Geometry and Quantization

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Abstract

We start with an exposition of basic gauge theory, from both the mathematical and physical perspectives. We describe how the Nahm equations appearing by dimensional reduction of the anti-self-dual Yang-Mills equations can be used to give an hyperkähler structure to the cotangent bundle of a complex Lie group $T^*K_{\mathbb{C}}$.

We develop the apparatus for studying the geometric quantization of the cotangent bundle of a complex Lie group $T^*K_{\mathbb{C}}$. In the case when K is Abelian, we are able to study the quantization of $T^*K_{\mathbb{C}}$ in different polarizations, and show that they are related by unitary BKS pairings, if the half-form correction is taken into account.

Keywords: Gauge theory, instantons, Nahm equations, geometric quantization, cotangent bundle of a complex Lie group

1 Introduction

We start by giving an introduction to the theory of connections on principal bundles. This is done in **Chapter 1**, where we also explain connections in vector bundles, state the main features of Riemannian holonomy like Berger's classification and end with the Chern-Weil theory of characteristic classes. The main reference for this chapter is [24], however there are many other classical books on differential geometry that contain the core of these subjects.

Chapter 2 explores some physical ideas of classical gauge theory using the mathematical apparatus developed in the first part. We will in particular see how connections, while natural from a mathematical point of view, fit well in this description of the physical world by identifying them with gauge fields. We shall also make some connections with physicists language and explore ideas such as gauge fields, gauge invariance and the Higgs mechanism. References for this are mainly [5], [28], [1] and for an exposition closer to the physics language check [15].

By exploring examples such as the Yang-Mills equations we will meet by the first time the notion of instantons, in particular monopoles and solutions to Nahm equations. These appear as solutions to the equations of Bogomolny and Nahm respectively and we will arrive at them by dimensional reduction of the anti-self dual Yang-Mills equations. For a more complete exposition the reader may want to consult [2].

The subject of **Chapter 3** is hyperkähler geometry. In this stage we shall assume that the reader is familiar with some symplectic geometry namely with the notion of moment map, however symplectic reduction is reviewed. For us the most important part of this chapter is hyperkähler reduction, that will be used later in the construction of moduli spaces, where we want the hyperkähler structure to descend to the quotient. References for hyperkähler geometry are [19] and [20]. We will also meet monopoles and Nahm equations for the second time, and give a unified formalism for them, using hyperkähler reduction as a way to construct their moduli spaces. For this the reader may check [18] and [2].

Chapter 4 is an application of hyperkähler reduction to show that the cotangent bundle of a complex Lie group is hyperkähler. This is a result due to Kronheimer [25], however some other useful references for what we develop here are [4] and [7]. Here we will meet again Nahm equations.

In **Chapter 5** we introduce and explain the methods of geometric quantization. Our exposition is motivated by quantum mechanics. We describe how geometric quantization attempts to construct a quantum system out of a classical one. So, given a symplectic manifold as the phase space of a classical system we are concerned with finding the quantum Hilbert space representing the corresponding quantum system. In this process one needs to introduce more structure and the concept of polarization comes in. Now it is no longer obvious how our quantum Hilbert spaces will depend on the choice of polarization. In fact the dependence of quantization on the choice of polarization is an important open problem in geometric quantization. The main references are [26] and the classics [23] and [30].

We then try to use geometric quantization in **Chapter 6** to quantize the cotangent bundle of a complex Lie group, which as we have seen is hyperkähler. Being hyperkähler, it is natural to look for quantization in different polarizations, associated to different holomorphic structures. However, it turns out to be difficult in general to find different polarizations for a given symplectic structure. In the case where the group is abelian we could use the tautological symplectic structure, that in the abelian case is Kähler with respect to a noncanonical complex structure. Besides that, this structure allows us to choose 3 different obvious polariza-

tions, while in the nonabelian case there was only 1. These features where an invitation to the study of geometric quantization in the Abelian case.
So we go ahead with our goal that we can reduce to the case of studying the geometric quantization of $T^*\mathbb{S}^1$ and this is the subject of the closing **Chapter 7**.

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