar, given by the inner product between them, and a bivector, which is the exterior product also between them. So, we have a multivector, \( u \),

\[
u = \mathbf{a} \mathbf{b} = \alpha + \mathbf{F} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \in \mathbb{R} \oplus \mathbb{R}^3
\]

(3)

where

\[
\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3
\]

(4)

and

\[
\mathbf{a} \wedge \mathbf{b} = 
\begin{vmatrix}
\mathbf{e}_{23} & \mathbf{e}_{31} & \mathbf{e}_{12} \\
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\
\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3
\end{vmatrix}.
\]

(5)

The unit trivector is the result of the exterior product of the three vectors of the orthonormal basis considered.

The unit trivector or oriented volume of this algebra is \( \mathbf{e}_{123} \). Formally we have \( \mathbf{e}_{123} = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \) and the square of this object is also \(-1\),

\[
\mathbf{i}^2 = \mathbf{e}_{123}^2 = \mathbf{e}_{12} = (\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3)(\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3) = -\mathbf{e}_1^2 \mathbf{e}_2^2 \mathbf{e}_3^2 = -1
\]

\[
\therefore \mathbf{e}_{123}^2 = -1.
\]

Only in \( \mathbb{R}^3 \) is possible to relate the exterior product with the cross product by \( \mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b})\mathbf{e}_{123} \). By analyze the exterior product we have that \( (\mathbf{a} \wedge \mathbf{b})^2 = (\mathbf{a} \cdot \mathbf{b})^2 - \mathbf{a}^2 \mathbf{b}^2 \) so

\[
|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta
\]

, where \( \theta = \varphi(\mathbf{a}, \mathbf{b}) \). We can see it represented in Figure 3.

Figure 3 – Representation of the exterior product between vectors \( \mathbf{a} \) and \( \mathbf{b} \).

An arbitrary element \( u \in \mathbb{C}^3 \) , which we call a multivector, is a (graded) sum of scalar, a vector, a bivector and a trivector: \( u = \alpha + \mathbf{a} + \mathbf{F} + \mathbf{V} \), \( \alpha = \langle u \rangle_0 \), \( \mathbf{a} = \langle u \rangle_1 \), \( \mathbf{F} = \langle u \rangle_2 \), \( \mathbf{V} = \langle u \rangle_3 \), denoting the operation of projecting onto the terms of a chosen grade \( k \) by \( \langle u \rangle_k \).
The geometric algebra in \( Cl_3 \) has the subspaces of scalars \( \mathbb{R} \), vectors \( \mathbb{R}^3 \), bivectors \( \wedge^2 \mathbb{R}^3 \) and trivectors \( \wedge^3 \mathbb{R}^3 \). This algebra is a linear space of dimension 1+3+3+1=8: adopting \( e_1, e_2, e_3 \) as an orthonormal basis for vector space \( \mathbb{R}^3 \), a suitable basis for corresponding linear \( Cl_3 \) is
\[
\{ 1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123} \}.
\]
(7)
where \( e_{12} = e_1 \wedge e_2 = e_2 e_1 \), \( e_{31} = e_3 \wedge e_1 = e_1 e_3 \) and \( e_{23} = e_2 \wedge e_3 = e_3 e_2 \) constitute a basis for the subspace \( \wedge^2 \mathbb{R}^3 \) of bivectors.

The subalgebra of scalars and trivectors is the center of \( Cl_3 \), i.e., it consists of those elements of \( Cl_3 \) which commute with every element in \( Cl_3 \):
\[
\text{Cen } Cl_3 = \mathbb{R} \oplus \wedge^3 \mathbb{R}^3.
\]
(8)
This algebra has an even part and an odd part. The even part, \( Cl_3^e = \mathbb{R} \oplus \wedge^2 \mathbb{R}^3 \), is isomorphic to the quaternions’ ring of Hamilton, and \( Cl_3^o = \mathbb{R} \oplus \wedge^3 \mathbb{R}^3 \) is the odd part which is isomorphic to the complexes.

2.1 Rotors and contractions

Rotors are defined as the geometric product of two unit vectors. Considering the linear space \( \mathbb{R}^3 \) and \( \mathbf{n}, \mathbf{m} \in \mathbb{R}^3 \), a rotor is,
\[
R = \mathbf{n} \mathbf{m}.
\]
(9)
From definition (9) we can observe that \( R \) is in fact a multivector defined as,
\[
R = \mathbf{n} \cdot \mathbf{m} + \mathbf{n} \wedge \mathbf{m} = \cos \theta + \mathbf{B} \sin \theta = \exp(\theta \mathbf{B}) \text{.}
\]
(10)
The rotor defined in (10) can handle a rotation of \( 2\theta \) in the plane of the corresponding bivector. So, if we would like to have a rotation of \( \theta \) in the corresponding bivector plane, we proceed as in

There are two types of contractions, the left contraction and the right contraction.

Considering the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \) and the bivector \( \mathbf{B} = \mathbf{b} \wedge \mathbf{c} \), the left contraction is by definition,
\[
\mathbf{a} \cdot \mathbf{B} = \frac{1}{2} \mathbf{aB} - \mathbf{Ba} \text{.}
\]
(11)
We obtain the fundamental rule of left contraction as
\[
\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} \text{.}
\]
(12)
In analogous way we have the right contraction as
\[
\mathbf{B} \cdot \mathbf{a} = \frac{1}{2}(\mathbf{B} - \mathbf{aB}) \text{.}
\]
(13)
Those two contractions can be related by
\[
\mathbf{a} \cdot \mathbf{B} = -\mathbf{B} \cdot \mathbf{a} \text{.}
\]
(14)
We can also write
\[
\mathbf{a} \mathbf{B} = \mathbf{a} \cdot \mathbf{B} + \mathbf{a} \wedge \mathbf{B} = \mathbf{B} \mathbf{a} - \mathbf{B} \cdot \mathbf{a} \mathbf{B} \text{.}
\]
(15)
So, we conclude that the left contraction of a vector with a bivector will step down the degree of the bivector into the degree of the vector. The right contraction have the same consequence, but the new vector is diametrically opposed to the one obtain in left contraction.

3 Anisotropic Media

3.1 Monochromatic plane waves

For electromagnetic field variation of the form
\[
\exp\left[i \mathbf{k} \cdot \mathbf{r} - \omega t \right] = \exp\left[i k_0 \mathbf{n} \cdot \mathbf{r} - \omega t \right]
\]
(16)
with
\[
\mathbf{k} = k_0 \mathbf{n}, \quad k_0 = \frac{\omega}{c}
\]
(17)
\[
\mathbf{n} = \frac{\mathbf{k}}{k_0}
\]
Maxwell equations in \( Cl_3 \) may be simply written, for source-free regions, as
\[
\mathbf{n} \wedge \mathbf{E} = -\varepsilon_0 \mathbf{cB} e_{123}
\]
\[
\mathbf{n} \wedge \mathbf{H} = -\mu_0 \mathbf{cD} e_{123}
\]
(18)
\[
\mathbf{n} \cdot \mathbf{D} = 0
\]
\[
\mathbf{n} \cdot \mathbf{H} = 0
\]
3.2 Electrical anisotropy

Anisotropy means that the magnitude of a property can only be defined along a given direction. Particularly, if we are speaking of electrical anisotropy this means that there is an angle between the electric field vector, \( \mathbf{E} \), and the electric displacement vector, \( \mathbf{D} \), that depends on the direction of the euclidean space \( \mathbb{R}^3 \) along which vector \( \mathbf{E} \) is applied. This means that it isn’t possible to write
\[
\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}
\]
where \( \varepsilon_0 \) is the permittivity of vacuum and
\( \varepsilon \) is a scalar called the (relative) dielectric permittivity of the medium. The common solution, with the dyadic analysis, consists in introducing a permittivity or dielectric tensor that, in a given coordinate system, may be written as a 3x3 matrix.

With geometric algebra \( C \), we simply write

\[
\mathbf{D} = \varepsilon \mathbf{E}
\]

where \( \varepsilon \mathbf{E} \) is a dielectric linear function \( \varepsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that maps vectors to vectors and it characterizes the medium only along a certain direction.

\[
\text{Figure 5 – Electrical anisotropy characterized by bivector } \mathbf{F} = \mathbf{E} \wedge \mathbf{D}.
\]

Henceforth, it will be considered that the media in study are not magnetic, lossless, unbounded and linear. In non-magnetic media we have \( \mathbf{B} = \mathbf{0} \), so the relationship between \( \mathbf{B} \) and \( \mathbf{H} \) is the same of vacuum, i.e.,

\[
\mathbf{B} = \mu_0 \mathbf{H}.
\]

Formally, considering the Figure 5, when an electric field \( \mathbf{E} = |\mathbf{E}| \mathbf{s} \) (in direction characterized by unit vector \( \mathbf{s} \)) is applied, the medium responds with an electric displacement field \( \mathbf{D} = |\mathbf{D}| \mathbf{t} \). The relation between \( \mathbf{s} \) and \( \mathbf{t} \) is such that \( \mathbf{s} \cdot \mathbf{t} = \cos \theta \) and \( \mathbf{s} \wedge \mathbf{t} = \hat{\mathbf{F}} \sin \theta \). Now, we are in conditions of characterizing the media correctly. So it follows that,

\[
\begin{align*}
D_1 &= \mathbf{s} \cdot \mathbf{D} = |\mathbf{D}| \cos \theta \\
D_2 &= \mathbf{r} \cdot \mathbf{D} = |\mathbf{D}| \sin \theta \\
\hat{\mathbf{F}} &= \mathbf{sr} = \frac{\mathbf{s} \wedge \mathbf{t}}{\sin \theta} \\
\mathbf{F} &= \mathbf{E} \wedge \mathbf{D} = |\mathbf{F}| \hat{\mathbf{F}}
\end{align*}
\]

\[
\mathbf{D} = \varepsilon \mathbf{E} \mathbf{s} \Rightarrow \varepsilon \mathbf{s} = \mathbf{s} \cdot \varepsilon \mathbf{s} = \varepsilon \mathbf{s} \cos \theta \Leftrightarrow D_1 = \varepsilon \mathbf{E} |\mathbf{E}| (22)
\]

The following considerations can also be made in accordance to Figure 5,

\[
\sigma = |\mathbf{E}| |\mathbf{D}|
\]

\[
\mathbf{F} = \mathbf{E} \wedge \mathbf{D} = |\mathbf{E}| |\mathbf{D}| \mathbf{s} \wedge \mathbf{t} = \hat{\mathbf{F}} \sigma \sin \theta
\]

\[
\mathbf{ED} = \mathbf{E} \cdot \mathbf{D} + \mathbf{E} \wedge \mathbf{D} = \sigma \mathbf{st}
\]

In expression (22), \( \varepsilon \mathbf{s} = \mathbf{s} \cdot \varepsilon \mathbf{s} \) is the permittivity along \( \mathbf{s} \) and \( \varepsilon \mathbf{s} \) is the dielectric function. Specifically, in an anisotropic medium, to each direction \( \mathbf{s} \) of the space corresponds a scalar \( \varepsilon \mathbf{s} = \mathbf{s} \cdot \varepsilon \mathbf{s} \).

To clarify these concepts presents an example.

Admitting that the linear operator \( \varepsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) has three real distinct eigenvalues \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 \). For convenience of future calculus, consider the following:

\[
\begin{align*}
\gamma_1^2 + \gamma_2^2 &= 1 \Rightarrow \mathbf{e}_1 = \alpha + 2 \beta \gamma_1^2 \\
\gamma_1^2 + \gamma_3^2 &= 1 \Rightarrow \mathbf{e}_2 = \alpha - 2 \beta \gamma_3^2
\end{align*}
\]

Considering the orthonormal basis, \( \mathbf{B} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3 \), formed by the eigenvectors of considered operator, such that:

\[
\begin{align*}
e_1 & \Rightarrow \mathbf{e}_1 = \varepsilon \mathbf{e}_1 \\
e_2 & \Rightarrow \mathbf{e}_2 = \varepsilon \mathbf{e}_2 \\
e_3 & \Rightarrow \mathbf{e}_3 = \varepsilon \mathbf{e}_3
\end{align*}
\]

\[
\mathbf{d}_1 = \gamma_1 \mathbf{e}_1 + \gamma_3 \mathbf{e}_3 \quad \mathbf{d}_2 = -\gamma_1 \mathbf{e}_1 + \gamma_3 \mathbf{e}_3.
\]

The relation between \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) is the following.
\[ \mathbf{d}_2 = R_\phi \mathbf{d}_1 \tilde{R}_\phi \]  

(29)

where \( R_\phi \) is a rotor given by

\[ R_\phi = \exp e_{31} \phi / 2 = \cos \phi / 2 + e_{31} \sin \phi / 2 \].

This equation means that \( \mathbf{d}_2 \) is the rotation of \( \mathbf{d}_1 \) in \( C\ell_3 \).

We are now in conditions of writing the linear operator in terms of the vectors defined in expression (28), so considering the generic vector \( \mathbf{a} \in \mathbb{R}^3 \) it follows,

\[ \varepsilon \mathbf{a} = \alpha \mathbf{a} + \beta \left[ \mathbf{a} \cdot \mathbf{d}_1 \mathbf{d}_1 + \mathbf{a} \cdot \mathbf{d}_2 \mathbf{d}_2 \right] \]  

(30)

where \( \alpha = \varepsilon_\parallel \) and \( \beta = \frac{1}{2} \varepsilon - \varepsilon_\parallel \).

A linear operator \( \varepsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) with three real positive and distinct eigenvalues, can be written in terms of two unit vectors \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) non parallel. For this reason it is said that this is a biaxial operator.

A uniaxial operator is a particular case of the biaxial operator. Continuing to consider the same example as before and considering now that \( \varepsilon_1 = \varepsilon_2 = \varepsilon_\perp \) and \( \varepsilon_3 = \varepsilon_\parallel \) we have the following,

\[ \mathbf{d}_1 \cdot \mathbf{d}_2 = \cos \phi = \frac{\varepsilon_3 - \varepsilon_\perp}{\varepsilon_3 - \varepsilon_\perp} = 1 \]

\[ \phi = 0 \]  

\[ \mathbf{d}_1 = \mathbf{d}_2 = \mathbf{c} \]  

(31)

In the uniaxial case, \( \varepsilon_\parallel \) is considered the non degenerate eigenvalue and corresponds to the eigenvector \( \mathbf{c} \) (axis of the crystal). The \( \varepsilon_\perp \) is the doubly degenerated eigenvalue corresponding to the other two eigenvectors which defining the orthogonal plane to the \( \mathbf{c} \) axis of the crystal.

Also in relation to the expression (32), it is said that the operator is an operator uniaxial because the two axes \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) of biaxial case, are reduced to only one \( \mathbf{c} \) axis.

The inverse of the dielectric function is the permeability function \( \eta = \varepsilon^{-1} \), such that \( \mathbf{E} = \eta \mathbf{D} / \varepsilon_0 \).

If \( \mathbf{E} = E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2 + E_3 \mathbf{e}_3 \) and \( \mathbf{e}_1 \), \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \) are the principal dielectric axes corresponding to the eigenvalues \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \varepsilon_3 \) (respectively), with \( \varepsilon_1 > \varepsilon_3 > \varepsilon_2 \), then

\[ \mathbf{D} = \varepsilon_1 E_1 \mathbf{e}_1 + \varepsilon_2 E_2 \mathbf{e}_2 + \varepsilon_3 E_3 \mathbf{e}_3 \]

and

\[ \mathbf{E} = \eta_1 D_1 \mathbf{e}_1 + \eta_2 D_2 \mathbf{e}_2 + \eta_3 D_3 \mathbf{e}_3 \]

where \( \eta_i = \varepsilon_i^{-1} \) (with \( i = 1,2,3 \)). One can readily show that, is \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) are the two unit vectors that characterize \( \varepsilon \), then \( \mathbf{c}_1 \) and \( \mathbf{c}_2 \) are the two unit vectors that characterize \( \eta \).

The process for obtaining the inverse of the dielectric function is similar to the process for the case of the dielectric function. So considering the Figure 7 we can write the following,

\[ \mathbf{c}_1 = r_1 \mathbf{e}_1 + r_3 \mathbf{e}_3 \]  

\[ \mathbf{c}_2 = -r_1 \mathbf{e}_1 + r_3 \mathbf{e}_3 \]  

(33)

where,

\[ r_1 = \frac{\varepsilon_1}{\varepsilon_2} \gamma \]  

\[ r_3 = \frac{\varepsilon_1}{\varepsilon_3} \gamma \]  

(34)

with

\[ r_1 = \sin \frac{\gamma}{2} \]  

\[ r_3 = \cos \frac{\gamma}{2} \]  

(35)

Similarly to equation (29), we can now write,

\[ \mathbf{c}_2 = R_\phi \mathbf{c}_1 \tilde{R}_\phi \]  

(36)

where \( \mathbf{c}_2 \) is the rotation of \( \mathbf{c}_1 \) in \( C\ell_3 \) originated by the rotor \( R_\phi = \exp e_{31} \phi / 2 \).

Whence,

\[ \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \sqrt{\alpha} \begin{pmatrix} \gamma & -\gamma \beta \\ -\gamma \beta & \gamma \end{pmatrix} \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} \]  

(37)

where

\[ \alpha = \frac{1}{2} (\varepsilon_1 - \varepsilon_3) \]  

(38)

\[ \beta = \frac{1}{2} (\varepsilon_1 - \varepsilon_2) \]  

(39)
\[
\beta = \frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{\xi^2 + \eta^2}} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (38)
\]

One should note that \( \gamma = \cosh \xi \), \( \gamma \beta = \sinh \xi \) and 
\( \alpha = \sqrt{\varepsilon_c \varepsilon_\parallel / \varepsilon_\perp} \), thus leading to \( \beta = \tanh \xi \). 
\[
\xi = \ln \left( \frac{1 + \beta}{1 - \beta} \right) = \ln \frac{\varepsilon_\perp}{\varepsilon_\parallel} \quad \text{and} \quad \delta / 2 = \sqrt{\varepsilon_\parallel / \varepsilon_\perp} \tan \phi / 2
\]

Accordingly, in comparison with expression (30), one has 
\[
\eta \mathbf{D} = \eta_\parallel \mathbf{D} + \left[ \eta_\perp - \eta_\parallel \right] \left[ \mathbf{D} \cdot \mathbf{c}_1 + \mathbf{D} \cdot \mathbf{c}_2 \right] \quad (39)
\]
The expression (39) is the inverse of the dielectric function for biaxial crystals. Now, doing the same that was made for equation (32), we can find the inverse of the dielectric function for uniaxial crystals. 
\[
\eta \mathbf{D} = \eta_\parallel \mathbf{D} + \left[ \eta_\perp - \eta_\parallel \right] \left[ \mathbf{D} \cdot \mathbf{c} \right] \quad (40)
\]

### 3.3 Uniaxial crystals

As we saw previously, for the case of a uniaxial medium we have the following dielectric function:
\[
\varepsilon \mathbf{E} = \varepsilon_\parallel \mathbf{E} + \varepsilon_\perp \mathbf{E} \quad (41)
\]

**Notation:** Normally, in uniaxial crystals is common consider \( \varepsilon_\perp = n_\perp^2 \) and \( \varepsilon_\parallel = n_\parallel^2 \).

These crystals are considered positive when \( \varepsilon_\parallel > \varepsilon_\perp \) (Figure 8) and negative if \( \varepsilon_\parallel < \varepsilon_\perp \) (Figure 9).

![Positive uniaxial crystal](image1)

![Negative uniaxial crystal](image2)

Analyzing the expression (42), we can infer the following,
\[
\eta \mathbf{E}_\parallel = \varepsilon \mathbf{E} \quad \Rightarrow \quad \mathbf{E} = n^2 \eta \mathbf{E}_\parallel
\]
\[
\mathbf{E}_\parallel = \mathbf{E} - \mathbf{E}_\parallel \quad \Rightarrow \quad \mathbf{E}_\parallel = n^2 \eta \mathbf{E}_\parallel - \mathbf{E}_\parallel
\]

The parallel component of electric field, \( \mathbf{E}_\parallel \), is defined as its component second \( \mathbf{k}_\parallel \), i.e., such that \( \mathbf{k} \cdot \mathbf{E}_\parallel = 0 \).
\[
E_\parallel = \frac{1}{n} \mathbf{n} \cdot \mathbf{E}
\]
\[
E_\parallel = \frac{1}{n^2} \mathbf{n} \cdot \mathbf{E} \quad (45)
\]

Considering the wave equation,
\[
\left[ \mathbf{k} \wedge \mathbf{E}_\parallel = 0 \right] \quad \mathbf{k} \wedge \mathbf{E}_\parallel = n^2 \left[ \mathbf{k} \wedge \eta \mathbf{E}_\parallel \right] - \mathbf{k} \wedge \mathbf{E}_\parallel = 0
\]

```
One should note that:

\[
\alpha_0 = \eta_1 = \frac{1}{n^2_0},
\]
\[
\beta_0 = \eta_1 - \eta_2 = \frac{1}{n^2_0} - \frac{1}{n^2_1},
\]

\[\eta E_\perp = \alpha_0 E_\perp + \beta_0 c \cdot E_\perp c\]
\[\mathbf{k} \wedge \eta E_\perp = \alpha_0 \mathbf{k} \wedge E_\perp + \beta_0 c \cdot E_\perp \mathbf{k} \wedge c\]

Finally, we are able to present the wave equation in a uniaxial crystal.

\[\alpha_0 n^2 - 1 \mathbf{k} \wedge E_\perp + \beta_0 n^2 c \cdot E_\perp \mathbf{k} \wedge c = 0.\]

(48)

wave equation for an uniaxial crystal

Applying the left contraction with \(\mathbf{k} \wedge c\) to equation (48) we obtain the following.

\[
\begin{bmatrix}
1 - \alpha_0 n^2 + \beta_0 n^2 \mathbf{k} \wedge c \\
\end{bmatrix} c \cdot E_\perp = 0.
\]

(49)

extraordinary wave

ordinary wave

Finally we can derive the two eigenwaves (or isonormal waves) that can propagate in a lossless nonmagnetic uniaxial crystal, i.e., in a medium characterized by \(D = \varepsilon_0 \varepsilon \mathbf{E}\) and \(B = \mu_0 \mathbf{H}\). Then, from equation (49), we get the equation of the ordinary wave,

\[n^2 = \frac{1}{\alpha_0} = n^2_0\]

(50)

and the equation of the extraordinary wave,

\[n^2 = \frac{n^2_0 n^2_1}{\mathbf{k} \cdot c^2 n^2_0 - \mathbf{k} \wedge c^2 n^2_1}.
\]

(51)

3.4 Biaxial crystals

Considering the equations (17), (18), (30), (39) and monochromatic plane waves propagating in a biaxial media, it comes immediately, like we saw for uniaxial crystals,

\[
n^2 E_\| = \varepsilon \mathbf{E}
\]
\[
E_\perp = \mathbf{E} - E_\| = \mathbf{E} - \mathbf{k} \mathbf{k}^* 
\]

(52)

Accordingly, in terms of impermeability function of equation (39) we may also write

\[\mathbf{k} \wedge \omega = 0 \quad \text{← wave equation}\]
\[\omega = n^2 \eta E_\perp - E_\perp
\]

(53)

or explicitly,

\[\alpha_0 n^2 - 1 \mathbf{k} \wedge E_\perp + \beta_0 n^2 c \cdot E_\perp \mathbf{k} \wedge c + E_\perp = 0
\]

(54)

where \(\alpha_0 = \eta_2\) and \(\beta_0 = \frac{\eta_1 - \eta_2}{2}.

Introducing the two following vectors,

\[
u = - \mathbf{k} \wedge c \mathbf{e}_{123}
\]
\[
v = - \mathbf{k} \wedge c \mathbf{e}_{213}
\]

(55)

and applying the left contraction of bivectors \(\mathbf{k} \wedge \mathbf{c}_1\) and \(\mathbf{k} \wedge \mathbf{c}_2\) with equation (54), we obtain,

\[
\begin{bmatrix}
1 - n^2 \beta_0 u^2 \\
\end{bmatrix} c \cdot E_\perp = 0.
\]

(56)

The eigenwaves corresponding to the direction of propagation \(\mathbf{k}\) (the wave normal) are characterized by two distinct refractive indexes (birefringence), such that \(n_{\perp}\) and \(n_{\parallel}\), such that

\[\frac{1}{n^2_\perp} = \alpha_0 + \frac{\beta_0}{2} u \cdot v \pm \sqrt{u^2 v^2}.
\]

(58)
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But then, according to the definition of optic axes, we conclude that the two unit vectors \( \mathbf{c}_1 \) and \( \mathbf{c}_2 \) that characterize the impermeability function are, in fact, the two optic axes of the biaxial crystal.

This two waves, contrarily to what happens for uniaxial crystals, cannot be divided into ordinary and extraordinary waves, because both of them have simultaneously characteristics of both ordinary and extraordinary waves as we can see in Figure 2.

3.5 Retarder plates

The retarder plates are optical components that allow, by transmission, change the state of polarization of a light wave. These plates are made of transparent anisotropic material (usually quartz-uniaxial positive or negative uniaxial-calcite) and are divided in full-wave plates, half-wave and quarter-wave, each one with their phase difference, defined.

The retarder plates may be zero-order or multiple order. The zero-order plates, are more difficult to produce and impractical with respect to polishing due to thicknesses too small.

Finally, it should be noted that there is a technique that allows to build zero-order retarder plates, without the disadvantages of multiple order plates, with thickness manageable. This technique consists in combine two multiple order plates.

4 CONCLUSIONS AND FUTURE WORK

The standard approach to anisotropic media has been the dyadic analysis where the problem is usually solved through the principal coordinate system of the dielectric tensor as it takes, in this specific system, its simplest diagonal form.

This paper presents a new approach to anisotropy, using the geometric algebra, specifically, we have applied geometric algebra \( Cl_3 \) to the problem of the electromagnetic waves propagation in uniaxial and biaxial crystals. This mathematics tool introduces a direct geometrical interpretation without the intervention of any coordinate system.

In this paper one can conclude that when handling with the propagation of electromagnetic waves in anisotropic media, particularly handling with monochromatic plane waves propagating in uniaxial and biaxial crystals, using an approach with geometric algebra that sort of media reveal physical invariants, that are the optic axes of the crystal in question. In the case of the biaxial crystals the dielectric axes and the optic crystals differ, however, for the uniaxial crystals the dielectric and the optic axe are the same.

There are three main aspects which are important to highlight in geometric algebra: the ease with which it define vector rotations using for the effect the rotors; the invertibility of all Clifford objects; and the interpretation associated to the operations made with that sort of objects. These three aspects make the manipulation of vectors more intuitive, which in turn make possible to define electric anisotropy in terms of the exterior product between the electric field and the electric displacement field, \( \mathbf{E} \wedge \mathbf{D} \).

With the approach using the geometric algebra, we have shown how this mathematics tool can provide a better mathematical framework for anisotropy than tensors and dyadics. Through the direct manipulation of coordinate-free objects such as vectors, bivectors and trivectors, geometric algebra is the most natural setting to study anisotropy, providing a deeper insight and simpler calculations, without loosing its direct geometrical interpretation.

In terms of future, we can refer works in the area of relativistic mechanics, particularly, applications of geometric algebra to the reformulation of classic electrodynamics in Minkowski’s space-time. There also some studies that can be made, using this new algebra, in the propagation of electromagnetic waves in some media like DNG, SNG, bianisotropic and bistorpic.

REFERENCES