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Resumo

Estudamos alguns aspectos básicos das teorias de gauge, tanto de uma perspectiva matemática como física. Neste estudo encontramos pela primeira vez as equações de Nahm, obtidas por redução dimensional das equações anti-auto-duais de Yang-Mills. Usamos as equações de Nahm para introduzir uma estrutura hyperkähler no fibrado cotangente a um grupo de Lie complexo $T^*K_{\mathbb{C}}$.
Apresentamos a quantização geométrica como um programa que a um sistema clássico tenta associar o correspondente sistema quântico, aqui com o objectivo de quantizar $T^*K_{\mathbb{C}}$. No caso em que K é um grupo de Lie Abeliano, estudamos a sua quantização relativamente a diferentes polarizações, incluindo a correcção por meias-formas, e mostramos que estas se relacionam unitariamente por emparelhamentos BKS.

Palavras-chave: Teoria de gauge, equações de Nahm, quantização geométrica, fibrado cotangente a um grupo de Lie complexo.

Abstract

We start with an exposition of basic gauge theory, from both the mathematical and physical perspectives. We describe how the Nahm equations appearing by dimensional reduction of the anti-self-dual Yang-Mills equations can be used to give an hyperkähler structure to the cotangent bundle of a complex Lie group $T^*K_{\mathbb{C}}$. We develop the apparatus for studying the geometric quantization of the cotangent bundle of a complex Lie group $T^*K_{\mathbb{C}}$. In the case when K is Abelian, we are able to study the quantization of $T^*K_{\mathbb{C}}$ in different polarizations, and show that they are related by unitary BKS pairings, if the half-form correction is taken into account.

Keywords: Gauge theory, instantons, Nahm equations, geometric quantization, cotangent bundle of a complex Lie group

Contents

Para o meu avô Ulisses que me ensinou a Tabuada.

Introduction

It is hard to say what the main object, goal, or even subject of this thesis is. But if pointed a gun to my head, I would say:

Object: Possibly Nahm equations is the most revisited object.

Goal: Relate different quantizations of the cotangent bundle of a complex Lie group. However we end up dealing only with the case where the Lie group is Abelian.

Subject: It starts with gauge theory and ends with geometric quantization.

The evolution of the thesis makes sense only as a story. It connects different areas of mathematics and physics entangled in a fascinating way. Here we will study some of these entanglements. We will use the Introduction as a guide to what is done in the different Chapters of the thesis.

We start by giving an introduction to the theory of connections on principal bundles. This is done in **Chapter 1**, where we also explain connections in vector bundles, state the main features of Riemannian holonomy like Berger's classification and end with the Chern-Weil theory of characteristic classes. The main reference for this chapter is [24], however there are many other classical books on differential geometry that contain the core of these subjects.

Chapter 2 explores some physical ideas of classical gauge theory using the mathematical apparatus developed in the first part. We will in particular see how connections, while natural from a mathematical point of view, fit well in this description of the physical world by identifying them with gauge fields. We shall also make some connections with physicists language and explore ideas such as gauge fields, gauge invariance and the Higgs mechanism. References for this are mainly [5], [28], [1] and for an exposition closer to the physics language check [15].

By exploring examples such as the Yang-Mills equations we will meet by the first time the notion of instantons, in particular monopoles and solutions to Nahm equations. These appear as solutions to the equations of Bogomolny and Nahm respectively and we will arrive at them by dimensional reduction of the anti-self dual Yang-Mills equations. For a more complete exposition the reader may want to consult [2].

The subject of **Chapter 3** is hyperkähler geometry. In this stage we shall assume that the reader is familiar with some symplectic geometry namely with the notion of moment map, however symplectic reduction is reviewed. For us the most important part of this chapter is hyperkähler reduction, that will be used later in the construction of moduli spaces, where we want the hyperkähler structure to descend to the quotient. References for hyperkähler geometry are [19] and [20]. We will also meet monopoles and Nahm equations for the second time, and give a unified formalism for them, using hyperkähler reduction as a way to construct their moduli spaces. For this the reader may check [18] and [2].

Chapter 4 is an application of hyperkähler reduction to show that the cotangent bundle of a complex Lie group is hyperkähler. This is a result due to Kronheimer [25], however some other useful references for what we develop here are [4] and [7]. Here we will meet again Nahm equations.

In **Chapter 5** we introduce and explain the methods of geometric quantization. Our exposition is motivated by quantum mechanics. We describe how geometric quantization attempts to construct a quantum system out of a classical one. So, given a symplectic manifold as the phase space of a classical system we are concerned with finding the quantum Hilbert space representing the corresponding quantum system. In this process one needs to introduce more structure and the concept of polarization comes in. Now it is no longer

obvious how our quantum Hilbert spaces will depend on the choice of polarization. In fact the dependence of quantization on the choice of polarization is an important open problem in geometric quantization. The main references are [26] and the classics [23] and [30].
We then try to use geometric quantization in **Chapter 6** to quantize the cotangent bundle of a complex Lie group, which as we have seen is hyperkähler. Being hyperkähler, it is natural to look for quantization in different polarizations, associated to different holomorphic structures. However, it turns out to be difficult in general to find different polarizations for a given symplectic structure.
In the case where the group is abelian we could use the tautological symplectic structure, that in the abelian case is Kähler with respect to a noncanonical complex structure. Besides that, this structure allows us to choose 3 different obvious polarizations, while in the nonabelian case there was only 1. These features where an invitation to the study of geometric quantization in the Abelian case.
So we go ahead with our goal that we can reduce to the case of studying the geometric quantization of $T^*\mathbb{S}^1$ and this is the subject of the closing **Chapter 7**.

Chapter 1

Bundles, Connections and Holonomy

1.1 Bundles

For a detailed account of the material covered in this section and in almost all of this chapter, the reader can find a very good exposition in [24]. All manifolds are assumed to be smooth.

1.1.1 Principal Bundles

In this section we motivate and define Principal Bundles and prove some of their properties. Let M be a manifold and G a Lie group. A principal bundle over M with structure group G must be thought of as another manifold, which we call P, that is obtained by putting a copy of G at every point of M, in a globally twisted way.

Definition 1 *A* ***Principal Bundle*** *with structure group G is a triple (π, P, M), such that*

1. *P and M are manifolds and G a Lie group acting freely and properly on the right on P*

2. *$\pi : P \longrightarrow M$ is the surjection on the quotient $M = P/G$*

3. *For each $x \in M$ it exists an open neighbourhood U such that $\pi^{-1}(U) \simeq U \times G$, via a map*

$$\psi : \pi^{-1}(U) \longrightarrow U \times G$$

$$p \longmapsto [(\pi(u), \phi(u))]$$

such that $\phi(ug) = \phi(u)g$ for all $g \in G$.

P is called the ***total space****, M the* ***base space*** *and the map $\psi : \pi^{-1}(U) \longrightarrow U \times G$ a local trivialization.*

Notice that in this definition, for all $x \in M$, the fibres are $\pi^{-1}(x) \simeq G$. The trivial principal bundle is $P = M \times G$ with G acting on the G factor by right multiplication.
For any principal bundle (π, P, M) with structure group G we can find a trivializing open cover $\{U_\alpha\}$ of M. This is an open cover such that on each open set U_α we have trivializing maps

$$\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$$
$$p \longmapsto [(\pi(u), \phi_\alpha(u))].$$

We can show that if $U_\alpha \cap U_\beta \neq \emptyset$, the maps $\psi_\alpha \circ \psi_\beta^{-1} : U_\alpha \cap U_\beta \times G \longrightarrow U_\alpha \cap U_\beta \times G$ are given by

$$\psi_\alpha \circ \psi_\beta^{-1}(x, g) = (x, g_{\alpha\beta}(x))$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ are called the **transition functions** of the bundle. They satisfy the cocycle condition, i.e. in $U_\alpha \cap U_\beta \cap U_\gamma$,

$$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}.$$

These transition functions depend on the choice of local trivializations, but they can be used to recover the whole principal bundle. One can think of them as ruling the gluing of local trivializations. In fact,

$$P = \bigsqcup_\alpha U_\alpha \times G/\sim,$$

where the equivalence relation is given by $(x, g) \sim (x', g')$ if $x = x'$ and it exists α, β such that $x \in U_\alpha \cap U_\beta$ and $g' = g_{\alpha\beta}(x)g$.

Definition 2 *A* ***section*** *is a smooth map $s : M \longrightarrow P$ such that $\pi \circ s = id_M$. A* ***local section*** *is defined on an open set U and is a section of the bundle $\pi^{-1}(U)$. We shall denote the set of (global) sections by $\Gamma(P)$ and the set of local sections on U by $\Gamma(U, P)$.*

Local sections on trivializing open sets $s : U_\alpha \longrightarrow P$ can be written as maps $s_\alpha : U_\alpha \longrightarrow G$, using the trivialization, by

$$\psi_\alpha \circ s(x) = (x, s_\alpha(x)).$$

In the intersection of two trivializing open sets $U_\alpha \cap U_\beta$ these maps clearly satisfy $s_\alpha(x) = s_\beta(x) g_{\alpha\beta}(x)$.

We shall now define maps between bundles, so in what follows we let (π, P, M) and (π', P', M') be principal bundles with structure groups G and G'.

Definition 3 *A* ***morphism of principal bundles*** *is a differentiable map $f : P \longrightarrow P'$ equivariant with respect to a morphism of Lie groups also denoted $f : G \longrightarrow G'$. It induces a differentiable map on the bases $f : M \longrightarrow M'$.*

A morphism of principal bundles is an **isomorphism** if the map f it induces is an isomorphism in the category of principal bundles, i.e. if it has an inverse map such that the compositions give the identity.
We say that two principal bundles are **equivalent** if there is an isomorphism between them. They are said to be **isomorphic** if there is an isomorphism in the subcategory of principal bundles over the same base and covering the identity on the base.

1.1.2 Vector Bundles

We now move to vector bundles. The motivation here is similar, we still need a base space M and want a vector bundle to be a manifold obtained by adding a copy of a vector space V at each point of M.
We could take a similar definition to the one made for principal bundles. However, since we already have principal bundles we will use them to define vector bundles using the associated bundle construction. This will also allow us to view the structure group G of the bundle as a group of automorphisms of the vector space V. In what follows let (π, P, M) be a principal bundle with structure group G, and $\rho : G \longrightarrow GL(V)$ be a left representation of G on V.

Definition 4 *We define the* ***Associated Bundle*** (π_E, E, M) ***with standard fibre*** *V, as the manifold*

$$E = P \times_G V = P \times V/G$$

where G acts on $P \times V$ via $g \cdot (p, v) \mapsto (pg, \rho(g^{-1})v)$, and the projection map is defined by

$$\pi_E : E \longrightarrow M$$
$$[(p, v)] \longmapsto \pi(p).$$

Remark 5 *This construction is more general and can be made for any manifold F on which G acts, giving bundles with fibre F; however we will not need this.*

1.2 The Theory of Connections

In this chapter we explore the theory of connections in principal and vector bundles. We will start by defining things for principal bundles and only later move on to vector bundles.

1.2.1 Connections on a Principal Bundle

Let P be a principal bundle over the base M with structure group G. The action of G on P, induces a map from the Lie Algebra $\mathfrak{g}$ into the vector fields on P. Choosing a point $u \in P$, this map is written $A \mapsto A^*_u$ and its image is called the vertical subspace at u. In this way we have a Lie algebra isomorphism $\mathfrak{g} \cong V_u$.
We think of a connection has the assignment of a horizontal subspace to each point of P, that gives a complement to this vertical subspace.

Definition 6 *Let P be a principal bundle over the base M with structure group G, T_uP the tangent space at u and V_u the tangent space to the fibre at u. An* ***Ehreshmann Connection*** *Γ is a smooth assignment of a subspace H_u of T_uP to each point $u \in P$, such that:*

1. *$T_uP = H_u \oplus V_u$*
2. *$H_{ug} = (R_g)_* H_u$, for $g \in G$*

We call H_u the ***horizontal subspace*** *and V_u the* ***vertical subspace****.*

As a consequence of this definition, a connection allows us to write any tangent vector X_u at u as a unique linear combination of a horizontal and a vertical part $X_u = hX_u + vX_u$. We shall now motivate the equivalent description of connections in terms of a 1-form. Given a vector field X in P, we would like to know how it stands relatively to the connection Γ. At $u \in P$, X_u is horizontal iff $vX_u = 0$, so we can walk backwards along the isomorphisms just described and go to the Lie Algebra $\mathfrak{g}$ in order to check how vertical X is. Hence we define the **connection 1-form** ω as a $\mathfrak{g}$ valued 1-form on P, such that:

$$\omega(X) = A \text{ with } vX = A^*.$$

Notice that in this way we have that $\forall u \in P$, $H_u = Ker(\omega_u)$. The next proposition justifies that we can identify Ehreshman connections Γ with connection 1-forms ω that satisfy $(R_g)^*\omega = ad(g^{-1})\omega$.

Proposition 7 *There is a one to one correspondence between Ehreshmann connections and connection 1-forms ω such that:*

1. *$\omega(A^*) = A, \forall A \in \mathfrak{g}$*
2. *$(R_g)^*\omega = ad(g^{-1})\omega$*

Proof: Let Γ be an Ehreshmann connection, and ω its connection 1-form. The first property is a direct consequence of the definition, so let us concern ourselves with the second one. Since both sides of the equality are linear we can just check it for horizontal and vertical vector fields. If X is a horizontal vector field so is $(R_a)^*X$ and both sides vanish. If $X = A^*$ is a vertical vector field, then:

$$((R_g)^*\omega)_u(A^*) = \omega_{gu}((R_g)_*A*) = (L_{g^{-1}})_*(R_g)_*A = ad(g^{-1})A = ad(g^{-1})\omega(A^*).$$

Conversely, given a 1-form ω with the properties stated, the distribution $u \mapsto H_u \equiv Ker(\omega_u)$ is smooth. The first property of an Ehreshmann connection follows from the fact that ω_u defines a surjective linear map $T_uP \longmapsto \mathfrak{g} \cong V_u$, hence $V_u \cong T_uP/Ker(\omega_u)$. The second property is a direct consequence of 2. □

So far we have been exploring the linear isomorphism $\mathfrak{g} \cong V_u$, we will now turn to another isomorphism that will allow us to define notion of horizontal lift.

Since at any $u \in P$ the projection $d_u\pi : T_uP \to T_{\pi(u)}M$ is surjective, and $Ker(d_u\pi) = V_u$ we have a linear isomorphism $T_uP \cong T_{\pi(u)}M$ that can be used to lift and project vector fields from and onto the base M. Let X be a vector field on M, we define the **horizontal lift** of X as being a horizontal vector field $\widetilde{X}$ on P such that $\pi_*\widetilde{X} = X$. It is easy to see that the horizontal lift of a vector field is invariant by the G-action on P. The next proposition states some immediate properties of the horizontal lift of vector fields.

Proposition 8 *Let $f \in C^\infty(M)$, X, Y be vector fields on M and $\widetilde{X}, \widetilde{Y}$ the corresponding lifts, then:*

1. $\widetilde{X} + \widetilde{Y} = \widetilde{X + Y}$
2. $(\pi^* f)\widetilde{X} = \widetilde{fX}$
3. $h[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}$

Remark 9 *The following hold:*

1. *Every horizontal lift of a vector field is invariant under the G-action on P and is the horizontal lift of its projection. This is an immediate consequence of $\pi_* : T_uP \to T_{\pi(u)}M$ being an isomorphism.*
2. *If $(U, x^1, ..., x^n)$ is a local coordinate chart for M, then $\left\{\widetilde{\frac{\partial}{\partial x^1}}, ..., \widetilde{\frac{\partial}{\partial x^n}}\right\}$ is a local basis to the distribution defined by the connection.*

We will now turn to the problem of expressing locally the connection 1-form. The goal is to get local 1-forms on M that represent the connection. Now let:

- U_α a open cover of M
- $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ the local trivializations
- $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ the transition functions
- $\sigma_\alpha : U_\alpha \to P$ local sections, such that $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$.

In U_α we define the local forms $\omega_\alpha = \sigma_\alpha^*\omega$. These are forms in U_α with values in the Lie Algebra $\mathfrak{g}$, that represent locally the connection 1-form and can be used to recover it. The next proposition states how this forms behave on the overlaps of open sets in the cover and shows that they do not match together to create a globally well defined 1-form on M.

Proposition 10 *If $U_\alpha \cap U_\beta \neq \emptyset$ then:*

$$\omega_\beta = ad(\psi_{\alpha\beta}^{-1})\omega_\alpha + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}. \tag{1.1}$$

And if ω_α is a set of 1-forms in M that satisfy the previous relation there is a unique connection 1-form ω that originates them.

Proof: The unique observation is that for $\pi(u) = x \in U_\alpha \cap U_\beta \neq \emptyset$, we have $\psi_\alpha(u) = (x, \phi_\alpha(u))$ where $\phi_\alpha : \pi^{-1}(U_\alpha) \to G$. In this situation we can write $\phi_\alpha(u) = \phi_\beta(u)\psi_{\alpha\beta}(x)$, and hence $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$. Now we just compute derivatives, however in order to more easily evaluate them rewrite the previous expressions in the form $\sigma_\beta(x) = R_{\psi_{\alpha\beta}(x)}\sigma_\alpha(x)$. Let X be a vector field on M.

$$d\sigma_\beta(X) = (R_{\psi_{\alpha\beta}(x)})_*(d\sigma_\alpha(X)) + d_{\sigma_\alpha(x)} R_{\psi_{\alpha\beta}(x)}(d\psi_{\alpha\beta}(X)).$$

Or in another fashion:

$$d\sigma_\beta(X) = (R_{\psi_{\alpha\beta}(x)})_*(d\sigma_\alpha(X)) + ((\psi_{\alpha\beta})_*(X))^*_{\sigma_\alpha(X)\psi_{\alpha\beta}(x)}.$$

Since $\omega(d\sigma_\beta(X)) = \omega_\beta(X)$ we get

$$\begin{aligned} \omega_\beta(X) &= \omega((R_{\psi_{\alpha\beta}(x)})_*(d\sigma_\alpha(X))) + \omega((d\psi_{\alpha\beta}(X))^*_{\sigma_\alpha(X)\psi_{\alpha\beta}(x)}) \\ &= ad(\psi_{\alpha\beta}(x))\omega_\alpha(X) + dL_{\psi_{\alpha\beta}(x)^{-1}}(d\psi_{\alpha\beta}(X)). \end{aligned} \tag{1.2}$$

□

Remark 11 *Above, the notation $\psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}$ means $\psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta} = \psi_{\alpha\beta}^*\theta$, where θ is the Maurer-Cartan form on G.*

The next question that can be asked is about the existence of connections and their extension properties, where by extension properties we mean: under which conditions can we extend a connection defined on a closed set. To answer these questions in a global way we shall need the following lemmas that give a local answer:

Lemma 12 *Let $x \in M$. Then there exists a neighbourhood U of x and a connection Γ defined in U.*

Proof: By local triviality define connections on $U \times G$ and pull them back to $\pi^{-1}(U)$. □

Lemma 13 *Let $x \in M$. Then there exists a neighbourhood U of x such that any connection Γ defined in a closed set $A \subset U$ can be extended to a connection on U.*

Proof: The proof is trivial since if (U, ϕ) is a coordinate chart we can get a connection form ω associated with Γ, write it in coordinates and reduce the problem to extending the smooth functions that are the components of ω. □

With these two lemmas we prove that connections always exist and can always be extended to a global connection.

Theorem 14 *Let (π, P, M) be a principal bundle with structure group G and A a closed subset of M. Then any connection in A can be extended to a connection in P, and in particular P admits a connection.*

Proof: Let U_i be a open cover of M, such that each $\pi^{-1}(U_i)$ is trivial and lies in the conditions of lemma 13. Define λ_i to be a partition of unity subordinate to the open cover U_i. If we have a connection defined in A, it defines connections in each $A \cap \pi^{-1}(U_i)$ where we have connection 1-forms ω_i that by lemma 13 can be extended to each $\pi^{-1}(U_i)$. In the case of $A \cap \pi^{-1}(U_i)$ being empty lemma 12 and 13 can give us a connection form in $\pi^{-1}(U_i)$. We define in P the connection 1-form given by:

$$\omega = \sum_i (\pi^* \lambda_j)\omega_i$$

It is then easy to check that ω is a connection 1-form in P that extends the original connection defined on A. $\square$

1.2.2 Curvature

Let (π, P, M) be a principal bundle with structure group G and ω a fixed connection on P, V a finite dimensional vector space and $\rho : G \longrightarrow Aut(V)$ a representation of G on V.

Definition 15 *We define $\tilde{\Omega}^k(P, V)$ to be the space of* ***V-valued horizontal k-forms on P*** *for the representation ρ, i.e $\varphi \in \tilde{\Omega}^k(P, V)$ iff:*

1. *$R_g^*\varphi = \rho(g^{-1}) \cdot \varphi$, for $g \in G$.*
2. *$\varphi(X_1, ..., X_k) = 0$ if one of the X_i is vertical.*

Notice that the space $\tilde{\Omega}^k(P, V)$ depends on the connection that we have fixed on P.

Remark 16 *A particular case of the above construction is the case of $\tilde{\Omega}^0(P, V)$. These are just functions on P with values on V such that for $g \in G$, and $p \in P$ we have that $\varphi(pg) = \rho(g^{-1}) \cdot \varphi(p)$. The importance of this case is that $\tilde{\Omega}^0(P, V)$ can be viewed as the space of sections of the associated vector bundle (π, E, M) with standard fibre V. This justifies the following definition.*

Definition 17 *Let G act on a manifold F on the left via ρ in such a way that we can construct the associated fibre bundle (π, E, M) with standard fibre F. Then we define $C(P, F)$ to be the space of maps $\tau : P \longrightarrow F$ such that $\tau(pg) = \rho(g^{-1}) \cdot \tau(p)$, for all $g \in G$.*

Proposition 18 *Let $\varphi \in \tilde{\Omega}^k(P, V)$, then:*

1. *$\varphi h \in \tilde{\Omega}^k(P, V)$, where $\varphi h(X_1, ..., X_k) = \varphi(hX_1, ..., hX_k)$ and hX_i being the horizontal component of X_i.*
2. *$d\varphi \in \Omega^{k+1}(P, V)$, with $R_g^* d\varphi = \rho(g^{-1}) \cdot d\varphi$.*
3. *$d\varphi h \in \tilde{\Omega}^{k+1}(P, V)$.*

The proof of this proposition is trivial, but it motivates the definition of the **exterior covariant derivative** as the operator

$$D : \tilde{\Omega}^k(P, V) \longrightarrow \tilde{\Omega}^{k+1}(P, V)$$

With D being given by 3 in the previous proposition as $D\varphi = d\varphi h$. Notice that the definition of D depends on the connection given.

We shall now focus on the case of $\mathfrak{g}$-valued horizontal k-forms on P, where ρ is the adjoint representation of G on $\mathfrak{g}$, i.e. $\tilde{\Omega}^k(P, \mathfrak{g})$. Notice that the connection 1-form ω is not horizontal, but even then $R_g^* d\omega = ad(g^{-1}) \cdot d\omega$. Hence, extending D to act on the whole $\Omega(P, \mathfrak{g})$ we have that $D\omega \in \tilde{\Omega}^2(P, \mathfrak{g})$, making the next definition valid.

Definition 19 *Let ω be a connection 1-form on P, we define the **curvature** of the connection ω, as being the $\mathfrak{g}$-valued horizontal 2-form $\Omega = D\omega$.*

Theorem 20 *Let (π, P, M) be a principal bundle with structure group G, and connection form ω, then the following identities hold:*

1. ***Cartan Structural Equation.***

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

2. ***Bianchi Identity.***

$$D\Omega = 0$$

Proof:

1. The equation is linear and antisymmetric in both sides, so it suffices to check it on $X, Y \in \chi(P)$, for the cases when they are both horizontal, both vertical and one vertical and the other horizontal.
(i) Let X, Y be vertical, then we can write $X = A^*, Y = B^*$. Obviously $\Omega(A^*, B^*) = 0$ and $[\omega, \omega](A^*, B^*) = [\omega(A^*), \omega(B^*)] - [\omega(B^*), \omega(A^*)] = 2[\omega(A^*), \omega(B^*)]$. For the $d\omega$ term notice that

$$d\omega(A^*, B^*) = A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega[A^*, B^*],$$

and the first two terms vanish, since $\omega(A^*) = A$ which is a constant.
(ii) Let X, Y be horizontal, then $\omega(X) = \omega(Y) = 0$ and we are left with $\Omega(X, Y) = d\omega(X, Y)$.
(iii) Let $X = A^*$ be vertical and Y horizontal, then $\Omega(A^*, Y) = 0$ because A^* is vertical, and $[\omega, \omega](A^*, Y) = 0$ because Y is horizontal. We are left with the $d\omega$ term, for which:

$$d\omega(A^*, Y) = A^*(\omega(Y)) - Y(\omega(A^*)) - \omega[A^*, Y] = -\omega[A^*, Y] = 0,$$

where we set the first two terms to zero because $\omega(A^*)$ is constant and $\omega(Y)$ is zero itself. The fact that the third term vanishes is a direct consequence of the definition of connection, since being X a horizontal vector field we have that $(R_g)_* X = X$ for all $g \in G$, hence:

$$[A^*, X] = \frac{d}{dt}|_{t=0}(R_{exp(-tA)})_* X = 0.$$

2. For the Bianchi identity, since $D\Omega \in \tilde{\Omega}^3(P, \mathfrak{g})$, it suffices to consider the case when the vector fields are horizontal, so $D = d$. Using the Cartan Structural Equation just proved:

$$d\Omega = d(d\omega + \frac{1}{2}[\omega, \omega]) = \frac{1}{2}d[\omega, \omega] = \frac{1}{2}[d\omega, \omega] - \frac{1}{2}[\omega, d\omega] = 0.$$

$\square$

Using the structure constants of the Lie Algebra $\mathfrak{g}$ associated with a fixed basis $X_1, ..., X_n$ we can rewrite the Cartan structural equation in a different way. To do this define real valued forms on P, such that $\Omega = \Omega^a X_a$ and $\omega = \omega^a X_a$.

$$\Omega^a = d\omega^a + c^a_{bc}\, \omega^b \wedge \omega^c$$

We now give a corollary of the Cartan structural equation (or its proof) and then use it to give an interesting geometrical characterization of the curvature on the principal bundle.

Corollary 21 *If X, Y are horizontal vector fields on P, then:*

$$\Omega(X, Y) = -\omega[X, Y]$$

For the promised geometrical interpretation of curvature let now X, Y be vector fields on M, that can be lifted to horizontal vector fields $\tilde{X}, \tilde{Y}$ on P. Using the previous corollary,

$$\Omega(\tilde{X}, \tilde{Y}) = -\omega[\tilde{(X)}, \tilde{(Y)}] =$$

$$= -\omega(v[\tilde{X}, \tilde{Y}]) =$$

$$= \omega\left(\widetilde{[X, Y]} - [\tilde{X}, \tilde{Y}]\right).$$

Since $h[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]}$, this means that the curvature evaluated on horizontal vectors, gives a measure of how much their commutator fails to be horizontal. Also as a corollary of these remarks and of the Frobenius Theorem we have

Corollary 22 *The distribution $p \longmapsto H_p$ defined by the connection is integrable iff the curvature of the connection vanishes.*

The same kind of reasoning used to prove the Cartan structural equation can be employed in proving a more general formula about the way how the covariant exterior derivative acts on $\tilde{\Omega}^k(P, V)$. In order to do so we introduce the following notation.
Let $\varphi \in \tilde{\Omega}^k(P, V)$ and $\psi \in \tilde{\Omega}^j(P, \mathfrak{g})$, and consider the Lie Algebra representation induced by $\rho : G \longrightarrow Aut(V)$. We then define $\psi \tilde{\wedge} \varphi \in \tilde{\Omega}^{k+j}(P, V)$, to be:

$$\begin{aligned} \psi \tilde{\wedge} \varphi(X_1, ..., X_{k+j}) = & \frac{1}{j!k!} \sum_{\sigma \in \Pi_{j+k}} sgn(\sigma) \cdot \\ & \cdot \ \rho(\psi(X_{\sigma(1)}, ..., X_{\sigma(j)}))(\varphi(X_{\sigma(j+1)}, ..., X_{\sigma(j+k)}))). \end{aligned} \tag{1.3}$$

Proposition 23 *Let $\varphi \in \tilde{\Omega}^k(P, V)$, then $D\varphi = d\varphi + \omega \tilde{\wedge} \varphi$.*

Notice that a special case of the above result is expressed by

Corollary 24 *For the adjoint representation $G \longrightarrow Aut(\mathfrak{g})$ and $\varphi \in \tilde{\Omega}^k(P, \mathfrak{g})$*

$$D\varphi = d\omega + [\omega, \varphi]. \tag{1.4}$$

Remark 25 *Be careful not to apply this to the connection form ω since it is not a horizontal form.*

Let us focus on the case of horizontal forms with values in $\mathfrak{g}$, i.e. $\tilde{\Omega}(P, \mathfrak{g})$. Now that we have a nice formula for D, a natural question arises:
When does the pair $(D, \tilde{\Omega}(P, \mathfrak{g}))$ define a chain complex

$$\cdots \longrightarrow \tilde{\Omega}^{k-1}(P, \mathfrak{g}) \longrightarrow \tilde{\Omega}^{k}(P, \mathfrak{g}) \longrightarrow \tilde{\Omega}^{k+1}(P, \mathfrak{g}) \longrightarrow \cdots$$

i.e. when does $D^2 = 0$?
In order to answer this, let's use the formula on the previous corollary to work out a formula for D^2 and then take our conclusions. Let $\varphi \in \tilde{\Omega}^k(P, \mathfrak{g})$, then

$$\begin{aligned} D^2\varphi &= d(D\varphi) + [\omega, D\varphi] \\ &= d(d\varphi + [\omega, \varphi]) + [\omega, d\varphi + [\omega, \varphi]] \\ &= d[\omega, \varphi] + [\omega, d\varphi] + [\omega, [\omega, \varphi]] \end{aligned}$$

Now we can easily check that $d[\omega, \varphi] = [d\omega, \varphi] - [\omega, d\varphi]$, and using the Jacobi identity for graded Lie algebras we have that $[\omega, [\omega, \varphi]] = -\dfrac{1}{2}[\varphi, [\omega, \omega]]$. Hence,

$$\begin{aligned} D^2\varphi &= [d\omega, \varphi] - \frac{1}{2}[\varphi, [\omega, \omega]] \\ &= [d\omega + \frac{1}{2}[\omega, \omega], \varphi] = [\Omega, \varphi]. \end{aligned}$$

For completeness we will state here this formula for D^2 as a corollary for the formula 1.4 for D.

Corollary 26 *For the adjoint representation $G \longrightarrow Aut(\mathfrak{g})$ and $\varphi \in \tilde{\Omega}^k(P, \mathfrak{g})$*

$$D^2\varphi = [\Omega, \varphi]. \tag{1.5}$$

Now we can directly read the answer to our question from this formula:

Theorem 27 *The pair $(D, \tilde{\Omega}(P, \mathfrak{g}))$ defines a chain complex if and only if the curvature of the connection vanishes.*

1.3 Holonomy in a Principal Bundle

1.3.1 Holonomy Groups

Let (π, P, M) be a principal bundle with structure group G, ω a connection on P, and $\gamma : I \longrightarrow M$ a sectionally C^1 path on M. We would like to obtain the **horizontal lift** of γ, i.e. a path $\widetilde{\gamma} : I \longrightarrow P$ whose velocity is horizontal or, in other words, such that $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}}(t)$.
With a finite open cover of γ, such that in each open set $\dot{\gamma}(t)$ can be extended to a vector field X, the uniqueness of the horizontal lift of X to $\tilde{X}$ and the Picard-Lindelöf theorem guarantee the existence and uniqueness of the horizontal lift of piecewise differentiable paths on M.
Given $\gamma : I \longrightarrow M$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ we can create a map between fibers

$$P(\gamma) : \pi^{-1}(x_0) \longrightarrow \pi^{-1}(x_1)$$

$$p_0 \longmapsto \widetilde{\gamma(1)}$$

that to each p_0 gives the time-one map of $\tilde{\gamma}$ starting at p_0. This map is called the **parallel transport** along γ.

Proposition 28 *The map $P(\gamma) : \pi^{-1}(x_0) \longrightarrow \pi^{-1}(x_1)$ is an isomorphism.*

Proof: By the definition of connection we know that the horizontal subspaces are invariant under right translations, so two lifts of γ, $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ starting respectively at p and pg are related by $\widetilde{\gamma_2} = R_g \widetilde{\gamma_1}$. This implies that $R_g \circ P(\gamma) = P(\gamma) \circ R_g$ and hence $P(\gamma)$ is injective and surjective. □

Remark 29 *The following hold*

1. *The parallel transport along $\gamma(t)$ has an inverse which is the parallel transport along $\gamma(1-t)$.*
2. *The composition of parallel transports is the parallel transport along concatenated paths. In formulas this means that $P(\gamma) \circ P(\beta) = P(\gamma \cdot \beta)$.*

The composition in the second remark is always possible if we restrict ourselves to loops based at a given point $x \in M$, hence denoting $C(x)$ as the space of loops based at x and $C^0(x)$ as the space of homotopically trivial loops based at x, the following definition makes sense.

Definition 30 *In the setup of the above constructions, we define the* ***holonomy group*** *of ω based at x, by*

$$\Phi(x) = \{P(\gamma) | \gamma \in C(x)\}$$

and the ***restricted holonomy group*** *of ω based at x, by*

$$\Phi^0(x) = \left\{P(\gamma) | \gamma \in C^0(x)\right\}.$$

Both these groups can be realized as subgroups of the structure group G, but for that we must think of them as based at points $p \in P$ rather than points $x \in M$. So, let $p \in \pi^{-1}(x)$, then $P(\gamma)(p) \in \pi^{-1}(x)$ as well, hence there is a $g \in G$ such that $P(\gamma)(p) = pg$. Since $P(\gamma) \circ P(\beta)(p) = P(\gamma)(ph) = P(\gamma)(p)h = pgh$ we have an injective group homomorphism of $\Phi(x)$ into G, whose image is

$$\Phi(p) = \{g \in G | \ \exists \gamma \in C(x) \text{ with } P(\gamma)(p) = pg\} \subset G.$$

The same construction is valid to the restricted holonomy group and we end up with $\Phi^0(p) \subset \Phi(p) \subset G$.

Proposition 31 *In the previous setting, the following hold*

1. *If q and p are on the same fibre, i.e. $q = pg \in P$ then $\Phi(q) = ad(g^{-1})\Phi(p)$ and $\Phi^0(q) = ad(g^{-1})\Phi^0(p)$.*
2. *If p and q can be joined by a horizontal curve then $\Phi(q) \simeq \Phi(p)$ and $\Phi^0(q) \simeq \Phi^0(p)$.*

Proof:

1. Let $h \in \Phi(q)$, then it exists a path γ such that $qh = P(\gamma)(q) = P(\gamma)(ph) = (P(\gamma)(p))h$, hence $pgh = (P(\gamma)(p))h$ and $ghg^{-1} \in \Phi(p)$.
2. Let $\tilde{\beta}$ be a horizontal curve joining p and q, then if $g \in \Phi(q)$ we have a path γ, such that $qg = P(\gamma)(q)$, then since $R_g \circ P(\beta^{-1}) = P(\beta^{-1}) \circ R_g$ we have that parallel transport along $\beta^{-1} \cdot \gamma \cdot \beta$ gives $g \in \Phi(p)$.

□

In the special case of M being connected, then any two fibres of P can be joined by a horizontal curve implying that the holonomy groups are all conjugate with each other, and hence isomorphic. We know by now that the holonomy groups are subgroups of the structure group G. The next theorem shows that in fact they are smooth, being Lie subgroups of G with identity component the restricted holonomy group.

Theorem 32 *Let (π, P, M) be a principal bundle with structure group G and M connected. If ω is a connection on P, then:*

1. *$\Phi^0(p)$ is a connected Lie subgroup of G.*

2. *$\Phi^0(p)$ is a normal subgroup of $\Phi(p)$ and $\Phi(p)/\Phi^0(p)$ is countable.*

Proof: See [24]. □

1.3.2 Connections induced by Morphisms

In this section (π, P, M) and(π', P', M') will be principal bundles with structure groups G and G' respectively, and $f : P' \longrightarrow P$ a principal bundle morphism. We shall also denote by f the maps induced in the bases $f : M \longrightarrow M'$ and on the fibres (the structure groups) $f : G \longrightarrow G'$. We explore the cases when a morphism between such bundles can carry connections between them and its consequences on the respective holonomies.

Proposition 33 *If f induces a diffeomorphism $f : M' \longrightarrow M$ on the bases, and we have a connection in P' with ω', Ω' being the connection and curvature forms, then:*

1. *There exists a unique connection ω on P, such that f maps the horizontal subspaces of P' into the horizontal subspaces of P.*

2. *The connection and curvature forms of this connection ω, Ω, satisfy $f^*\omega = f_* \circ \omega'$ and $f^*\Omega = f_* \circ \Omega'$.*

3. *$f : G' \longrightarrow G$ maps $\Phi(p')$ surjectively onto $\Phi(f(p'))$*

Proof:

1. To construct the connection we define the horizontal subspaces.
Since $f : M' \longrightarrow M$ is a difeomorphism, then for all $p \in P$ we have $x' \in M'$, such that $f(x') = \pi(p)$ and hence we can pick up $p' \in \pi^{-1}(x')$ and we have that $f(p') = pg$, for some $g \in G$. Then we define
$$H_p = (R_g)_* f_* H'_{p'}$$
It can be checked that the horizontal subspaces are indeed well defined and define a connection.

2. To check that this formulas hold we treat the cases of X' being vertical and horizontal separately and the formula hold by linearity.
In the case of X' being vertical, we can write $X' = (A')^*$, and hence $\omega(f_*(A')^*) = \omega((f_*A')^*) = f_*A' = f_*\omega'((A')^*)$. The case of X' being horizontal is easier since f_*X' is also horizontal by a) and both sides vanish.
For the curvature this is a direct consequence of the formula for the connection 1-forms and the Cartan structural equation.

3. Let $\tilde{\gamma}$ be the lift of γ starting at p, with $\pi(p) = x$. Since $f : M' \longrightarrow M$ is a diffeomorphism it exists x' such that $f(x') = x$ and we have that $f^{-1} \circ \gamma$ is a loop based at x', it's horizontal lift $\widetilde{f^{-1} \circ \gamma}$ represents an element of the holonomy group and by the first part it is mapped by f to a horizontal curve $f(\widetilde{f^{-1} \circ \gamma})$ that coincides with γ by uniqueness of the horizontal lift of γ.

□

In the case when the bases coincide we say that the connection ω in P is reducible to the connection ω' in P'.

Proposition 34 *If $f : G' \longrightarrow G$ is an isomorphism of the structure groups, and we have a connection in P with ω, Ω being the connection and curvature forms, then:*

1. *There exists a unique connection ω' on P', such that f maps the horizontal subspaces of P' into the horizontal subspaces of P.*

2. *The connection and curvature forms of this connection ω', Ω', satisfy $f^*\omega = f_* \circ \omega'$ and $f^*\Omega = f_* \circ \Omega'$.*

3. *$f : G' \longrightarrow G$ maps $\Phi(p')$ injectively into $\Phi(f(p'))$.*

Proof:

1. In this case we define the connection on ω' on P' by it's connection 1-form. Since $f_* : \mathfrak{g}' \longrightarrow \mathfrak{g}$ is an isomorphism we can define
$$\omega' = (f_*^{-1}) \circ (f^*\omega)$$
We can check that ω' defines indeed a connection in P', the horizontal subspaces of P' are mapped into the horizontal subspaces of P, since $Ker(\omega) = f_* Ker(\omega)$.

2. Follows from the definition of ω' and the Cartan Structural Equation.

3. $f : \Phi(p') \longrightarrow \Phi(f(p'))$ since horizontal curves are mapped to horizontal curves, and this map is injective since it is the restriction of $f : G' \longrightarrow G$ which is injective, in fact an isomorphism by hypothesis.

□

In this case we say that ω' is **induced** by ω.

Remark 35 *If $G = G'$ and $f : G' \longrightarrow G$ is the identity map on G, then the theorem gives $\omega' = f^*\omega$ as a naturally defined connection on the pullback bundle f^*P.*

Theorem 36 *(Reduction Theorem) Let (π, P, M) be a principal bundle with structure group G and M connected, ω a connection on P and $p_0 \in P$. Then the set*

$$P(p_0) = \{p \in P | \; \exists \textit{ an horizontal path } \gamma \textit{ with } \gamma(0) = p_0 \textit{ and } \gamma(1) = p\} :$$

1. *Is a reduced subbundle of P, with structure group $\Phi(p_0)$.*

2. *The connection ω on P is reducible to a connection in $P(p_0)$.*

Proof:

1. The first thing to do is to prove that $P(p_0)$ is a reduced subbundle of P with structure group $\Phi(p_0)$, for that we shall use the

   Lemma 37 *If $Q \subset P$, H is a subgroup of G, and the following conditions are satisfied:*

   a) $\pi : P \longrightarrow M$ maps Q surjectively onto M.

   b) $R_h Q = Q \; \forall h \in H$.

   c) $\forall p, q \in Q$ with $\pi(p) = \pi(q)$ there exists $h \in H$ such that $p = qh$.

   d) $\forall x \in M$, $\exists$ a neighbourhood $x \in U$ and a section $\sigma : U \longrightarrow P$ such that $\sigma(U) \subset Q$.

Then, (π, Q, M) is a principal bundle with structure group H.

Proof:(of the lemma) The proof of the lemma amounts to constructing local trivializations for Q. That can be done using the neighbourhoods U and the local sections $\sigma : U \longrightarrow P$ to define something like an origin for the trivialization. More explicitly we have that if $q \in Q$, then there exists $x \in M$ with $q \in \pi^{-1}(x)$, and we also know that $q = \sigma(x)h$, for some $h \in H$. The definition of the local trivialization $\Psi : \pi^{-1}(U) \cap Q \longrightarrow U \times H$ is $q = \sigma(x)h \longmapsto (x, h)$. It can be seen that such Ψ is indeed an isomorphism and that we can equip Q with a differentiable structure for which Ψ is a diffeomorphism and Q is a submanifold of P. □

Now we need to check that the pair $P(p_0), \Phi(p_0)$ satisfies the conditions 1) to 4).
The only nontrivial part is showing that $\pi : P(p_0) \longrightarrow M$ is surjective which is a consequence of M being connected and hence we can join every fibre with the fibre $\pi^{-1}(p_0)$. The existence of the local sections also needs some construction and amounts to lift a local coordinates centred in $\pi(p_0)$ to P.

2. For this part we will also reduce our proof to

Lemma 38 *Let (π, Q, M) with structure group H be a subbundle of (π, P, M) with structure group G and ω a connection on (π, P, M). If $\forall p \in Q$ we have $H_p \subset T_pQ$, then ω is reducible to a connection in Q.*

Proof:(of the lemma) Just set $H'_p = H_p$. □

Using this lemma we just need to check that H_p is tangent to $P(p_0)$, for $p \in P(p_0)$, but that is a consequence of

$$H_p = span\left\{\widetilde{\frac{\partial}{\partial x^1}}, ..., \widetilde{\frac{\partial}{\partial x^n}}\right\}$$

and all of the $\widetilde{\frac{\partial}{\partial x^1}}$ admit horizontal integral curves.

□

Thus we know that when M is connected $P(p_0)$ is in fact a bundle known as the **holonomy bundle through** p_0. When p and q can be joined by a horizontal curve in P then $P(p) = P(q)$, and otherwise we have that $P(p) \cap P(q) = \emptyset$, hence P is the disjoint union of holonomy bundles. Also notice that when M is connected we can join every fibers by horizontal curves and we have a diffeomorphism $R_g : P(p) \longrightarrow P(pg)$.

1.3.3 Holonomy Theorems

In this section we will see some important theorems regarding holonomy in principal bundles, giving us a concrete relation between structure groups, connections and holonomy bundles. Throughout this section (π, P, M) will be a principal bundle with structure group G, equipped with a connection ω.
The first result that we will see is the Ambrose-Singer theorem which gives a description of the holonomy obtained from infinitesimally small loops, by relating it to the curvature of the connection.
To motivate the theorem, pick $u \in P$ and in a sufficiently small open neighbourhood in M take X, Y commuting vector fields (we could use $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$). Let ϕ^s_X, ϕ^s_Y be their fluxes and define for each t the loop $\gamma_t : [0; 4\sqrt{t}] \longrightarrow M$ by

$$\gamma_t(s) = \begin{cases} \phi^s_X & \text{if } s \leq \sqrt{t}, \\ \phi^{s-\sqrt{t}}_Y \circ \phi^{\sqrt{t}}_X & \text{if } \sqrt{t} < s \leq 2\sqrt{t}, \\ \phi^{s-2\sqrt{t}}_{-X} \circ \phi^{\sqrt{t}}_Y \circ \phi^{\sqrt{t}}_X & \text{if } 2\sqrt{t} < s \leq 3\sqrt{t}, \\ \phi^{s-3\sqrt{t}}_{-Y} \circ \phi^{\sqrt{t}}_{-X} \circ \phi^{\sqrt{t}}_Y \circ \phi^{\sqrt{t}}_X & \text{if } 3\sqrt{t} < s \leq 4\sqrt{t}. \end{cases}$$

Then its horizontal lift starting at $u \in P$ is the path $\tilde{\gamma_t} : [0, 4\sqrt{t}] \longrightarrow P$ that induces the following map in holonomy

$$\tilde{\gamma_t}(4\sqrt{t}) = \phi_{\tilde{X}}^{-\sqrt{t}} \circ \phi_{\tilde{Y}}^{-\sqrt{t}} \circ \phi_{\tilde{X}}^{\sqrt{t}} \circ \phi_{\tilde{Y}}^{\sqrt{t}}$$

Letting t vary, this defines a curve $g(t)$ in the Holonomy Group $\Phi(u)$ such that $\tilde{\gamma_t}(4\sqrt{t}) = ug(t)$ and $g(0) = e$ the identity on G, hence

$$\frac{d}{dt}|_{t=0} ug(t) = A^*$$

with A in the Lie algebra of $\Phi(u)$. Let us compute this.

$$A^* = \frac{d}{dt}|_{t=0}\tilde{\gamma_t}(4\sqrt{t}) = \frac{d}{dt}|_{t=0}\phi_{\tilde{X}}^{-\sqrt{t}} \circ \phi_{\tilde{Y}}^{-\sqrt{t}} \circ \phi_{\tilde{X}}^{\sqrt{t}} \circ \phi_{\tilde{Y}}^{\sqrt{t}} = [\tilde{X}, \tilde{Y}].$$

So $A = \omega(A^*) = \omega([\tilde{X}, \tilde{Y}]) = -\Omega(\tilde{X}, \tilde{Y})$ by the corollary 1. This result is indeed more general and the curvature really describes infinitesimal holonomy. The following theorem sates it.

Theorem 39 *(Ambrose-Singer) Let M be a connected manifold and ω a connection in (π, P, M) with curvature Ω. Then for $u \in P$, the Lie algebra of the holonomy group $\Phi(u)$ is*

$$\{\Omega_v(X, Y) \in \mathfrak{g} | \quad v \in P(u) \text{ and } X, Y \in H_v\}$$

where $P(u)$ is the holonomy bundle of ω through u.

For a proof check [24] where it can be also found a very clever proof of the following theorem that we state here as a curiosity.

Theorem 40 *If P is connected and $dimM \geq 2$ then there exists a connection in P such that for all $p \in P$ the bundles P and $P(p)$ coincide.*

Notice that this curiosity has however some important consequences, for example it allows us to realize any connected Lie Group as the holonomy group of a connection in the trivial bundle $M \times G$ with $dimM \geq 2$

1.4 Flat Connections

Definition 41 *A connection in the principal bundle (π, P, M) with structure group G is called a **flat connection** if its curvature vanishes.*

We now define the **canonical flat connection** in the trivial bundle $P = M \times G$ by defining its horizontal subspace at $p = (x, a)$ as

$$H_p = T_p(M \times a).$$

In what follows let $\pi_2 : M \times G \longrightarrow G$ be the projection on the second factor.
In order to get a more useful way of dealing with this connection let us find its 1-form. If $X_u \in T_pP = T_p(M \times a) \oplus T_aG$, then $X_p = Y + A_a$ with $Y \in T_p(M \times a)$ and $A_a \in T_aG$, then

$$\omega(X_p) = \omega(A_a) = dL_{a^{-1}}A_a = \theta(A_a) = \theta(d\pi_2 Xpu) = \pi_2^*\theta(X_p)$$

where θ is the **Maurer-Cartan form** on G, i.e. $\theta(A) = A$, $\forall A \in \mathfrak{g}$. We conclude from this calculation that the canonical flat connection 1-form is $\omega = \pi_2^*\theta$. We now have to check that this connection deserves being called flat. To do so we have to show that its curvature vanishes, but this is a direct consequence of the **Maurer-Cartan equation** $d\theta = -\frac{1}{2}[\theta, \theta]$.

$$\begin{aligned}\Omega = d\omega + \frac{1}{2}[\omega, \omega] &= d(\pi_2^*\theta) + \frac{1}{2}[\pi_2^*\theta, \pi_2^*\theta] \\ &= \pi_2^*\left(d\theta + \frac{1}{2}[\theta, \theta]\right) = 0.\end{aligned}$$

In fact this canonical flat connection is really important and as we will see it is a prototype of all flat connections locally.

Theorem 42 *A connection in a principal bundle (π, P, M) with structure group G is flat iff $\forall x \in M$ there is a neighbourhood U of x such that the connection induced in $\pi^{-1}(U)$ is isomorphic to the canonical flat connection in $U \times G$.*

Proof: If the connection induced in $\pi^{-1}(U)$ is isomorphic to the canonical flat connection via a map $f : \pi^{-1}(U) \longrightarrow U \times G$, then by the theorems on mappings of connection we have that their curvature are related by $\Omega = f^*\Omega_{flat} = 0$ and we are done.
Conversely if $\Omega = 0$, then for $x \in M$ choose a neighbourhood U sufficiently small to be simply connected, hence for $p \in \pi^{-1}(U)$ we have $\Phi(p, U) = \Phi^0(p, U)$. In this way $\Phi(p, U)$ only has the identity component, being connected. By the Ambrose-Singer theorem the Lie Algebra of $\Phi(p, U)$ is 0 and since $\Phi(p, U)$ is connected we have that $\Phi(p, U) = \{e\}$. $\square$

As a consequence of the second part of the proof in a principal bundle (π, P, M) with simply connected base we have $\Phi(p) = \Phi^0(p)$ and hence the same reasoning proves

Corollary 43 *Let (π, P, M) be a principal bundle with structure group G where the base M is simply connected. Then $P \cong M \times G$ and all connections in P are isomorphic to the canonical flat connection in $M \times G$.*

We will now see how flat connections give us representations of the fundamental group.
Let M be a connected manifold and (π, P, M) a principal bundle over M with structure group G and equipped with a flat connection ω.
Fix a point $p_0 \in P$ and consider $P(p_o)$ the holonomy bundle through p_0, by the reduction theorem it is a principal bundle with structure group $\Phi(p_0)$, which by the Ambrose-Singer theorem is discrete ($\Omega = 0$). Since $P(p_0)$ is connected, it is a covering of M and for $x_0 = \pi(p_0)$ we define the following map

$$\Pi(x_0, M) \longrightarrow \Phi(p_0)$$

$$\gamma \longmapsto P(\gamma),$$

that to each class of paths associate the element of the holonomy group associated with parallel transport along one of those paths. This map is indeed well defined since we are using a flat connection and hence by the Ambrose-Singer theorem $\Phi^0(p_0) = \{e\}$ and hence homotopic paths give the same parallel transport. In fact this map is a surjective group homomorphism by the properties of parallel transport (see 4.1).

1.5 Connections and Holonomy in a Vector Bundle

1.5.1 Connections

We will use this section to transport some of the previous notions into the context of vector bundles.
We will always be thinking of vector bundles (π, E, M) as associated bundles of a principal bundle with structure group G, where V is the standard fibre of E and $\rho : G \longrightarrow GL(V)$ is a representation of G on V.

For vector bundles, connections are introduced in order to differentiate sections, but in the framework we have been describing we already have a way to do so. Notice that, as stated in remark 16, we have an identification of the sections of (π, E, M) with $C(P, V)$ and when we have a connection on (π, P, M) we know how to differentiate elements of $C(P, V)$ with respect to vector fields on the base (we will have to lift them). In order to proceed with the described strategy we will state:

Proposition 44 *Let (π, P, M) be a principal bundle with structure group G, (π, E, M) an associated vector bundle with standard fiber V. Then the sections of the associated vector bundle are in one to one correspondence with $C(P, V)$*

Proof: By the construction of the associated bundle we know that $E = P \times_G V$ and hence any element of E_p reached by a section of (π, E, M) is of the form $[(u, \tau(u))]$ with $h : P \longrightarrow V$ and since it must not depend on $u \in \pi^{-1}(p) \subset P$ we have that

$$[(u, \tau(u))] = [(ug, \tau(ug))]$$

and (remember the way how G acts on $P \times V$) we conclude that $[(ug, \tau(ug))] = [(ug, \rho(g^{-1})\tau(u))]$, so τ is G-equivariant and hence $\tau \in C(P, V)$.
Conversely, given $\tau \in C(P, V)$ we can define a section of the vector bundle by:

$$s : M \longrightarrow E = P \times_G V$$

$$p \longmapsto [(u, \tau(u))]$$

This maps are easily seen to be inverses of each other. □

Following the previous proof we shall now give a name to those maps. Let $T : C(P, V) \longrightarrow \Gamma(M, E)$, be the map that to each element of $C(P, V)$ associates a section of the associated vector bundle. Now we are able to differentiate sections of the vector bundle using this map.

Definition 45 *Let $s : M \longrightarrow E = P \times_G V$ be a section of the associated vector bundle, and X a vector field in M, then we define the **covariant derivative** of s along X, to be section of E given by*

$$\nabla_X s = T((DT^{-1}(s))(\tilde{X}))$$

where D is the covariant exterior derivative and $\tilde{X}$ the horizontal lift of X.

This definition is too abstract and we shall now describe connections is a vector bundle in an easier way. In fact the above definition shows that a connection in a vector bundle gives a mapping

$$\nabla : \Gamma(M, E) \longrightarrow \Omega^1(M) \otimes E$$

that is Leibnitzian, i.e. for all $f \in C^\infty(M)$ and section s,

$$\nabla(fs) = df \otimes s + f\nabla s$$

This shows that a connection is local, i.e. it only depends on the values of the sections in a neighbourhood of any point. So with a local trivialization we can write a local section $s : U \longrightarrow P$ as $s = s^i e_i$, where e_i is a local basis of sections, then

$$\nabla_X s = (ds^i(X) + \theta^i_j(X)s^j)e_i$$

where $\theta^i_j \in \Omega^1(M)$ are the coeficients of $\theta \in \Omega^1(M) \otimes E \otimes E^*$. When we have a connection in the principal bundle used to create the associated bundle we have a canonical choice for this θ's as the local connection 1-forms $\rho(\omega_\alpha)$ in the cover $\{U_\alpha\}$. In fact, this agrees with the definition given above for the connection induced in a vector bundle by the connection on the principal bundle.
With this discussion we can summarize things in the following way:

Summary 46 *If we have a principal bundle P with a connection whose local 1-forms are ω_α in an open cover $\{U_\alpha\}$, and E is an associated vector bundle, then we can introduce a connection on E, by representing the local 1-forms ω_α and setting*

$$\nabla_X s = ds(X) + \omega_\alpha(X)s. \tag{1.6}$$

Now it is trivial to show the following

Proposition 47 *Let $s, s_1, s_2 : M \longrightarrow E = P \times_G V$ be sections of the associated vector bundle, X, Y vector fields in M and $f \in C^\infty(M)$, then:*

1. $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$
2. $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$
3. $\nabla_{fX} s = f\nabla_X s$
4. $\nabla_X fs = (X \cdot f)s + f\nabla_X s$

There are other ways of transporting the notion of connection on the principal bundle to the associated vector bundles. We now describe one of them. This will introduce the notion of parallel transport in the a vector bundle and improve our geometric intuition on the meaning of covariant differentiation.
Let $v \in E$ we define the **vertical subspace** V_v to be the tangent space to the fiber, and the **horizontal subspace** H_v to be the image of the horizontal subspace defined by the connection on P under the map $P \times V \longrightarrow E = P \times_G V$, it can be easily checked that this is well defined.
A path in E is said to be horizontal if its velocity is a horizontal vector, as in the principal bundle case we can use the uniqueness theorem of Picard on the solution of ODE's to state that we can uniquely lift a path on the base $\gamma : I \longrightarrow M$ to a horizontal path $\tilde{\gamma} : I \longrightarrow E$ that projects on γ. We will call $\tilde{\gamma}$ the **horizontal lift** of γ.
This setting is now suitable to define the notion of **parallel transport** along a path $\gamma : I \longrightarrow M$, as a linear map

$$P(\gamma(t)) : E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$$

that to each $v \in E_{\gamma(0)}$ associates $\tilde{\gamma}(t) \in E_{\gamma(t)}$, where $\tilde{\gamma} : I \longrightarrow E$ is the only lift of γ starting at v.
The parallel transport gives another way of understanding covariant differentiation. We will skip the proof of the following

Theorem 48 *Let X be a vector field in M and s a section of E, then:*

$$\nabla_X s = \frac{d}{dt}|_{t=0} P(\gamma(t))^{-1}(s(\gamma(t))).$$

In several books, connections are first introduced on vector bundles via covariant derivatives which are defined axiomatically. That is the converse treatment of the one followed here.

1.5.2 Tensor Bundles and Riemannian Holonomy

Associated to every n dimensional manifold M there is a principal $GL(n, \mathbb{R})$-bundle called the **frame bundle** and usually denoted F, its construction is as follows.
The fibre over a any $p \in M$ is the set of frames for T_pM and in fact $GL(n, \mathbb{R})$ acts freely and transitively on this set, so that both are naturally identified. The local triviality is easily checked by fixing local frames on a family of open sets, which are obviously related by elements of $GL(n, \mathbb{R})$ on the intersections.

Given a finite dimensional vector space V with a representation of $GL(n,\mathbb{R})$ we canonically have an associated vector bundle $E = F \times_{GL(n,\mathbb{R})} V$ with standard fibre V. Using a Riemannian metric on M we have a reduction of the frame bundle to a $O(n)$ bundle. In this way we can produce for example:

1. with $V = \mathbb{R}^n$ we get the tangent bundle TM, whose sections are vector fields.
2. with $V = (\mathbb{R}^n)^*$ we get the cotangent bundle T^*M, whose sections are 1-forms.
3. with $V = \bigwedge_{i=1}^k (\mathbb{R})^*$ we get $\bigwedge_{i=1}^k T^*M$, whose sections are k-forms.

In this way, a connection in the frame bundle F also gives rise to a connection in this associated vector bundles by the constructions of the previous subsections.

In the case of TM the situation is particularly symmetric since a connection differentiates vector fields with respect to vector fields. Fix a metric g for M, a connection ∇ is said to be **compatible with the metric** g, if $\nabla g = 0$. Moreover the symmetry of the situation allows us to define the **torsion tensor** T, of the connection by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and the connection is said to be torsion free if the torsion tensor vanishes on M.

Theorem 49 *(Levi-Civita) Let (M,g) be a Riemannian manifold, then it there is a unique compatible and torsion free connection on TM, called the* ***Levi-Civita connection*** *on (M,g).*

The curvature of the Levi-Civita connection is denoted R and called **Riemannian curvature**. Now, on a Riemannian manifold (M,g), we can think about the holonomy of the Levi-Civita connection ∇ which is usually called **Riemannian holonomy**, we will also denote the Riemannian holonomy group by Φ and by Φ^0 in the restricted case, remembering that this last one is obtained by considering only trivial loops and is in fact the connected component of the identity of Φ.
A tensor field ϕ is said to be **parallel** if $\nabla\phi = 0$, and our goal is now to describe a result of Berger which relates the holonomy group with the structures on the manifold that are parallel.

We shall restrict ourselves to

- Simply connected manifolds, then $\Phi = \Phi^0$ is a connected Lie group.
- A Riemannian manifold is said to be **irreducible** if the representation of the Riemannian holonomy group is irreducible.
  By a theorem of de Rham, every Riemannian manifold can be decomposed as a product of irreducible Riemannian manifolds, being the Riemannian holonomy group the product of the Riemannian holonomy groups of the irreducible components of the first decomposition. So we may also restrict our attention to irreducible Riemannian manifolds.
- Our final restriction is to consider locally non-symmetric Riemannian manifolds. **Locally symmetric** Riemannian manifolds are manifolds with covariant constant Riemannian Curvature, i.e. $\nabla R = 0$. Cartan proved that these manifolds have the same Riemannian holonomy as symmetric manifolds and used Lie group theory to classify them. We will skip this.

Now, with all restrictions made, we will state.

Theorem 50 *(Berger) The Riemannian holonomy group Φ of an irreducible, simply-connected and locally non-symmetric Riemannian manifold M is one of the following:*

Φ	*dim(M)*	*Name*	*Structures*
SO(n)	n	*Orientable manifold*	*Orientation*
U(n)	*2n*	*Kähler manifold*	*Kähler*
SU(n)	*2n*	*Calabi-Yau manifold*	*Kähler + Ricci-Flat*
Sp(n)·Sp(1)	*4n*	*Quaternion-Kähler manifold*	*Einstein*
Sp(n)	*4n*	*Hyperkähler manifold*	*Hyperkähler*
G2	*7*	*G2 manifold*	*Ricci-flat*
Spin(7)	*8*	*Spin(7) manifold*	*Ricci-flat*

The proof of this result amounts to use the Ambrose-Singer theorem to identify the Lie algebra of the holonomy group with values of the Riemannian curvature. Then by studying the symmetries of the Riemannian curvature it is possible to reduce the possibilities to the ones in the list.

Remark 51
- *In fact all the groups in Berger's list are realised as the holonomy groups of some Riemannian manifold.*
- *By de Rham's theorem, the holonomy group of any Riemannian manifold is a product of the ones on Berger's list and the ones on Cartan's list.*

1.6 Characteristic Classes

In this section our goal is to find invariants of a given principal bundle (π, P, M) with structure group G. The idea is to construct some cohomology classes of the base M with the help of a connection ω in P, or its curvature Ω to be more precise. In the end we show that those cohomology classes do not depend on the choice of connection.

Let $\{U_\alpha\}$ be a open cover of M where P trivializes. As in section 1.2.1 we can define local sections $\sigma_\alpha : U_\alpha \to P$ in order to pull the connection form down to U_α and get locally defined forms $\omega_\alpha = \sigma_\alpha^* \omega \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$, that do not match to give globally defined forms on M, since as we saw on 3.1. on the overlap $U_\alpha \cap U_\beta$ we have

$$\omega_\beta = ad(\psi_{\alpha\beta}^{-1})\omega_\alpha + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta} \tag{1.7}$$

where $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are the transition functions of the bundle.
This is a good time to introduce the analogous construction for the curvature by defining $\Omega_\alpha = \sigma_\alpha^* \Omega \in \Omega^2(U_\alpha) \otimes \mathfrak{g}$ and checking that on the overlap $U_\alpha \cap U_\beta$, we get:

$$\begin{aligned}
\Omega_\beta &= \sigma_\beta^* \Omega \\
&= (R_{\psi_{\alpha\beta}} \sigma_\alpha)^* \Omega \\
&= \sigma_\alpha^* R_{\psi_{\alpha\beta}}^* \Omega \\
&= ad(\psi_{\alpha\beta}^{-1}) \sigma_\alpha^* \Omega \\
&= ad(\psi_{\alpha\beta}(x)^{-1}) \Omega_\alpha.
\end{aligned}$$

Where we used the fact that the curvature is a horizontal form. We proved

Proposition 52 *In the previous setup, if $U_\alpha \cap U_\beta \neq \emptyset$ then:*

$$\Omega_\beta = ad(\psi_{\alpha\beta}^{-1})\Omega_\alpha. \tag{1.8}$$

This proposition tells us that if we want to construct globally well defined cohomology classes of M from the curvature we have to introduce some kind of Ad(G)-invariant functions. This motivates us to introduce $I^k(G)$ as the set of polynomials

$$P : \mathfrak{g} \times ... \times \mathfrak{g} \longrightarrow \mathbb{R}$$

that are multilinear, symmetric and Ad(G)-invariant. By this last condition we mean that $\forall g \in G$,

$$P(ad(g)X_1, ..., ad(g)X_k) = P(X_1, ..., X_k).$$

Define the **Weil Algebra** as

$$I(G) = \bigoplus_{k=0}^{+\infty} I^k(G),$$

which is a graded ring when equipped with the product defined by: for $P_1 \in I^j(G)$ and $P_2 \in I^k(G)$

$$P_1P_2(X_1, ..., X_{j+k}) = \frac{1}{(j+k)!} \sum_{\sigma \in \Pi_{j+k}} P_1(X_{\sigma(1)}, ..., X_{\sigma(j)})P_2(X_{\sigma(j+1)}, ..., X_{\sigma(j+k)}),$$

where Π_n is the group of symmetric permutations of order n. It can be shown that it there is a one-to-one correspondence between $I^k(G)$ and the set of homogeneous, degree-k, Ad(G)-invariant polynomials in $\mathfrak{g}$.
In fact, giving $\tilde{P}$ a homogeneous, degree-k, Ad(G)-invariant polynomial function in $\mathfrak{g}$ and fixing a basis $\{\xi^1, ..., \xi^r\}$ of $\mathfrak{g}^*$

$$\tilde{P}(X) = \sum_{i_1,...,i_k} a_{i_1...i_k} \xi^{i_1}(X)...\xi^{i_k}(X)$$

where $a_{i_1...i_k}$ are symmetric constants, we can construct an element of $I^k(G)$ given by

$$P(X_1, ..., X_k) = \sum_{i_1,...,i_k} a_{i_1...i_k} \xi^{i_1}(X_1)...\xi^{i_k}(X_k).$$

Remark 53 *Let $Y, X_1, ..., X_k \in \mathfrak{g}$. Then, for $t \in \mathbb{R}$ we have $P(X_1, ..., X_k) = P(Ad(e^{tY})X_1, ..., Ad(e^{tY})X_k)$ by Ad(G)-invariance and taking $\frac{d}{dt}$ at $t = 0$ we have:*

$$P([Y, X_1], X_2, ..., X_k) + P(X_1, [Y, X_2], ..., X_k) + ... + P(X_1, X_2, ..., [Y, X_k]) = 0$$

This will be useful in future computations.

Definition 54 *Let (π, P, M) be a principal bundle with structure group G, equipped with a connection whose curvature is Ω, and have local representatives Ω_α in U_α, for each α. Given $P \in I^k(G)$ we define*

$$\begin{aligned} P(\Omega_\alpha) &= P(\Omega_\alpha, ..., \Omega_\alpha) \\ &= \sum_{i_1,...,i_k} \Omega_\alpha^{i_1} \wedge ... \wedge \Omega_\alpha^{i_k} P(X_{i_1}, ..., X_{i_k}) \end{aligned}$$

Here we understand the X_i's as forming a basis of the Lie algebra, so that $\Omega_\alpha = \sum_i \Omega_\alpha^i X_i$.

Notice that this is a consistent definition, since we can define these forms for all U_α, as P is Ad(G)-invariant and since the local representatives of curvature transform by an adjoint action, we conclude that they coincide in the overlap $U_\alpha \cap U_\beta$ and give a globally well defined $2k$-form on M, which we will call $P(\Omega)$. But we are searching for cohomology classes, so the questions now are:

1. What else do we have to do in order to get closed $2k$-forms on M?
2. And to make them independent of the connection?

The next theorem answers the question.

Theorem 55 *(Chern-Weil) In the previous setup we have that:*

1. *$dP(\Omega) = 0$ and hence $[P(\Omega)] \in H^{2k}(M)$.*

2. *$[P(\Omega)]$ is independent of the connection.*

Proof:

1. When we write $P(\Omega)$ we actually mean $P(\Omega, ..., \Omega)$ for an $\mathrm{Ad}(G)$-invariant, homogeneous polynomial of degree k. Since the curvature is a 2-form
$$d\ P(\Omega, ..., \Omega) = \sum P(\Omega, ..., d\Omega, ..., \Omega)$$
But since by the Bianchi identity $0 = D\Omega = d\Omega + [\omega, \Omega]$ we have
$$d\left(P(\Omega, ..., \Omega)\right) = P(-[\omega, \Omega], \Omega, ..., \Omega) + ... + P(\Omega, ..., -[\omega, \Omega]) = 0$$
by the trick explained in remark 53.

2. Let ω_0, ω_1 be two connection 1-forms, given by two different connections in the bundle P. Define the path of connections
$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)$$
whose curvature is
$$\Omega_t = D\omega_t = d\left(\omega_t + \frac{1}{2}[\omega_t, \omega_t]\right).$$
We will prove that the polynomials associated with Ω_0 and Ω_1 differ by an exact form, in the following way
$$P(\Omega_1) - P(\Omega_0) = d\ \left(k \int_0^1 dt\ P(\omega_1 - \omega_0, \Omega_t, ..., \Omega_t)\right), \tag{1.9}$$
so that they define the same class in cohomology.
Start by computing
$$\begin{aligned}\frac{d\Omega_t}{dt} &= \frac{d}{dt}\left(d\omega_t + + \frac{1}{2}[\omega_t, \omega_t]\right) \\ &= d\Big(\frac{d\omega_t}{dt}\Big) + \left[\frac{d\omega_t}{dt}, \omega_t\right] \\ &= d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t].\end{aligned}$$
So, inserting this in each entry of P we get that
$$\begin{aligned}\frac{d}{dt}P(\Omega_t) &= P(d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t], \Omega_t, ..., \Omega_t) + P(\Omega_t, d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t], ..., \Omega_t) \\ &= kP(d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t], \Omega_t, ..., \Omega_t)\end{aligned} \tag{1.10}$$
We now just need to show that
$$P(d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t], \Omega_t, ..., \Omega_t) = d(P(\omega_1 - \omega_0, \Omega_t, ..., \Omega_t))$$
then by equation (1.10), the result we are trying to prove (equation 1.9) will follow. To do this we work out the right hand side.
$$\begin{aligned}d(P(\omega_1 - \omega_0, \Omega_t, ..., \Omega_t)) &= P(d(\omega_1 - \omega_0), \Omega_t, ..., \Omega_t) - P(\omega_1 - \omega_0, d\Omega_t, ..., \Omega_t) - ... \\ &\quad - P(\omega_1 - \omega_0, \Omega_t, ..., d\Omega_t) \\ &= P(d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t], \Omega_t, ..., \Omega_t) \\ &\quad \underbrace{- P([\omega_1 - \omega_0, \omega_t], \Omega_t, ..., \Omega_t) + (k-1)P(\omega_1 - \omega_0, d\Omega_t, ..., \Omega_t)}_{=0 \text{ by the trick in remark 53}} \\ &= P(d(\omega_1 - \omega_0) + [\omega_1 - \omega_0, \omega_t], \Omega_t, ..., \Omega_t).\end{aligned}$$

Now integrating (1.10) from 0 to 1 we get the desired result.

□

In the proof we saw that, given any two connections ω_0, ω_1 and a polynomial $P \in I^k(G)$, there exists a $(2k-1)$-form $P_k(\omega_0, \omega_1)$ called the k^{th} **Chen-Simons transgression form**, such that

$$P(\Omega_1) - P(\Omega_0) = dP_k(\omega_0, \omega_1). \tag{1.11}$$

Moreover, we also found (in the proof of the Chern-Weil theorem) that this transgression form depends on ω_0, ω_1, P in the following way

$$P_k(\omega_0, \omega_1) = k \int_0^1 dt \; P(\omega_1 - \omega_0, \Omega_t, ..., \Omega_t). \tag{1.12}$$

The Chern-Weil theorem has lots of consequences which we will now explore. First, we will organize the information that it gives in the following:

Definition 56 *Given a connection ω with curvature Ω on a principal bundle over M we define the* ***Chern-Weil homomorphism*** *to be the morphism of graded algebras (however doubling the degree) , given by*

$$\begin{aligned} I^k(G) &\longrightarrow H^{2k}(M) \\ P &\longmapsto [P(\Omega)]\,. \end{aligned} \tag{1.13}$$

By the Chern-Weil theorem, it is an invariant of the principal bundle not depending on the connection chosen. Moreover the elements $[P(\Omega)]$ are called ***characteristic classes****.*

Remark 57 *Note that this is in fact a morphism of algebras thanks to the definition 54 of $P(\Omega)$ and the product on the Weil algebra. Let $\{X_i\}$ be a basis for the Lie algebra and take $P_1 \in I^m$, $P_2 \in I^n$, then*

$$\begin{aligned} (P_1P_2)(\Omega) &= \sum_{i_1,...,i_{n+m}} \Omega^{i_1} \wedge ... \wedge \Omega^{i_{n+m}} (P_1P_2)(X_{i_1}, ..., X_{i_{n+m}}) \\ &= \sum_{i_1,...,i_{n+m}} \Omega^{i_1} \wedge ... \wedge \Omega^{i_{n+m}} \frac{1}{(n+m)!} \sum_{\sigma \in \Pi_{n+m}} P_1(X_{i_1}, ..., X_{i_m}) P_2(X_{i_{m+1}}, ..., X_{i_{n+m}}) \\ &= \frac{1}{(n+m)!} \sum_{\sigma \in \Pi_{n+m}} P_1(\Omega, ..., \Omega) \wedge P_2(\Omega, ..., \Omega) \\ &= P_1(\Omega, ..., \Omega) \wedge P_2(\Omega, ..., \Omega) \end{aligned}$$

This proves our claim that the Chern-Weil homomorphism is indeed a morphism of algebras.

Notice that the Chern-Weil homomorphism is functorial. In the sense that if $f : P \longrightarrow P'$ is a bundle morphism and ω' is a connection in P', then $\omega = f^*\omega'$ is a connection in P, and

$$P(f^*\Omega) = f^*P(\Omega).$$

From this it follows that

Theorem 58 *Equivalent bundles give rise to the same Chern-Weil homomorphism. In particular for trivial bundles the Chern-Weil homomorphism vanishes.*

Theorem 59 *Let $i : N \hookrightarrow M$ be a $2k$ dimensional, compact and orientable submanifold and $P \in I^k(G)$, then:*

$$\int_N i^*P(\Omega)$$

is independent of the connection ω with curvature Ω.

Proof: Since for any two connections ω_0 and ω_1, we have $P(\Omega_1) - P(\Omega_0) = d\ P_k(\omega_0, \omega_1)$, then

$$\begin{aligned}\int_N i^*P(\Omega_0) &= \int_N i^*P(\Omega_0) + di^*P_k(\omega_0, \omega_1) \\ &= \int_N i^*P(\Omega_1)\end{aligned}$$

by Stokes theorem. □

We will now see two particular cases that give rise to the Prontrjagin and Chern classes. In the first case we consider $G = GL(n, \mathbb{R})$, so $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ and

$$\det(I + sA) = I + s\sigma_1(A) + ... + s^n\sigma_n(A). \tag{1.14}$$

Where the σ_k's give us elements of $I^k(GL(n, \mathbb{R}))$. We can define the **Pontrjagin classes** as the characteristic classes of the principal $GL(n, \mathbb{R})$ bundle, given by

$$p_k = \frac{1}{(2\pi)^{2k}} [\sigma_{2k}(\Omega)] \in H^{4k}(M). \tag{1.15}$$

Moreover, we also define the **total Pontrjagin class** by

$$p = 1 + p_1 + ... + p_n \in H(M). \tag{1.16}$$

Now let $G = GL(n, \mathbb{C})$, so that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. We we will also use (1.14) and define the **Chern classes** by

$$c_k = \frac{1}{(2\pi i)^k} [\sigma_k(\Omega)] \in H^{2k}(M). \tag{1.17}$$

As before the **total Chern class** is defined via

$$c = 1 + c_1 + ... + c_n \in H(M). \tag{1.18}$$

Chapter 2

Classical Yang-Mills Theory

Our goal in this chapter is to use the geometry studied in the first half to explore some features of physical Yang-Mills theory. Yang-Mills theory was first discovered (or constructed depending on the philosophical point of view) in the 1950's by physicists, and it was a surprise that the same kind of objects they where working with had already been studied by mathematicians. Basically the same objects have appeared in two very different ways. These objects are the notions of connection, curvature and gauge transformation.
Our notation will be now closer to the physics one, for example instead of using ω for a connection we will use a more common A. We will describe the ingredients of Yang-Mills theories:

- A principal bundle P with structure group G over the space-time M which is in most cases a pseudo-Riemannian manifold,
- A vector bundle E with standard fibre V, associated to a representation ρ of G. We also require V to have a G-invariant hermitian inner product $\langle \cdot, \cdot \rangle$,
- Matter will be described by fields which are sections of E,
- Classically, over each point of M such fields represent the internal state of a particle. As the particle moves in the space-time M, this internal state must be transported and this is where the notion of a connection comes in,
- Curvature is interpreted as a force field.

Later we shall see that in fact a connection is needed in order to have the physical meaningful symmetries in the system. This will be in fact our approach to their introduction. These symmetries are the so called gauge symmetries.

2.1 Fields and Lagrangians

2.1.1 Matter Fields and Gauge Transformations

We will be faced with two different kinds of fields, these are matter fields and gauge fields.

Matter fields must be thought of as the physical object which encapsulates information on the internal state of a particle. To each point of the space-time a matter field must associate an element of the vector space V. So we will define a matter field as a section of an associated bundle E. Remembering the relation between sections of an associated vector bundle and G-invariant functions $P \longrightarrow V$ we can give the following

Definition 60 *A* ***matter field*** ψ *is an element of* $C(P, V)$.

A gauge transformation roughly corresponds to a symmetry of the internal space of states under which we must require the physics to be invariant, this is

Definition 61 *A **gauge transformation** $f : P \longrightarrow P$ is an automorphism of the principal bundle P, which preserves the identity on the base, i.e. $\pi = f \circ \pi$, where π is the projection onto the base M.*

Later we will see how the imposition of gauge invariance (i.e the invariance of physics under gauge transformations) leads to the necessity of introducing another kind of fields called gauge fields. This will be identified with a procedure that in physics is usually called minimal coupling.
We shall denote the space of gauge transformations by $\mathcal{G}$. This is usually called the **gauge group** since it is a group under composition. In fact, we can prove that given $\tau \in C(P, G)$ the automorphism

$$\begin{aligned} f : P &\longrightarrow P \\ p &\longmapsto p\tau(p) \end{aligned}$$

is a gauge transformation and this gives a bijection $\mathcal{G} \simeq C(P, G)$ with the adjoint action of G onto itself. Continuing in this line of thought we can go further and view $\mathcal{G}$ as a Lie group with Lie algebra $C(P, \mathfrak{g})$, with the adjoint representation of G on $\mathfrak{g}$. This last one is called the **gauge algebra**.
$C(P, \mathfrak{g})$ is in fact a Lie algebra, with Lie bracket given by

$$[H, H'](p) = [H(p), H'(p)],$$

and exponential map defined by $(Exp(H))(p) = exp(H(p))$, where $H, H' \in C(P, \mathfrak{g})$ and exp is the exponential map in $\mathfrak{g}$.

By the theorems on mappings of connections we know that there is a well defined action of $\mathcal{G}$ on the **space of connections** $\mathcal{A}$ by pullback.

Theorem 62 *The space of connections is an affine space $\mathcal{A}$ with tangent space $\tilde{\Omega}^1(P, \mathfrak{g})$.*

Proof: Let A be a connection on P, then the map

$$\begin{aligned} \tilde{\Omega}^1(P, \mathfrak{g}) &\longrightarrow \mathcal{A} \\ \tau &\longmapsto A + \tau \end{aligned}$$

is a bijection since it is obviously injective. To check surjectivity, let A' be another connection on P, then $A' = A + (A' - A)$ and $(A' - A) \in \tilde{\Omega}^1(P, \mathfrak{g})$. $\square$

By the same kind of computation done for getting local forms for the connection, one can show that if $f \in \mathcal{G}$ is a gauge transformation, represented by an element $\tau \in C(P, G)$, then:

$$f^*A = ad(\tau^{-1})A + \tau^{-1}d\tau,$$

where $\tau^{-1}d\tau = \tau^*\theta$ and θ is the Maurer-Cartan form. We also have an action of the gauge group $\mathcal{G}$ on $\tilde{\Omega}^k(P, V)$ by pullback. With the same conventions this action is given by

$$f^*\varphi = \rho(\tau^{-1})(\varphi).$$

By differentiating the action of the one parameter family $Exp(tH)$, we can get infinitesimal versions of this actions given by

$$\begin{aligned} \frac{d}{dt}(Exp(tH)^*A) &= DH \\ \frac{d}{dt}(Exp(tH)^*\varphi) &= -\rho(H)\varphi \end{aligned}$$

where D is the covariant exterior derivative induced by the connection A.

Remark 63 *Physicists usually talk of local gauge transformations and global transformations. To illustrate these concepts, let $P = M \times G$ be the trivial bundle, and define the left action of G on P by:*

$$L_g : M \times G \longrightarrow M \times G$$
$$(x, h) \longmapsto (x, gh).$$

*If $g \in G$ dos not depend on the space-time point $x \in M$ we speak of a **global transformation**, otherwise we can have $g : M \longrightarrow G$ depending on the space-time point, in this case we speak of a **local gauge transformation**.*

2.1.2 Gauge Fields and the Lagrangian Formulation

From now on we shall omit the ρ's denoting the representation of G on V whenever the situation is explicit enough.

In classical mechanics the physical path is the one that minimizes the action functional, which is the integral along the path of the Lagrangian. In this case, the Lagrangian was a function of both the position and the velocity, i.e. a function on the space of 1-jets of functions $[0, 1] \longrightarrow M$, where M is the configuration space of the problem.
In the case of classical field theory, we have a similar creed, however now we must minimize an action functional which is the integral over the space-time M of a Lagrangian density.
For now, this Lagrangian density depends at each point on the value of the matter fields and their first derivatives, i.e. on the space of 1-jets of functions $P \longrightarrow V$. As usual we shall denote this space by $J^1(P, V)$. However, in fact we are only interested on the subset of 1-jets that are represented by elements of $C(P, V)$, so the **Lagrangian density** or **Lagrangian** is a function $\mathcal{L} : C(P, V) \longrightarrow C^\infty(M)$ such that

$$\mathcal{L}\left[\psi\right](x) = L(j^1_p\psi) = L(p, \psi_p, d\psi_p)$$

for any $p \in \pi^{-1}(x)$ and $L : J^1(P, V) \longrightarrow \mathbb{R}$, i.e. it only depends on the information that is contained on the 1-jet class of ψ. Following this we will now impose that $L(j^1_p\psi) = L(j^1_{pg})$ for $g \in G$. Since

- $\psi(pg) = g^{-1} \cdot \psi(p)$
- $d\psi_{pg} = g^{-1} \cdot (d\psi_p \circ dR_{g^{-1}})$,

we have

$$L(p, \psi_p, d\psi_p) = L(p, g^{-1} \cdot \psi(p), g^{-1} \cdot d\psi_p \circ dR_{g^{-1}}).$$

This is the simplest natural invariance that a Lagrangian can have. In fact later we shall see that one needs to add to this Lagrangian an explicit dependence on what we will call gauge fields, in order to make it gauge invariant.

Definition 64 *A Lagrangian $L : J^1(P, V) \longrightarrow \mathbb{R}$ or $\mathcal{L} : C(P, V) \longrightarrow C^\infty(M)$ is said to be*

1. *G-**invariant** if $L(p, g \cdot \psi_p, g \cdot d\psi_p) = L(p, \psi_p, d\psi_p)$ for all $g \in G$*
2. ***gauge invariant** if $\mathcal{L}(f^*\psi) = \mathcal{L}(\psi)$, for all $f \in \mathcal{G}$.*

It is not true that a G-invariant Lagrangian is gauge invariant. To see this, suppose that $f \in \mathcal{G}$ such that $f(p) = p\tau(p)$, for $\tau \in C(P, G)$. Since $\psi \in C(P, V) \simeq \tilde{\Omega}^0(P, V)$ we have $f^*\psi = \tau^{-1} \cdot \psi$, so if γ is a path passing through $p \in P$ with $\dot{\gamma}(0) = X$

$$\begin{aligned}
d_p(f^*\psi)(X) &= d_p(\tau^{-1} \cdot \psi)(X) \\
&= \frac{d}{dt}\Big|_{t=0} (\tau^{-1}(\gamma(t))) \cdot (\psi(\gamma(t))) \\
&= \frac{d}{dt}\Big|_{t=0} (\tau^{-1}(\gamma(t))) \cdot (\psi(p)) + \frac{d}{dt}\Big|_{t=0} \tau^{-1}(p) \cdot \psi(\gamma(t)) \\
&= d_p(\tau^{-1})(X) \cdot \psi(p) + \tau^{-1}(p) \cdot d_p\psi(X).
\end{aligned}$$

Hence we got that

$$d_p(f^*\psi) = d_p(\tau^{-1})(X) \cdot \psi(p) + \tau^{-1}(p) \cdot d_p\psi(X)$$

and we needed that $d_p(f^*\psi) = \tau^{-1}(p) \cdot d_p\psi$ for a G-invariant Lagrangian depending on derivatives of ψ to be gauge invariant. The following theorem gives a way of taking a G-invariant Lagrangian into a gauge invariant one. This amounts to adding a dependence on a connection on P which the physicists call **gauge potential**.

Theorem 65 *If $L : J^1(P, V) \longrightarrow \mathbb{R}$ is a G-invariant Lagrangian and $\mathcal{A}$ the space of connections on P, define:*

$$\begin{aligned}
\mathcal{L} : J^1(P, V) \times \mathcal{A} &\longrightarrow C^\infty(M) \\
(\psi, A) &\longmapsto \mathcal{L}(\psi, A)(x) = L(p, \psi(p), D\psi_p)
\end{aligned}$$

for $p \in \pi^{-1}(x)$ and D the covariant exterior derivative defined by the connection A. Then $\mathcal{L}$ is well defined and gauge invariant, in the sense that for $f \in \mathcal{G}$ $\mathcal{L}(f^*\psi, f^*A) = \mathcal{L}(\psi, A)$.

Proof: Let $\tau \in C(P, G)$ defined by $f \in \mathcal{G}$, let's check the theorem by parts:

1. (Well defined) Since $R_g^* D_p\psi = D_{pg}\psi \circ (R_g)_*$ we have $D_{pg}\psi = g^{-1} \cdot (D_p\psi \circ (R_{g^{-1}})_*)$, so

$$\begin{aligned}
L(pg, \psi(pg), D_{pg}\psi) &= L(pg, g^{-1} \cdot \psi(p), g^{-1} \cdot D_p\psi \circ (R_{g^{-1}})_*) \\
&= L(p, \psi(p), D_p\psi)
\end{aligned}$$

   So, indeed it gives a well defined function on the space-time M defining a Lagrangian.

2. (Gauge invariant) Just compute with $p \in \pi^{-1}(x)$, and remember the way how $f \in \mathcal{G}$ acts

$$\begin{aligned}
\mathcal{L}(f^*\psi, f^*A)(x) &= L(p, f^*\psi(p), d_p f^*\psi + (f^*A)_p \cdot f^*\psi(p)) \\
&= L(p, \tau^{-1}(p) \cdot \psi(p), f^*(d\psi + A \cdot \psi)(p) \\
&= L(p, \tau^{-1}(p) \cdot \psi(p), f^*(D_p\psi)) \\
&= L(p, \tau^{-1}(p) \cdot \psi(p), \tau^{-1}(p) \cdot D_p\psi) \\
&= L(p, \psi(p), D_p\psi) \\
&= \mathcal{L}(\psi, A)(x),
\end{aligned}$$

   which proves the gauge invariance of this Lagrangian.

□

Remark 66 *The theorem justifies the procedure to which physicists call **minimal coupling**, in which partial derivatives are changed into covariant ones*

$$\partial_\mu \psi^i \longrightarrow D_\mu \psi^i = \partial_\mu \psi^i + A^i_{\mu j} \psi^j$$

making the physics invariant under gauge transformations via the gauge invariance of the Lagrangian. This is why physicists introduced gauge potentials. Only later it was realized that gauge potentials and connections are the same thing.

The curvature of the gauge potential is the **field strength** and it is interpreted as the strength of a field propagating a given interaction. The way how this interaction propagates and interacts with matter is contained in the Lagrangian, and we usually add to the Lagrangian a kinetic term, quadratic in the field strength.

1. There is no problem in introducing a term quadratic in the field strength F, since $F \in \tilde{\Omega}^2(P, \mathfrak{g})$ is a horizontal form and hence G-equivariant. So when acted on by a gauge transformation given by $\tau \in C(P, G)$ we have $f^*F = ad(\tau^{-1})F$, so the only requirement we must have is that this quadratic term must be ad-invariant, this is in fact the case.

2. The interaction terms between matter and gauge fields, must also mix both of them in a gauge-invariant way. This can be interpreted as a restriction on the interaction terms that can exist in nature (if we believe we are on the right way...).

Later we shall illustrate this for electromagnetism by deriving the QED Lagrangien.

2.1.3 The Equations of Motion

As we made transparent in the previous section we must now derive the equations of motion of the classical field theory, through the **minimal action principle**. The action functional is $\mathcal{S} : C(P, V) \times \mathcal{A} \longrightarrow \mathbb{R}$ given by

$$\mathcal{S}[\psi, A] = \int_M \mathcal{L}[\psi, A],$$

where the integral is with respect to a pseudo-Riemannian volume element in the space-time M, and where it is assumed that the integral exists. We implement the minimum action principle by searching for critical points of this functional

$$\delta S = \frac{d}{dt}\Big|_{t=0} \mathcal{S}[\psi + t\phi, A + ta] = 0,$$

with $\phi \in C(P, V)$ and $a \in \tilde{\Omega}^1(P, \mathfrak{g})$. We shall now find some more handy formulas for the equations of motion, which come expressed as Euler-Lagrange equations for the matter and gauge fields. However we shall not explore the the variation of the gauge potential until the next section.
Do not get afraid of the following:

$$\begin{aligned}
\delta S &= \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi + t\phi, A + ta] \\
&= \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi + t\phi, A] + \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi, A + ta] \\
&= \frac{d}{dt}\Big|_{t=0} \int_M L(p, \psi(p) + t\phi(p), D_p\psi + tD_p\phi) + \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi, A + ta] \\
&= \frac{d}{dt}\Big|_{t=0} \int_M L(p, \psi(p) + t\phi(p), D_p\psi) + \frac{d}{dt}\Big|_{t=0} \int_M L(p, \psi(p), D_p\psi + tD_p\phi) \\
&\quad + \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi, A + ta] \\
&= \frac{d}{dt} \int_M \langle \frac{\partial L}{\partial \psi}, \phi \rangle + \int_M \langle \frac{\partial L}{\partial D\psi}, D\phi \rangle + \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi, A + ta] \\
&= \int_M \langle \frac{\partial L}{\partial \psi} + D^* \frac{\partial L}{\partial D\psi}, \phi \rangle + \frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}[\psi, A + ta].
\end{aligned}$$

Here, $\langle \cdot, \cdot \rangle$ is a metric in $C(P, V)$ induced by the metric in V and D^* is the adjoint of D for the pairing $\int_M \langle \cdot, \cdot \rangle$. Since ϕ and a are arbitrary we can write the equations of motion in the form

$$\frac{\partial L}{\partial \psi} + D^* \frac{\partial L}{\partial D\psi} = 0,$$
$$\frac{d}{dt}\Big|_{t=0} \int_M \mathcal{L}\left[\psi, A + ta\right] = 0.$$

2.1.4 Examples of Physical Yang-Mills Theories

2.1.4.1 Yang-Mills Equations

Let P be a G-principal bundle over the space time M, and E an associated vector bundle with standard fibre V. The Yang Mills Lagrangian is

$$\mathcal{L}_{YM}\left[A\right] = \frac{1}{2} Tr(F \wedge *F)$$

where $*$ is the Hodge star operator and $F = dA + \frac{1}{2}\left[A, A\right]$ is the curvature of the connection A, or the gauge field in a physics language. This Lagrangian describes a gauge field propagating with no matter and its action is a functional in the space of connections $S_{YM} : \mathcal{A} \longrightarrow \mathbb{R}$, whose critical points are calculated in the next theorem.

Theorem 67 *(Yang Mills Equations) The equations of motion to the Yang-Mills Lagrangian are*

$$DF = 0,$$
$$D * F = 0.$$

Proof: The first equation is just the Bianchi Identity. To calculate the second equation of motion we consider the following variational problem

$$\frac{d}{dt}\Big|_{t=0} \int_M \frac{1}{2} Tr(F(t) \wedge *F(t)) = 0,$$

where $F(t)$ is the curvature of $A + ta$ with $a \in \tilde{\Omega}^1(P, \mathfrak{g})$. Doing things slowly,

$$\begin{aligned} F(t) &= d(A + ta) + \frac{1}{2}\left[A + ta, A + ta\right] \\ &= F + t(da + \frac{1}{2}\left[a, A\right] + \frac{1}{2}\left[A, a\right]) + \frac{1}{2} t^2 \left[a, a\right] \\ &= F + tDa + \frac{1}{2} t^2 \left[a, a\right] \end{aligned}$$

where F is the curvature of A and D its covariant exterior derivative.

$$\begin{aligned} Tr(F(t) \wedge *F(t)) &= Tr((F + tDa + \frac{1}{2}t^2\left[a, a\right]) \wedge *(F + tDa + \frac{1}{2}t^2\left[a, a\right])) \\ &= Tr(F \wedge *F) + 2tTr(F \wedge *Da) + o(t^2) \end{aligned}$$

So that we are just left with

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_M \frac{1}{2} Tr(F(t) \wedge *F(t))) &= \int_M Tr(F \wedge *Da) \\ &= \int_M Tr(D * F \wedge a) \end{aligned}$$

and since $a \in \tilde{\Omega}^1(P, \mathfrak{g})$ is arbitrary we conclude that the equation is $D * F = 0$. $\square$

The fact that the first equation is just the Bianchi identity tell us that in fact it is not physics, but a consequence of the geometry of the object (curvature) used to model the field strength.

Remark 68 *Notice that to write the integral of the Yang-Mills action*

$$S_{YM} = \int_M \frac{1}{2} Tr(F(t) \wedge *F(t))$$

we needed to pull-back the form F to M. In fact this can be done globally since $Tr(F(t) \wedge *F(t))$ is ad-invariant and hence the pullback to M does not depend on the local sections chosen.

Just for fun, let's now get Maxwell equations (in vacua) from here. To achieve this we must describe **pure electromagnetism** propagating in vacua, this amounts to choose P a $U(1)$-bundle over the space time M which is the Minkowski space-time. The Yang-Mills Lagrangian is now the Maxwell Lagrangian

$$\mathcal{L}[A] = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

since $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. We interpret the components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the electric and magnetic field in the following way

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

So that the Yang-Mills equations reduce to the **Maxwell equations**

$$\begin{cases} \nabla \cdot B = 0 \\ \nabla \times E + \dfrac{\partial B}{\partial t} = 0 \\ \nabla \cdot E = 0 \\ \nabla \times B - \dfrac{\partial E}{\partial t} = 0. \end{cases}$$

2.1.4.2 A Lagrangian for QED

We shall now describe electrodynamics coupled to matter, which is modelled as a matter field with values in an associated vector bundle.
The setting is:

- A principal $U(1)$-bundle P over the Minkowski space-time M.
- An associated vector bundle $E = P \times_{U(1)} \mathbb{C}^4$ with fibre $\mathbb{C}^4$, where we have the usual inner product

$$\langle z, w \rangle = z^* \cdot w = \begin{pmatrix} \overline{z_1} & \overline{z_2} & \overline{z_3} & \overline{z_4} \end{pmatrix} . \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 . \end{pmatrix}.$$

This is called the **Dirac space**, where we also have a representattion of the Clifford Algebra $Cl(1,3)$ in terms of the **γ-matrices**, given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

- A matter field $\psi \in C(P, \mathbb{C}^4)$ can be interpreted as a section of this associated vector bundle E.
- The representation of $U(1)$ in $\mathbb{C}^4$ is just given by complex multiplication, i.e $e^{i\theta} \cdot z = e^{i\theta} z$, so the representation of $\mathfrak{u}(1) \simeq i\mathbb{R}$ is just multiplication.

We must now create a gauge invariant Lagrangian that propagates the gauge field for electromagnetism, and the matter field, and also couples electromagnetism with matter, i.e it must have an interaction term.
We do this by parts, first let us just focus on the matter. We must propagate matter, so part of the Lagrangian is

$$\mathcal{L} = i\langle \psi, \gamma^\mu \partial_\mu \psi \rangle - m\|\psi\|^2.$$

The first term is a kinetic term propagating the matter field and the second is a massive term, where m is the mass of the particle described by the matter field. However, in order to impose gauge invariance we need to implement the minimal coupling procedure,

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + ieA^i_{\mu j}.$$

This introduces a connection, which is interpreted as the gauge potential for electromagnetism describing the physical **photon field**, e is the **electric charge** of the electron. So, in fact, the interaction is required in order to achieve gauge invariance. The Lagrangian has evolved to

$$\mathcal{L} = i\langle \psi, \gamma^\mu D_\mu \psi \rangle - m\|\psi\|^2$$

or, writing it in such a way that the interaction term is more explicit,

$$\mathcal{L} = i\langle \psi, \gamma^\mu \partial_\mu \psi \rangle - m\|\psi\|^2 - e\langle \psi, \gamma^\mu A_\mu \psi \rangle.$$

In this way, we can see explicitly the interaction term of the photon with the matter field describing the electron. This term is proportional to the electric charge of the electron e, in general such constants of proportionality are called **coupling constants**. We are almost done in succeeding to get the Lagrangian for QED. The final step is to add a kinetic term propagating the gauge field, which is the electromagnetic field. The Lagrangian for QED is hence

$$\mathcal{L} = i\langle \psi, \gamma^\mu D_\mu \psi \rangle - m\|\psi\|^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where $F = ie\, dA$ is the gauge field strength, i.e the curvature of the connection A.
As a final remark notice that a mass term $m^2 A_\mu A^\mu$ for the photon field cannot be added while keeping gauge invariance. This is why the photon is massless.

2.1.4.3 The Higgs Mechanism

Remember that when we derived a Lagrangian for QED we noticed that it was not possible to add a massive term to the photon field in order to keep gauge invariance. However, some gauge felds that appear in Nature actually have mass. To get massive term for the given gauge field without violating gauge invariance, one uses the so called the **Higgs mechanism**, which is related to the phenomenon of **spontaneous symmetry breaking**.
We will give a general presentation of the Higgs Mechanism. For this, our setting will be

- A principal bundle (π, P, M) over a pseudo-Riemannian space time M and with structure group G.
- An orthogonal representation $\rho : G \longrightarrow O(V)$ of G on an inner-product $\langle \cdot, \cdot \rangle$ vector space V with $dimV = m < \infty$.
- A Lagrangian $\mathcal{L} : J^1(P, V) \longrightarrow \mathbb{R}$ that can be written in the form

$$\mathcal{L}[\psi] = \frac{1}{2} \langle \partial_\mu \psi, \partial^\mu \psi \rangle - W(\psi)$$

where ψ is the matter field and $W : V \longrightarrow \mathbb{R}$ is a G-invariant function interpreted as a potential energy, where G-invariance is necessary to make the Lagrangian G-invariant. Notice that in this way we can get a gauge invariant Lagrangian by the minimal coupling procedure, which gives

$$\mathcal{L}[\psi, A] = \frac{1}{2}\langle D_\mu \psi, D^\mu \psi\rangle - W(\psi) - \frac{1}{2}(F, F),$$

where we added a kinetic term propagating the gauge field. Here $(\cdot,\cdot)$ is an *ad*-invariant inner product in $\mathfrak{g}$.

We call an element $v_0 \in V$ a vacuum state if it is a local minimum of the potential W. Let $G \cdot v_0$ be the orbit of the vacuum under the G-action via ρ and G_0 its isotropy subgroup, that is usually called the **unbroken subgroup**. Notice that

$$G \cdot v_0 \simeq G/G_0$$

is a submanifold of the sphere of radius $\|v_0\|$, since the inner product is G-invariant.
The next theorem gives the notion of mass near a vacuum.

Theorem 69 *Let $d^2_{v_0}W : T_{v_0}V \times T_{v_0}V \longrightarrow \mathbb{R}$ be the Hessian of W in a vacuum v_0 and $d = dim(G \cdot v_0)$. Then there exists an orthonormal basis $\{u_1, ..., u_d, u_{d+1}, ..., u_m\}$ of $T_{v_0}V$ such that*

1. *$span\{u_1, ..., u_d\} = T_{v_0}(G \cdot v_0\ )$;*

2. *$M_{ij} := d^2_{v_0}W(u_i, u_j)$ is diagonal and $M_{11} = \cdots = M_{dd} = 0$.*

Proof: We need to get an orthonormal basis such that M can be written in the form

$$M = \begin{pmatrix} 0 & 0 \\ 0 & diag \end{pmatrix}.$$

To do so we just use the metric to split $T_{v_0}V = T_{v_0}G \cdot v_0 \oplus T_{v_0}(G \cdot v_0)^\perp$, now choose an orthonormal basis for each one such that M is diagonal in the second part (this can be done since the representation is orthogonal), use Gram-Schmidt to change the first one if needed in order to get a full orthonormal basis.
To prove that we have the zeros above it suffices to show that

$$d^2_{v_0}W(A^*_{v_0}, w) = 0$$

for all $A \in \mathfrak{g}$ and $w \in T_{v_0}V$, but this is obvious since if $\gamma(t)$ passes in v_0 at $t = 0$ with velocity w, then

$$\begin{aligned} d^2_{v_0}W(A^*_{v_0}, w) &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} W(exp(tA)\gamma(s)) \\ &= \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} W(\gamma(s)) \end{aligned}$$

Where we used that W is G-invariant. $\square$

The name spontaneous symmetry breaking arises from the fact that now we move to the vacuum v_0 and study matter fields perturbatively near it, though breaking G-symmetry. We write near v_0,

$$\psi = v_0 + \psi'$$

and we call ψ' the shifted field of ψ. In fact, this means that the gauge symmetry is apparently broken because we are writing the Lagrangian perturbatively near a vacuum, however the gauge symmetry is still present but to view it we should look at the original Lagrangian.
Note that ψ' is not in $C(P, V)$. As V is a vector space we can think of ψ' as being in the tangent space $T_{v_0}V \simeq V$. In this way, we may write

$$\psi' = \xi^1 u_1 + ... + \xi^d u_d + \eta^{d+1} u_{d+1} + ... + \eta^m u_m$$

where the functions $\xi^i, \eta^i : P \longrightarrow \mathbb{R}$ are called the **Higgs fields**. More specifically it is common in physics to call the ξ^i's the **Goldstone bosons** and η^i's **scalar mesons**.
The mass of the Higgs field associated with u_i is defined as

$$m_i^2 = d_{v_0}^2 W(u_i, u_i).$$

Expanding the potential near v_0 we can see that the scalar mesons acquire a mass, while the Goldstone bosons are massless.

$$\begin{aligned}
W(\psi) &= W(v_0 + \psi') = W(v_0) + \frac{1}{2} d_{v_0}^2 W(\psi', \psi') + ... \\
&= W(v_0) + \frac{1}{2} d_{v_0}^2 W(\xi^i u_i + \eta^{d+i} u_{d+i}, \xi^i u_i + \eta^{d+i} u_{d+i}) + ... \\
&= W(v_0) + (\eta^{d+i})^2 \frac{1}{2} d_{v_0}^2 W(u_{d+i}, u_{d+i}) + ... \\
&= W(v_0) + \frac{1}{2} (\eta^{d+i})^2 m_{i+d}^2 + ...
\end{aligned}$$

Where we used the last theorem to throw away the nondiagonal terms and those terms in u_i for $i \leq d$. In fact we where already expecting this result, notice that the Goldstone bosons are responsible for perturbing the potential in the directions of the orbit $G \cdot v_0$ and we have that the potential is G-invariant.
We shall now show how the procedure of breaking the symmetry and perturbing around a vacuum, gives the gauge potential a mass out of the Lagrangian. Up to third order terms

$$\begin{aligned}
\mathcal{L}[\psi, A] &= \mathcal{L}[v_0 + \psi', A] \\
&= \frac{1}{2} \langle D_\mu (v_0 + \psi'), D^\mu (v_0 + \psi') \rangle - W(v_0 + \psi') - \frac{1}{2}(F, F) \\
&= \frac{1}{2} \langle \partial_\mu \psi' + A_\mu v_0 + A_\mu \psi', \partial^\mu \psi' + A^\mu v_0 + A^\mu \psi' \rangle \\
&\quad - W(v_0) - \frac{1}{2} (\eta^{d+i})^2 m_{i+d}^2 - \frac{1}{2}(F, F) + ... \\
&= \frac{1}{2} \langle \partial_\mu \psi', \partial^\mu \psi' \rangle + \frac{1}{2} \langle A_\mu v_0, A^\mu v_0 \rangle + \langle \partial_\mu \psi', A^\mu \psi' \rangle - W(v_0) \\
&\quad - \frac{1}{2} (\eta^{d+i})^2 m_{i+d}^2 - \frac{1}{2}(F, F) + ... \\
&= \Big\{ \frac{1}{2} \sum_i \partial_\mu \xi^i \partial^\mu \xi^i \Big\} + \frac{1}{2} \Big\{ \sum_i \partial_\mu \eta^{d+i} \partial^\mu \eta^{i+d} - (\eta^{d+i})^2 m_{i+d}^2 \Big\} \\
&\quad + \underbrace{\frac{1}{2} \langle A_\mu v_0, A^\mu v_0 \rangle}_{\text{massive term}} + \langle \partial_\mu \psi', A^\mu \psi' \rangle - W(v_0) - \frac{1}{2}(F, F) + ...
\end{aligned}$$

So we can see that the gauge potential A (i.e. the connection) acquires a mass and the matter field had been devided in Klein-Goron type Lagrangians for $m - d$ massive scalar mesons and d massless Goldston bosons. The last ones are seen as spurious since they correspond to the freedom of making a gauge transformation, in fact they can be eliminated by working with local unitary gauges that we shall not explore.

2.2 Monopoles and Instantons

2.2.1 (Anti)-Instantons

Remember the Yang-Mills equations

$$\begin{aligned}
DF &= 0, \\
D * F &= 0.
\end{aligned}$$

Here, we shall analyse some special solutions to these equation that arise in the case of 4-dimensional space-time M. In fact we are interested in the case $M \simeq \mathbb{R}^4$, but this case reduces to the case of its compactification

$M \simeq \mathbb{S}^4$, when we impose that the integral of the Yang-Mills action must converge, and hence at infinity the connection must be constant.
The Hodge star operator $*$ restricted to horizontal 2-forms $\tilde{\Omega}^2(P, \mathfrak{g}) \simeq \Omega^2(M) \otimes \mathfrak{g}$ satisfies $*^2 = 1$, so there is a splitting

$$\tilde{\Omega}^2(P, \mathfrak{g}) \simeq \tilde{\Omega}^2(P, \mathfrak{g})_+ \oplus \tilde{\Omega}^2(P, \mathfrak{g})_-$$

into the ± 1-eigenspaces. We call the forms on $\tilde{\Omega}^2(P, \mathfrak{g})_+$ **self-dual**, and the ones in $\tilde{\Omega}^2(P, \mathfrak{g})_-$ **anti-self-dual**. Notice that this allows us to split every horizontal 2-form uniquely as the sum of a self-dual and an anti-self-dual part.
A connection is called (anti-)self-dual if its curvature is an (anti-)self-dual form.

Theorem 70 *(Anti-)self-dual connections are always solutions to the Yang-Mills equations and are respectively called* ***(anti-)instantons****.*

Proof: The first equation is the Bianchi identity and hence it is always satisfied. The (anti-)self-dual conditions turns the second equation into the first, since $*F = \pm F$ and then

$$D * F = \pm DF = 0.$$

□

The following theorem shows that the topology of P gives, via the sign of its second Chern number c_2, a way of knowing if the absolute minimum of the action is an instanton or an anti-instanton.

Theorem 71 *Let $c = 8\pi c_2$ and $S_\pm = \int_M Tr(F_\pm \wedge *F_\pm)$, where $F_\pm$ is the (anti-)self-dual part of F, then*

$$S = S_+ + S_-,$$

$$c = S_+ - S_-.$$

and so

$$S = 2S_+ - c = 2S_- + c.$$

Then, the absolute minimum of the Yang-Mills action functional is an instanton if $c > 0$, and an anti-instanton if $c < 0$.

Proof: Just compute

$$\begin{aligned} S &= \int_M Tr((F_+ + F_-) \wedge (*F_+ * F_-)) \\ &= \int_M Tr(F_+ \wedge *F_+) + Tr(F_- \wedge *F_-) + 2Tr(F_+ \wedge *F_-) \\ &= S_+ + S_-, \end{aligned} \tag{2.1}$$

where we used that $Tr(F_+ \wedge *F_-) = 0$, since the self-dual and anti-self-dual forms are orthogonal relative to the inner product $Tr(\cdot \wedge *\cdot)$. We can check this since:

$$Tr(F_+ \wedge *F_-) = -Tr(F_+ \wedge F_-),$$

$$Tr(F_+ \wedge *F_-) = Tr(F_- \wedge *F_+) = Tr(F_+ \wedge F_-).$$

To check the last equation in the theorem we have

$$\begin{aligned}
c &= \int_M Tr(F\wedge F) = Tr((F_+ + F_-)\wedge(F_+ + F_-)) \\
&= \int_M Tr(F_+\wedge F_+) + Tr(F_-\wedge F_-) + 2Tr(F_+\wedge F_-) \\
&= \int_M Tr(F_+\wedge *F_+) - Tr(F_-\wedge *F_-) \\
&= S_+ - S_-.
\end{aligned} \tag{2.2}$$

Summing these two equations we have

$$S = 2S_+ - c = 2S_- + c$$

and can conclude that in the case of $c > 0$ then the connection A is an absolute minimum of the action iff A is an instanton, in the case of $c < 0$ then the absolute minimum is an anti-instanton. □

2.2.2 The Dirac Monopole

Here we want to study a solution to Maxwell equations (8.4) in $M = \mathbb{R}\times\mathbb{R}^3 - \{0\}$, that correspond to a Coulomb magnetic field. So we want

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_z & -B_y \\ 0 & -B_z & 0 & B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix},$$

where magnetic field B is given by

$$B = \frac{g}{4\pi}\frac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z).$$

Here g is the magnetic charge of the particle placed at the origin. Let us go to spherical coordinates and view this as a problem in $M = \mathbb{R}\times(\mathbb{R}^+\times\mathbb{S}^2)$, if ϕ is the azimuthal angle, then

$$F = -\frac{g}{4\pi} sen\phi \; d\theta\wedge d\phi.$$

We can now restrict ourselves to $\mathbb{S}^2$ and find the gauge potential responsible for the field strength $-ieF$, or in other words the connection $-ieA$ that has this curvature. Since we are in a $U(1)$-bundle we have that $-ieF = -iedA$. If we write $A = A_\theta d\theta + A_\phi d\phi$, then the equation $F = dA$ becomes

$$\frac{\partial A_\phi}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi} = -\frac{g}{4\pi} sen\phi.$$

We will now solve this equation in two different open sets: the north one $N = \mathbb{S}^2 - \{\text{south pole}\}$ and the south one $S = \mathbb{S}^2 - \{\text{north pole}\}$. If we try a solution with $A_\phi = 0$ we have the following solutions

$$\begin{cases} A_S = \frac{g}{4\pi}(1+cos\phi)d\theta, \text{ in S} \\ A_N = \frac{g}{4\pi}(-1+cos\phi)d\theta, \text{ in N} \end{cases}$$

Notice that in fact these solutions are only defined in the indicated open sets since $d\theta$ is not defined on the poles, so in order to have well defined forms their $d\theta$ component must vanish on the poles.
In fact these two 1-forms define a connection. For this we must have

$$ieA_S - ieA_N = ie\frac{g}{4\pi}2d\theta = \psi^{-1}d\psi$$

for some transition function $\psi : N \cap S \longrightarrow U(1)$ of the bundle. So we conclude that ieA_S and ieA_N define a connection in a $U(1)$-bundle over $\mathbb{S}^2$ trivial over N and S and whose transition function is

$$\psi(\theta,\phi) = \exp\left(i\frac{ge}{2\pi}\theta\right).$$

From this we can impose the periodicity of the transition function in θ and get the **charge quantization** condition of Dirac (in units where $\hbar = 1$)

$$ge = 2\pi k \tag{2.3}$$

with $k \in \mathbb{Z}$. So the electric and magnetic charge must appear in discrete values related by the above condition. We can also calculate the integral over $\mathbb{S}^2$ of the first Chern class of this bundle,

$$\begin{aligned}
\int_{\mathbb{S}^2} c_1 &= \frac{1}{2\pi i}\int_{\mathbb{S}^2} -ieF \\
&= \frac{1}{2\pi i}\int_{\mathbb{S}^2} i\frac{eg}{4\pi} sen(\phi)\ d\theta d\phi \\
&= \frac{eg}{2\pi} \\
&= k,
\end{aligned} \tag{2.4}$$

using the charge quantization (2.3). So, we got an integer as expected.

2.2.3 The Bogomolny and Nahm Equations

2.2.3.1 Bogomolny Equations

The Bogomolny equations describe monopoles in $\mathbb{R}^3$ that arise by minimizing the functional

$$S_{YMH} = \int_{\mathbb{R}^3} \frac{1}{2}Tr(F \wedge *F) + \frac{1}{2}Tr(\nabla\phi \wedge *\nabla\phi)$$

where F is as usually, the curvature of a connection A in a G-bundle over $\mathbb{R}^3$ and ϕ is a matter field, i.e. a section of an associated vector bundle with standard fibre $V = \mathfrak{g}$ where G acts by adjoint representation. ϕ is usually called the **Higgs field**.

Such type action arises while one tries to give mass to the Yang-Mills particle, as in the case of electromagnetism where we have seen that the photon cannot be massive (see the chapter on QED), in the Yang-Mills case we are also forbidden to had a mass term in order to keep gauge invariance.

However, we have seen that there is way to make a give mass to the gauge fields, this is through the Higgs mechanism and involves the introduction of a potential for the Higgs field in the above Lagrangian. We shall call this theory **Yang-Mills-Higgs in 3-dimensions**.

Notice that we can start with Yang-Mills theory in 4 dimensions and impose that the gauge field is invariant under translations in the x_3 direction (using coordinates x_0, x_1, x_2, x_3 in Minkowski space), so

$$\tilde{A} = A_0 dx_0 + A_1 dx_1 + A_2 dx_2 + \phi dx_3$$

with A_i, ϕ independent of x_3, so we can write $\tilde{A} = A + \phi dx_3$ where A and ϕ are now defined in $\mathbb{R}^3$, the curvature of $\tilde{A}$ is

$$\begin{aligned}
\tilde{F} &= d\tilde{A} + \frac{1}{2}\left[\tilde{A},\tilde{A}\right] \\
&= dA + d\phi \wedge dx_3 + \frac{1}{2}\left[A + \phi dx_3, A + \phi dx_3\right] \\
&= dA + \frac{1}{2}[A,A] + d\phi \wedge dx_3 + [A, \phi dx_3] \\
&= F + \nabla\phi \wedge dx_3
\end{aligned}$$

So the Lagrangian is

$$\mathcal{L}_{YMH} = \frac{1}{2}Tr(F \wedge *F) + \frac{1}{2}Tr(\nabla\phi \wedge *\nabla\phi) + Tr(F \wedge *\nabla\phi),$$

since $Tr(F \wedge *\nabla\phi) = dTr(\phi, *F)$, this term is irrelevant in the action and the Lagrangian only depends in 3 dimensions and has the form

$$\mathcal{L}_{YMH} = \frac{1}{2}Tr(F \wedge *F) + \frac{1}{2}Tr(\nabla\phi \wedge *\nabla\phi).$$

We conclude that

Theorem 72 *Yang-Mills-Higgs theory in 3-dimensions is equivalent to Yang-Mills theory in 4-dimensions invariant under translations in one direction.*

Using this result we can use the (anti-)self-dual Yang-Mills equations for $\tilde{A}$ to get special solutions of the equations of motion of the Yang-Mills-Higgs theory in 3-dimensions. So, $*\tilde{F} = \pm\tilde{F}$ gives

$$F = \pm * \nabla\phi.$$

These are the so called **Bogomolny equations**.

2.2.3.2 Nahm's Equations

Also by dimensional reduction of the anti-self-dual Yang-Mills equations we can get the Nahm equations. These are obtained by considering a connection

$$A = T_1 dx_1 + T_2 dx_2 + T_3 dx_3$$

in which the T_i's only depend on x_0. The curvature is

$$F = dA + \frac{1}{2}[A, A] = \frac{dT_i}{ds} ds \wedge dx_i + \frac{1}{2}[T_i, T_j]\, dx_i \wedge dx_j,$$

where summation over repeated indexes is understood. Since $*(ds \wedge dx_i) = \epsilon_{ijk} dx_j \wedge dx_k$ and $*(dx_i \wedge dx_j) = \epsilon_{ijk} ds \wedge dx_k$ we have

$$*F = \frac{1}{2}\epsilon_{ijk}\frac{dT_i}{ds} dx_j \wedge dx_k + \frac{1}{2}\epsilon_{ijk}[T_i, T_j]\, ds \wedge dx_i.$$

The anti-self dual Yang-Mills equations $F = - * F$ in four dimensions give now the **baby Nahm equations**, which can be written

$$\frac{dT_i}{ds} + \epsilon_{ijk}[T_j, T_k] = 0. \tag{2.5}$$

The same procedure of dimensional reduction of the anti-self-dual Yang-Mills equations, but now of a connection $A = A_0 dx_0 + T_1 dx_1 + T_2 dx_2 + T_3 dx_3$, gives the full **Nahm equations**

$$\frac{dT_i}{ds} + [T_0, T_i] + \epsilon_{ijk}[T_j, T_k] = 0. \tag{2.6}$$

Chapter 3

Hyperkähler Geometry and Moduli Spaces

3.1 Hyperkähler Geometry

3.1.1 Quaternionic Vector Spaces

Here we list some facts about the quaternions, since these will be a good motivation for hyperkähler manifolds.

- The quaternions are a four dimensional algebra over $\mathbb{R}$, given by

  $$\mathbb{H} = \{a + ib + jc + kd,\ a, b, c, d \in \mathbb{R}\}$$

  with the multiplication rules

  $$i^2 = j^2 = k^2 = ijk = -1.$$

- We can think of the vector space $\mathbb{H}^n$ as $4n$-dimensional real vector space, or as an $\mathbb{H}$-module with multiplication on the right

  $$q \cdot (q_1, ..., q_n) = (q_1 q, ..., q_n q)$$

  by elements of $\mathbb{H}$.
  With such definitions, the right multiplication commutes with the linear action of $GL(n, \mathbb{H})$ acting by matrix multiplication on the left.

- $\mathbb{H}^n$ has a natural Euclidean metric arising from the fact that it is a $4n$-dimensional real vector space. Giving the global coordinates x_i^μ on $\mathbb{H}^n$, such that $q_i = x_i^0 + ix_i^1 + jx_i^2 + kx_i^3$, the metric is

  $$g = \sum_{i=0}^{n} dx_i^0 \otimes dx_i^0 + dx_i^1 \otimes dx_i^1 + dx_i^2 \otimes dx_i^2 + dx_i^3 \otimes dx_i^3.$$

- multiplication by i, j, k gives us 3 complex structures in $\mathbb{H}^n$, which will be denoted I, J, K. These are compatible with the metric, so we have 3 symplectic forms, given by

  $$\begin{aligned}
  \omega_1 &= \sum_{i=0}^{n} dx_i^0 \wedge dx_i^1 - dx_i^2 \wedge dx_i^3 \\
  \omega_2 &= \sum_{i=0}^{n} dx_i^0 \wedge dx_i^2 - dx_i^3 \wedge dx_i^1 \\
  \omega_3 &= \sum_{i=0}^{n} dx_i^0 \wedge dx_i^3 - dx_i^1 \wedge dx_i^2.
  \end{aligned}$$

  So, we can see $\mathbb{H}^n$ as a Kähler vector space in 3 different ways.

- Define the quaternionic unitary group $Sp(n)$ as the subgroup of $GL(n, \mathbb{H})$ preserving the metric g. It can be viewed as a subgroup of $GL(4n, \mathbb{R})$ and it also preserves the complex structures I, J, K and the symplectic forms $\omega_1, \omega_2, \omega_3$.

- Using each one of the i, j, k we can identify $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ by writing the holomorphic coordinates

$$\begin{aligned} a + ib + jc + kd &= (a + ib) + k(d + ic) \\ &= (a + jc) + i(b + jd) \\ &= (a + kd) + j(c + kb). \end{aligned}$$

  In these cases we have respectively chosen I, J, K to be the the complex structure under which the identification is made.

3.1.2 Hyperkähler Manifolds

We want to define hyperkähler manifolds to have Riemannian holonomy in $Sp(n)$. As pointed out previously, this group when represented in a vector space preserves: a Riemannian metric, 3 almost complex structures and 3 skew-symmetric forms. Since tensors preserved by the holonomy and parallel tensors are the same thing, this motivates

Definition 73 *An* ***hyperkähler manifold*** *is a Riemannian manifold together with* 3 *compatible and parallel complex structures I, J, K satisfying*

$$I^2 = J^2 = K^2 = IJK = -1.$$

Notice that this definition immediately implies that hyperkähler manifolds have holonomy in $Sp(n)$ being placed where we wanted in Berger's list.
Also notice that in an hyperkähler manifold we can define the 2-forms $\omega_1(\cdot, \cdot) = g(I\cdot, \cdot)$ and so on. Since they are defined in terms of g and I, J, K all parallel, the 2-forms ω_i are also parallel and hence also closed, to see this just use

$$\begin{aligned} d\omega_i(X, Y, Z) &= X \cdot \omega_i(Y, Z) - Y \cdot \omega_i(X, Z) + Z \cdot \omega_i(X, Y) \\ &\quad - \omega_i([X, Y], Z) + \omega_i([X, Z], Y) + \omega_i([Y, Z], X) \end{aligned}$$

and plug in the condition for the ω_i's being parallel

$$X \cdot \omega_i(Y, Z) = \omega_i(\nabla_X Y, Z) + \omega_i(Y, \nabla_X Z)$$

and use the fact that the Levi-Civita connection is torsion free to conclude that $d\omega_i = 0$. Just to have it stated

Lemma 74 *If $\nabla \omega_i = 0$, then $d\omega_i = 0$.*

We can conclude that an hyperkähler manifold is Kähler in 3 different ways, in fact as we shall see later there is a whole $\mathbb{S}^2$ of Kähler structures in an hyperkähler manifold.
Checking the integrability of I, J, K may be a hard thing to do, however there is the following nice result

Proposition 75 *Let (M, g) be a Riemannian manifold with almost complex structures I, J, K satisfying the quaternionic relations. Then M is hyperkähler if and only if $d\omega_i = 0$ for $i = 1, 2, 3$.*

Proof: If $IX = iX$ and $IY = iY$, then the integrability of I, i.e. the vanishing of the Nijhenius tensor is equivalent to

$$I\,[X,Y] = i\,[X,Y]\,,$$

but since $\iota_{[X,Y]}\omega_2 = \iota_{I[X,Y]}\omega_3$ the nondegeneracy of the ω_i's implies that the condition for the integrability of I is equivalent to

$$\iota_{[X,Y]}\omega_2 = i\iota_{[X,Y]}\omega_3.$$

We must now show that this condition is equivalent to the closeness of the 2-forms ω_2 and ω_3. Just compute

$$\iota_{[X,Y]}\omega_2 = [L_X, \iota_Y]\,\omega_2 = \underbrace{L_X\iota_Y\omega_2}_{=iL_X\iota_Y\omega_3} - \iota_Y L_X\omega_2$$

where we used $L_X\iota_Y\omega_2 = iL_X\iota_Y\omega_3$, since $IY = iY$. To evaluate the other term we use this and Cartan's magic formula

$$\begin{aligned}
\iota_Y L_X\omega_2 &= \iota_Y(\iota_X d + d\iota_X)\omega_2 \\
&= \iota_Y\iota_X d\omega_2 + i\iota_Y d\iota_X\omega_3 + i\iota_Y\iota_X d\omega_3 - i\iota_Y\iota_X d\omega_3 \\
&= i\iota_Y L_X\omega_3 + \iota_Y\iota_X d\omega_2 - i\iota_Y\iota_X d\omega_3.
\end{aligned}$$

If we group everything together, we get

$$\begin{aligned}
\iota_{[X,Y]}\omega_2 &= iL_X\iota_Y\omega_3 - i\iota_Y L_X\omega_3 - \iota_Y\iota_X d\omega_2 + i\iota_Y\iota_X d\omega_3 \\
&= i\,[L_X, \iota_Y]\,\omega_3 - \iota_Y\iota_X d\omega_2 + i\iota_Y\iota_X d\omega_3 \\
&= i\iota_{[X,Y]}\omega_3 - \iota_Y\iota_X d\omega_2 + i\iota_Y\iota_X d\omega_3
\end{aligned}$$

from this we conclude that $\iota_{[X,Y]}\omega_2 = i\iota_{[X,Y]}\omega_3$ if and only if $d\omega_2 = d\omega_3 = 0$. Doing the same thing for J, K we conclude that I, J, K are integrable if and only if $d\omega_1 = d\omega_2 = d\omega_3 = 0$. □

Remark 76 *This is a notorious difference between Kähler and hyperkähler geometry.*

Notice that in the Kähler case it is possible to have a Riemannian manifold (M, g) together with a compatible almost complex structure I, such that $d\omega = 0$ and hence M is symplectic, but if $\nabla I \neq 0$, then I is not integrable and (M, I) is not a complex manifold, this is what we call almost Kähler geometry.

The previous proposition shows that there is no analogue in hyperkähler geometry. Looking at the proof we understand that the quaternionic algebra that I, J, K must obey, is telling us that the integrability of one of the almost complex structures is related to the closeness of the other two 2-forms.

Notice that

$$Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$$

where $U(2n)$ is the group preserving an almost complex structure and $Sp(2n, \mathbb{C})$ the group preserving a non-degenerate skew-symmetric complex form.

In fact, if (M, g) is an hyperkähler manifold we can in particular choose the complex structure given by I and verify that the complex symplectic form $\omega_2 + i\omega_3$ is holomorphic with respect to I (i.e. of type $(2, 0)$). This construction explicitly gives the above decomposition of $Sp(n)$ and shows that every hyperkähler manifold is a **holomorphic symplectic manifold**, i.e. a complex manifold with a symplectic holomorphic 2-form, i.e. of type $(2, 0)$.

In fact there is a converse result to the one just showed, but we shall not go that way.

3.1.3 Constructions in Hyperkähler Geometry

Our goal here is to describe hyperkähler reduction as a technique to obtain hyperkähler manifolds, however for completeness we will also describe the twistor space construction. These are two different ways to construct hyperkähler manifolds, in the twistor space the emphasis of the construction is placed on the complex structures I, J, K while in the hyperkähler reduction the emphasis is placed in the symplectic structures $\omega_1, \omega_2, \omega_3$.

3.1.3.1 Twistor Space

In the previous section we showed that an hyperkähler manifold (M, g) is a Kähler manifold with respect to 3 different complex structures I, J, K with corresponding symplectic structures $\omega_1, \omega_2, \omega_3$. However, if we have real numbers b, c, d such that $b^2 + c^2 + d^2 = 1$, then

$$(bI + cJ + dK)^2 = -1$$

and $bI + cJ + dK$ is another almost complex structure on M. It is obviously integrable and compatible with the hyperkähler metric g, having symplectic form $b\omega_1 + c\omega_2 + d\omega_3$, so we may conclude that

Summary 77 *An hyperkähler manifold is Kähler in a whole $\mathbb{S}^2$ of ways.*

Let $u = (b, c, d) \in \mathbb{S}^2$, define the complex structure on M, $I_u = bI + cJ + dK$. Since $\mathbb{S}^2 \simeq \mathbb{CP}^1$ it is also a complex manifold with canonical complex structure denoted by I_0.

Definition 78 *The product $Z = M \times \mathbb{CP}^1$ together with the integrable complex structure*

$$I(X, Y) = (I_u X, I_0 Y)$$

*for $(X, Y) \in T_{(x,u)}Z$ defines a $2n + 1$ complex dimensional manifold called the **twistor space of M.***

In fact the projection $Z \longrightarrow \mathbb{CP}^1$ is holomorphic with respect to the complex structures I in Z and I_0 in $\mathbb{CP}^1$.
There are some results that state conditions for holomorphic projections onto $\mathbb{CP}^1$ to come from a twistor space construction like the one above. In those cases each fibre is not only a complex manifold (M, I_u), but also an hyperkähler manifold.

3.1.3.2 Kähler Reduction

The hyperkähler reduction is similar to Kähler reduction, so to describe it we will start by reviewing Kähler reduction.

The following theorem is due to Mardsen, Weinstein and Meyer and gives a way of obtaining a symplectic manifold out of another, by fixing some degrees of freedom which are symmetries of a hamiltonian Lie group action.

Theorem 79 *(Symplectic Reduction) Let a compact Lie Group G act in an hamiltonian way on a symplectic manifold (M, ω), with moment map $\mu : M \longrightarrow \mathfrak{g}^*$. Assume that G acts freely on $\mu^{-1}(0)$. Then the quotient*

$$\mu^{-1}(0)/G$$

*is a symplectic manifold of dimension $dimM - 2dimG$, called the **symplectic quotient** or **symplectic reduction**.*

Proof: We divide the proof in three parts:

- To show that $\mu^{-1}(0)$ is an embedded submanifold of dimension $dimM - dimG$, we must show that 0 is a regular value of μ.
  If $p \in \mu^{-1}(0)$, then
  $$im\ d\mu_p \quad = \quad \{d\mu_p(v)(\cdot) = \omega(\cdot, v) \text{ with } v \in T_pM\},$$
  but since G acts freely on $\mu^{-1}(0)$ and G is compact, the infinitesimal stabilizer of the action at p in just $\{0\}$, so for all $\xi \in \mathfrak{g}$ we have that $\xi^*_p \neq 0$. By nondegeneracy of ω, there exists $v \in T_pM$ with $\omega(\xi^*_p, v) \neq 0$, so we conclude that $imd\mu_p = \mathfrak{g}^*$ and that that $\mu^{-1}(0)$ is an embedded submanifold of dimension $dimM - dimG$.

- Since the G-action is free in $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ is a manifold of dimension $dimM - 2dimG$.
- To see that the symplectic structure descends to the quotient, consider the diagram

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"'] & M \\ \mu^{-1}(0)/G & \end{tikzcd}$$

We shall show that there exists a symplectic form ω^{red} in $\mu^{-1}(0)/G$ such that $\pi^*\omega^{red} = i^*\omega$.
Since for $p \in \mu^{-1}(0)$, $T_p\mu^{-1}(0) = kerd\mu_p$ we have for $\xi \in \mathfrak{g}$

$$\iota_{\xi^*} i^*\omega = i^* d\mu^{\xi} = 0.$$

So $i^*\omega$ vanishes along the orbits of G in $\mu^{-1}(0)$, since it is obviously invariant under the G-action (it is an hamiltonian action). We can define a symplectic form in the quotient

$$\omega^{red}(X,Y) = i^*\omega(\tilde{X},\tilde{Y})$$

where $X, Y \in T_xM$ and $\tilde{X},\tilde{Y}$ are arbitrary lifts to T_pM with $p \in \pi^{-1}(x)$. This is easily checked to be well defined and it is obvious that $\pi^*\omega^{red} = i^*\omega$, and the closedness of ω^{red} is a consequence of the exterior derivative commuting with the pull-back.

□

In the case where M is Kähler, and the G-action preserves the complex structure as well, then it also comes down to the quotient, and $\mu^{-1}(0)/G$ is a Kähler manifold.

3.1.3.3 Hyperkähler Reduction

For the hyperkäher case we start with an hyperhamiltonian action of a Lie group G on M. This is an action which is hamiltonian with respect to each of the Kähler forms ω_i with corresponding moment map μ_i. We can then think of an hyperkähler moment map as

$$\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

with $\mu = (\mu_1, \mu_2, \mu_3)$. The ananologous to the Mardsen-Weinstein-Meyer theorem is

Theorem 80 *Let a compact Lie group G act on a hyperkähler manifold M in an "hyperhamiltonian" way. If, in addition, the G-action preserves the 3 complex structures I, J, K and acts freely on $\mu^{-1}(0) = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$, we have that the hyperkähler quotient*

$$\mu^{-1}(0)/G = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/G,$$

is an hyperkähler manifold of dimension $dimM - 4dimG$.

Proof: Using the Kähler case it is immediate that the hyperkähler quotient is contained in a Kähler manifold with respect to each one of the descending Kähler structures ω_i^{red}. What remains to be shown is that the quaternionic relations between these Kähler structures are well defined on the quotient. We shall construct a model for the tangent space in the quotient, by identifying it with a subspace of the tangent space upstairs. Then one needs to show that this is actually invariant under I, J, K.
Let $x \in \mu^{-1}(0)/G$ and $p \in \pi^{-1}(x) \hookrightarrow M$. We shall identify $T_x\mu^{-1}(0)/G$ with the orthogonal complement to both the G-orbit and to $(kerd\mu_p)^\perp$, i.e.

$$(T_x\mu^{-1}(0)/G)^\perp = T_pG \cdot p + (kerd\mu_p)^\perp$$

or more explicitly

$$(T_x\mu^{-1}(0)/G)^{\perp} = \{\xi^*, \nabla\mu_i^{\xi} \text{ for } i = 1,2,3 \text{ and } \xi \in \mathfrak{g}\}.$$

Our goal is to show that $T_x\mu^{-1}(0)/G$ is preserved by I, J, K, but this is equivalent to showing that its orthogonal complement is itself preserved by I, J, K

$$d\mu_1^{\xi}(\cdot) = \omega(\xi^*, \cdot) = g(I\xi^*, \cdot).$$

So we conclude that $\nabla\mu_1^{\xi} = I\xi^*$, doing the same thing for the other moment maps we get that:

$$(T_x\mu^{-1}(0)/G)^{\perp} = \{\xi^*, I\xi^*, J\xi^*, K\xi^* \text{ for all } \xi \in \mathfrak{g}\}$$

and so it is preserved by I, J, K as we wanted to show. $\square$

Notice that we can think of the holomorphic symplectic structure of M given by the complex structure I and the holomorphic symplectic form $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$. It is preserved hamiltonianly by the complex $G_{\mathbb{C}}$-action with holomorphic moment map $\mu_2 + i\mu_3$. One can reformulate the hyperkähler quotient as

$$\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)/G.$$

In the next chapter we shall see a case where exploring the action of $G_{\mathbb{C}}/G$ in $\mu_{\mathbb{C}}^{-1}(0)$ and taking into account the intersections with $\mu_1^{-1}(0)$, then it may suffice to study the holomorphic version of the symplectic quotient:

$$\mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}.$$

3.1.4 Nontrivial Examples of Hyperkähler Metrics

An hyperkähler manifold must have dimension a multiple of 4, and in the case of a 4 dimensional hyperkähler manifold M, Berger's classification theorem implies that M has holonomy $Sp(1) = SU(2)$ and hence an hyperkähler metric is the same thing as a Ricci-flat Kähler metric.
Here, we shall focus on hyperkähler metrics on a special kind of 4-manifolds M that can be obtained as $\mathbb{S}^1$-bundles over an open set of $\mathbb{R}^3$. Moreover we assume that the natural $\mathbb{S}^1$-action is "hyperhamiltonian". We then have an hyperkähler moment map $\mu : M \longrightarrow \mathbb{R}^3$ such that

$$d\mu_i = \iota_{\frac{\partial}{\partial\theta}}\omega_i,$$

where $\frac{\partial}{\partial\theta}$ is the infinitesimal generator of the $\mathbb{S}^1$-action.
We get that $d\mu_1(\cdot) = g(I\frac{\partial}{\partial\theta}, \cdot)$, and so on. We conclude that

$$\begin{aligned}
\nabla\mu_1 &= I\frac{\partial}{\partial\theta} \\
\nabla\mu_2 &= J\frac{\partial}{\partial\theta} \\
\nabla\mu_3 &= K\frac{\partial}{\partial\theta},
\end{aligned}$$

and hence $\frac{\partial}{\partial\theta}, \nabla\mu_1, \nabla\mu_2, \nabla\mu_3$ are all linearly independent. This shows that away from the vanishing points of $\frac{\partial}{\partial\theta}$, the moment maps μ_1, μ_2, μ_3 are in fact local coordinates on the base. The Hawking-Gibbons ansatz to write the metric in M is

$$g = V\sum_{i=1}^{3} d\mu_i \otimes d\mu_i + \frac{1}{V}(d\theta + \sum_{i=1}^{3} a_i d\mu_i)^2$$

where V is a function on the base and $d\theta + \sum_{i=1}^{3} a_i d\mu_i$ is a connection 1-form on the $\mathbb{S}^1$-bundle.
To calculate the symplectic forms, notice that we may write

$$g = \eta_0 \otimes \eta_0 + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3$$

with

$$\eta_0 = \frac{1}{\sqrt{V}}\left(d\theta + \sum_{i=1}^{3} a_i d\mu_i\right) \ , \ \eta_i = \sqrt{V} d\mu_i$$

satisfying $I^*\eta_0 = \eta_1$, $I^*\eta_2 = \eta_3$, then one conclude that

$$\begin{aligned}\omega_1 &= \eta_0 \wedge \eta_1 + \eta_2 \wedge \eta_3 \\ &= \left(d\theta + \sum_{i=1}^{3} a_i d\mu_i\right) \wedge d\mu_1 + V d\mu_2 \wedge d\mu_3.\end{aligned}$$

The same thing can be done with respect to J, K and we get a similar expression, however the conditions to impose on each of the symplectic forms will lead us to simmilar cnclusions regarding the form of $a = \sum_{i=1}^{3} a_i d\mu_i$ and V, so we may proceed only with ω_1.
To check that this is in fact a hyperkähler metric we just need the ω_i's to be closed, this leads to

$$\begin{aligned}0 &= d\left(\sum_{i=1}^{3} a_i d\mu_i\right) \wedge d\mu_1 + dV \wedge d\mu_2 \wedge d\mu_3 \\ &= \sum_{i=1}^{3} \varepsilon_{1ji} \frac{\partial a_i}{\partial \mu_j} d\mu_1 \wedge d\mu_2 \wedge d\mu_3 + \frac{\partial V}{\partial \mu_1} d\mu_1 \wedge d\mu_2 \wedge d\mu_3\end{aligned} \tag{3.1}$$

and so we got that $\frac{\partial V}{\partial \mu_1} = \sum_{i=1}^{3} \varepsilon_{1ij} \frac{\partial a_i}{\partial \mu_j}$, this together with the condition for the other two symplectic forms yields $dV = *da$ or

$$grad\ V = curl\ a$$

where we regard a as a vector field on $\mathbb{R}^3$. In particular V is harmonic.
With the general form of the metric, provided that $dV = *d\alpha$, we now state some important cases regarding the choice of V.

- $V = \dfrac{1}{r}$ with $r = \sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}$ is the Euclidean metric on $\mathbb{R}^4$
- $V = \dfrac{1}{r} + c$ is the so called Taub-NUT metric on $\mathbb{R}^4$.

3.2 Moduli Spaces

3.2.1 Nahm and Bogomolny Equations Revisited

We will now construct a moment map that will be very useful in the next chapter. Here this construction will be used to give a unified formalism for both the Nahm equations and the Bogomolny equations. In fact what we shall use is an infinite dimensional generalization of this.
Let G be a Lie group with a bi-invariant inner product $\langle \cdot, \cdot \rangle$ for the adjoint action on $M = \mathfrak{g} \otimes \mathbb{H}$. Notice that M is a quaternion vector space whose elements are of the form $A = A_0 + iA_1 + jA_2 + kA_3$. We equip M with the hyperkähler structure given by

- the metric

$$g(A, A') = \sum_{i=0}^{3} \langle A_i, A'_i \rangle$$

- the complex structures I, J, K given by multiplication by i, j, k.

Let $\xi \in \mathfrak{g}$, then

$$\begin{aligned} \xi_A^* &= [\xi, A_0] + i[\xi, A_1] + j[\xi, A_2] + k[\xi, A_3] \\ &= [\xi, A]\,. \end{aligned}$$

We may now identify $\mathfrak{g}^* \simeq \mathfrak{g}$ via the bi-invariant inner product so that we may write

$$\langle \xi, (d\mu_i)_A(\cdot) \rangle = \omega_i([\xi, A]\,, \cdot),$$

to calculate the moment maps. Evaluating the right hand side we have

$$\begin{aligned} \omega_1([\xi, A]\,, B) &= g(I[\xi, A]\,, B) \\ &= \langle [\xi, A_0]\,, B_1 \rangle - \langle [\xi, A_1]\,, B_0 \rangle + \langle [\xi, A_2]\,, B_3 \rangle - \langle [\xi, A_3]\,, B_2 \rangle \\ &= \langle \xi, [A_0, B_1] - [A_1, B_0] + [A_2, B_3] - [A_3, B_2] \rangle \end{aligned}$$

So we can conclude that $\mu_1(A) = [A_0, A_1] + [A_2, A_3]$. Doing the same thing but now for $i = 2, 3$ we have

$$\begin{aligned} \mu_1(A) &= [A_0, A_1] + [A_2, A_3] \\ \mu_2(A) &= [A_0, A_2] + [A_3, A_1] \\ \mu_3(A) &= [A_0, A_3] + [A_1, A_2]\,. \end{aligned} \tag{3.2}$$

The holomorphic moment map of $\omega_2 + i\omega_3$, is

$$\mu_2 + i\mu_3 = [A_0 + iA_1, A_2 + iA_3]\,.$$

By viewing $M = \mathfrak{g} \otimes \mathbb{H}$ as $\mathfrak{g} \otimes \mathbb{C}^2$ we can write complex coordinates $\alpha = A_0 + iA_1$ and $\beta = A_2 + iA_3$ and get

$$\begin{aligned} \mu_{\mathbb{C}}(A) &= [\alpha, \beta] \\ \mu_r(A) &= \frac{1}{2i}[\alpha^*, \alpha] + \frac{1}{2i}[\beta^*, \beta]\,. \end{aligned}$$

3.2.1.1 Nahm Equations

We must now consider the infinite dimensional affine space $\mathcal{A}$ of maps $I \longrightarrow \mathfrak{g} \otimes \mathbb{H}$, and $\mathcal{G}$ is the gauge group of maps $I \longrightarrow G$. This is in fact the group of gauge transformations of a trivial G principal bundle over I. Using the previous formalism, we can obtain the Nahm equations by plugging in the the moment map (3.10),

$$\begin{cases} A_0 = \frac{d}{ds} + T_0(s) \\ A_1 = T_1(s) \\ A_2 = T_2(s) \\ A_3 = T_3(s), \end{cases} \tag{3.3}$$

where the $T_i : I \longrightarrow \mathfrak{g}$ for $i = 0, 1, 2, 3$ are $\mathfrak{g}$ valued functions on the interval $I \subset \mathbb{R}$. This gives

$$\frac{dT_i}{ds} + [T_0, T_i] + \epsilon_{ijk}[T_j, T_k] = 0. \tag{3.4}$$

for $i = 1, 2, 3$. And we can in fact interpret $T_0 ds$ as a connection 1-form on the trivial principal G-bundle over I. Moreover, the group of gauge transformations acts by gauge transformations in T_0 and by conjugation in T_1, T_2, T_3. This shows that the space of solutions to the Nahm equations (modulo gauge transformations), can be obtained by hyperkähler reduction from an infinite dimensional affine space $\mathcal{A}$. In fact, for some particular boundary conditions this moduli space can be proved to be a smooth, finite dimensional hyperkähler manifold, see [2] for a detailed construction, or in a more expository style [18]. In the next chapter we will study one of these examples due to Kronheimer, check [25].

3.2.1.2 Bogomolny Equations

In a similar way we can get the Bogomolny equations from this formalism. For that we replace M by the affine space of maps $\mathbb{R}^3 \longrightarrow \mathfrak{g} \otimes \mathbb{H}$, this is an hyperkähler infinite dimensional vector space as before and we shall denote it by $\mathcal{A}$ also. The gauge group $\mathcal{G}$ is now the group of maps $\mathbb{R}^3 \longrightarrow G$. To get the Bogomolny equation we set

$$\begin{cases} A_0 = \phi(x^1, x^2, x^3) \\ A_1 = \nabla_{\frac{\partial}{\partial x^1}} = \frac{\partial}{\partial x^1} + T_1(x^1, x^2, x^3) \\ A_2 = \nabla_{\frac{\partial}{\partial x^2}} = \frac{\partial}{\partial x^2} + T_2(x^1, x^2, x^3) \\ A_3 = \nabla_{\frac{\partial}{\partial x^3}} = \frac{\partial}{\partial x^3} + T_3(x^1, x^2, x^3) \end{cases} \tag{3.5}$$

with $T_i : \mathbb{R}^3 \longrightarrow \mathfrak{g}$ for $i = 0, 1, 2, 3$. In this way we can interpret $T_1 dx^1 + T_2 dx^2 + T_3 dx^3$ as being a connection 1-form on a trivial principal G-bundle over $\mathbb{R}^3$. The gauge group acts on ϕ by conjugation and on this connection by gauge transformations. We get

$$F = - * \nabla\phi.$$

Now in the same way as before, the moduli space of solutions of the Bogomolny equations up to gauge transformations and with particular boundary conditions can be shown to be a smooth finite dimensional hyperkähler manifold, see [2].

3.2.2 Equivalence of the Moduli Spaces

We have just described how to get both the Nahm and the Bogomolny equations out of the same formalism. We also saw that when the moduli spaces of solutions to these equations are well defined smooth manifolds they have hyperkähler structures. In fact is a result due to Hitchin in [17], that with certain extra conditions, both these moduli spaces can be identified. More recently Nakajima proved in [27] that there is an hyperkäler isometry between both these moduli spaces.
Hitchin's approach encoded a connection with algebraic geometry. This was done by showing that the same spectral curve in $T\mathbb{CP}^1$ can arise from both the Nahm and the Bogomolny equations.

In order to easily obtain the connection with the spectral curve in $T\mathbb{CP}^1$, let us describe a Lax pair form of the moment map (3.10). This is related to the twistor space description, by introducing the moment map

$$(\mu_3 + i\mu_2) + 2\mu_1\zeta - (\mu_3 - i\mu_2)\zeta^2, \tag{3.6}$$

where ζ is the usual coordinate in $\mathbb{CP}^1 - \{\infty\} \simeq \mathbb{C}$. The vanishing of the moment map (3.10) is equivalent to the vanishing of the moment map (3.6) for all ζ's. This moment map can be written

$$\left[A_0 + iA_1 + (A_3 - iA_2)\zeta, A_3 + iA_2 - 2iA_1\zeta + (A_3 - iA_2)\zeta^2\right]. \tag{3.7}$$

3.2.2.1 Spectral Curve from Nahm

Start by gauging T_0 to 0, i.e. we choose a gauge transformation $g : I \longrightarrow G$, such that $g \cdot T_0 = 0$. This can easily been done solving

$$g^{-1}\frac{dg}{ds} = T_0.$$

Then the Nahm equations (3.4) reduce in this gauge to the baby Nahm equations

$$\frac{dT_i}{ds} + \epsilon_{ijk}\left[T_j, T_k\right] = 0, \tag{3.8}$$

for $i = 1, 2, 3$ and we have summation implicit over repeated indexes. We now introduce the usual coordinates (ζ, η) and $(\tilde{\zeta}, \tilde{\eta})$ in $T\mathbb{CP}^1$, related in $\zeta, \tilde{\zeta} \in \mathbb{C} - \{0\}$ by

$$(\zeta, \eta) = \left(\frac{1}{\zeta}, -\frac{\eta}{\zeta^2}\right).$$

This are just the usual trivializations of $T\mathbb{CP}^1$. We now define

$$\begin{aligned} A(\zeta) &= T_3 + iT_2 - 2iT_1\zeta + (T_3 - iT_2)\zeta^2 \\ A_+(\zeta) &= A_0 + iT_1 + (T_3 - iT_2)\zeta, \end{aligned}$$

which allows us to write the baby Nahm equations in the form of a Lax pair

$$\frac{dA}{ds} + [A_+, A] = 0. \tag{3.9}$$

Remark 81 *This can be directly obtained from the Lax pair form (3.7).*

A general property of Lax pairs is that all coefficients of the characteristic polynomial are conserved, i.e. do not depend on s.

Lemma 82

$$\frac{d}{ds} tr(A^n) = 0.$$

Proof: Just compute,

$$\begin{aligned} \frac{d}{ds} tr(A^n) &= tr\left(n\frac{dA}{ds}A^{n-1}\right) \\ &= tr\left(n\left[A_+, A\right]A^{n-1}\right) \\ &= 0, \end{aligned}$$

since the trace is cyclic. □

This allows us to construct a well defined curve in $T\mathbb{CP}^1$ given by

$$det(\eta - A(\zeta)) = 0, \tag{3.10}$$

this is the so called **spectral curve** as obtained from the Nahm equations.

3.2.2.2 Spectral Curve from Bogomolny

We shall now see how to obtain the spectral curve from the monopole data. Here we also use the Lax pair formalism (3.7) and the Lax form of the Bogomolny equation

$$\left[-\phi + i\nabla_{\frac{\partial}{\partial x^1}} + (\nabla_{\frac{\partial}{\partial x^3}} - i\nabla_{\frac{\partial}{\partial x^2}})\zeta, \nabla_{\frac{\partial}{\partial x^3}} + i\nabla_{\frac{\partial}{\partial x^2}} - 2i\nabla_{\frac{\partial}{\partial x^1}}\zeta + (\nabla_{\frac{\partial}{\partial x^3}} - i\nabla_{\frac{\partial}{\partial x^2}})\zeta^2\right] = 0. \tag{3.11}$$

Introducing complex coordinates in $T\mathbb{CP}^1$ via a parametrization of the lines in $\mathbb{R}^3$, we can put

$$\eta = x^3 + ix^2 - 2ix^1\zeta + (x^3 - ix^2)\zeta^2.$$

Here (ζ, η) are as before coordinates on an open set of $T\mathbb{CP}^1$. This allows us to interpret equation (3.11) as the kernel of the operator $\nabla_u + i\phi$ along each line u in $\mathbb{R}^3$. For $SU(2)$ monopoles this kernel may be viewed as a rank-2 vector bundle over $T\mathbb{CP}^1$, where the fibre over each point $u \in T\mathbb{CP}^1$ is the 2 dimensional vector space generated by the solutions of

$$\nabla_u + i\phi = 0.$$

However, these solutions may or not be L^2 along the line u. To be L^2, they must decay at $\pm\infty$. We get two line bundles $L^\pm$ whose fibre at each u is spanned by the solution to $\nabla_u + i\phi = 0$ which decays at the respective end of the line u.
The **spectral curve** is precisely given by the set of lines u in $\mathbb{R}^3$, or points in $T\mathbb{CP}^1$, where these two line bundles agree and where we have solutions which decay at both ends.

Chapter 4

Cotangent Spaces of Lie Groups as Moduli Spaces

In this chapter, let the Latin indexes i, j, k run from 1 to 3 and the Greek ones run from 0 to 3.

Our goal in this chapter is to explicitly illustrate the construction of an hyperkähler moduli space. We will look at two examples of moduli spaces :

1. The complexification $K_{\mathbb{C}}$ of a compact Lie group K, can be identified with a moduli space for solutions to the baby Nahm equations (2.5) via Kähler reduction. We will not go over the details, in this case, but just present the setup for the construction.

2. The cotangent bundle $T^*K_{\mathbb{C}}$ can be identified with a moduli space of solutions to the Nahm equations (3.4). The construction uses hyperkähler reduction via the formalism presented in section 3.2.1 which will be repeated here. In this case we will give all the details.

4.1 A complex Lie Group is Kähler

Let K be a compact Lie group and $K_{\mathbb{C}}$ its complexification. Fix $\langle \cdot, \cdot \rangle$ an Ad-invariant inner product on the Lie algebra $\mathfrak{k}$. Then

$$T^*K \simeq TK \simeq K \times \mathfrak{k} \simeq K_{\mathbb{C}},$$

where the first isomorphism comes from the inner product, the second one by left invariance, and the last one via the polar decomposition

$$\begin{aligned} \Phi : K \times \mathfrak{k} &\longrightarrow K_{\mathbb{C}} \\ (g, Y) &\longmapsto g \exp iY. \end{aligned} \tag{4.1}$$

So, in fact $K_{\mathbb{C}}$ has both its natural complex structure and the tautological symplectic structure from T^*K. They are compatible and hence $K_{\mathbb{C}}$ is Kähler. Here we shall obtain this Kähler structure via Kähler reduction from an infinite dimensional vector space.

- The vector space is $\mathcal{A} = \{A_0 + iA_1\}$ and can be identified with a space of maps $T : [0,1] \longrightarrow \mathfrak{k} \otimes \mathbb{C}$ by

$$\begin{aligned} T_0(s) &\longmapsto A_0(s) = \frac{d}{ds} + T_0(s) \\ T_1(s) &\longmapsto A_1(s). \end{aligned}$$

- In $\mathcal{A}$ we consider the the metric

$$h(A, A') = \int_0^1 \left(\langle T_0(s), T_0'(s) \rangle + \langle T_1(s), T_1'(s) \rangle \right) \, ds. \tag{4.2}$$

with $A=(T_0,T_1), A'=(T_0',T_1')\in\mathcal{A}$. This, together with the complex structure given by multiplication by i, give $\mathcal{A}$ a Kähler structure. The Kähler form ω is given by

$$\omega(A,A') = \int_0^1 \left(\langle T_0(s), T_1'(s)\rangle - \langle T_1(s), T_0'(s)\rangle\right)\, ds \tag{4.3}$$

and we will sometimes use the short notation $\omega = \int_0^1 dT_0 \wedge dT_1$.

- Let $\mathcal{G}_0 = \left\{g:[0,1]\longrightarrow K, \text{ of class } C^2 \text{ with } g(0)=g(1)=1_K\right\}$, i.e. the space of C^2 loops in G based at the identity 1_K. We let $\mathcal{G}_0$ act on $\mathcal{A}$ by gauge transformations

$$g\cdot(T_0,T_1) = (gT_0g^{-1} + g\frac{dg^{-1}}{ds}, gT_1g^{-1}).$$

  The Lie algebra of $\mathcal{G}_0$ is the space of loops in the Lie algebra $\mathfrak{k}$, i.e

$$Lie\mathcal{G}_0 = \left\{g:[0,1]\longrightarrow \mathfrak{k}\ , \text{ of class } C^2 \text{ with } g(0)=g(1)=0\right\}.$$

- This action is hamiltonian for the symplectic structure $\omega(\cdot,\cdot)=g(I\cdot,\cdot)$. We use the inner product in $Lie\mathcal{G}_0$ given by

$$(T(s),T'(s)) = \int_0^1 \langle T(s),T'(s)\rangle\ ds$$

  to identify $Lie\mathcal{G}_0$ with its dual. Hence, the moment map must satisfy

$$h(\xi, d\mu(\cdot)) = \omega(\cdot,\xi^*),$$

  for $\xi\in Lie\mathcal{G}_0$. One finds,

$$\mu(A) = [A_0,A_1] = \frac{dT_1}{ds} + [T_0,T_1]\,. \tag{4.4}$$

  So notice that the equation $\mu(A)=0$ is just a baby Nahm equation. Hence the quotient $\mu^{-1}(0)/\mathcal{G}_0$ is also the moduli space of solutions to the baby Nahm equations up to gauge transformations.

The claim is that we can identify the Kähler quotient $\mu^{-1}(0)/\mathcal{G}_0$, with $K_{\mathbb{C}}$. In fact, this can be easily understood, since the baby Nahm equation (4.4) has a trivial solution in each $\mathcal{G}$-orbit, where $\mathcal{G}=\{g: [0,1]\longrightarrow K\}$.
There is only one $g\in\mathcal{G}$ such that $g(0)=1_K$ and $g\cdot T_0=0$. With this gauge transformation, the baby Nahm equation is now $\frac{d}{ds}(g\cdot T_1)=0$, so that $g\cdot T_1 = gT_1g^{-1}=\eta\in\mathfrak{k}$ is constant. Then the solution is such that $g\cdot(T_0,T_1)=(0,\eta)$, and this allows the identification:

$$\begin{array}{rcl} \psi:\mu^{-1}(0)/\mathcal{G}_0 & \longrightarrow & K\times\mathfrak{k}\simeq K_{\mathbb{C}} \\ (T_0,T_1) & \longmapsto & (g(1),T_1(0)) \end{array}$$

Identifying $T_{[A]}(\mu^{-1}(0)/\mathcal{G}_0)$ with $T_A(\mathcal{O}_A)^{\perp}$ and transporting the metric (4.2) to T^*K, together with the natural complex structure on $K_{\mathbb{C}}$, gives origin to a Kähler structure.
Let's now compute the Kähler form, in any point $(1_K,\eta)\in\{1_K\}\times\mathfrak{k}$, and extend it later to the whole of T^*K. Let $(A,B),(A',B')\in T_{(1_k,\eta)}T^*K$. We want to calculate

$$(\psi^{-1})^*\omega((A,B),(A',B')) = \omega(\psi_*^{-1}(A,B),\psi_*^{-1}(A',B')). \tag{4.5}$$

Remark 83 *1. Note that $\psi_*^{-1}(A,B)\in T_{\psi^{-1}(1_k,\eta)}\mu^{-1}(0)/\mathcal{G}_0$, so in (4.5) we mean any lift of this vector to $\mu^{-1}(0)\hookrightarrow\mathcal{A}$. This is no problem since ω is degenerate along the $\mathcal{G}_0$-orbit.*

2. *Since we need to compute $\psi_*^{-1}(A,B)$, it is useful to have an explicit formula for ψ^{-1}. For that notice that we can write $g\cdot(T_0,T_1)=(0,\eta)$ as*

$$\begin{cases} T_0 = -\frac{dg^{-1}}{ds}g \\ T_1 = g^{-1}\eta g. \end{cases} \tag{4.6}$$

So we can use these equations to construct the inverse map

$$\begin{aligned} \psi^{-1} : K\times\mathfrak{k} &\longrightarrow \mu^{-1}(0)/\mathcal{G}_0 \\ (u,\eta) &\longmapsto (T_0,T_1),\ \textit{using (4.6)}, \end{aligned}$$

with the gauge element g, an arbitrary path starting at 1_k and ending at u. This can be checked to give the wanted inverse map.

Writing $(A,B)=\frac{d}{dt}\Big|_{t=0}\gamma(t)$ with $\gamma(t)=(\exp(tA),\eta+tB)$, the pushforwarded vectors are

$$\begin{aligned} \psi_*^{-1}(A,B) &= \frac{d}{dt}\Big|_{t=0}\psi^{-1}(\exp(tA),\eta+tB) \\ &= \frac{d}{dt}\Big|_{t=0}(-\frac{dg^{-1}}{ds}g, g^{-1}(\eta+tB)g) \end{aligned}$$

with $g\in\mathcal{G}$ a path with $g(0)=1_K$ and $g(1)=\exp(tA)$. We pick $g(s)=\exp(stA)$ and proceed with the calculation

$$\begin{aligned} \psi_*^{-1}(A,B) &= \frac{d}{dt}\Big|_{t=0}(-\frac{de^{-stA}}{ds}e^{stA}, e^{-stA}(\eta+tB)e^{stA}) \\ &= (A, B-s\,[\eta,A]). \end{aligned}$$

Finally, plugging this back in (4.5) gives

$$\begin{aligned} (\psi^{-1})^*\omega((A,B),(A',B')) &= \int_0^1 ds\,\langle A, B'-s\,[\eta,A']\rangle - \langle A', B-s\,[\eta,A]\rangle \\ &= \langle A,B'\rangle - \langle A',B\rangle + \langle \eta,[A,A']\rangle\,. \end{aligned} \tag{4.7}$$

This agrees with the tautological symplectic structure in T^*K given by seeing it as a cotangent bundle. To extend to the whole of T^*K one just needs to notice that the map ψ is anti-equivariant by the $\mathcal{G}_R$ action on $\mu^{-1}(0)/\mathcal{G}_0$ and the right K action on T^*K. Let us explain in more detail what this actions are:

- $\mathcal{G}_R=\{g:[0,1]\to K \mid g(0)=1_K\}$ is a normal subgroup of $\mathcal{G}$ with the right endpoint free. It acts on $\mu^{-1}(0)/\mathcal{G}_0$ by gauge transformations.
- K is the compact Lie group and acts on $T^*K\simeq K\times\mathfrak{k}_{\mathbb{C}}$ by right translations on the first factor.
- The anti-equivariance of ψ means that it intertwines the action of $h\in\mathcal{G}_R$ with the action of $h(1)^{-1}\in K$.

Since both ω and the tautological symplectic forms are invariant under the respective actions we conclude that $(\psi^{-1})^*\omega$ equals this last one.

4.2 The Cotangent Bundle of a Complex Lie Group is Hyperkähler

As before, let K be a compact Lie group, $K_{\mathbb{C}}$ its complexification, $K_{\mathbb{C}}\simeq T^*K$. In this section, we show that the cotangent bundle $T^*K_{\mathbb{C}}$ of the complex Lie group is hyperkähler.

4.2.1 Setup for Reduction

We follow very closely [25] and the strategy is to use hyperkähler reduction from an infinite dimensional quaternionic vector space. Remember from 3.2.1, that from an Ad-invariant inner product on $\mathfrak{k}$ and the quaternionic vector space $M = \mathfrak{k} \otimes \mathbb{H}$ we could construct finite dimensional hyperkähler quotients. This was done by considering, for example, a space of paths in M, i.e functions $[0,1] \longrightarrow M$ in the case of Nahm equations, and functions $\mathbb{R}^3 \longrightarrow M$ in the case of Bogomolny equations.
Here, we shall also use a description in terms of Nahm equations. We are interested in doing hyperkähler reduction from a space $\mathcal{A}$ which can be identified with a space of paths in M. This is done via

$$\begin{aligned} T_0(s) &\longmapsto A_0(s) = \frac{d}{ds} + T_0(s) \\ T_i(s) &\longmapsto A_i(s),\ i = 1,2,3. \end{aligned}$$

The procedure allows us to identify $T^*K_{\mathbb{C}}$ with an hyperkähler moduli space for the Nahm equations. The main difference between this and the previous chapter (where we mentioned "monopole" solutions of the Nahm equations) is that here we will require the paths T_μ to be C^1 and the gauge group to have fixed endpoints. We now recall the main setup in which we will do hyperkähler reduction.

- Define
$$\mathcal{A} = \left\{ A_0 + iA_1 + jA_2 + kA_3 \mid A_0(s) = \frac{d}{ds} + T_0(s) \text{ and } A_i(s) = T_i(s) \text{ with } T_\mu : [0,1] \to \mathfrak{k} \right\}.$$
We also require that the $T'_\mu s : [0,1] \longrightarrow \mathfrak{k}$ are of class C^1.
- $\mathcal{A}$ is a quaternionic vector space with a metric
$$h(A, A') = \sum_{\mu=0}^{3} \int_0^1 \langle T_\mu(s), T'_\mu(s) \rangle \, ds$$
with $A = (T_\mu), A' = (T'_\mu) \in \mathcal{A}$. With the same notation as in the previous case we have now in $\mathcal{A}$ the hyperkähler structure, whose symplectic forms are
$$\begin{aligned} \omega_1 &= \int_0^1 dT_0 \wedge dT_1 - dT_2 \wedge dT_3 \\ \omega_2 &= \int_0^1 dT_0 \wedge dT_2 - dT_3 \wedge dT_1 \\ \omega_3 &= \int_0^1 dT_0 \wedge dT_3 - dT_1 \wedge dT_2. \end{aligned}$$
- Let $\mathcal{G}_0 = \{ g : [0,1] \longrightarrow K \text{ of class } C^2 \text{ with } g(0) = g(1) = 1_K \}$, i.e. the space of C^2 loops in G based at the identity 1_K. We let $\mathcal{G}_0$ act on $\mathcal{A}$ by gauge transformations
$$g \cdot (T_0, T_i) = (gT_0g^{-1} + g\frac{dg^{-1}}{ds}, gT_ig^{-1}).$$
- This action is hyperhamiltonian, in the sense that it preserves the 3 symplectic structures $\omega_1(\cdot,\cdot) = g(I\cdot,\cdot), \omega_2(\cdot,\cdot) = g(J\cdot,\cdot), \omega_3(\cdot,\cdot) = g(K\cdot,\cdot)$ in an hamiltonian way, i.e. it there is a moment map for each one of them. As in 3.2.1 we use the inner product in $Lie\mathcal{G}_0$ given by
$$(T(s), T'(s)) = \int_0^1 \langle T(s), T'(s) \rangle \, ds \tag{4.8}$$
to identify $Lie\mathcal{G}_0$ with its dual. Hence, having the moment maps
$$\begin{aligned} \mu_1(A) &= [A_0, A_1] + [A_2, A_3] = \frac{dT_1}{ds} + [T_0, T_1] + [T_2, T_3] \\ \mu_2(A) &= [A_0, A_2] + [A_3, A_1] = \frac{dT_2}{ds} + [T_0, T_2] + [T_3, T_1] \\ \mu_3(A) &= [A_0, A_3] + [A_1, A_2] = \frac{dT_3}{ds} + [T_0, T_3] + [T_1, T_2]. \end{aligned} \tag{4.9}$$

This give us a hyperkahler moment map $\mu : \mathcal{A} \longrightarrow \mathfrak{k}^* \otimes \mathbb{R}^3$, given by $\mu = (\mu_1, \mu_2, \mu_3)$.

- Our claim, which we shall prove in this chapter is that

$$T^* K_{\mathbb{C}} = \mu^{-1}(0)/\mathcal{G}_0. \tag{4.10}$$

It is now obvious that this hyperkähler quotient is a moduli space for the Nahm equations.

4.2.2 Reduction

Since we are doing hyperkahler reduction from an infinite dimensional vector space, the proof we gave cannot be applied to conclude that the quotient $\mu^{-1}(0)/\mathcal{G}_0$ is a smooth hyperkähler manifold. Here we give a proof that this is indeed the case. We split the analysis in three steps:

1. show that $\mu^{-1}(0)$ is a smooth Banach manifold.

2. show that $\mu^{-1}(0)/\mathcal{G}_0$ is a smooth finite dimensional manifold.

3. show that the hyperkähler structure descends to the quotient.

4.2.2.1 $\mu^{-1}(0)$ is a Smooth Banach Manifold

Since $\mathcal{A}$ is an infinite dimensional vector space with an Euclidean metric it is a Banach space. In the same way $Lie\mathcal{G}_0$ equipped with (4.8), is also a Banach vector space. Using the implicit function theorem on Banach spaces [22] it is enough to show that $d\mu$ is surjective along $\mu^{-1}(0)$.
To show this, let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathfrak{k}^* \otimes \mathbb{R}^3$. We need to show that for all $A \in \mu^{-1}(0)$ there exists $Y \in T_A\mathcal{A}$ such that $d\mu_A(Y) = \gamma$. This gives for $i = 1, 2, 3$

$$\begin{aligned} \gamma_i &= \frac{d\mu_i}{dt}\Big|_{t=0}(A + tY) \\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}(T_i + tY_i) + [T_0 + tY_0, T_i + tY_i] + \varepsilon_i^{jk}\left[T_j + tY_j, T_k + tY_k\right] \\ &= \frac{dY_i}{ds} + [T_0, Y_i] + [T_0, T_i] + \varepsilon_i^{jk}\left\{[T_j, Y_k] + [Y_j, T_k]\right\} \end{aligned} \tag{4.11}$$

where summation over repeated indices is understood. So we just need to find solutions for these linearised and nonhomogeneous Nahm equations. Since these are are first order linear ODE's these solutions exist. In fact if we further impose that $Y(0) = Y(1) = 0$ this solution is unique, hence $d\mu$ is surjective at 0.

4.2.2.2 $\mu^{-1}(0)/\mathcal{G}_0$ is a Smooth Finite Dimensional Manifold

We want to identify the tangent space in the quotient with the orthogonal complement to the orbit upstairs, i.e. $T_{[A]}(\mu^{-1}(0)/\mathcal{G}_0) \simeq (T_A\mathcal{O}_A)^\perp$. To show that, we proceed as follows:

1. Fix $A \in \mu^{-1}(0)$ and a neighbourhood of A there.

2. Show that there exists a slice for the orbit $\mathcal{O}_A$ of $\mathcal{G}_0$, such that in that neighbourhood that slice intersects all $\mathcal{G}_0$-orbits only once.

So in this setting, the slice S we are searching for coincides at A with the orthogonal complement to $\mathcal{O}_A$.

$$S_A = \left\{A + Y \text{ with } Y \text{ small such that } Y \in (T_A\mathcal{O}_A)^\perp\right\}$$

Let $u = u(s) \in Lie\mathcal{G}_0$, so $u(s)$ is a loop in $\mathfrak{k}$ with $u(0) = u(1) = 0$, and $u \cdot A = \frac{d}{dt'}\Big|_{t'=0} \exp(t'u) \cdot A$. So $Y \in (T_A\mathcal{O}_A)^{\perp}$ if and only if for all $u \in Lie\mathcal{G}_0$

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} h\left(\frac{d}{dt'}\Big|_{t'=0} \exp(t'u) \cdot A, A + tY\right) \\
&= h\left(\left([u, T_0] - \frac{du}{ds}, [u, T_i]\right), Y\right) \\
&= \int_0^1 \left(\langle -\frac{du}{ds}, Y_0\rangle + \sum_{\mu=0}^{3} \langle [u, T_\mu], Y_\mu\rangle\right) ds \\
&= \int_0^1 \langle u, \frac{dY_0}{ds} + \sum_{\mu=0}^{3} [T_\mu, Y_\mu]\rangle \, ds. \qquad (4.12)
\end{aligned}$$

We conclude that $Y \in (T_A\mathcal{O}_A)^{\perp}$ if and only if

$$\frac{dY_0}{ds} + \sum_{\mu=0}^{3} [T_\mu, Y_\mu] = 0. \qquad (4.13)$$

Let $A + \tilde{Y}$ be in that neighbourhood of A and $g \in \mathcal{G}_0$; we need to show that its orbit $\mathcal{O}_{A+\tilde{Y}}$ intersects the slice only once.

$$\begin{aligned}
g \cdot (A + \tilde{Y}) &= \left(gT_0g^{-1} + g\tilde{Y}_0g^{-1} + g\frac{dg^{-1}}{ds}, gT_ig^{-1} + g\tilde{Y}_ig^{-1}\right) \\
&= (T_0, T_i) + \left(gT_0g^{-1} - T_0 + g\tilde{Y}_0g^{-1} + g\frac{dg^{-1}}{ds}, gT_ig^{-1} - T_i + g\tilde{Y}_ig^{-1}\right) \\
&= A + Y
\end{aligned}$$

with

$$Y = (gT_0g^{-1} - T_0 + g\tilde{Y}_0g^{-1} + g\frac{dg^{-1}}{ds}, gT_ig^{-1} - T_i + g\tilde{Y}_ig^{-1}).$$

Plugging this Y into equation (4.13), we get a second order nonlinear ODE for g with the boundary conditions $g(0) = g(1) = 1_{K_{\mathbb{C}}}$. Its linearisation via the first order terms of $g = e^u$, with $u \in Lie\mathcal{G}_0$ is hence a linear second order ODE for u with boundary conditions $u(0) = u(1) = 0$, which always has a unique solution. Once again we have a map between Banach spaces whose derivative is surjective and hence $\mu^{-1}(0)/\mathcal{G}_0$ is a smooth Banach manifold, see [22].
We conclude that $\mu^{-1}(0)/\mathcal{G}_0$ is a smooth Banach manifold and furthermore we have a model for the tangent space. The fact that it is finite dimensional is a conclusion of the still to come identification $\mu^{-1}(0)/\mathcal{G}_0 \simeq T^*K_{\mathbb{C}}$.

Remark 84 *We have also showed that the equations that identify $T_{[A]}(\mu^{-1}(0)/\mathcal{G}_0) \simeq (T_A\mathcal{O}_A)^{\perp}$ are the ones obtained here and the previous one for $Y \in Ker\ d\mu$, i.e.*

$$\begin{aligned}
\frac{dY_0}{ds} + \sum_{\mu=0}^{3} [T_\mu, Y_\mu] &= 0 \qquad (4.14) \\
\frac{dY_i}{ds} + [T_0, Y_i] + [T_0, T_i] + \varepsilon_i^{jk} \{[T_j, Y_k] + [Y_j, T_k]\} &= 0 \qquad (4.15)
\end{aligned}$$

This gives a model for the tangent space on the quotient.

4.2.2.3 The Hyperkähler Structure Descends to $\mu^{-1}(0)/\mathcal{G}_0$

Checking this is similar to proving the finite dimensional case in theorem 80 in the previous chapter. However we will just recall the steps for that in this case:

- I, J, K preserve $T_{[A]}(\mu^{-1}(0)/\mathcal{G}_0) \simeq (T_A\mathcal{O}_A)^{\perp}$. This is easily seen since for example $I(T_0, T_1, T_2, T_3) = (-T_1, T_0, -T_2, T_3)$ and the equations (4.14) are still satisfied. The same holds for J and K.

- Since each of the symplectic forms is degenerate along the orbits of $\mathcal{G}_0$, they descend well to the quotient to give ω_i^{red}, and the hyperkähler metric in the quotient is defined via $g(\cdot,\cdot) = \omega_1^{red}(\cdot, I\cdot) = \omega_2^{red}(\cdot, J\cdot) = \omega_3^{red}(\cdot, K\cdot)$.

- The ω_i^{red}'s are closed hence by proposition 75, we have that $\mu^{-1}(0)/\mathcal{G}_0$ is hyperkähler.

4.2.3 Identification with $T^*K_{\mathbb{C}}$

Recall the discussion at the end of sections 3.1.3.3 and 3.2.1, about the complex viewpoint on reduction. We will apply this here, to show that with respect to any of the complex structures (here we pick I) $\mu^{-1}(0)/\mathcal{G}_0 \simeq T^*K_{\mathbb{C}}$ as Kähler manifolds. As in 3.1.3.3 notice that $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$ is a complex symplectic form of type $(2,0)$ with moment map $\mu_{\mathbb{C}} = \mu_2 + i\mu_3$. With the complex coordinates $\alpha = A_0 + iA_1$ and $\beta = A_2 + iA_3$ we have

$$\mu_c(A) = [\alpha, \beta] \tag{4.16}$$
$$\mu_r(A) = \frac{1}{2i}[\alpha^*, \alpha] + \frac{1}{2i}[\beta^*, \beta], \tag{4.17}$$

where $\mu_r = \mu_1$ and we call (4.17) and (4.17) respectively the complex and the real equation. Equating them to zero we get

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0 \tag{4.18}$$
$$\frac{d}{ds}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0 \tag{4.19}$$

Now consider the complexified gauge group

$$\mathcal{G}_0^{\mathbb{C}} = \left\{ g : [0,1] \longrightarrow K_{\mathbb{C}} \text{ a } C^2 \text{ loop with } g(0) = g(1) = 1_{K_{\mathbb{C}}} = (1_K, 0) \right\}.$$

The action is the extension of the previous one for $\mathcal{G}_0$, and in terms of α, β is given by

$$g \cdot (\alpha, \beta) = \left(g\alpha g^{-1} + \frac{1}{2} g \frac{dg^{-1}}{ds}, g\beta g^{-1} \right).$$

This action still preserves the complex equation since $\mu_{\mathbb{C}}$ is equivariant for the $\mathcal{G}_0^{\mathbb{C}}$ and $K_{\mathbb{C}}$ actions by conjugation (i.e. gauge transformations and *ad* to be more specific). However the real equation is not preserved by it. This is not a problem since as Donaldson shows in [8]: for each (α, β) satisfying the complex equation there is only one element in $\mathcal{H} = \mathcal{G}_0^{\mathbb{C}}/\mathcal{G}_0$ satisfying the real equation. This leads to the conclusion that

$$\mu^{-1}(0)/\mathcal{G}_0 = \mu_1^{-1}(0) \cap \mu_c^{-1}(0)/\mathcal{G}_0 \simeq \mu_c^{-1}(0)/\mathcal{G}_0^{\mathbb{C}}.$$

So, we can regard the hyperkähler quotient as the space of solutions to the complex equation modulo complex gauge transformations. This is useful since the solution of the complex equation is locally trivial and invariant under the action of the total complex gauge group. The total complex gauge group is $\mathcal{G} = \{u : [0,1] \longrightarrow K_{\mathbb{C}}\}$ the space of paths in the complex Lie group. It acts by gauge transformations

$$u \cdot (\alpha, \beta) = (u\alpha u^{-1} + \frac{1}{2} u \frac{du^{-1}}{ds}, u\beta u^{-1}).$$

Notice that if we can find a gauge transformation in $\mathcal{G}^{\mathbb{C}}$ such that $u \cdot \alpha = 0$ then the complex equation turns into $\frac{d}{ds} u \cdot \beta = 0$ and hence $u \cdot \beta$ is constant. In this way a solution (α, β) is given by

$$\begin{cases} \alpha = -\frac{1}{2} \frac{du^{-1}}{ds} u \\ \beta = u^{-1} \eta u \ , \ \eta \in \mathfrak{k}_{\mathbb{C}} \end{cases} \tag{4.20}$$

However in this way we are overcounting the solutions. To avoid this notice that:

- If $a \in K_{\mathbb{C}}$, then a and au give rise to the same solution. Hence we are interested in this solutions up to translation. We can cut this out by fixing the initial point to be $u(0) = 1_{K_{\mathbb{C}}}$.
  This is the same as saying that there is only one $u \in \mathcal{G}^{\mathbb{C}}$ such that $u(0) = 1_{K_{\mathbb{C}}}$ solving $\alpha = -\frac{1}{2} \frac{du^{-1}}{ds} u$.

- When $\alpha = -\frac{1}{2} \frac{du^{-1}}{ds} u$, the action of the complex gauge group $\mathcal{G}_0^{\mathbb{C}}$ gives

$$\begin{aligned} g \cdot \alpha &= g \alpha g^{-1} + \frac{1}{2} g \frac{dg^{-1}}{ds} = -\frac{1}{2} g \frac{du^{-1}}{ds} u g^{-1} + \frac{1}{2} g \frac{dg^{-1}}{ds} \\ &= \frac{1}{2} \left(g u^{-1} \frac{du^{-1}}{ds} g^{-1} + g \frac{dg^{-1}}{ds} \right) = \frac{1}{2} (g u^{-1}) \frac{d}{ds} \left((g u^{-1})^{-1} \right), \end{aligned}$$

  hence the complex gauge group replaces u by $g^{-1} u$ and since $g(1) = 1_{K_{\mathbb{C}}}$, each $\mathcal{G}_0^{\mathbb{C}}$ is completely determined by the endpoint $u(1) \in K_{\mathbb{C}}$ and the element $\eta \in \mathfrak{k}_{\mathbb{C}}$.

This gives Kronheimer's map (see [25]):

$$\begin{aligned} \Psi : \mu^{-1}(0)/\mathcal{G}_0 &\longrightarrow T^* K_{\mathbb{C}} \\ (\alpha, \beta) &\longmapsto (u(1), \eta). \end{aligned}$$

Note that this is a biholomorphic map relative to I on $\mu^{-1}(0)/\mathcal{G}_0$ and the natural complex structure on $T^* K_{\mathbb{C}}$.

4.2.4 Computation of $(\Psi^{-1})^* \omega_{\mathbb{C}}$

We will now illustrate how to bring the holomorphic structure to $T^* K_{\mathbb{C}}$, using this map, this is done in [7]. We will bring it to the submanifold $1_{K_{\mathbb{C}}} \times \mathfrak{k}_{\mathbb{C}}$. As in the previous section, we will compute $(\Psi^{-1})^* \omega_{\mathbb{C}}$ in this submanifold. Using the identification $T^* K_{\mathbb{C}} \simeq K_{\mathbb{C}} \times \mathfrak{K}_{\mathbb{C}}$, take $(A, B) \in T_{(1_{K_{\mathbb{C}}}, \eta)} T^* K_{\mathbb{C}}$. We can proceed in a similar way as for $T^* K$ and get

$$\Psi_*^{-1}(A, B) = (\frac{1}{2} A, B - s\, [\eta, A]).$$

Remark 85 *As before, $\omega_{\mathbb{C}}$ is degenerate along the $\mathcal{G}_0^{\mathbb{C}}$-orbit. So there is freedom to take any lift of $\Psi_*^{-1}(A, B)$ and we will work with the one obtained.*

The only thing left is to compute $(\Psi^{-1})^* \omega_{\mathbb{C}}((A, B), (A', B'))$. In order to do this, we must find a formula for $\omega_{\mathbb{C}}$ in terms of the complex coordinates (α, β). Since $\omega_{\mathbb{C}} = \omega_2 + i \omega_3$ and

$$\begin{cases} dT_0 = \frac{1}{2} (d\alpha + d\bar{\alpha}) \\ dT_1 = -\frac{i}{2} (d\alpha - d\bar{\alpha}) \\ dT_2 = \frac{1}{2} \left(d\beta + d\bar{\beta} \right) \\ dT_3 = -\frac{i}{2} \left(d\beta - d\bar{\beta} \right), \end{cases} \tag{4.21}$$

we have

$$\omega_{\mathbb{C}} = \int_0^1 d\alpha \wedge d\beta. \tag{4.22}$$

Working this together with our formula for the pushforward vectors, we get

$$
\begin{aligned}
(\Psi^{-1})^*\omega_{\mathbb{C}}((A,B),(A',B')) &= \int_0^1 \left(\frac{1}{2} \langle A, B' - s\,[A',\eta] \rangle - \frac{1}{2} \langle B - s\,[A,\eta]\,, A' \rangle \right) \, ds \\
&= \frac{1}{2} \{ \langle A, B' \rangle - \langle B, A' \rangle \} \\
&\quad - \frac{1}{2} \int_0^1 s\,(\langle A, [A',\eta] \rangle - \langle [A,\eta]\,, A' \rangle) \, ds \\
&= \frac{1}{2} (\langle A, B' \rangle - \langle B, A' \rangle + \langle \eta, [A', A] \rangle) , \qquad (4.23)
\end{aligned}
$$

for $(A,B),(A',B') \in T^{1,0}(\mu^{-1}(0)/\mathcal{G}_0)$. So, we conclude that the complex holomorphic form induced in $T^*K_{\mathbb{C}}$ via its hyperkähler structure coincides with the tautological symplectic form in $T^*K_{\mathbb{C}}$. Note in particular that this is not Kähler with respect to I, since it has a nonvanishing $(2,0)$ component. Later in chapter 6 we will study this tautological structure.

Remark 86 *It would be nice to find an explicit formula for the hyperkähler metric in $T^*K_{\mathbb{C}}$. However there is a problem that was not present in computing the holomorphic symplectic form. This is basically because $\omega_{\mathbb{C}}$ is degenerate in the directions of the $\mathcal{G}_0^{\mathbb{C}}$-orbits, so it does not matter which lifts of the vector we use to compute it. In the the metric, this is not the case and one needs to use the identification $T_{[A]}(\mu^{-1}(0)/\mathcal{G}_0) \simeq (T_A\mathcal{O}_A)^{\perp}$, that gives the model for the tangent space in the quotient, as explained in remark 84. In fact the obtained lift is not in the orthogonal complement of the $\mathcal{G}_0$-orbit.*

In fact in [7] the hyperkähler map is computed in a submanifold using a map constructed there. This map is an alternative to the Kronheimer's identification and allows the identification of $\mu^{-1}(0)/\mathcal{G}_0$ with a large open set in $T^*K_{\mathbb{C}} \simeq K \times \mathfrak{k}^3$ in a different way

Chapter 5

Geometric Quantization

In this chapter we expose geometric quantization and describe some of the difficulties arising when one tries to implement it, namely the need to introduce a polarization and the way how geometric quantization may depend on this choice of polarization.
The classical reference for geometric quantization is [30]. Very good exposition of the mathematical technology needed can be found in [9] and mainly in [26].

5.1 The Idea of Quantization

In this section we shall motivate what is meant by quantization of a classical system. Ideally, quantization would be the procedure that to each classical system associates a quantum system in such a way that all the structure needed to do that was already present in the classical description.

5.1.1 From Classical to Quantum Mechanics

We are trying to find a map between a classical system and the corresponding quantum system. This must associate to each classical state a quantum state and to each classical observable a quantum observable, these are:

- A classical system is characterized by a symplectic manifold (M, ω) also called phase space, classical states are represented by points in M, and classical observables are represented by real functions in M, i.e. $C^\infty(M)$ is the Poisson algebra of observables, with the Poisson bracket induced by ω.
  In fact in statistical mechanics the uncertainty on the state of the system is encapsulated in saying that the classical states are represented by a probability measure on the phase space.

- A quantum system is characterized by an Hilbert space $\mathcal{H}$. States are vectors in $\mathcal{H}$ and the observables are self-adjoint operators in $\mathcal{H}$, denoted by $L_{sa}(\mathcal{H})$. The self-adjointness is needed to ensure that they have real eigenvalues which are interpreted as the measurable quantities.

This is summarized in the following table

	Classical Mechanics	Quantum Mechanics
Space of States	Symplectic manifold (M, ω)	Hilbert space $\mathcal{H}$
Space of Operators	Real functions $C^\infty(M)$	Self-adjoint operators $L_{sa}(\mathcal{H})$

We must now make some remarks on the dynamics, as these will be helpful to establish the axioms for quantization. Both in classical mechanics and quantum mechanics the dynamics can be described in two different ways that are associated to what physicists call the active and passive way of looking at a transformation. In the active way, states depend on time and observables do not, while in the passive way the opposite is true.
For our purposes it suffices to examine the passive way, which is usually called the Hamiltonian picture of classical mechanics and the Heisenberg picture of quantum mechanics. In classical mechanics an observable $f \in C^\infty(M)$ evolves via the "Hamilton" equation

$$\frac{df}{dt} = \{H, f\},$$

where $H \in C^\infty(M)$ is the energy observable, called the Hamiltonian. In quantum mechanics the situation is very similar and the dynamics of the system is also contained in a observable $\hat{H}$ called the Hamiltonian which is now a self-adjoint operator in the quantum space $\mathcal{H}$. Any other observable $\hat{f}$ will evolve via the Von-Neumann equation

$$\frac{d\hat{f}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{f}\right]$$

where $\hbar$ is Planck's constant.
This certainly motivates a physical requirement that is natural to impose on quantization: preserve the time evolution of the systems, i.e. map $\frac{df}{dt}$ to $\frac{d\hat{f}}{dt}$ whenever f is mapped to $\hat{f}$. This can be interpreted as a functorial property of quantization.
On the mathematical side, it is natural to require that G-hamiltonian group actions on a symplectic manifold (M, ω) representing a phase space should be mapped to unitary representations on the associated quantum Hilbert space. This is another functorial property that to symmetries of the classical system associates symmetries of the quantum one. An interesting situation is when the G-action on M is transitive, then we must be able to find a relation between homogeneous symplectic manifolds and irreducible unitary representations of G. Now that we have motivated quantization, we can state Dirac's axioms.

Axiom 87 *(Dirac) A quantization of the classical system (M, ω) is a map that to M associates an Hilbert space $\mathcal{H}$ and to the Poisson algebra $C^\infty(M)$ associates a subalgebra of $L_{sa}(\mathcal{H})$, via a map*

$$\begin{aligned} C^\infty(M) &\longrightarrow L_{sa}(\mathcal{H}) \\ f &\longmapsto \hat{f} \end{aligned}$$

that satisfies

1. *$af + bg \mapsto a\hat{f} + b\hat{g}$, with $a, b \in \mathbb{R}$.*
2. *$1 \mapsto I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$.*
3. *$\{f, g\} \mapsto \frac{i}{\hbar}\left[\hat{f}, \hat{g}\right]$.*
4. *Let G be a group acting on (M, ω) by symplectomorphisms and on $\mathcal{H}$ by unitary transformations. If the G-action on M is transitive, then its action on $\mathcal{H}$ must be irreducible.*

Remark 88 *This axioms can be summarized by:*

1. *Linearity over $\mathbb{R}$.*

2. If the only expected value of the observable is 1, so the same must hold for it's quantization and all the eigenvalues of $\hat{1}$ must be 1, so $\hat{1} = I_{\mathcal{H}}$.

3. This is the property that we have been motivating and can be interpreted physically as imposing that the quantum dynamics must be the same as the classical one. From the mathematical point of view this can also be interpreted as demanding the map $C^\infty(M) \longrightarrow L_{sa}(\mathcal{H})$ to be a Lie Algebra morphism.

4. This is called the irreducibility axiom.

The reader can find a good motivation for this axioms in [9]. From now on we shall regard every symplectic manifold as the phase space of a certain classical system.

5.1.2 The Guiding Example

We will now state what we mean by quantization of the symplectic manifold $M = \mathbb{R}^{2n}$, with the standard symplectic form $\omega_{st} = \sum_{i=1}^n dx^i \wedge dy^i$, where $\{x^i, y^i\}_{i=1}^n$ are the Euclidean coordinates. We can think of this as the phase space of a particle in the configuration space $Q = \mathbb{R}^n$ with coordinates $\{x^i\}_{i=1}^n$. Then $M = T^*Q$ and the $\{y^i\}_{i=1}^n$ are coordinates along the fibres.
A wave function is a complex valued function

$$\psi : Q \longrightarrow \mathbb{C},$$

whose square $|\psi|^2$ is proportional to the probability density of finding the particle in a given position of the configuration space Q.
The quantum Hilbert space is the space of all wave functions, i.e. is the space of square integrable $\mathbb{C}$-valued functions

$$\mathcal{H} = L^2(Q, dx),$$

where dx denotes Lebesgue measure in Q, the inner product is given by integration

$$\langle \psi | \phi \rangle = \int_Q \bar{\psi}\phi \; dx.$$

In this thesis we shall focus on the construction of the quantum Hilbert space and disregard the role of the algebra of observables, however for completeness we shall state what is the desirable result in this simplest case.
The coordinate functions are observables and the x's are interpreted as position and the y's as momenta. These are quantized by

$$\begin{aligned} x^i &\mapsto x^i \\ y^i &\mapsto -i\hbar \frac{\partial}{\partial x^i}. \end{aligned}$$

Notice that in fact

$$\{x^i, y^j\} = \delta^{ij} = \frac{i}{\hbar}\left[\hat{x^i}, \hat{y^j}\right]$$

The Hamiltonian $H \in C^\infty(M)$, given by

$$H = \sum_{i=1}^n \frac{(y^i)^2}{2} + V(x)$$

where $V : Q \to \mathbb{R}$ is the potential, is quantized to the operator

$$\hat{H} = \sum_{i=1}^n -\frac{\hbar^2}{2}\frac{\partial^2}{\partial (x^i)^2} + V(x)$$

which can be seen to be self-adjoint, by integrating by parts.

5.2 Prequantization

Prequantization is a first attempt to get a Hilbert space out of a symplectic manifold (M,ω). The idea is to find wave functions, however as we will see, this give us a bigger than supposed Hilbert space, which will be called prequantum space. The subject of quantization, in the next section, will then be to find the quantum space as a subspace of this Hilbert space.

5.2.1 Attempts at Quantization

The first attempt would be to say that a wave function is a $\mathbb{C}$-valued function on M in the Hilbert space $L^2(M,\varepsilon)$, with

$$\varepsilon = \frac{\omega^n}{n!}.$$

This is called the Liouville form, which is a volume form in every symplectic manifold. The inner product between two wave functions would be

$$\langle\psi|\phi\rangle = \int_M \bar{\psi}\phi \; \varepsilon.$$

For the observables one could try to set

$$\hat{f} = -i\hbar X_f,$$

where X_f is the Hamiltonian vector field generated by f. But in this way a constant function is mapped to the null operator and axiom 2 would not be satisfied. We can try to fix this by setting

$$\hat{f} = -i\hbar X_f + f$$

however we have now spoiled the axiom 3. This can be fixed by picking up a local symplectic potential θ (in our conventions $\omega = -d\theta$) and set

$$\hat{f} = -i\hbar\left[X_f + \frac{i}{\hbar}\theta(X_f)\right] + f$$

this in fact solves the problem and is straightforward to check it satisfies Dirac's axioms.
Notice that we solved a problem but created another, i.e. we are getting a quantization prescription that depends on the symplectic potential we are picking, in fact if θ is a local symplectic potential so is $\theta' = \theta + d\alpha$ with α a real function defined locally. The way to get rid of this ambiguity is to notice that quantization of the observable f using the potential θ' gives

$$\hat{f}' = \hat{f} - X_f(\alpha),$$

where $\hat{f}$ denotes quantization of f using θ. Hence

$$\hat{f}'(e^{i\frac{\alpha}{\hbar}}\psi) = e^{i\frac{\alpha}{\hbar}}\hat{f}(\psi)$$

and we conclude that we must impose that the wave functions ψ and $e^{i\frac{\alpha}{\hbar}}\psi$ represent the same element in the Hilbert space. In this way the operator $\hat{f}$ is well defined, i.e. independent of the symplectic potential.
To understand what happened to the Hilbert space we must now look at our new wave functions, and observe that ψ and $e^{i\frac{\alpha}{\hbar}}\psi$ are the same. Regarding $e^{i\frac{\alpha}{\hbar}}$ as a gauge transformation, i.e. as a transition function of a complex line-bundle over M, we can interpret ψ as a section of that line bundle. In this line-bundle one can regard $X_f + \frac{i}{\hbar}\theta(X_f)$ as a covariant derivative using a locally defined connection 1-form proportional to θ and hence with curvature proportional to ω. This motivates

Definition 89 *A **prequantization** of (M,ω) is an hermitian line-bundle (L,h) (h is the hermitian structure) over M, with a compatible connection ∇ whose curvature is $\frac{\omega}{i\hbar}$. If a prequantization (L,∇,h) exists, then (M,ω) is said to be **prequantizable**.*

Remark 90 *In the setup of the definition*

1. *L is called the* ***prequantum line-bundle*** *and ∇ the* ***prequantum connection****.*

2. *The* ***prequantum space*** *$\mathcal{H}^{prQ}$ is the Hilbert space of square integrable sections of L,*

$$\mathcal{H}^{prQ} = \overline{\{\psi \in \Gamma(L) \mid \int_M h(\psi, \psi)\ \varepsilon < \infty\}}$$

   with the inner product

$$\langle \psi | \phi \rangle = \int_M h(\psi, \phi)\ \varepsilon.$$

3. *The observables are quantized via*

$$\hat{f} = -i\hbar \nabla_{X_f} + f.$$

4. *The condition on the curvature of ∇ is clear from $d\left(\frac{i}{\hbar}\theta\right) = \frac{\omega}{i\hbar}$.*

5.2.2 When is a Symplectic Manifold Prequantizable?

We now try to find sufficient and necessary conditions for a symplectic manifold (M, ω) to be prequantizable. Suppose (M, ω) is prequantizable, let $U \subset M$ be an open set where we have a symplectic potential θ, such that $\frac{i}{\hbar}\theta$ is a connection 1-form of the prequantum line bundle L. Further, let γ be a trivial loop in U based at a point x. Then parallel transport arround γ gives

$$P(\gamma) = \exp \oint_\gamma \frac{i}{\hbar}\theta,$$

and we can pick two surfaces Σ and Σ', in U with $\partial\Sigma = -\partial\Sigma' = \gamma$ such that $\Sigma \cup \Sigma'$ is a 2-cycle. By the Stokes theorem and taking account of orientations

$$P(\gamma) = \exp \int_\Sigma \frac{\omega}{i\hbar} = \exp\left(-\int_{\Sigma'} \frac{\omega}{i\hbar}\right).$$

Notice that this formula is still valid for all loops γ in M and surfaces spanning them, because a general surface can be divided into smaller ones, such that their borders are loops in small open sets $U \simeq \mathbb{R}^{2n}$ where a symplectic potential exists. We may now conclude that $\exp \int_{\Sigma\cup\Sigma'} \frac{\omega}{i\hbar} = 1$, or more generally that

$$\int \frac{\omega}{\hbar} \in 2\pi\mathbb{Z},$$

where the integration is carried over any 2-cycle in M. This gives the necessary condition:

Proposition 91 *(from [30]) If (M, ω) is prequantizable, then $\int_\Sigma \frac{\omega}{\hbar} \in 2\pi\mathbb{Z}$ for all 2-cycles Σ in M.*

To search for a stronger condition we continue assuming that (M, ω) is prequantizable and motivated by the result of the previous proposition we study the isomorphism

$$\check{H}^2(M, \mathbb{R}) \simeq H^2(M, \mathbb{R})$$

between the Čech cohomology and the de Rham cohomology. In particular, for a locally finite good cover $\{U_j\}$ of M, in degree 2 this isomorphism applies to $\frac{\omega}{\hbar}$ to give the Čech 2-cocycle that we now describe. Since $\frac{\omega}{i\hbar}$ is the curvature of the prequantum connection ∇, its connection forms are obtained by $\frac{\omega}{i\hbar}\Big|_{U_j} = d\frac{i}{\hbar}\theta_j$, for the symplectic potentials θ_j. Furthermore, in $U_{ij} = U_i \cap U_j$ we have

$$\frac{i}{\hbar}(\theta_i - \theta_j) = (g_{ij})^{-1} dg_{ij} = dlog(g_{ij}),$$

where g_{ij} are the transition functions of the prequantum bundle L. Then the cocycle condition for the transition functions g_{ij}'s implies that

$$\frac{1}{2\pi}\left(log(g_{ij}) + log(g_{jk}) + log(g_{ki})\right) \in \mathbb{Z}.$$

So we conclude that the image of $\frac{\omega}{2\pi\hbar}$ under this isomorphism lies in $\check{H}^2(M,\mathbb{Z})$. This is in fact not only a necessary but also a sufficient condition.

Theorem 92 *(from [30]) (M,ω) is prequantizable if and only if $\frac{\omega}{2\pi\hbar} \in H^2(M,\mathbb{Z})$. The inequivalent choices of prequantizations are parametrized by $H^1(M,\mathbb{S}^1)$.*

Proof: The first implication was already checked and we just need to check the converse, i.e. we will show how to construct a prequantization of M knowing that $\frac{\omega}{2\pi\hbar} \in H^2(M,\mathbb{Z})$.
As before, pick a good cover $\{U_j\}$ of M and define α_{ij} by $d\alpha_{ij} = \frac{1}{\hbar}(\theta_i - \theta_j)$ via a choice of symplectic potentials in this cover. By hypothesis the image of $\frac{\omega}{2\pi\hbar}$ under the isomorphism with Čech cohomology lies in $\check{H}^2(M,\mathbb{Z})$, so we can take

$$\frac{1}{2\pi}\left(\alpha_{ij} + \alpha_{jk} + \alpha_{ki}\right) \in \mathbb{Z}.$$

We will now define prequantum line bundle L to be the bundle with transition functions $g_{ij} = e^{i\alpha_{ij}}$, they are well defined since $g_{ij}g_{jk}g_{ki} = 1$ and since

$$(g_{ij})^{-1}dg_{ij} = \frac{i}{\hbar}(\theta_i - \theta_j),$$

the $\frac{i}{\hbar}\theta_i$'s can be used to define a connection ∇ in L which will be the prequantum connection. This is because the curvature is $\frac{\omega}{i\hbar}$ and since the connection forms are imaginary there exists in L an hermitian structure with which ∇ is compatible.
The nonequivalent prequantizations of (M,ω) come in the choice of the α's since we could have picked $\alpha_{ij}+\beta_{ij}$ instead, where the β_{ij} are antisymmetric constants such that $\frac{1}{2\pi}\left(\beta_{ij} + \beta_{jk} + \beta_{ki}\right) \in \mathbb{Z}$. The effect of this is to replace the transitions functions by $g_{ij}e^{i\beta_{ij}}$, i.e. L is replaced by the line bundle $L \otimes F$ where F is the line bundle with transition functions $e^{i\beta_{ij}}$, and these are constant so F is a flat bundle.
The result then follows, since connections on flat line bundles having an hermitian structure are parametrized by $H^1(M,\mathbb{S}^1)$. For further details the reader may want to check [30], or [9] for a longer discussion on this. □

5.3 Quantization

In the last section, we have defined the prequantum space associated with a symplectic manifold (M,ω). This involved selecting a prequantization, i.e. an hermitian line bundle L over M with a connection whose curvature was $\frac{\omega}{i\hbar}$, then the prequantum space was defined as the Hilbert space of square integrable sections. From now on we are always working a prequantizable symplectic manifold (M,ω) with a prequantization (L,∇,h).

Remark 93 *In our guiding example, the case of $(\mathbb{R}^{2n},\omega_{st})$ we can pick the trivial line bundle $L = M \times \mathbb{C}$ and the global symplectic potential $\theta = \sum_{i=1}^n y^i dx^i$. Then the prequantum Hilbert space is $L^2(\mathbb{R}^{2n}, dxdy)$ which is bigger than the result we are searching for, $L^2(\mathbb{R}^n, dx)$.*
This was obtained since we noticed that $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, where $\mathbb{R}^n$ could be viewed as the configuration space of the problem, and then the wave functions were function defined only on $\mathbb{R}^n$ with the coordinates $\{x^i\}$. We can select the quantum space as a subspace of the prequantum space by restricting ourselves to functions such that

$$\frac{\partial\psi}{\partial y^i} = 0,$$

i.e. they do not depend on the directions of the fibres.

In this section we apply this kind of reasoning in order to get the quantum space; we use polarizations as a way of selecting half of the directions of M, and then select from the prequantum space the wave functions constant along those directions.

5.3.1 Polarizations

To select half of the directions in M we will pick a half dimensional subbundle P of the complexified cotangent bundle $TM^{\mathbb{C}}$. A condition to impose on a distribution is for it to be involutive, then the Frobenius theorem ensures that its real part is integrable. Remember that we are interested in finding the subspace of the wave functions that are constant along the P directions, for that we use the prequantum connection ∇ and search for covariant constant wave functions in the directions of P, i.e. solutions to the equations

$$\nabla_X \psi = 0,$$

for all $X \in P$. We shall call this covariantly constant wave functions, the **P-polarized sections** of L. Then, if ψ is a P-polarized section and we pick $X, Y \in P$ then $[X, Y] \in P$ and have $\nabla_{[X,Y]}\psi = 0$. This gives

$$\omega(X, Y)\psi = i\hbar \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) \psi = 0.$$

The conclusion is that a for us to find P-polarized sections, then P must be isotropic, in fact a Lagrangian distribution since it is half dimensional. This discussion motivates

Definition 94 *A subbundle $P \subset TM^{\mathbb{C}}$ of the complexified cotangent bundle is called a **polarization** if*

1. *P is Lagrangian*

2. *P is involutive*

3. *$dim P \cap \bar{P} \cap TM$ is constant.*

Given a polarization P we consider the subbundles of TM,

$$E = (P + \bar{P}) \cap TM,$$
$$D = P \cap \bar{P} \cap TM.$$

Condition 3 in the definition ensures that D, the real part of the polarization, is a C^∞ distribution, then by condition 2 the Frobenius theorem implies that it is integrable. However we do not know in general if the base of this distribution $\mathcal{D}$ (i.e. the space of leafs of the foliation $\mathcal{D} = M/D$) is or not a manifold. This is why we introduce the following definition.

Definition 95 *We say that a polarization P is **strongly integrable** if E is involutive, the bases of the distributions $\mathcal{E} = M/E$ and $\mathcal{D} = M/D$ are manifolds and we have a submersion*

$$\pi : \mathcal{D} \longrightarrow \mathcal{E}.$$

We will now restrict ourselves to the case of strongly integrable polarizations P. Introduce an hermitian form on P defined by

$$b(X, Y) = i\omega(X, \bar{Y})$$

for $X, Y \in P$. Notice that since P is Lagrangian the kernel of b is $D^{\mathbb{C}}$. For a strongly integrable polarization we have that $dimD = n-k$ and $dimE = n+k$ for $k \in \mathbb{N}$ constant, and we can in the neighbourhood of each point introduce coordinates $x_1, ..., x_{n-k}, y_1, ..., y_{n-k}, u_1, ..., u_k, v_1, ...v_k$ such that

$$E = \left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{n-k}}, \frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_k} \right\}$$
$$D = \left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{n-k}} \right\}$$

where $z_i = u_i + iy_i$. We shall now focus on the two limit cases:

Real When $P = D$, the form b vanishes identically on P. We say that P is **real**, in this case the integral submanifolds of P give us a Lagrangian foliation of M, and locally we have coordinates $x_1, ..., x_n, y_1, ..., y_n$ such that

$$P = \left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right\};$$

Pseudo-Kähler When $D = 0$, the polarization is called **pseudo-Kähler** and b is now nondegenerate on P, besides we have coordinates $u_1, ..., u_n, v_1, ...v_n$ such that

$$P = \left\{ \frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n} \right\},$$

for $z_i = u_i + iy_i$. The most interesting special case is when the form b is positive, in this case we say that P is **Kähler**. There is a correspondence between Kähler polarizations and Kähler structures, given by

Theorem 96 *Let (M, ω) be a symplectic manifold.*

1. *if J is a complex structure such that (M, ω, J) is Kähler, then $P = T^{0,1}M$ is a Kähler polarizations on M, called the holomorphic polarization.*
2. *if P is a Kähler polarization on M, then there exists a complex structure J such that (M, ω, J) is a Kähler manifold.*

Proof: The first assertion is obvious. For the second one we shall construct the complex structure J. First notice that $TM^{\mathbb{C}} = P \oplus \bar{P}$, so we can uniquely write $X \in TM^{\mathbb{C}}$ as $X = Y + Z$ with $Y \in P$ and $Z \in \bar{P}$. Define $JX = iY - iZ$. It is now easy to verify that J is compatible with ω (remember that b is positive) and J is integrable by its definition. □

Remark 97 *Remember that our goal in defining polarizations is to find the wave functions that are covariantly constant along its directions. In the cases where the form b is not positive it can be shown that there are no globally defined wave functions [30].*

We could apply immediately the procedure of selecting the quantum space as the space of wave functions that are covariantly constant along P, however motivated by some examples this will end up needing a correction by introducing half-form bundles.

5.3.2 Half-forms

Here we follow Pedro Matias' thesis [26]. There is, as far as I know, no easy way to motivate the introduction of half-forms besides that it gives the correct quantization of the harmonic oscillator. There is however a way in which we can understand that introducing them may be useful, specially when we are dealing with real polarizations.

When we restrict to the wave functions that do not depend on P, they end up depending on the directions that are complementary to P, so one would like to keep on carrying some information about these.
When P is a real polarization it makes sense to say that the polarized sections get down to sections of a line bundle over $\mathcal{D} = M/P$, the base of the fibration. What makes sense is to define an inner product via

$$\langle \psi | \phi \rangle = \int_{\mathcal{D}} h(\psi, \phi)\ d\mu,$$

where $d\mu$ is a measure in $\mathcal{D}$. In this case the half-form bundles carry the information regarding this measure.
Given a polarization P in M we define the subbundle P^0 of the complexified cotangent bundle $T^*M^{\mathbb{C}}$ as the annihilator of P. Then the **canonicall bundle associated with** P is the line bundle $K_P = \bigwedge^n P^0$.

Definition 98 *A* ***half-form bundle*** *associated with P is a line bundle δ_P such that $\delta_P \otimes \delta_P \simeq K_P$.*

Notice that if K_P has a half-form bundle δ_P then $c_1(K_P) = 2c_1(\delta_P)$ and hence a necessary condition for K_P to have a half-form bundle is for it to have even first Chern class. In fact, we have the following result.

Proposition 99 *Let P be a polarization in M, then:*

1. *K_P has a half-form bundle δ_P if and only if $c_1(K_P)$ is even.*

2. *There is a bijective correspondence between isomorphism classes of half-form bundles and $H^1(M, \mathbb{Z}_2)$.*

Proof: Check [30]. □

We denote a sections of δ_P by $\sqrt{\lambda}$, where λ is the section of K_P to which $\sqrt{\lambda} \otimes \sqrt{\lambda}$ is mapped via the map $\Gamma(\delta_P) \otimes \Gamma(\delta_P) \simeq \Gamma(K_P)$; this section is defined up to sign.
The two special cases when P is a:

Real polarization we can identify $K_P \simeq \bigwedge^n T^*\mathcal{D}$ and so elements of K_P are volume forms in $\mathcal{D}$ and so we can think of elements of δ_P as carrying half of the measure in $\mathcal{D}$. In conclusion, we have a map $\mu : \Gamma(\delta_P) \otimes \Gamma(\delta_P) \longrightarrow \Gamma(\bigwedge^n T^*\mathcal{D})$. For more details about this map in the flat case check [23].

Kähler polarization it can be seen as the holomorphic polarization of a Kähler manifold and hence we can identify $K_P \simeq \bigwedge^{n,0} T^*M^{\mathbb{C}}$ and define an hermitian structure on this tensor bundle K_P, induced by the metric on M, given by

$$(\alpha, \beta)_{K_P} = \frac{\overline{\alpha} \wedge \beta}{b\omega}$$

where $\alpha, \beta \in \Gamma(K_P)$ and $b = (2i)^n(-1)^{n(n-1)/2}$. Using this we get an hermitian structure on δ_P, given by $(\sqrt{\alpha}, \sqrt{\beta})_{\delta_P} = \sqrt{(\alpha, \beta)_{K_P}}$.
Also notice that we can define a map

$$\begin{aligned} \Gamma(\delta_P) \otimes \Gamma(\delta_P) &\longrightarrow \Gamma(\bigwedge^{2n} T^*M^{\mathbb{C}}) \\ (\sqrt{\alpha}, \sqrt{\beta}) &\longmapsto (\sqrt{\alpha}, \sqrt{\beta})_{\delta_P} \varepsilon, \end{aligned} \tag{5.1}$$

which will be useful in the following.

Let P be a polarization such that K_P admits a half-form δ_P. To introduce a connection in δ_P, take a connection $\tilde{\nabla}$ in K_P such that in the directions of P it coincides with the the Lie derivative, i.e. for $\lambda \in \Gamma(K_P)$ we have

$$\tilde{\nabla}_X \lambda = L_X \lambda,$$

for all $X \in P$. Then, by imposing the Leibnitz rule we get a connection $\nabla^{1/2}$ in δ_P such that

$$\nabla = \nabla^{1/2} \otimes 1 + 1 \otimes \nabla^{1/2}.$$

For a more detailed exposition on half-forms bundles the reader may want to check [9].

5.3.3 The Quantum Space

Let (M,ω) be a symplectic manifold with a prequantization (L,∇,h). We shall now study the quantum space associated to both real and Kähler polarizations.
Given a polarization P, with K_P admitting a half-form δ_P we will substitute the prequantum line bundle L by the **quantum line bundle** $L\otimes\delta_P$, whose sections shall be the new wave functions $\psi\otimes\sqrt{\lambda}$.
Using the connection $\nabla^{1/2}$ in δ_P and the prequantum connection ∇ in L we define the quantum connection ∇^Q in $L\otimes\delta_P$, by

$$\nabla^Q=\nabla\otimes 1+1\otimes\nabla^{1/2}.$$

From now on we may refer to the quantum connection as just ∇. Define the P**-polarized sections** of $L\otimes\delta_P$ as the ones that are covariantly constant relative to the quantum connection, i.e.

$$\nabla_X(\psi\otimes\sqrt{\lambda})=0$$

for all $X\in P$. The definition of the quantum space will basically select the square integrable P-polarized sections in a proper way.

5.3.3.1 Quantization in a Real Polarization

Let P be a real polarization in M. In this case, a section $\sqrt{\lambda}\in\Gamma(\delta_P)$ carries information about a measure in $\mathcal{D}$, and we have a map

$$\mu:\Gamma(\delta_P)\otimes\Gamma(\delta_P)\longrightarrow\Gamma(\bigwedge^n T^*\mathcal{D}) \tag{5.2}$$

that allows us to define an inner product of wave functions.

Definition 100 *Let P be a real polarization such that $\mathcal{D}$ is orientable. The **quantum space** $\mathcal{H}^Q$ associated with P is the closure of*

$$\left\{\psi\otimes\sqrt{\lambda}\in\Gamma(L\otimes\delta_P)\mid\nabla_X(\psi\otimes\sqrt{\lambda})=0,\ \forall_{X\in P},\ \textit{and}\ \int_{\mathcal{D}}h(\psi,\psi)\ \mu(\sqrt{\lambda},\sqrt{\lambda})<\infty\right\}$$

with the inner product

$$\langle\psi\otimes\sqrt{\lambda}|\phi\otimes\sqrt{\nu}\rangle=\int_{\mathcal{D}}h(\psi,\phi)\ \mu(\sqrt{\lambda},\sqrt{\nu}).$$

Remark 101 *This gives the correct quantization of our guiding example when we choose $P=\{\frac{\partial}{\partial x^i}\}_{i=1}^n$. However in general the map (5.2) is defined up to a constant; this makes the definition of the inner product a bit arbitrary.*

5.3.3.2 Quantization in a Kähler Polarization

The case of the Kähler polarization changes from the previous one in the sense that there are no leaves of any foliation, so the integration must be taken over the whole of M.

Recall that we can identify K_P with the holomorphic n-forms on M, for a suitable complex structure J such that (M,ω,J) is a Kähler manifold. Besides that, remember that we have a map $\Gamma(\delta_P)\otimes\Gamma(\delta_P)\to\Gamma(\bigwedge^{2n}T^*M^{\mathbb{C}})$ which we shall use to integrate over M in the definition of the inner product.

Definition 102 *Let P be a Kähler polarization, the **quantum space** $\mathcal{H}^Q$ associated with P is the closure of*

$$\left\{\psi\otimes\sqrt{\alpha}\in\Gamma(L\otimes\delta_P)\mid\nabla_X(\psi\otimes\sqrt{\alpha})=0,\ \forall_{X\in P},\ \textit{and}\ \int_M h(\psi,\psi)\ (\sqrt{\alpha},\sqrt{\beta})_{\delta_P}\ \varepsilon<\infty\right\}$$

with the inner product

$$\langle \psi \otimes \sqrt{\alpha} | \phi \otimes \sqrt{\beta} \rangle = \int_M h(\psi, \phi) \ (\sqrt{\alpha}, \sqrt{\beta})_{\delta_P} \ \varepsilon$$

Remark 103 1. *There is a nice feature that occurs here and not in a real polarization, in fact here we have a natural choice of constant in the map (5.1). So, we get an unambiguous definition of the inner product in $\mathcal{H}^Q$.*

2. *The canonical bundle $K_P = \bigwedge^{n,0} T^*M^{\mathbb{C}}$ is in the Kähler case a holomorphic line bundle, with a holomorphic structure, since its first Chern class is of type $(1,1)$. So we can find in δ_P a holomorphic structure that makes the quantum connection a holomorphic structure for $L \otimes \delta_P$. In this way we can interpret the quantum space associated with the Kähler quantization as the global holomorphic sections of this line bundle.*

5.3.4 The Quantum Operators

We will be brief in this section since we are mainly interested in the definition of the quantum space made previously. Let (M, ω) be a symplectic manifold with quantization $\mathcal{H}^Q$ with respect to a polarization P and with quantum line bundle $L \otimes \delta_P$. We make also the further assumption that K_P is trivial, in this case we are able to explicitly find the algebra of observables.

The first thing we need to do is to find a quantization of observables compatible with the introduction of half-forms. To do so, we just need to replace the prequantum connection in L by the quantum connection in $L \otimes \delta_P$ which we still denote by ∇. This gives the map

$$\hat{f} = -i\hbar \nabla_{X_f} + f,$$

for $f \in C^\infty(M)$ an observable. Since our quantum space $\mathcal{H}$ is restricted to the polarized sections this quantization procedure will no longer be well defined in general (unless we seriously restrict the algebra of observables).

To see this, let $\psi' = \psi \otimes \sqrt{\lambda} \in \mathcal{H}^Q$ a wave function and $f \in C^\infty(M)$ an observable. We want to know if $\hat{f}(\psi') \in \mathcal{H}^Q$, in which case we must have that $\hat{f}(\psi')$ is a P-polarized section. But, for $X \in P$,

$$\begin{aligned} \nabla_X(\hat{f}(\psi')) &= \nabla_X(-i\hbar \nabla_{X_f} \psi' + f\psi') \\ &= -i\hbar \nabla_X \nabla_{X_f} \psi' + df(X)\psi' + f \nabla_X \psi'. \end{aligned}$$

Since $\frac{\omega}{i\hbar}$ is the curvature of ∇ (this is where the assumption that K_P is trivial gets in, since we can pick a flat connection on K_P and hence in δ_P) and have

$$\left[\nabla_X, \nabla_{X_f}\right] = \nabla_{[X, X_f]} + \frac{\omega}{i\hbar}(X, X_f).$$

Substituting this back and using the definition of X_f gives

$$\begin{aligned} \nabla_X(\hat{f}(\psi')) &= -i\hbar \nabla_{X_f} \nabla_X \psi' - i\hbar \nabla_{[X, X_f]} \psi' - \omega(X, X_f)\psi' + df(X)\psi' + f \nabla_X \psi' \\ &= (-i\hbar \nabla_{X_f} + f) \nabla_X \psi' - i\hbar \nabla_{[X, X_f]} \psi' \\ &= \hat{f}(\nabla_X \psi') - i\hbar \nabla_{[X, X_f]} \psi' \end{aligned}$$

and since $\psi' \in \mathcal{H}^Q$, it is P-polarized and we get

$$\nabla_X(\hat{f}(\psi')) = -i\hbar \nabla_{[X, X_f]} \psi'. \tag{5.3}$$

This condition must hold true for all $X \in P$ and $\psi' \in \mathcal{H}^Q$ so we conclude that

Proposition 104 *Let P be a polarization in (M,ω) with trivial canonical bundle K_P. Let $f \in C^\infty(M)$, then $\hat{f} = -i\hbar\nabla_{X_f} + f$ defines an operator in $\mathcal{H}^Q$ only if X_f preserves the polarization P, that is $[X_f, X] \in P$ for all $X \in P$.*

We conclude that the current definition of $\mathcal{H}_P$ implies some restrictions on the algebra of classical observables that can be quantized. The problem is that in general our prescription for the quantization of an observable gives an operator that does not preserve the polarization.
In the special case we have analysed of K_P being trivial the algebra of quantizable observables is restricted to those whose hamiltonian vector field preserves the polarization.

5.4 Pairing Maps

The definition of quantum space given is depending on the polarization chosen. That situation is unsatisfactory and we would like to have unitary isomorphisms between the different quantum spaces obtained when using different polarizations. Currently, there is no general result of this kind and the knowledge is limited to a list of examples which include complex Lie groups, toric manifolds and complex tori.
From now on, to avoid confusion, we shall make a small change in notation. Let (M,ω) be a symplectic manifold with prequantum line bundle L and two different polarizations P_1 and P_2. We shall denote the respective quantum spaces by $\mathcal{H}^Q_{P_1}$ and $\mathcal{H}^Q_{P_2}$.
One of the tools used to study the relationship between $\mathcal{H}^Q_{P_1}$ and $\mathcal{H}^Q_{P_2}$ is the Blattner-Kostant-Sternberg (BKS) pairing. The idea is create a pairing $\langle\cdot|\cdot\rangle^{BKS}$

$$\mathcal{H}^Q_{P_1} \times \mathcal{H}^Q_{P_2} \longrightarrow \mathbb{C},$$

that when nondegenerate allows us to get bijective maps between quantum spaces and whose unitarity we are interested in studying. So given a BKS pairing we define the **BKS pairing maps** as

$$U_{ij} : \mathcal{H}^Q_{P_j} \longrightarrow \mathcal{H}^Q_{P_i}$$

such that for $\psi'_j \in \mathcal{H}^Q_{P_j}$ we have

$$\langle U_{ij}(\psi'_j)|\phi'_i\rangle^{Q_i} = \langle\psi'_j|\phi'_i\rangle^{BKS}$$

for all $\psi'_i \in \mathcal{H}^Q_{P_i}$. Since the construction of the BKS pairing depends on how the polarizations intersect we will split our analysis in different cases.

5.4.1 Transversal Intersection

In the case of $P_1 \cap \bar{P}_2 = 0$ we have a map

$$\mu_{12} : \Gamma(\delta_{P_1}) \otimes \Gamma(\delta_{P_2}) \longrightarrow \Gamma(\bigwedge^{2n} T^*M^{\mathcal{C}}),$$

that allows us to define the **BKS pairing** between P_1 and P_2 as

$$\langle\psi \otimes \sqrt{\lambda_1}|\phi \otimes \sqrt{\lambda_2}\rangle^{BKS} = \int_M h(\psi,\phi)\ \mu_{12}(\sqrt{\lambda_1}, \sqrt{\lambda_2}),$$

for $\psi \otimes \sqrt{\lambda_1} \in \Gamma(L \otimes \delta_{P_1})$ and $\phi \otimes \sqrt{\lambda_2} \in \Gamma(L \otimes \delta_{P_2})$. However, in general we cannot fix the exact form of the map μ_{12} and it is defined up to a constant. But there is a situation where we can do more and define an unambiguous BKS pairing, this is the case of two Kähler polarizations. Notice that two different Kähler polarizations are always transversal and sections of the associated half-forms can be identified with square roots of holomorphic n-forms and we have an explicit map

$$\begin{aligned} \Gamma(\delta_{P_1}) \otimes \Gamma(\delta_{P_2}) &\longrightarrow \Gamma(\bigwedge^{2n} T^*M^{\mathcal{C}}) \\ (\sqrt{\alpha_1}, \sqrt{\alpha_2}) &\longmapsto \sqrt{\frac{\alpha_1 \wedge \overline{\alpha_2}}{b\omega}}\ \varepsilon \end{aligned} \tag{5.4}$$

with $b = (2i)^n(-1)^{n(n-1)/2}$.
So in this case the BKS pairing is given by

$$\langle \psi \otimes \sqrt{\lambda_1} | \phi \otimes \sqrt{\lambda_2} \rangle^{BKS} = \int_M h(\psi, \phi) \sqrt{\frac{\bar{\alpha_1} \wedge \alpha_2}{b\omega}} \ \varepsilon,$$

and we define the corresponding BKS pairing maps.

5.4.2 Non-Transversal Intersection

In this case the difficulty increases a lot and since it does not appear in our presentation we will be very brief and describe a procedure that can only be applied in some special cases.
For our treatment to work out we need that the distribution $D_{12} = P_1 \cap \bar{P}_2 \cap TM$ is integrable and the space of leafs $\mathcal{D}_{12} = M/D_{12}$ to be a manifold. Then if $dim \mathcal{D}_{12} = k$, we have a map

$$\mu : \Gamma(\delta_{P_1}) \otimes \Gamma(\delta_{P_2}) \longrightarrow \Gamma(\bigwedge^k T^* \mathcal{D}_{12}^{\mathbb{C}}).$$

We also require that for all $X \in D_{12}^{\mathbb{C}}$ and $\psi_i' = \psi_i \otimes \sqrt{\lambda_i} \in \mathcal{H}_{P_i}$ for $i = 1, 2$ we have

$$X \cdot h(\psi_1, \psi_2) = 0.$$

This condition is not true in general, but when it holds makes sense to define the BKS pairing by integration in $\mathcal{D}_{12}$, so when the previous setup holds we define it by

$$\langle \psi_1 \otimes \sqrt{\lambda_1} | \psi_2 \otimes \sqrt{\lambda_2} \rangle^{BKS} = \int_{\mathcal{D}_{12}} h(\psi_1, \psi_2) \ \mu(\sqrt{\lambda_1}, \sqrt{\lambda_2}).$$

We can then define the BKS pairing map in the usual way. Further details can be found in [30].

Chapter 6

Symplectic Geometry on the Cotangent Bundle of a Complex Lie Group

Our goal in this chapter would be to quantize $T^*K_{\mathbb{C}}$, i.e. the cotangent bundle of a complex Lie group. In order to do that we must in a first place give it the structure of a classical system, i.e. we must choose a symplectic structure on $T^*K_{\mathbb{C}}$. Then one needs to find a polarization, preferably more than one in order to compare the different quantizations. So in this chapter we describe symplectic structures and polarizations of $T^*K_{\mathbb{C}}$. Except in the case where K is Abelian, it is hard to find more than one polarization for the different symplectic structures studied.
So, we study the quantization $T^*K_{\mathbb{C}}$ in the case where K is Abelian. Then by analysing carefully the problem we reduce this problem to the case of quantizing $T^*\mathbb{S}^1$, which is done in the next chapter.

6.1 Differential Geometry of $T^*K_{\mathbb{C}}$

We now study some of the differential geometry of $T^*K_{\mathbb{C}}$. Start with the triviality of this cotangent bundle to identify $T^*K_{\mathbb{C}} \simeq K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$. Fix an Ad-invariant inner product in $\mathfrak{K}$, denoted by $\langle \cdot, \cdot \rangle$ and extend it to an inner product in $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$ compatible with the complex structure also denoted $\langle \cdot, \cdot \rangle$. Now pick $\{X_1, ..., X_n\}$ an orthonormal basis of $\mathfrak{k}$ for $\langle \cdot, \cdot \rangle$. Then

$$\{X_1, ..., X_n, X_{n+1}, ..., X_{2n}\}$$

with $X_{n+j} = iX_j$ for $j \in \{0, ..., n\}$ is an orthonormal basis for $\mathfrak{k}_{\mathbb{C}}$. Moreover, with the dual 1-forms satisfying $\theta^\mu(X_\nu) = \delta^\mu_\nu$, for $\mu, \nu \in \{0, ..., 2n\}$ we have that

$$\left\{\theta^1, ..., \theta^n, \theta^{n+1}, ..., \theta^{2n}\right\}$$

is an orthonormal basis for the induced metric in the dual $\mathfrak{k}_{\mathbb{C}}^*$. In this chapter we will use the Latin indexes to run from $0, ..., n$ and the Greek ones to run from $0, ..., 2n$.
Now we put global linear coordinates $\{y^\mu\}$ in $\mathfrak{k}_{\mathbb{C}}^*$, so that any $\alpha \in \mathfrak{k}_{\mathbb{C}}^*$ can be written as $\alpha = y^\mu \theta^\mu$ using the Einstein summation convention. Extending $\{X_1, ..., X_n\}$ to left invariant vector fields in K we have now an orthonormal basis of the tangent space at any point of $T^*K_{\mathbb{C}}$, which is

$$\left\{X_1, ..., X_{2n}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^{2n}}\right\}.$$

Once again we remark that we are extending the previous inner products in $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}^*$ in the obvious way.

6.2 Tautological and Hyperkähler Symplectic Structures

We now describe the tautological symplectic structure in $T^*K_{\mathbb{C}}$ and see how it fits relatively to the natural complex structure. We further relate this symplectic structure with the ones that arrise by hyperkähler reduction, see section 4.2 in chapter 4.

Definition 105 *The* ***tautological symplectic structure*** *of $T^*K_{\mathbb{C}}$ (as a cotangent bundle) is the one induced by the global symplectic potential*

$$\theta = y^\mu \theta^\mu. \tag{6.1}$$

The symplectic form is then given by

$$\omega_{taut} = -d\theta = -dy^\mu \wedge \theta^\mu - y^\mu d\theta^\mu$$

using that

$$\begin{aligned} d\theta^\mu(X_\nu, X_\lambda) &= \overbrace{X_\nu \cdot (\theta^\mu(X_\lambda))}^{=0} - \overbrace{X_\lambda \cdot (\theta^\mu(X_\nu))}^{=0} - \theta^\mu([X_\nu, X_\lambda]) \\ &= -\theta^\mu([X_\nu, X_\lambda]) \\ &= -\theta^\mu(c^\epsilon_{\nu\lambda} X_\epsilon) \\ &= -c^\mu_{\nu\lambda}, \end{aligned}$$

where the $c^\mu_{\nu\lambda}$ are the structure constants and we have $d\theta^\mu = -\frac{1}{2}c^\mu_{\nu\lambda}\theta^\nu \wedge \theta^\lambda$, giving the following form for the symplectic form.

Proposition 106 *The symplectic form for the tautological symplectic structure on $T^*K_{\mathbb{C}}$ is given by*

$$\omega_{taut} = -dy^\mu \wedge \theta^\mu + \frac{1}{2} y^\mu c^\mu_{\nu\lambda} \theta^\nu \wedge \theta^\lambda. \tag{6.2}$$

Remark 107 *Using the identification of $K_{\mathbb{C}} \times \mathfrak{k}^*_{\mathbb{C}} \simeq K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}$, provided by the inner product on $\mathfrak{k}_{\mathbb{C}}$, it is easy to check that*

$$\omega_{taut} = (\Psi^{-1})^*\omega_{\mathbb{C}} + \overline{(\Psi^{-1})^*\omega_{\mathbb{C}}}, \tag{6.3}$$

where $(\Psi^{-1})^*\omega_{\mathbb{C}}$ comes from Kronheimer's construction, described in chapter 4. So, the tautological symplectic form is related by (6.3) the symplectic holomorphic form (4.23) induced by hyperkähler reduction studied in 4.2.4, back on chapter 4. This is a result in [7].

The complex structure of $K_{\mathbb{C}}$ extends naturally to a complex structure on $T^*K_{\mathbb{C}}$, by

Definition 108 *The* ***natural complex structure*** *on $T^*K_{\mathbb{C}}$ is the the involution J of the tangent bundle extended by linearity from*

$$\begin{cases} JX_i = X_{i+n} \\ JX_{i+n} = -X_i \\ J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial y^{i+n}} \\ J\frac{\partial}{\partial y^{n+i}} = \frac{\partial}{\partial y^i}. \end{cases}$$

Remark 109 *The minus sign in $J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial y^{i+n}}$ may seem strange at first, however it is in fact needed in order to make this the natural complex structure in $T^*K_{\mathbb{C}}$. This is thanks to the fact that the $\{y^\mu\}$ are coordinates in the dual $\mathfrak{k}^*_{\mathbb{C}}$. And from $JX_i = X_{i+n}$ we may conclude that $J\theta^i = -\theta^{i+n}$.*

We now have the natural question: is $(T^*K_{\mathbb{C}}, \omega_{taut}, J)$ Kähler? Surprisingly (at least for me) the answer is: no! This is due to the fact that the complex structure J does not mix the directions of the base with the ones of the fibres, in the sense of adapted complex structures.
In fact we can easily see that $(T^*K_{\mathbb{C}}, \omega_{taut}, J)$ is not Kähler. Since $\Psi : \mu^{-1}(0)/\mathcal{G}_0 \longrightarrow T^*K_{\mathbb{C}}$ is an holomorphic map and $\omega_{\mathbb{C}}$ is a form of type $(2,0)$ in $\mu^{-1}(0)/\mathcal{G}_0$ and hence $\Psi^{-1}\omega_{\mathbb{C}}$ is also of type $(2,0)$ in $T^*K_{\mathbb{C}}$. Now by equation (6.3) we have that ω_{taut} is of type $(2,0)+(0,2)$. A direct proof of this is given in the following.

Lemma 110

$$J^*\omega_{taut} = -\omega_{taut} \tag{6.4}$$

Proof: The proof is an annoying computation. To ease it, notice that

$$\begin{cases} c^{i+n}_{jk} = c^i_{j+n\ k} = c^i_{j\ k+n} = 0 \\ c^{i+n}_{j+n\ k} = c^{i+n}_{j\ k+n} = c^i_{jk} \\ c^i_{j+n\ k+n} = -c^i_{jk} = 0 \\ c^{i+n}_{j+n\ k+n} = 0 \end{cases} . \tag{6.5}$$

Hence if in our formula (6.2) for ω_{taut} we write the sums only up to n, and then use (6.5) we get

$$\begin{aligned} \omega_{taut} &= -dy^i \wedge \theta^i + -dy^{i+n} \wedge \theta^{i+n} + \frac{1}{2} y^i c^i_{jk} \theta^j \wedge \theta^k \\ &\quad - \frac{1}{2} y^i c^i_{jk} \theta^{j+n} \wedge \theta^{k+n} + \frac{1}{2} y^{i+n} c^i_{jk} \theta^{j+n} \wedge \theta^k + \frac{1}{2} y^{i+n} c^i_{jk} \theta^j \wedge \theta^{k+n}. \end{aligned} \tag{6.6}$$

We can know calculate $J^*\omega_{taut}$. For that notice that from the definition of the natural complex structure (108) we have

$$\begin{cases} J\theta^i = -\theta^{i+n} \\ Jdy^i = dy^{i+n} \end{cases} .$$

Using this in our formula (6.6) for ω_{taut}, we get

$$\begin{aligned} J^*\omega_{taut} &= -dy^{i+n} \wedge (-\theta^{i+n}) - (-dy^i) \wedge \theta^i + \frac{1}{2} y^i c^i_{jk} (-\theta^{j+n}) \wedge (-\theta^{k+n}) \\ &\quad - \frac{1}{2} y^i c^i_{jk} \theta^j \wedge \theta^k + \frac{1}{2} y^{i+n} c^i_{jk} \theta^j \wedge (-\theta^{k+n}) + \frac{1}{2} y^{i+n} c^i_{jk} (-\theta^{j+n}) \wedge \theta^k \\ &= -\omega_{taut}. \end{aligned}$$

□

Has we have already said this lemma has an obvious consequence:

Corollary 111 $(T^*K_{\mathbb{C}}, \omega_{taut}, J)$ *is* ***not*** *Kähler.*

6.3 A Zoo of Symplectic Structures

We will now give different symplectic structures to $T^*K_{\mathbb{C}}$. We further remark on possible choices of polarizations that could be used to carry on quantization.

6.3.1 Tautological Structure

We have seen in 6.2, that by considering $T^*K_{\mathbb{C}}$ as a cotangent bundle, the tautological symplectic structure is

$$\omega_{taut} = -dy^\mu \wedge \theta^\mu + \frac{1}{2} y^\mu c^\mu_{\nu\lambda} \theta^\nu \wedge \theta^\lambda.$$

It has a symplectic potential, but is not Kähler with respect to any known complex structure. This makes it difficult to find polarizations for it, besides the obvious vertical polarization, and that is why we will not study the quantization of $(T^*K_{\mathbb{C}}, \omega_{taut})$.

6.3.2 Hyperkähler Structure

From 4.2 we know that the $(\Psi^{-1})^*\omega_1$ is Kähler with respect to the complex structure $(\Psi)_* I (\Psi^{-1})_*$. However we do not have an explicit formula for this symplectic structure and it does not seem easy to get any polarization other than the holomorphic one. Furthermore, it is easily checked that none of the hyperkähler complex structures, other than $\pm I$, give Lagrangian distributions with respect to this symplectic structure.

6.3.3 Almost Kähler Form

We will call canonical almost Kähler form, to the one canonically related with natural complex structure J. The J-holomorphic 1-forms are

$$\begin{cases} dz^i = \theta^i - iJ^*\theta^i = \theta^i + i\theta^{i+n} \\ dw^i = dy^i - iJ^*dy^i = dy^i - idy^{i+n} \end{cases} \tag{6.7}$$

and we can define a 2-form Kähler form, by

$$\omega_c = \frac{i}{2}\left(dz^i \wedge d\bar{z}^i + dw^i \wedge d\bar{w}^i\right). \tag{6.8}$$

This form is compatible with the symplectic structure, since it is constructed in order to be of type $(1,1)$. One can also check that expanding the holomorphic 1-forms we get the formula

$$\omega_c = \theta^i \wedge \theta^{i+n} - dy^i \wedge dy^{n+i}. \tag{6.9}$$

In fact, this form is not closed unless K is Abelian. We will not use it in quantization, even in the Abelian case. The reason for this is that the natural complex structure (with which ω is compatible) is not that interesting as one would expect, since it does not mix the cotangent direction with the ones on the base. So quantization would amount to separately quantizing each factor the product $K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}$. The case of $K_{\mathbb{C}}$ was done in [12], [10] and [11], while the case of $\mathfrak{k}_{\mathbb{C}}$ is just the quantization of complex finite dimensional vector space.

6.3.4 Mixed Almost Kähler Form

We can find a complex structure that mixes the base and fiber directions defining:

$$\begin{cases} \tilde{J} X_\mu = \frac{\partial}{\partial y^\mu} \\ \tilde{J} \frac{\partial}{\partial y^\mu} = -X_\mu \end{cases}.$$

We now search for a nondegenerate 2-form ω_m compatible with $\tilde{J}$. From analogy with the flat case we see that

$$\omega_m = \theta^\mu \wedge dy^\mu \tag{6.10}$$

works. However this is only a Kähler form in the Abelian case. Moreover, in the Abelian case it has a symplectic potential

$$\Theta = y^\mu \theta^\mu \tag{6.11}$$

and $\omega_m = -d\Theta$. Also notice that in the Abelian case we have 3 obvious polarzations for this structure, these are

$$\begin{aligned}
\mathcal{P}_H &= \{X_1, ..., X_{2n}\} \\
\mathcal{P}_V &= \left\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^{2n}}\right\} \\
\mathcal{P}_J &= \left\{X_1 + i\frac{\partial}{\partial y^1}, ..., X_{2n} + i\frac{\partial}{\partial y^{2n}}\right\},
\end{aligned}$$

which we call the horizontal, the vertical and the holomorphic polarization respectively. The existence of all these polarizations makes it interesting to study the quantization of $T^*K_{\mathbb{C}}$ in the Abelian case.

Remark 112 *There is whole family $\tilde{J}_s$, for $s \in \mathbb{R}^+$, of complex structures in $T^*K_{\mathbb{C}}$, given by*

$$\begin{cases} \tilde{J}_s X_\mu = \frac{1}{s}\frac{\partial}{\partial y^\mu} \\ \tilde{J}_s \frac{\partial}{\partial y^\mu} = -sX_\mu \end{cases}.$$

The holomorphic polarization given by these complex is equivalent to the one given by $\tilde{J}$ for $s = 1$. Moreover in the limits $s \longrightarrow 0, +\infty$, this polarization converges to the vertical and horizontal polarizations respectively.

6.4 The Abelian Case $K = \mathbb{T}^n$

Here, we give some preliminaries to the quantization of $(T^*K_{\mathbb{C}}, \omega_m)$ in the case where K is a compact Abelian Lie group. In this case we must have $K = \mathbb{T}^n$ for some $n \in \mathbb{N}$.
Recall that since we are always working in $T^*K_{\mathbb{C}}$ via its identification with $K_{\mathbb{C}} \times \mathfrak{K}^*_{\mathbb{C}}$ we may write

$$T^*K_{\mathbb{C}} = K_{\mathbb{C}} \times (\mathfrak{k}_{\mathbb{C}})^*.$$

Now remember the identification of $K_{\mathbb{C}} \simeq T^*K \simeq K \times \mathfrak{k}$, substituting this gives

$$T^*K_{\mathbb{C}} \simeq (K \times \mathfrak{k}) \times (\mathfrak{k} \times \mathfrak{k})^*. \tag{6.12}$$

So the basis for the tangent space at a given point can be written

$$\left\{\underbrace{X_1, ...X_n, X_{n+1}, ..., X_{2n}}_{\text{in } K\times\mathfrak{k}}, \underbrace{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \frac{\partial}{\partial y^{n+1}}, ..., \frac{\partial}{\partial y^{2n}}}_{\text{in } (\mathfrak{k}\times\mathfrak{k})^*}\right\} \tag{6.13}$$

and the complex structure $\tilde{J}$ mixes the vectors on the left part with the ones on the right part. In our case $K = \mathbb{T}^n$ and $\mathfrak{k} = \mathbb{R}^n$ and the isomorphism (6.12) becomes

$$T^*\mathbb{T}^n_{\mathbb{C}} \simeq (\mathbb{T}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)^*. \tag{6.14}$$

Since $\mathbb{T}^n = (\mathbb{S}^1)^n$, and using the inner product $\sum_i d\theta^i \otimes d\theta^i$, where the θ^i's are the angular coordinates in the different $\mathbb{S}^1$'s. We get for (6.13)

$$\left\{\underbrace{\frac{\partial}{\partial \theta^1}, ...\frac{\partial}{\partial \theta^n}, \frac{\partial}{\partial \theta^{n+1}}, ..., \frac{\partial}{\partial \theta^{2n}}}_{\text{in } (\mathbb{T}^n\times\mathbb{R}^n)}, \underbrace{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \frac{\partial}{\partial y^{n+1}}, ..., \frac{\partial}{\partial y^{2n}}}_{\text{in } (\mathbb{R}^n\times\mathbb{R}^n)^*}\right\} \tag{6.15}$$

and the complex structure $\tilde{J}$ is sending each one of the elements in the left side to an element on the right side. The symplectic structure ω_m also restricts well to each of these two groups of directions. We conclude that we can split

$$T^*\mathbb{T}^n_{\mathbb{C}} \simeq \left(\mathbb{S}^1 \times \mathbb{R}^*\right)^n \times \left(\mathbb{R} \times \mathbb{R}^*\right)^n \tag{6.16}$$

and the Kähler structure restricts well to each of the factors. Hence we can resytict ourselves to quantize:

- $(\mathbb{R} \times \mathbb{R}^*)^n \simeq \mathbb{C}^n$ with its canonical Kähler structure. This is easy and was in fact our guiding example 5.1.2 in chapter 5, when we discussed the method of geometric quantization.

- $(\mathbb{S}^1 \times \mathbb{R}^*)^n \simeq (T^*\mathbb{S}^1)^n$ also with its canonical Kähler structure. This is the subject of the next chapter.

Chapter 7

Quantization of $T^*\mathbb{S}^1$

7.1 Kähler structure, Polarizations and Prequantization

The goal is to quantize $T^*\mathbb{S}^1$ with the Kähler structure that makes sense in the discussion of the previous chapter, i.e. the one that is obtained by restriction when we are on the cotangent bundle of an Abelian complex Lie group. So we pick

$$\omega = -dy \wedge d\theta = -d\Theta,$$

with symplectic potential $\Theta = yd\theta$ (so the θ is the angular coordinate and Θ the symplectic potential). The complex J structure is given by

$$J\frac{\partial}{\partial\theta} = \frac{\partial}{\partial y} \quad , \quad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial\theta}$$

and as we saw in the previous chapter $(T^*\mathbb{S}^1, \omega, J)$ is a Kähler manifold.
Since $T^*\mathbb{S}^1$ is 2-dimensional any nonvanishing vector field gives rise to a polarization. However considering an arbitrary vector field with nonconstant coefficients gives complicated equations for the covariantly constant sections. We have 3 interesting polarizations

$$P_V = \left\{\frac{\partial}{\partial y}\right\} \quad , \quad P_H = \left\{\frac{\partial}{\partial\theta}\right\} \quad , \quad P_J = \left\{\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial y}\right)\right\}$$

these are respectively the vertical, the horizontal and the holomorphic polarization with $z = \theta + iy$. Later, we shall see it is useful to consider also the family of complex structures J_s with $s \in \mathbb{R}_+$, given by

$$J_s\frac{\partial}{\partial\theta} = \frac{1}{s}\frac{\partial}{\partial y} \quad , \quad J_s\frac{\partial}{\partial y} = -s\frac{\partial}{\partial\theta}.$$

For each $s > 0$, the corresponding holomorphic polarization is given by

$$P_s = \left\{\frac{\partial}{\partial\bar{z}_s} = \frac{1}{2}\left(\frac{\partial}{\partial\theta} + \frac{i}{s}\frac{\partial}{\partial y}\right)\right\} = \left\{\frac{1}{2}\left(\frac{\partial}{\partial y} - is\frac{\partial}{\partial\theta}\right)\right\}$$

with $z_s = \theta + isy$. This polarization agrees with P_J, for $s = 1$ and gives back P_V and P_H in the limits when s goes to 0 and $+\infty$ respectively. Note that different choices of s give equivalent complex structures, but different Kähler structures, as the Kähler metric changes with s.

We now turn to the prequantization of $T^*\mathbb{S}^1$. We pick the trivial prequantum line bundle $L = T^*\mathbb{S}^1 \times \mathbb{C}$, so that $\Gamma(L)$ is just the set of complex valued functions in $T^*\mathbb{S}^1$.

Remark 113 *In order to treat the horizontal polarization, we shall admit distributional sections.*

The prequantum connection ∇ is defined by a global connection 1-form which is $\frac{i}{\hbar}\Theta$ and the compatible hermitian structure is given by $h(\psi,\phi)=\bar{\psi}\phi$ with $\psi,\phi\in\Gamma(L)$.
In this setting, the prequantum space is the space $L^2(T^*\mathbb{S}^1,\omega)$ of square integrable complex valued functions on M with respect to the Liouville measure $\varepsilon=\omega$

$$\mathcal{H}^{prQ}=\left\{\psi\in C^\infty(M,\mathbb{C})\mid \int_{T^*\mathbb{S}^1}|\psi|^2\,\omega<\infty\right\}$$

with the usual L^2-inner product

$$\langle\psi|\phi\rangle=\int_{T^*\mathbb{S}^1}\bar{\psi}\phi\;\omega.$$

7.2 Quantization

We will start our discussion on the quantization of $T^*\mathbb{S}^1$ with respect to each of the 3 polarizations P_V,P_H,P_J. Their canonical bundles K_V,K_H,K_P are trivialized by $d\theta,dy,dz=d\theta+idy$ respectively. So their Chern class is trivially even and all of them admit a half-form bundle. The half-form bundles are also trivial and will be denoted by $\delta_V,\delta_H,\delta_J$, we can take the obvious trivializations $\sqrt{d\theta},\sqrt{dy},\sqrt{dz}=\sqrt{d\theta+idy}$.
We equip each of the half-form bundles with the flat connection $\nabla^{1/2}$ coinciding with the Lie derivative along the polarization, and then

$$\nabla^{1/2}_{\frac{\partial}{\partial y}}\sqrt{d\theta}=\nabla^{1/2}_{\frac{\partial}{\partial\theta}}\sqrt{dy}=\nabla^{1/2}_{\frac{\partial}{\partial\bar{z}}}\sqrt{dz}=0$$

So we can write any covariantly constant sections of the half-form bundles (relative to the respective polarizations) in the form

$$f(\theta)\otimes\sqrt{d\theta}\;,\;\; g(y)\otimes\sqrt{dy}\;,\;\; h(z)\otimes\sqrt{dz}.$$

for $\delta_V,\delta_H,\delta_J$ respectively and where f,g,h are functions such that f just depends on θ, g just depends on y and h is J-holomorphic.
We now consider the quantum bundles $L\otimes\delta_V,L\otimes\delta_H,L\otimes\delta_J$ and equip each of them with the corresponding quantum connection

$$\nabla^Q=\nabla\otimes 1+1\otimes\nabla^{1/2}.$$

7.2.1 The Equations of Covariant Constancy

We now want to find the quantum spaces associated with each one of the polarizations. So we look for the polarized sections of each of the quantum bundles, the equations of covariant constancy are

$$\nabla^Q_{X_i}(\psi\otimes\sqrt{\lambda_i})=0,$$

for the corresponding quantum connections and $i=1,2,3$ with

$$\lambda_i(s)=\begin{cases} d\theta & \text{if } i=1,\\ dy & \text{if } i=2,\\ dz & \text{if } i=3.\end{cases}\quad\text{and}\quad X_i(s)=\begin{cases}\frac{\partial}{\partial y} & \text{if } i=1,\\ \frac{\partial}{\partial\theta} & \text{if } i=2,\\ \frac{\partial}{\partial z} & \text{if } i=3.\end{cases}$$

Since the $\sqrt{\lambda_i}$'s are already covariantly constant along X_i for $i=1,2,3$ the covariant constancy equations become:

$$\begin{aligned}0&=\nabla^Q_{X_i}(\psi\otimes\sqrt{\lambda_i})\\&=\nabla_{X_i}\psi\otimes\sqrt{\lambda_i}+\psi\otimes\underbrace{\nabla^{1/2}_{X_i}\sqrt{\lambda_i}}_{=0}\\&=\nabla_{X_i}\psi\otimes\sqrt{\lambda_i}.\end{aligned}$$

So we are reduced to studying the equations $\nabla_{X_i}\psi\otimes\sqrt{\lambda_i}=0$ for each $i=1,2,3$. We have organized the results in a series of lemmas.

Lemma 114 $(i=1)$ *For the polarization P_V, the covariantly constant sections are of the form*

$$\sigma_V = \psi(\theta) \otimes \sqrt{d\theta},$$

with $\psi : \mathbb{S}^1 \longrightarrow \mathbb{C}$.

Proof: In this case we just need to solve $\nabla_{\frac{\partial}{\partial y}} \psi = 0$, but since the connection 1-form is $\frac{i}{\hbar}\Theta$ this is

$$\begin{aligned} 0 = \nabla_{\frac{\partial}{\partial y}} \psi &= \frac{\partial \psi}{\partial y} + \frac{i}{\hbar} y d\theta (\frac{\partial}{\partial y}) \psi \\ &= \frac{\partial \psi}{\partial y} \end{aligned}$$

and the result follows. □

Lemma 115 $(i=2)$ *For the polarization P_H, the covariantly constant sections are of the form*

$$\sigma_H = \sum_{k \in \mathbb{Z}} a_k \delta(y - \hbar k) e^{-ik\theta} \otimes \sqrt{dy}, \tag{7.1}$$

where $\delta(y)$ denotes the Dirac δ-function along the fibres, and the a_k's are complex coefficients.

Proof: The equation of covariant constancy is

$$\begin{aligned} 0 = \nabla_{\frac{\partial}{\partial \theta}} \psi &= \frac{\partial \psi}{\partial y} + \frac{i}{\hbar} y d\theta (\frac{\partial}{\partial \theta}) \psi \\ &= \frac{\partial \psi}{\partial \theta} + \frac{i}{\hbar} y \psi, \end{aligned}$$

which is a linear ODE. Now, notice that $\nabla_{\frac{\partial}{\partial \theta}} e^{-\frac{i}{\hbar} y\theta} = 0$ and hence any solution must be of the form $\psi = f e^{-\frac{i}{\hbar} y\theta}$, with $f : T^*\mathbb{S}^1 \longrightarrow \mathbb{C}$ such that

$$\begin{aligned} 0 &= \nabla_{\frac{\partial}{\partial \theta}} f e^{-\frac{i}{\hbar} y\theta} \\ &= \frac{\partial f}{\partial \theta} e^{-\frac{i}{\hbar} y\theta} + f \underbrace{\nabla_{\frac{\partial}{\partial \theta}} e^{-\frac{i}{\hbar} y\theta}}_{=0} \\ &= \frac{\partial f}{\partial \theta} e^{-\frac{i}{\hbar} y\theta}. \end{aligned}$$

We conclude that $\frac{\partial f}{\partial \theta} = 0$, and hence f must not depend on the angular direction. Imposing the general form of the solution $\psi(\theta, y) = f(y) e^{-\frac{i}{\hbar} y\theta}$ to be 2π periodic in θ, gives $e^{-\frac{i}{\hbar} y 2\pi} = 1$, i.e.

$$y = \hbar k \text{ with } k \in \mathbb{Z}. \tag{7.2}$$

So, f must be distributional and supported on the set $\mathbb{S}^1 \times \bigcup_{k\in\mathbb{Z}} \{\hbar k\}$. It is a theorem in analysis that such a distribution can be written as a finite sum of Dirac δ's and its derivatives

$$f(y) = \sum_{k \in \mathbb{Z}} \sum_{n \geq 0} a_k^n \delta^{(n)}(y - \hbar k).$$

We shall now see that the coefficients a_k^n vanish for $n \neq 0$. Write each term of the solution of the covariant constancy equation as

$$\psi(\theta, y) = \delta^{(n)}(y - \hbar k) e^{-\frac{i}{\hbar} y\theta},$$

acting on a C^∞ test function g with compact support by

$$\psi(g) = \int_0^{2\pi} (-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k} \left(e^{-\frac{i}{\hbar} y\theta} g \right) \, d\theta.$$

In what follows let g be a test function which is constant on a neighbourhood of $y = \hbar k$ and decaying to 0, so that it has compact support. Then for each $k \in \mathbb{Z}$,

$$\begin{aligned} 0 &= \left(\frac{\partial \psi}{\partial \theta} + \frac{i}{\hbar} y \psi\right)(g) \\ &= -\psi\left(\frac{\partial g}{\partial \theta}\right) + \frac{i}{\hbar}\psi\,(yg) \\ &= -\int_0^{2\pi} \frac{\partial g}{\partial \theta}(-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k}\left(e^{-\frac{i}{\hbar}y\theta}\right)\, d\theta + \frac{i}{\hbar}\int_0^{2\pi}(-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k}(ye^{-\frac{i}{\hbar}y\theta})g\, d\theta \end{aligned} \tag{7.3}$$

Integrating the first term by parts equation (7.3) can be written in all its "splendour"

$$\begin{aligned} 0 &= -\left(g(-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k} e^{-\frac{i}{\hbar}y\theta}\right)\Big|_{\theta=0}^{\theta=2\pi} + \int_0^{2\pi} g\ (-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k}\left(-\frac{i}{\hbar}ye^{-\frac{i}{\hbar}y\theta}\right)\, d\theta \\ &\quad + \frac{i}{\hbar}\int_0^{2\pi} g\ (-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k}(ye^{-\frac{i}{\hbar}y\theta})\, d\theta \\ &= -\left(g\ (-1)^n \frac{\partial^n}{\partial y^n}\Big|_{y=\hbar k} e^{-\frac{i}{\hbar}y\theta}\right)\Big|_{\theta=0}^{\theta=2\pi} \\ &= -\left(-\frac{i\theta}{\hbar}\right)^n e^{-ik\theta} g(\theta)\Big|_{\theta=0}^{\theta=2\pi}. \end{aligned}$$

This equals 0 for $n = 0$, and for $n \neq 0$ this gives

$$0 = -\left(-\frac{i\theta}{\hbar}\right)^n g(2\pi) = -\left(-\frac{i\theta}{\hbar}\right)^n g(0).$$

So all a_k^n's must vanish for $n \neq 0$, the solution follows. □

Remark 116 *Note that, since test functions have compact support, we are allowed to take an infinite sum in k in our expression (7.1), for σ_H.*

This result illustrates a phenomenon which is the appearance of **Bohr-Sommerfeld leaves**. In fact, these appear thanks to the existence problem of solutions to the covariant constancy equations (as in our case).

To see how they occur more generally, let P be a polarization with real directions, and γ a closed path in a leaf of $D = P \cap \bar{P} \cap TM$. The covariant constant sections are parallel transported along these paths γ, and so we must have

$$P(\gamma) = \exp \oint_\gamma \frac{i}{\hbar}\theta = 1.$$

If we have $P(\gamma) \neq 1$ for a given closed path γ, then there are no solutions of the covariant constancy equations on that leaf. The Bohr-Sommerfeld set is formed by the set of leaves where ∇ has trivial holonomy, hence being the support of the polarized sections that form the quantum space. In our case this is $\mathbb{S}^1 \times \bigcup_{k\in\mathbb{Z}}\{\hbar k\}$.

Remark 117 *In fact if we do not use distributional sections then (in these cases) we would be condemned to have no polarized sections. A way of correcting this without the use of distributional sections is to consider higher Cěch cohomology groups. This is usually called the* ***cohomological correction*** *and goes like this: Let $\mathcal{F}$ be the sheaf of smooth locally polarized sections. Then one can define a quantum space as*

$$\mathcal{H} = \bigoplus_{k \leq 2n} H^k(M, \mathcal{F}).$$

When the Lagrangian distribution defined by the polarization is regular Śniatycki proved in [29] that in fact one gets the same dimension for the quantum space in both cases.

Back to the solutions of the covariant constancy equations to get the polarized sections. The only polarization still missing is the holomorphic one.

Lemma 118 *($i = 3$) For the polarization P_J, the covariant constant sections are of the form*

$$\sigma_J = f(z)e^{-\frac{y^2}{2\hbar}} \otimes \sqrt{dz},$$

where $f(z)$is a holomorphic function.

Proof: We have

$$0 = \nabla_{\frac{\partial}{\partial \bar{z}}}\psi = \frac{\partial \psi}{\partial \theta} + i\frac{\partial \psi}{\partial y} + \frac{i}{\hbar}y\psi. \tag{7.4}$$

Now notice that $\nabla_{\frac{\partial}{\partial \bar{z}}} e^{-\frac{y^2}{2\hbar}} = 0$ and hence we can write any solution to (7.4) as $\psi(z, \bar{z}) = f(z, \bar{z})e^{-\frac{y^2}{2\hbar}}$ if and only if

$$\begin{aligned} 0 &= \nabla_{\frac{\partial}{\partial \bar{z}}} \left(f(z, \bar{z})e^{-\frac{y^2}{2\hbar}} \right) \\ &= \frac{\partial f}{\partial \bar{z}} e^{-\frac{y^2}{2\hbar}} + f\nabla_{\frac{\partial}{\partial \bar{z}}} e^{-\frac{y^2}{2\hbar}} \\ &= \frac{\partial f}{\partial \bar{z}} e^{-\frac{y^2}{2\hbar}}. \end{aligned}$$

□

7.2.2 The Quantum Spaces

We may now define the quantum spaces associated with these polarizations. As we saw in 5.3.3 the inner product on these quantum spaces is only defined up to some arbitrary constant. So, in this section some magical constants will appear, they are responsible for the unitarity of the BKS pairings. In a later section we will explain how these constants can be naturally obtained via degeneration of complex structures.

The base for the vertical polarization P_V is $\mathcal{D}_V = \mathbb{S}^1$ and so elements of the quantum space associated with it must be polarized sections $\sigma_V = \psi(\theta) \otimes \sqrt{d\theta}$ that are square integrable in $\mathbb{S}^1$.

Definition 119 *(Vertical) The* ***quantum space associated with the vertical polarization*** *is $\mathcal{H}_V = L^2(\mathbb{S}^1, d\theta)$ with the inner product*

$$\langle \psi | \phi \rangle = \sqrt{\pi \hbar} \int_{\mathbb{S}^1} \bar{\psi}\phi \; d\theta. \tag{7.5}$$

For the horizontal polarization one has $\mathcal{D}_H = \mathbb{R}$ and the polarized sections as $\sigma_H = \sum_{k\in\mathbb{Z}} a_k \delta(y - \hbar k)e^{-ik\theta} \otimes \sqrt{dy}$. These are completely determined by the sequence $(a_k)_{k\in\mathbb{Z}}$. Moreover pairing two polarized sections we get

$$\langle \sum_{k\in\mathbb{Z}} a_k \delta(y - \hbar k)e^{-ik\theta} | \sum_{l\in\mathbb{Z}} b_l \delta(y - \hbar l)e^{-il\theta} \rangle = \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \bar{a_k} b_l \langle \delta(y - \hbar k) | \delta(y - \hbar l) \rangle$$

and it is natural to think that the δ's supported at distinct points are orthogonal, so that we may write $\langle \delta(y - \hbar k) | \delta(y - \hbar l) \rangle = \delta_{lk} \|\delta(y - \hbar k)\|^2$. Hence, the inner product would be defined up to the choice of the norm of the δ's by

$$\langle (a_k)_{k\in\mathbb{Z}} | (b_k)_{k\in\mathbb{Z}} \rangle = \sum_{k\in\mathbb{Z}} \bar{a_k} b_k \|\delta(y - \hbar k)\|^2.$$

Definition 120 *(Horizontal) The* ***quantum space associated with the horizontal polarization*** *is* $\mathcal{H}_H = \ell^2$ *with the inner product*

$$\langle (a_k)_{k\in\mathbb{Z}} | (b_k)_{k\in\mathbb{Z}} \rangle = \sqrt{\frac{\pi}{\hbar}} \sum_{k\in\mathbb{Z}} \bar{a_k} b_k. \tag{7.6}$$

Remark 121 *In the previous definition we take* $\|\delta(y - \hbar k)\| = \left(\frac{\pi}{\hbar}\right)^{1/4}$.

For the holomorphic polarization one can use the map (5.1), and get $\sqrt{dz} \otimes \sqrt{dz} \mapsto \sqrt{\frac{d\bar{z}\wedge dz}{2i\omega}}\ \omega$. So for two polarized sections one has

$$\begin{aligned}
\langle f(z)e^{-\frac{y^2}{2\hbar}} \otimes \sqrt{dz} | g(z)e^{-\frac{y^2}{2\hbar}} \otimes \sqrt{dz} \rangle &= \int_{T^*\mathbb{S}^1} \overline{f(z)} g(z) e^{-\frac{y^2}{\hbar}} \sqrt{\frac{d\bar{z}\wedge dz}{2i\omega}}\ \omega \\
&= \int_{T^*\mathbb{S}^1} \overline{f(z)} g(z) e^{-\frac{y^2}{\hbar}}\ \omega
\end{aligned}$$

since $d\bar{z} \wedge dz = 2i\omega$. Then we have,

Proposition 122 *(Holomorphic) The* ***quantum space associated with the holomorphic polarization*** *is* $\mathcal{H}_J = \mathcal{HL}^2(T^*\mathbb{S}^1, e^{-\frac{y^2}{\hbar}}\omega)$ *with the inner product*

$$\langle f(z) | g(z) \rangle = \int_{T^*\mathbb{S}^1} \overline{f(z)} g(z) e^{-\frac{y^2}{\hbar}}\ \omega \tag{7.7}$$

By $\mathcal{HL}^2(T^*\mathbb{S}^1, e^{-\frac{y^2}{\hbar}}\omega)$ *we mean the Hilbert space of holomorphic functions in* $T^*\mathbb{S}^1$ *that are holomorphic with respect to* J *and square integrable with respect to the measure* $e^{-\frac{y^2}{\hbar}}\ \omega$.

Remark 123 *Notice that* $\mathcal{HL}^2(T^*\mathbb{S}^1, e^{-\frac{y^2}{\hbar}}\omega)$ *is unitarily isomorphic to the Segal-Bargman space* $\mathcal{HL}^2(T^*\mathbb{S}^1, d\nu_\hbar)$, *where* $d\nu_\hbar = \frac{1}{\sqrt{\pi\hbar}} e^{-\frac{y^2}{\hbar}}\ \omega$ *is the* $\mathbb{S}^1$*-averaged heat kernel measure, see [14]. The isomorphism is given by*

$$\begin{aligned}
\mathcal{HL}^2(T^*\mathbb{S}^1, e^{-\frac{y^2}{\hbar}}\omega) &\longrightarrow \mathcal{HL}^2(T^*\mathbb{S}^1, d\nu_\hbar) \\
f &\longmapsto (\pi\hbar)^{1/4} f
\end{aligned}$$

Remark 124 *In all the previous definitions of the quantum spaces we write the inner products in a way that may suggest that we are forgetting information about the polarized sections, but this is not the case. We use this notations only to ease the writing. In fact, by* $\langle f(z)|g(z)\rangle$ *we are thinking about pairing* $\sigma_J = f(z)e^{-\frac{y^2}{\hbar}} \otimes \sqrt{dz}$ *with* $\sigma'_J = g(z)e^{-\frac{y^2}{\hbar}} \otimes \sqrt{dz}$. *Similar observations apply to the other quantum spaces.*

Each of the quantum spaces is a separable Hilbert space and so has a countable Hilbert basis. We now search for orthonormal Hilbert basis for each one of the quantum spaces.

- In $\mathcal{H}_V$ we just need to write square integrable functions f in $\mathbb{S}^1$. These can be identified with 2π-periodic functions in $\mathbb{R}$. Using the Fourier series of f we have that $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is a basis. Compute

$$\langle e^{in\theta} | e^{ik\theta} \rangle = \sqrt{\pi\hbar} \int_{\mathbb{S}^1} e^{i(k-n)\theta}\ d\theta = 2\pi\sqrt{\pi\hbar}\delta_{nk}$$

  and then we conclude that $\{\frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{in\theta}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}_V$.

- For $\mathcal{H}_H$ we start with the sequence $(\delta_{nk})_{k\in\mathbb{Z}}$ which is 0 everywhere except in the n^{th} position where it is 1

$$\langle(\delta_{nk})_{k\in\mathbb{Z}}|(\delta_{lk})_{k\in\mathbb{Z}}\rangle = \sqrt{\frac{\pi}{\hbar}}\sum_{k\in\mathbb{Z}}\delta_{nk}\delta_{lk} = \sqrt{\frac{\pi}{\hbar}}\delta_{nl}$$

We conclude that an orthonormal basis is given by $\{\left(\frac{\hbar}{\pi}\right)^{1/4}(\delta_{nk})_{k\in\mathbb{Z}}\}_{n\in\mathbb{Z}}$.

- It remains $\mathcal{H}_J$. Since it is obtained from a subspace of the J-holomorphic functions on T^*S^1 we know that $\{e^{inz}\}_{n\in\mathbb{Z}}$ is a generating set. As before compute

$$\begin{aligned}
\langle e^{inz}|e^{ikz}\rangle &= \int_{T^*\mathbb{S}^1} e^{i(k-n)z}e^{-\frac{y^2}{\hbar}}\ \omega \\
&= \int_{\mathbb{S}^1} d\theta\ e^{i(k-n)\theta}\int_{\mathbb{R}} dy\ e^{-(n+k)y-\frac{y^2}{\hbar}} \\
&= 2\pi\delta_{nk}\int_{\mathbb{R}} dy\ e^{-(n+k)y-\frac{y^2}{\hbar}} \\
&= 2\pi\delta_{nk}\int_{\mathbb{R}} dy\ e^{-\frac{1}{\hbar}\left(y+\frac{(n+k)\hbar}{2}\right)^2+\frac{(n+k)^2\hbar}{4}} \\
&= 2\pi\delta_{nk}\sqrt{\hbar}\ e^{\frac{(n+k)^2\hbar}{4}}\int_{\mathbb{R}} dy\ e^{-y^2} \\
&= 2\pi\sqrt{\pi\hbar}\ e^{\frac{(n+k)^2\hbar}{4}}\delta_{nk} \qquad (7.8)
\end{aligned}$$

Then $\{\frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}}e^{inz-\frac{\hbar n^2}{2}}\}$ is an orthonormal basis for $\mathcal{H}_J$.

7.3 Pairing maps

We begin this section by remembering Remark 124, since now we are going to pair elements of different quantum spaces. In order to avoid confusion we will now write everything down. In the following table we summarise the list of corresponding orthonormal basis, but now with everything written.

Quantum Space	Orthonormal Basis
$\mathcal{H}_V$	$\left\{\sigma_V^k = \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}}e^{ik\theta}\otimes\sqrt{d\theta}\right\}_{k\in\mathbb{Z}}$
$\mathcal{H}_H$	$\left\{\sigma_H^k = \sqrt{i}\left(\frac{\hbar}{\pi}\right)^{1/4}\delta(y+\hbar k)e^{ik\theta}\otimes\sqrt{dy}\right\}_{k\in\mathbb{Z}}$
$\mathcal{H}_J$	$\left\{\sigma_J^k = \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}}e^{ikz-\frac{\hbar k^2}{2}}e^{-\frac{y^2}{2\hbar}}\otimes\sqrt{dz}\right\}_{k\in\mathbb{Z}}$

Remark 125 *Notice that:*

1. *We have reordered the basis in $\mathcal{H}_H$, the reason is that with this reordering we will get diagonal pairing maps in what follows.*

2. *We also added an $\sqrt{i}$ term to the basis elements of $\mathcal{H}_H$.*

7.3.1 Vertical and Horizontal

We now study the pairing between two basis elements σ_V^k and σ_H^n. This gives

$$
\begin{aligned}
\langle \sigma_V^k | \sigma_H^n \rangle^{BKS} &= \int_{T^*\mathbb{S}^1} \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{-ik\theta} \sqrt{i} \left(\frac{\hbar}{\pi}\right)^{1/4} \delta(y+\hbar n) e^{in\theta} \sqrt{\frac{d\theta \wedge dy}{2i\omega}}\ \omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} dy\ \delta(y+n\hbar) \int_{\mathbb{S}^1} d\theta\ e^{i(k-n)\theta} \\
&= \delta_{n,k}
\end{aligned}
$$

and we conclude that: the BKS pairing is nondegenerate, and that the BKS pairing map is a unitary isomorphism. In fact the BKS pairing map is the extension by linearity of the map

$$
\begin{aligned}
\mathcal{H}_V &\longleftrightarrow \mathcal{H}_H \\
\sigma_V^k &\ \ \leftrightarrow\ \ \sigma_H^k.
\end{aligned}
$$

Just to illustrate, we now show how to find explicitly formulas for the the pairing maps. Let $\sigma_V = f(\theta)\otimes\sqrt{d\theta} \in \mathcal{H}_V$ and $\sigma_H = \sum_{k\in\mathbb{Z}} a_k \delta(y+\hbar k)e^{ik\theta}$. Writing the Fourier series of f we get

$$
\begin{aligned}
\sigma_V &= \sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} e^{ik\theta} \otimes \sqrt{d\theta} \\
&= \sum_{k\in\mathbb{Z}} \sqrt{\frac{1}{2}\sqrt{\frac{\hbar}{\pi}}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ik\theta} \otimes \sqrt{d\theta} \\
&= \sum_{k\in\mathbb{Z}} \sqrt{\frac{1}{2}\sqrt{\frac{\hbar}{\pi}}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} \sigma_V^k.
\end{aligned}
$$

Now since the BKS pairing map is a unitary isomorphism which is diagonal in the basis $\{\sigma_V^k\}$ and $\{\sigma_H^k\}$, we get

$$
\begin{aligned}
U_{HV}(\sigma_V) &= \sum_{k\in\mathbb{Z}} \sqrt{\frac{1}{2}\sqrt{\frac{\hbar}{\pi}}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} U_{HV}(\sigma_V^k) \\
&= \sum_{k\in\mathbb{Z}} \sqrt{\frac{1}{2}\sqrt{\frac{\hbar}{\pi}}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} \sigma_H^k \\
&= \sum_{k\in\mathbb{Z}} \sqrt{\frac{1}{2}\sqrt{\frac{\hbar}{\pi}}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} \sigma_H^k \\
&= \sqrt{i} \sum_{k\in\mathbb{Z}} \sqrt{\frac{1}{2}\sqrt{\frac{\hbar}{\pi}}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt} \left(\frac{\hbar}{\pi}\right)^{1/4} \delta(y+\hbar k)e^{ik\theta} \otimes \sqrt{dy} \\
&= \sqrt{i}\sqrt{\frac{\hbar}{2\pi}} \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{S}^1} dt\ f(t)e^{-ikt}\right) \delta(y+\hbar k)e^{ik\theta} \otimes \sqrt{dy}.
\end{aligned}
$$

So, the BKS pairing map from the vertical to the horizontal polarization gives a sequence $\left(\sqrt{\frac{\hbar}{2\pi}} \int_{\mathbb{S}^1} dt\ f(t)e^{-ikt}\right)_{k\in\mathbb{Z}}$, i.e. is the sequence of Fourier coeficients up to a constant. From the horizontal to the vertical quantum spaces one has

$$\begin{aligned}
U_{VH}(\sigma_H) &= \sum_{k\in\mathbb{Z}} a_k \delta(y+\hbar k) U_{VH}\left(e^{ik\theta}\otimes\sqrt{dy}\right)\\
&= \sum_{k\in\mathbb{Z}} a_k U_{VH}\left(\delta(y+\hbar k)e^{ik\theta}\otimes\sqrt{dy}\right)\\
&= \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{i}}\left(\frac{\pi}{\hbar}\right)^{1/4} a_k U_{VH}\left(\sigma_H^k\right)\\
&= \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{i}}\left(\frac{\pi}{\hbar}\right)^{1/4} a_k \sigma_V^k\\
&= \sum_{k\in\mathbb{Z}} \frac{a_k}{\sqrt{i2\pi\hbar}} e^{ik\theta}\otimes\sqrt{dy}
\end{aligned}$$

Then the BKS pairing map sends a sequence $(a_k)_{k\in\mathbb{Z}}$ to a function whose Fourier coefficients are $\frac{a_k}{\sqrt{2\pi i\hbar}}$.

7.3.2 Vertical and Holomorphic

As in the previous case we will pair the basis elements. Let

$$\sigma_V^k = \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ik\theta}\otimes\sqrt{d\theta},$$

$$\sigma_J^n = \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{inz-\frac{\hbar n^2}{2}} e^{-\frac{y^2}{2\hbar}}\otimes\sqrt{dz},$$

and compute their pairing

$$\begin{aligned}
\langle\sigma_V^k|\sigma_J^n\rangle^{BKS} &= \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}}\int_{T^*\mathbb{S}^1} e^{-ik\theta}e^{inz-\frac{\hbar n^2}{2}}\ e^{-\frac{y^2}{2\hbar}}\ \sqrt{\frac{d\theta\wedge dz}{2i\omega}}\ \omega\\
&= \delta_{nk}\frac{e^{-\frac{\hbar n^2}{2}}}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} dy\ e^{-ny-\frac{y^2}{2\hbar}}\\
&= \delta_{nk}\frac{e^{-\frac{\hbar n^2}{2}}}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} dy\ e^{-\frac{1}{2\hbar}(y+\hbar n)^2+\frac{\hbar n^2}{2}}\\
&= \delta_{n,k}.
\end{aligned}$$

We again conclude that also in this case the pairing map is a unitary isomorphism between $\mathcal{H}_V$ and $\mathcal{H}_J$. Furthermore we got that in this basis this unitary isomorphism is diagonal, just as before.

Remark 126 *In fact this case falls into a class studied in [12], since $T^*\mathbb{S}^1$ is the cotangent bundle of a compact Lie group. Hall showed that in the general class of complex Lie groups the pairing between the vertical and holomorphic polarizations is unitary and given by the Coherent State Transform (up to some constants).*

7.3.3 Horizontal and Holomorphic

Proceeding in the same way as done before we will study the pairing between σ_J^n and σ_H^k

$$\begin{aligned}
\langle\sigma_J^n|\sigma_H^k\rangle^{BKS} &= \frac{1}{\pi\sqrt{2}}\int_{T^*\mathbb{S}^1}\delta(y+\hbar n)e^{-ik\theta}e^{inz-\frac{\hbar n^2}{2}}e^{-\frac{y^2}{2\hbar}}\sqrt{i}\sqrt{\frac{d\bar{z}\wedge dy}{2i\omega}}\ \omega\\
&= \delta_{nk}e^{-\frac{\hbar n^2}{2}}\int_{\mathbb{R}} dy\ \delta(y+k\hbar)e^{-\frac{y^2}{2\hbar}-ny}\\
&= \delta_{n,k}
\end{aligned}$$

and as in the previous cases the BKS pairing map is a unitary isomorphism, which is diagonal in this basis.

Remark 127 *The fact that all the pairing maps are unitary and diagonal with respect to the same basis for the quantum spaces implies that they are transitive, i.e.*

$$U_{ij} \circ U_{jk} \circ U_{ki} = 1_i. \tag{7.9}$$

Where $i, j, k \in \{H, J, K\}$ and 1_i is the identity operator in the quantum space $\mathcal{H}_i$.

7.4 Degenerating Complex Structures

In all the cases studied we were respectively pairing two real polarizations and a real with an holomorphic one. As explained in 5.4.1 this is done in a somewhat arbitrary basis. Let us give a natural justification for the magical constants that were introduced in the pairing maps. We always consider pairings between holomorphic polarizations and obtain the other ones by degenerating the complex structures. This is adapted from [10] and [11].

We consider the complex structures J_s introduced in (7.1). These form a family of complex structures for all $s \in \mathbb{R}^+$. Here we must consider the polarization P_s which is holomorphic with respect to the complex structure J_s. We will repeat it here

$$P_s = \left\{ \frac{\partial}{\partial \bar{z}_s} = \frac{1}{2}\left(\frac{\partial}{\partial \theta} + \frac{i}{s}\frac{\partial}{\partial y} \right) \right\} = \left\{ \frac{\partial}{\partial y} - is\frac{\partial}{\partial \theta} \right\},$$

with $z_s = \theta + isy$. The importance of P_s is that for $s = 0, 1, +\infty$ it coincides with P_V, P_J, P_H. We will be able two work unambiguously with P_s and then obtain well defined limits that give results concerning the other polarizations.

7.4.1 Quantization with P_s

The first thing we need to do is to fix δ_s, the half-form bundle of the canonical bundle K_s, in fact it is unique and trivialized by $\sqrt{dz_s}$. Now we need to search for polarized sections of the bundle $L \otimes \delta_s$, these are

Lemma 128 *The P_s-polarized sections of $L \otimes \delta_s$ are the solutions of the covariant constancy equations and are given by*

$$\sigma_s = f(z_s) e^{-s\frac{y^2}{2\hbar}} \otimes \sqrt{dz_s}$$

with f a J_s-holomorphic function.

Proof: By the triviality of the bundle we can focus on solutions to $\nabla_{\frac{\partial}{\partial \bar{z}_s}} \psi = 0$.

$$\nabla_{\frac{\partial}{\partial \bar{z}_s}} \psi = \frac{1}{2}\left(\frac{\partial \psi}{\partial \theta} + \frac{i}{s}\frac{\partial \psi}{\partial y} + \frac{iy}{\hbar}\psi \right) = 0$$

Now notice that $\nabla_{\frac{\partial}{\partial \bar{z}_s}} e^{-s\frac{y^2}{2\hbar}} = 0$ and that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial \bar{z}_s}} \left(f(z_s, \bar{z}_s) e^{-s\frac{y^2}{2\hbar}} \right) &= \frac{\partial f}{\partial \bar{z}_s} e^{-s\frac{y^2}{2\hbar}} + f \nabla_{\frac{\partial}{\partial \bar{z}_s}} e^{-s\frac{y^2}{2\hbar}} \\ &= \frac{\partial f}{\partial \bar{z}_s} e^{-s\frac{y^2}{2\hbar}} \end{aligned}$$

So we conclude that a solution must be of the form $\psi(z_s, \bar{z}_s) = f(z_s) e^{-s\frac{y^2}{2\hbar}}$ with $\frac{\partial f}{\partial \bar{z}_s} = 0$. $\square$

We now study how to pair two sections $\sigma_s = f(z_s)e^{-s\frac{y^2}{2\hbar}} \otimes \sqrt{dz_s}$ and $\sigma_t = g(z_t)e^{-t\frac{y^2}{2\hbar}} \otimes \sqrt{dz_t}$, for $s, t \in \mathbb{R}^+$. This will allow us to give an expression for the BKS pairing map and the inner product on the quantum spaces. The first thing is the pairing of half-forms $(\sqrt{dz_s}, \sqrt{dz_t}) \mapsto \sqrt{\frac{d\bar{z}_s \wedge dz_t}{2i\omega}}\ \omega = \sqrt{\frac{s+t}{2}}\ \omega$ (check 5.4.1). The **BKS pairing** is then

$$\langle \sigma_s | \sigma_t \rangle^{BKS} = \int_{T^*\mathbb{S}^1} \overline{f(z_s)} g(z_t) e^{-(s+t)\frac{y^2}{2\hbar}} \sqrt{\frac{s+t}{2}}\ \omega.$$

We now use this to define the inner product on the quantum spaces.

Proposition 129 *For all $s \in \mathbb{R}_+$ the **quantum space $\mathcal{H}_s$ associated with the holomorphic polarization** P_s is the Hilbert space $\mathcal{HL}^2(T^*\mathbb{S}^1, e^{-s\frac{y^2}{\hbar}}\omega)$, with the inner product*

$$\langle f(z_s) | g(z_s) \rangle = \int_{T^*\mathbb{S}^1} \overline{f(z_s)} g(z_s) e^{-s\frac{y^2}{\hbar}} \sqrt{s}\ \omega.$$

Here $\mathcal{HL}^2(T^*\mathbb{S}^1, e^{-s\frac{y^2}{\hbar}}\omega)$ is the space of square integrable functions in $T^*\mathbb{S}^1$ which are J_s-holomorphic.

Note that this definition agrees with the definition made previously for the quantum space $\mathcal{H}_J$, we just need to take $s = 1$.

7.4.2 Degenerating to the Real Polarizations

We will now show that it there is a well defined limit of the quantum spaces $\mathcal{H}_s$ as s goes to $+\infty, 0$. Here we shall study these limits and show they agree with $\mathcal{H}_H, \mathcal{H}_V$ respectively. The strategy is as follows.

- Find an orthonormal basis $\{\sigma_s^k\}_{k\in\mathbb{Z}}$ of $\mathcal{H}_s$ for each s.
- Show that this basis converges to the orthonormal basis of $\mathcal{H}_H, \mathcal{H}_V$ in the respective limits.

We start by finding the orthonormal basis $\{\sigma_s^k\}_{k\in\mathbb{Z}}$ of $\mathcal{H}_s$. To do that we just need to do some very similar computations as the ones in (7.8) and find

$$\begin{aligned}
\langle e^{ikz_s} | e^{ikz_s} \rangle &= \int_{T^*\mathbb{S}^1} e^{i\bar{k}\bar{z}_s} e^{ikz_s} e^{-s\frac{y^2}{\hbar}} \sqrt{s}\ \omega \\
&= \int_{\mathbb{S}^1} d\theta \int_{\mathbb{R}} dy\ e^{-2sky - s\frac{y^2}{\hbar}} \sqrt{s} \\
&= 2\pi e^{s\hbar k^2} \int_{\mathbb{R}} dx\ e^{-\frac{sx^2}{\hbar}} \sqrt{s} \\
&= 2\pi\sqrt{\pi\hbar} e^{s\hbar k^2}.
\end{aligned}$$

Hence, an orthonormal basis is

$$\left\{ \sigma_s^k = \frac{e^{-s\frac{\hbar k^2}{2}}}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ikz_s} e^{-s\frac{y^2}{2\hbar}} \otimes \sqrt{dz_s} \right\}_{k\in\mathbb{Z}}. \tag{7.10}$$

7.4.2.1 Horizontal

We are now ready to use (7.10) to get the limit in question, here this is $s \to +\infty$, just compute:

$$
\begin{aligned}
\sigma_s^k &= \frac{e^{-s\frac{\hbar k^2}{2}}}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ikz_s} e^{-s\frac{y^2}{2\hbar}} \otimes \sqrt{dz_s} \\
&= \sqrt{s} e^{ik\theta} \frac{e^{-s\frac{\hbar k^2}{2}}}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{s\frac{\hbar k^2}{2}} e^{-\frac{s}{2\hbar}(y+\hbar k)^2} \otimes \sqrt{idy + \frac{1}{s} d\theta} \\
&= \left(\frac{\hbar}{\pi}\right)^{1/4} \underbrace{\frac{1}{\sqrt{2\pi\frac{\hbar}{s}}} e^{-\frac{s}{2\hbar}(y+\hbar k)^2}}_{\to\delta(y+\hbar k)} e^{ik\theta} \otimes \underbrace{\sqrt{idy + \frac{1}{s} d\theta}}_{\to\sqrt{i}\sqrt{dy}} \\
&\to \sqrt{i} \left(\frac{\hbar}{\pi}\right)^{1/4} \delta(y+\hbar k) e^{ik\theta} \otimes \sqrt{dy}.
\end{aligned} \tag{7.11}
$$

7.4.2.2 Vertical

To study the limit $s \to 0$ is easier and we get

$$
\begin{aligned}
\sigma_s^k &= \frac{e^{-s\frac{\hbar k^2}{2}}}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ikz_s} e^{-s\frac{y^2}{2\hbar}} \otimes \sqrt{dz_s} \\
&= \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ik\theta} \underbrace{e^{-s(\hbar k^2 + \frac{y^2}{2\hbar})}}_{\to 1} \otimes \underbrace{\sqrt{d\theta + isdy}}_{\to\sqrt{d\theta}} \\
&\to \frac{1}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ik\theta} \otimes \sqrt{d\theta}.
\end{aligned} \tag{7.12}
$$

7.4.3 Unitarity of the BKS Pairing Map

We will now show that the pairing map between two holomorphic polarizations $s \neq t$ is a unitary isomorphism. The strategy is as in the previous cases to show that the BKS pairing map is diagonal with entries 1 with respect to orthonormal basis for $\mathcal{H}_s, \mathcal{H}_t$ that we have fixed.

$$
\begin{aligned}
\langle \sigma_s^k | \sigma_t^n \rangle &= \int_{T^*\mathbb{S}^1} \overline{\frac{e^{-s\frac{\hbar k^2}{2}}}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{ikz_s} e^{-s\frac{y^2}{2\hbar}}} \frac{e^{-t\frac{\hbar n^2}{2}}}{\sqrt{2\pi\sqrt{\pi\hbar}}} e^{inz_t} e^{-t\frac{y^2}{2\hbar}} \sqrt{\frac{s+t}{2}}\ \omega \\
&= \frac{1}{2\pi\sqrt{\pi\hbar}} \int_{\mathbb{S}^1} d\theta\ e^{i(n-k)\theta} \int_{\mathbb{R}} dy\ e^{-(s+t)\left(\frac{y^2}{2\hbar} + ky + \frac{\hbar k^2}{2}\right)} \sqrt{\frac{s+t}{2}}\ \omega \\
&= \frac{\delta_{nk}}{\sqrt{\pi\hbar}} \int_{\mathbb{R}} dy\ e^{-(s+t)\left(\frac{y^2}{2\hbar} + ky + \frac{\hbar k^2}{2}\right)} \sqrt{\frac{s+t}{2}} \\
&= \frac{\delta_{nk}}{\sqrt{\pi\hbar}} \int_{\mathbb{R}} dy\ e^{-\frac{(s+t)}{2\hbar}(y+\hbar k)^2} \sqrt{\frac{s+t}{2}} \\
&= \delta_{nk}.
\end{aligned}
$$

Remark 130 *The case of quantization of $T^*\mathbb{S}^1$ is also important for: the quantization of locally toric manifolds (see [3] and [16]) and the case of complex Lie groups (see [12] and [10],[11]).*

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