ABSTRACT

The study moving media in relativistic electrodynamics is a very important topic in electromagnetics research. Any subject studied within this field must be addressed using a four-dimensional space, where the regular cross product is no longer valid. Therefore, this study is done, in general, using algebraic manipulation with tensors. To avoid the use of this complicated tensor algebra, a new mathematical formalism, called geometric algebra or Clifford algebra, is addressed in this thesis. The geometric algebra has given its first steps in the late nineteenth century, although it remained almost forgotten in the years that followed. In 1963, the geometric algebra retrieves its prominence with the work developed by David Hestenes, who reformulated it and gave it the modern consistency. Since then, it has been gaining importance, and its strength is recognized by the way one structure can serve as a basis for such unrelated topics in physics and mathematics, in a clear and concise way. This thesis begins with the fundamental properties of geometric algebra and its application to the three-dimensional space with Euclidean metric.

Gradually, in a comprehensive way, one shifts into the Minkowski’s space-time, and its application in relativistic electrodynamics which is the core of this dissertation. The study of Minkowski’s space-time becomes more simpler due to the use of certain tools such as the boost (or active Lorentz transformation) and vacuum form reduction. In space-time algebra, Maxwell equation’s are reduced to only two, the homogeneous and the inhomogeneous equations and, in vacuum, they can be written as a single equation. In vacuum, as in other isotropic medium, the space-time constitutive relation, written in terms of the Faraday and Maxwell bivectors, is the same either in the lab frame as in the proper frame, since these two bivectors are not relative. By itself, the theoretical formulation of geometric algebra proofs that this kind of algebra can become, in the future, the unified language of modern physics.

Index Terms – Geometric algebra; Bivector; Relativistic electrodynamics; Boost; Maxwell bivector; Faraday bivector; Vacuum form reduction; Minkowski’s space-time; Moving media.

1. INTRODUCTION

The main objective of this work is to study several electrodynamic and relativistic effects, using the most appropriate algebra for that meaning – STA (spacetime algebra). In order to do so, one must first understand the basic concepts of Euclidean algebra, which is the foundation of spacetime algebra.

The history of geometric algebra got its start in Ancient Greece ([1] and [2]) with the writing of algebraic relations in geometric form. However, the formalism of geometric algebra as we know it today had its beginning only in nineteenth century.

In the early nineteenth century the mathematical representation of rotations in three dimensions still remained an unsolved problem. In 1843, Sir William Rowan Hamilton, wanted to extend the concept of complex numbers to other dimensions, and succeeded when he presented the quaternions of Hamilton, who successfully represented the rotations in three dimensions. These quaternions of Hamilton are used today to model problems in most applied sciences: computer vision, robotics, virtual reality, navigation, object-oriented programming [5]. Almost at the same time, in 1844, Grassmann presented the exterior product, which is the key operation in algebra is now known as “external algebra.” It also noted that the exterior product of Grassmann, exists in any dimension, with the added advantage of not relying on any metric adopted. The exterior product is as so a generalization of the Gibbs product which is confined to three dimensions.

In 1878, Clifford unified the findings of Grassmann and Hamilton in one unique algebraic structure, which gave origin to geometric algebra. However the introduction of vector calculus by Gibbs in the beginning of the 19 century that simplified largely the study of electromagnetism and the premature death of Clifford led his work to be forgotten.

Only in the beginning of the XX century when Einstein published the restrict-relativity theory, which
was formulated in four dimensions, experts started to question the validity of Gibbs product when applied to more than three dimensions and begun to give credit to Clifford’s work. So in 1920 geometric algebra was used in quantum mechanics to solve the problem of spin matrixes. Though this geometric algebra continued to be sub valorized and critics didn’t even attributed the geometric significance to geometric algebra.

In 1960 David Hestenes in his works regarding quantum mechanics, realized that geometric algebra was a powerful mathematical tool that was able to unify all areas of quantum mechanics.

Nowadays there are several experts that study geometric algebra and its applications to several of the areas already spoken. Emphasis on the works of Pertti Lounesto, Chris Doran, Anthony Lasenby, Leo Dorst, Daniel Fontijne e Stephen Mann.

2. BASIC CONCEPTS OF GEOMETRIC ALGEBRA $C\ell_3$

The geometric product or Clifford’s product is the key definition of geometric algebra. Consider the linear space $\mathbb{R}^3$ with the Euclidean metric where the orthonormal basis of this space is $B = \{e_1, e_2, e_3\}$.

$$j, k \in \{1, 2, 3\} \rightarrow e_j \cdot e_k = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad (1)$$

Where $|e_1| = |e_2| = |e_3| = 1$. Given the vector $r = xe_1 + ye_2 + ze_3 \in \mathbb{R}^3$ its respective length is $|r| = \sqrt{r \cdot r} = \sqrt{x^2 + y^2 + z^2}$, and the geometric product of this vector with itself is so that it is equal to the square of its length,

$$r^2 = |r|^2. \quad (2)$$

The definition given in (2) is the fundamental axiom of the euclidean geometric algebras ($C\ell_2$ and $C\ell_3$).

In consequence of this axiom the geometric product is non commutative, associative, invertible and depends on a metric.

This algebra had an object who square negatively, and is called a bivector. A bivector is a directed plane segment. $\hat{F} = e_{12} = e_1 e_2$ represents a unit bivector as shown in Figure 1.

Given two generic vectors $a, b \in \mathbb{R}^3$ the geometric product, $ab$, is the graduate sum of a scalar, given by the inner product between them, and a bivector, which is the exterior product also between them. As so, we have a multivector, $u$.

$$u = ab = a \cdot b + a \wedge b \in \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}^3 \quad (3)$$

where

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (4)$$

and

$$a \wedge b = \begin{vmatrix} e_{12} & e_{13} & e_{23} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (5)$$

The unit trivector is the result of the exterior product of the three vectors of the orthonormal basis considered. The trivector or oriented volume of this algebra is $e_{123}$. Formally we have $e_{123} = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3$ and the square of this object is also $-1$,

$$i^2 = e_{123}^2 = e_{123} e_{123} = (e_1 e_2 e_3) (e_1 e_2 e_3) = -e_1^2 e_2^2 e_3^2 = -1 \quad (6)$$

$\therefore (e_{123})^2 = -1$.
Only in $\mathbb{R}^3$ is possible to relate the exterior product with the cross product by $\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b})$. By analyzing the exterior product we have that:

$$a_b = \mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$$

so

$$|a_b| = |\mathbf{a}||\mathbf{b}||\sin \theta|,$$

where $\theta = \angle(\mathbf{a}, \mathbf{b})$, we can see it represented in Figure 3.

The geometric algebra in $\mathbb{C}_3$ has the subspaces of scalars $\mathbb{R}$, vectors $\mathbb{R}^3$, bivectors $2\mathbb{R}^3$ and trivectors $3\mathbb{R}^3$.

This algebra is a linear space of dimension $1+3+3+1=8$: adopting $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as an orthonormal basis for vector space $\mathbb{R}^3$, a suitable basis for corresponding linear $\mathbb{C}_3$ is

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{123}\}.$$

where $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_2 \wedge \mathbf{e}_1$, $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_1 \wedge \mathbf{e}_3$ and $\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_3 \wedge \mathbf{e}_2$ constitute a basis for the subspace $2\mathbb{R}^3$ of bivectors.

This structure is associated with an even part and an odd part. The even part, $\mathbb{C}_3^+ = \mathbb{R} \oplus 2\mathbb{R}^3$, is isomorphic to the quaternion’s ring of Hamilton, and $\mathbb{C}_3^- = \mathbb{R} \oplus 3\mathbb{R}^3$ is the odd part which is isomorphic to the complexes. Besides, there exist the algebra center, $\text{Cent}(\mathbb{C}_3) = \mathbb{R} \oplus 3\mathbb{R}^3$, who is the set of the elements that commute with all elements of this algebra.

### 2.1 Rotors

The rotors are the operators of geometric algebra responsible for the generation of rotations. These rotations can be rotations in the plane (in $\mathbb{C}_2$ algebra), spatial rotations (in $\mathbb{C}_3$ algebra) or spatial rotations and Lorentz transformations or boosts active (in $\mathbb{C}_{1,3}$ algebra).

Rotors are defined as the geometric product of two unitary vectors. Considering the linear space $\mathbb{R}^3$ and $\mathbf{n}, \mathbf{m} \in \mathbb{R}^3$ a rotor is,

$$R = \mathbf{n} \mathbf{m}.$$

From definition we can observe that $R$ is a multivector defined as,

$$R = \mathbf{n} \mathbf{m} + \mathbf{n} \wedge \mathbf{m} = \cos \theta + \mathbf{B} \sin \theta = \exp(\theta \mathbf{B}).$$

The rotor defined in (7) can handle a rotation of $2\theta$ in the plane of the corresponding bivector. So, if we would like to have a rotation of $\theta$ in the corresponding bivector plane, we proceed as in

$$\mathbf{u} = -\mathbf{B} \mathbf{e}_{123}.$$

Figure 4 – Rotation of the vector $\mathbf{a}$ to $\mathbf{a}'$.  

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It may be concluded that when performing a rotation, the parallel component of the vector is changes, while the perpendicular component remains unchanged.

2.2 Contractions

Another important operation which will be introduced is the contraction - it is possible to define two types of contractions:
- Left Contraction.
- Right Contraction.

Considering the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \) and the bivector \( \mathbf{B} = \mathbf{b} \wedge \mathbf{c} \), the left contraction is by definition,
\[
a \lrcorner \mathbf{B} = \frac{1}{2} (\mathbf{aB} - \mathbf{Ba})
\]
(8)

It follows immediately the fundamental rule of contraction on the left
\[
\mathbf{a} \lrcorner (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b}
\]
(9)

In analogous way we obtain the right contraction as
\[
\mathbf{B} \rhd \mathbf{a} = \frac{1}{2} (\mathbf{Ba} - \mathbf{aB})
\]
(10)

It is an anti-symmetric operation, hence
\[
\mathbf{a} \rhd \mathbf{B} = -\mathbf{B} \rhd \mathbf{a}
\]
(11)

We can also write
\[
\begin{align*}
\mathbf{aB} &= \mathbf{a} \lrcorner \mathbf{B} + \mathbf{a} \wedge \mathbf{B} \\
\mathbf{B} \mathbf{a} &= \mathbf{B} \rhd \mathbf{a} + \mathbf{B} \wedge \mathbf{a}
\end{align*}
\]
(12)

So, we conclude that left contraction of a vector with a bivector will step down the degree of the bivector into the degree of the vector. The right contraction have the same consequence, but the new vector is diametrically opposed to the one obtain in left contraction. In Fig. 5 is the geometrical representation of a left contraction of a vector with a bivector.

3. GEOMETRIC ALGEBRA APPLIED TO MINKOWSKI SPACETIME

It was Minkowski who, in 1908, taking the work of the special theory of relativity of Einstein, worked on the concept of time and space, to create an entity where to stay highlighted their interdependence.

The Einstein’s theory firms in two postulates. The first one is that all inertial frame of reference are equivalent. The second says that the speed of light is the same to all inertial frame of reference.

To spacetime studies the euclidean algebra is unacceptable, it drive us to impossible physics results. So, the spacetime algebra is useful to handle certain kinds of problems which cannot be addressed by Euclidean algebras.

This algebra differs from the latter ones on a basic principle: the square of the vectors is not always equal to 1, as follows
\[
\mathbf{e}_0^2 = \mathbf{e}_i^2 = \mathbf{e}_j^2 = \mathbf{e}_k^2 = 1, \quad \mathbf{e}_0^2 = -1.
\]

This algebra is defined in \( \mathbb{R}^{1,3} \) and is spanned by the basis set \( \mathbf{B} = \{ \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} \).

It is constituted by one scalar, four vectors, six bivectors, four trivectors and the pseudoscalar – quadrivector, hence the dimension is equal to 16.

In Minkowski spacetime an event is composed by a temporal component \( (c^2 t) \) and a spatial component \( (\mathbf{r}) \):
\[
\mathbf{r} = (c^2 t) \mathbf{e}_0 + \mathbf{r}.
\]
(13)

We get
\[
\begin{align*}
\mathbf{r}^2 &= (c^2 t)^2 + \mathbf{r}^2 \\
(\mathbf{r})^2 &= -[\mathbf{r}]^2
\end{align*}
\]
(14)

Due to the fact that Lorentz metric is semi-defined negatively an event can be
\[
\begin{align*}
|\mathbf{r}|^2 = 0 &\iff (c^2 t)^2 = (\mathbf{r})^2 &\rightarrow \text{Parabolic or light like} \\
|\mathbf{r}|^2 > 0 &\iff (c^2 t)^2 > (\mathbf{r})^2 &\rightarrow \text{Hyperbolic or time like.} \\
|\mathbf{r}|^2 < 0 &\iff (c^2 t)^2 < (\mathbf{r})^2 &\rightarrow \text{Elliptic or space like}
\end{align*}
\]

The bivectors are no longer only simple bivectors. It can be simple or not simple. An example of a non simple bivector is:
\[
\mathbf{F} = \mathbf{e}_4 \wedge \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3 \Rightarrow \mathbf{F}^2 = 2 \mathbf{1} \in \mathbb{R}^{4,1,3}.
\]
(15)

One should note that the simple bivectors can be classified as
- Parabolic \( \rightarrow \mathbf{F}^2 = 0 \)
- Elliptic \( \rightarrow \text{Rotation} \quad \rightarrow \exp(\theta \mathbf{e}_3) = \cos \theta + \mathbf{e}_3 \sin \theta \)
- Hyperbolic \( \rightarrow \text{Boost} \quad \rightarrow \exp(\zeta \mathbf{e}_0) = \cosh \zeta + \mathbf{e}_0 \sinh \zeta \)
The most important highlight is that elliptical bivectors originate rotations \( (\exp(\theta\mathbf{e}_2) = \cos \theta + \mathbf{e}_3 \sin \theta) \), while hyperbolic bivectors originate boosts \( (\cosh \zeta + \mathbf{e}_0 \sinh \zeta) \). A boost is a similar transformation to rotation, although not as simple. A boost is also known as active Lorentz transformation.

### 3.1 Lorentz Transformation

There are active and passive Lorentz transformations. \( \zeta \) is the rapidity parameter, which controls the intensity of the boost, so the goal is to expose that tool. A boost performs a transformation of a certain frame defined by \( \mathbf{f}_0, \mathbf{f}_1 \) to another defined by \( \mathbf{e}_0, \mathbf{e}_1 \). \( \gamma = \cosh \zeta = \frac{1}{\sqrt{1 - \beta^2}} \) is the correction factor, which depends directly on the velocity of a certain particle. \( \beta = \frac{v}{c} = \tanh \zeta \) is the normalized velocity hence \( \gamma \beta = \sinh \zeta \). When the rapidity tends to infinite, the normalized velocity tends to \( \tanh \left( \frac{\pi}{4} \right) = 1 \), which means that \( \mathbf{f}_0 \rightarrow \mathbf{e}_0 + \mathbf{e}_t \) and \( \mathbf{f}_1 \rightarrow \mathbf{e}_0 + \mathbf{e}_t \), therefore tending to the bisection, as represented next,

![Figure 6 - Representation of a boost](image)

Boosts result Minkowski diagrams, which will be the used tool to study relativistic effects. For a purpose of simplicity, one will use a \( C_{1,1} \) diagram instead of \( C_{\ell,3} \) for a matter of convention, the first approach shall be taken.

![Figure 7 – Minkowski diagram](image)

An event is described by a person standing on a frame, which can be represented on the Minkowski diagram according to the expression which links both time and space. An observer located on the rest frame \( S \) shall interpret an event \( \mathbf{r} = (ct)\mathbf{e}_0 + \mathbf{e}_t \), while another observer located on the \( \overline{S} \) will have another interpretation \( \mathbf{r} = (ct')\mathbf{f}_0 + \mathbf{f}_1 \), which means that the same event has diverse point of view, therefore it is not possible to define an absolute time or space. Simultaneity is a relative concept, and the passive Lorentz transformation explains that. While the active transformation performs an actual transformation of vectors, the passive merely shows the diverse points of view for a particular event. Any two events \( A \) and \( B \) which occur on a line parallel (also known as line of simultaneity in \( \overline{S} \)) to axis \( \mathbf{x} \), happen simultaneously regarding an observer on the \( S \) frame. This is not valid for another observer who is placed on the \( \overline{S} \) frame.

The difference in time events regarding \( S \) frame is given by \( \Delta ct_{AB} = ct_B - ct_A \) which is different from zero. Following the same thread of thought, any two events \( A \) and \( B \) which occur on a line parallel to axis \( x \), defined as \( ct_0 \), also known as line of simultaneity in \( S \) happen simultaneously regarding an observer on the \( S \) frame. This is not valid for another observer who is placed on the \( \overline{S} \) frame, which is shown on next figure. The difference in time events regarding \( S \) frame is given by \( \Delta \tilde{ct}_{AB} = \tilde{ct}_B - \tilde{ct}_A \) which is different from zero.

Several application shall be addressed on this paper, starting with time dilation and space contraction. Either effect may be performed considering the rest frame as \( S \) or \( \overline{S} \), as the result is the same. From the previous figure it can be inferred that vector \( \overline{AB} \) is equal to the sum of other two vectors, as follows:

![Figure 8 – Time dilation](image)
\[ \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}. \]  
(16)

This results on the following equation
\[ cT_f = cTe + vTe. \]  
(17)

According with the equation (15) the time dilation is given by:
\[ T_0 y = \frac{T_0}{\sqrt{1 - \beta^2}} = T. \]  
(18)

Now, analyzing the space contraction, the geometric equation which translates the problem, is given by
\[ \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}. \]  
(19)

\[ \frac{L_0}{\gamma} = L_0 \sqrt{1 - \beta^2} = L. \]  
(20)

It may be concluded that the correction factor is the parameter that allows transforming the time or length observed on a specific frame to another which may not be at rest.

4. RELATIVISTIC OPTICS

The Minkowski spacetime algebra in relativistic approach avoids the use of a rather complex mathematical tensor. The use of this science, although recent, shows striking advantages, on mathematical simplicity involved. The Doppler Effect is a classic example where we can apply the algebra of spacetime to reduce the complexity of the problem, relating the same Doppler Effect with the phenomenon of aberration.

We define in \( C^{1,3} \), an wave vector as \( \overrightarrow{k} = k_0 \overrightarrow{e_0} + \overrightarrow{k} \in \mathbb{R}^{1,3} \) where \( k_0 = \omega / c \) and \( \overrightarrow{k} = k \overrightarrow{s} \), and consider a plane wave propagation of the form \( \exp[-i(\overrightarrow{k} \cdot \overrightarrow{r})] \).

For an electromagnetic wave propagating in a stationary media characterized by refractive index \( n_0 \), where \( \overrightarrow{k} = k \overrightarrow{s} = n_0 k_0 \overrightarrow{s}_0 = n_0 (\omega / c) \overrightarrow{s}_0 \) we have
\[ \overrightarrow{k} = \left( \frac{\omega}{c} \right) (\overrightarrow{e}_0 + n_0 \overrightarrow{s}_0). \]  
(21)

To climb to Doppler Effect, we consider an emission of a plane wave between two observers, an emitter and a receiver. The wave vector is describe by
\[ \overrightarrow{k}_{emitter} \rightarrow \overrightarrow{k} = \frac{\omega}{c} (\overrightarrow{e}_0 + \overrightarrow{s}_e) \]  
(22)
\[ \overrightarrow{k}_{receiver} \rightarrow \overrightarrow{k} = \frac{\omega}{c} (\overrightarrow{f}_0 + \overrightarrow{s}_r). \]

The emitter is associate with a base \( \mathcal{B}_e = \{ \overrightarrow{e}_0, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3 \} \) while the receiver has a base \( \mathcal{B}_r = \{ \overrightarrow{f}_0, \overrightarrow{f}_1, \overrightarrow{f}_2, \overrightarrow{f}_3 \} \). The vector \( \overrightarrow{k} \) is seen equally by both reference and frequency of the emitter is \( \omega_e = \overrightarrow{k} \cdot \overrightarrow{v} \) and the frequency of the receiver is \( \omega_r = \overrightarrow{k} \cdot \overrightarrow{u} \), where \( \overrightarrow{v} = c \overrightarrow{e}_0 \) and \( \overrightarrow{u} = c \overrightarrow{f}_0 \).

The relation between the bases of the observers can be given by a simple boost, where \( \mathcal{B}_r = \mathcal{B}_e = \mathcal{B}_e \wedge \overrightarrow{e}_0 \):
\[ \overrightarrow{e}_0 \mapsto \overrightarrow{f}_0 = \exp(\zeta \mathcal{B}_r) \overrightarrow{e}_0 = \cosh \zeta \overrightarrow{e}_0 + \mathcal{B}_r \sinh (\zeta) \overrightarrow{e}_0. \]

We can have \( \overrightarrow{s} \neq \overrightarrow{s}_e \), so is defined that \( \overrightarrow{s} \cdot \overrightarrow{s}_e = -\cos \phi \). A boost is enough only when \( \phi = 0 \).

In general case we have a spatial rotation \( \mathcal{U} \), followed by a boost, \( \mathcal{L} \) as shown in Fig. 10.

Rotation
\[ \mathcal{U} \]

Boost
\[ \mathcal{L} \]

Figure 10 – The relationship between the emitter and the receiver, through a rotor and a boost.

Rotation U
\[ \mathcal{U} \]

Boost L
\[ \mathcal{L} \]

Figure 9 – Space contraction
\[ \frac{L_0}{\gamma} = L_0 \sqrt{1 - \beta^2} = L. \]
Boost:
\[
\mathbf{e}_0 \mapsto f_0 = L(\mathbf{e}_0) = L^2 \mathbf{e}_0 = \exp (\xi \hat{\mathbf{B}}) \mathbf{e}_0,
\]
\[
\hat{\mathbf{B}} = \mathbf{s} \times f_0 = (L \mathbf{s} L) (L \mathbf{e}_0 L) = \mathbf{s} \mathbf{e}_0 .
\]
(24)

It is possible obtain
\[
\mathbf{u} = c f_0 \mapsto \mathbf{u} = c \gamma (\mathbf{e}_0 + \beta \mathbf{s}).
\]
(25)

As the wave vector is the same, independently the observer, we have
\[
\mathbf{k} \cdot \mathbf{u} = [c \gamma (\mathbf{e}_s + \beta \mathbf{s})] \left[ \frac{\omega}{c} (\mathbf{e}_0 + \mathbf{s}) \right] = \omega \gamma (1 - \beta \cos \phi).
\]
(26)

But \(\omega_s = \mathbf{k} \cdot \mathbf{u} \), so we get the relativistic Doppler Effect:
\[
\frac{\omega_s}{\omega} = \gamma (1 - \beta \cos \phi) = \frac{1 - \beta \cos \phi}{\sqrt{1 - \beta^2}^\circ}.
\]
(27)

It is the general formula for Doppler shift – when angle \(\phi = 0\left(\phi = \frac{\pi}{2}\right)\) results on the longitudinal (transverse) Doppler Effect. The first situation degenerates on \(\frac{\omega_s}{\omega} = \frac{(1 - \beta)}{(1 + \beta)}\) thus on a redshift, as \(\omega_s < \omega\); the second situation results on \(\frac{\omega_s}{\omega} = \frac{1}{\sqrt{1 - \beta^2}}\) or a blueshift, which happens when \(\omega_s > \omega\). Doppler shift may be used to make a spectral analysis of the sun’s light, for instance. To say that an image is blueshifted (where a blueshift occurs) means an observer is looking at that part of the sun that is moving towards him, or the light is compressed to shorter wavelengths, so the frequency is increasing. Likewise, the opposite can be said about a red image, where the opposite takes place, also called as a redshift.

When the medium is a cold plasma without collisions in which
\[
n(\omega) = \sqrt{\frac{1 - \omega_s^2}{\omega^2}},
\]
(28)

Being the plasma frequency \(\omega_p\), then we obtain
\[
\omega = \gamma \left( \omega - \beta \sqrt{\omega^2 - \omega_p^2} \cos \theta \right).
\]
(29)

5. RELATIVISTIC ELECTRODYNAMICS

As shown, geometric algebra possesses innumerous techniques for studying problems in electrodynamics and electromagnetism. This chapter proposes several applications regarding the matters referred above, which will emphasize the power of geometric compared to traditional algebras. A big improvement of this treatment, is a more recent and compact formulation on Maxwell’s equations. It will permit to transform Maxwell’s four equations into astonishing individual one. In order to study electromagnetism and electrodynamics, one must introduce a fundamental operator – del.
\[
\nabla = \mathbf{e}_x \frac{\partial}{\partial x_1} + \mathbf{e}_y \frac{\partial}{\partial x_2} + \mathbf{e}_z \frac{\partial}{\partial x_3} \in \mathbb{R}^{0,3}.
\]
(30)

First, it’s important to consider that \(\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}\) are vectors that belong to \(\mathbb{R}^{0,3}\), they represent the electric field, magnetic field, electric displacement field and magnetizing field, respectively. \(\mathbf{J} \in \mathbb{R}^{0,3}\) and \(\mathbf{\rho} \in \mathbb{R}\) are the sources of electromagnetic field: current density and charge density, respectively.

It is possible to represent the Maxwell equations in \(\mathcal{Cl}_3\) on a most known structure, where one may distinguish two groups of equations - Faraday group, which includes Maxwell-Faraday’s equation, also known as the Faraday’s law of induction and Gauss’s law for magnetism
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.
\]
(31)
\[
\nabla \cdot \mathbf{B} = 0.
\]
(32)

The second pair of equations is identified as the Maxwell group – both Ampère’s circuital law with Maxwell’s correction and Gauss’s law are present, as the following set of equations show
\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.
\]
\[
\nabla \cdot \mathbf{D} = -\mathbf{\rho}.
\]

In order to characterize Maxwell equations in \(\mathcal{Cl}_{3}\) algebra, one must introduce their relative vectors. Those bivectors are hyperbolic or time like, they square positively:
\[
\begin{align*}
\mathbf{E} &= \hat{\mathbf{E}} \mathbf{e}_0 = \hat{\mathbf{E}} \wedge \mathbf{e}_0 \in \wedge^2 \mathbb{R}^{0,3} \\
\mathbf{B} &= \hat{\mathbf{B}} \mathbf{e}_0 = \hat{\mathbf{B}} \wedge \mathbf{e}_0 \in \wedge^2 \mathbb{R}^{0,3} \\
\mathbf{D} &= \hat{\mathbf{D}} \mathbf{e}_0 = \hat{\mathbf{D}} \wedge \mathbf{e}_0 \in \wedge^2 \mathbb{R}^{0,3} \\
\mathbf{H} &= \hat{\mathbf{H}} \mathbf{e}_0 = \hat{\mathbf{H}} \wedge \mathbf{e}_0 \in \wedge^2 \mathbb{R}^{0,3}
\end{align*}
\]
(33)

Besides that, the Dirac operator and electric charge – current densities are also necessary,
\[ J = \rho \, e_0 + \frac{1}{c} \, \mathbf{J} \in \mathbb{R}^{1,3}, \quad \partial = \frac{1}{c} \, \partial_t + \mathbf{V} \in \mathbb{R}^{1,3}. \tag{34} \]

Now it is possible to define the essential bivectors of the electromagnetic field, which are absolute vectors – Faraday and Maxwell bivectors

\[
\begin{align*}
\text{Maxwell bivector} & \rightarrow \mathbf{F} = \frac{1}{c} \mathbf{E} + \mathbf{IB} \\
\text{Faraday bivector} & \rightarrow \mathbf{G} = \mathbf{D} + \frac{1}{c} \mathbf{IH}.
\end{align*} \tag{35} \]

Finally we introduce the two Maxwell equations in Minkowski spacetime as

(i) homogeneous equation \[ \partial \wedge \mathbf{F} = 0 \]

(ii) inhomogeneous equation \[ \partial \mathbf{G} = \mathbf{J}. \tag{36} \]

The homogeneous equation expresses the magnetic flux conservation whereas the inhomogeneous equation expresses the electric charge conservation.

We consider plane wave propagation, in vacuum the spacetime constitutive relation is

\[ \mathbf{G}_0 = \frac{1}{\eta_0} \mathbf{F}_0 \tag{37} \]

where \( \eta_0 = \sqrt{\mu_0 / \varepsilon_0} \). The zero indexes in the equation above that indicates plane wave propagation will be omitted from now.

We can write the geometric product \( \partial \mathbf{F} = \partial \mathbf{G} + \partial \wedge \mathbf{F} \). With this constitutive relations and geometric product it is possible to have a unique Maxwell equation

\[ \partial \mathbf{F} = \eta_0 \, \mathbf{J}. \tag{38} \]

To the particularly case for source-free regions:

\[ \partial \mathbf{F} = 0. \tag{39} \]

### 5.1. Moving media

The \( \mathcal{C}_{1,3} \) algebra is useful to study a great variety of problems. One of the classic electrodynamics problems is moving media. Considering a simple isotropic medium (limited, linear and lossless), in \( \mathcal{C}_{3} \) algebra is defined by the following constitutive relations (using the Space-Time Algebra):

\[ \mathbf{D} = \varepsilon_0 \, \varepsilon \, \mathbf{E} \]
\[ \mathbf{H} = \frac{1}{\mu_0} \, \mu \, \mathbf{H}. \tag{40} \]

Then, one may determine vector \( \mathbf{G} \)

\[ \mathbf{G} = \frac{1}{2 \eta_0} \left( \varepsilon + \frac{1}{\mu} \right) \mathbf{F} - \frac{1}{2 \eta_0} \left( \varepsilon - \frac{1}{\mu} \right) \mathbf{F}_v \tag{42} \]

As \( \mathbf{v} = c \, e_0 \) represents the frame where the media is at rest, one obtains

\[ \mathbf{F}_v = -\frac{1}{c} \mathbf{E} + \mathbf{IB} \tag{43} \]

This represents vector \( \mathbf{F} \mathbf{F} \) seen by an observer on frame \( e_0 \). For a certain linear combination of \( \mathbf{F} \) with \( \mathbf{F}_v \) comes

\[ \mathbf{G} = \frac{1}{2 \eta_0} \left( \varepsilon + \frac{1}{\mu} \right) \mathbf{F} - \frac{1}{2 \eta_0} \left( \varepsilon - \frac{1}{\mu} \right) \mathbf{F}_v \tag{44} \]

Then

\[ \mathbf{G} = \frac{1}{\eta} \exp(-\xi \mathbf{r}_u) \mathbf{F} \tag{45} \]

With

\[ \mathbf{r}_u (\mathbf{F}) = \mathbf{r}_u = \mathbf{u}^{-1} \mathbf{F} \mathbf{u} = \frac{1}{c^2} \mathbf{u} \mathbf{F} \mathbf{u} \tag{46} \]

The local form of the constitutive relation represents a projection of the constitutive relation on a particular observer, who is at rest regarding the media. In other words, it could be the media’s own observer. It is possible to split the constitutive relation \( \mathbf{G} = \mathbf{G}(\mathbf{F}) \mathbf{G} = \mathbf{G}(\mathbf{F}) \) into two constitutive relations \( \mathbf{D} = \mathbf{D} \left( \mathbf{E}, \mathbf{B} \right) \) and \( \mathbf{H} = \mathbf{H} \left( \mathbf{E}, \mathbf{B} \right) \). This means that the media at hand is bianisotropic, and both \( \mathbf{D} \) and \( \mathbf{H} \) depend, not only on the electric field intensity \( \mathbf{E} \), but also on magnetic field intensity \( \mathbf{B} \). According to these results it can be concluded that an isotropic media on its own frame is considered as a bianisotropic medium on the rest frame.

### 5.2. Vacuum Form Reduction

The vacuum form reduction will lead us to a new perspective on relativistic optics in moving media. Let us consider the following transformation
\[
\begin{align*}
F &= \exp \left( \frac{\xi}{2} \tau_0 \right) F' \\
G &= \exp \left( -\frac{\xi}{2} \tau_0 \right) G' \\
&\Rightarrow \\
F' &= \exp \left( -\frac{\xi}{2} \tau_0 \right) F \\
G' &= \exp \left( \frac{\xi}{2} \tau_0 \right) G.
\end{align*}
\]

Then, formally, equation (43) reduces to
\[
G' = \frac{1}{\eta} F'.
\] (48)

With this transformation, we reduce the spacetime constitutive relation of an isotropic medium to a spacetime constitutive relation just as in vacuum.

One should note that with this transformation we are in another Minkowski spacetime, now the spacetime is fictitious – equivalent spacetime. This change takes consequences in the spacetime algebraic structure. The new structure has
\[
\mathcal{B} = \{ e_0, e_1, e_2, e_3 \} \mapsto \mathcal{B}' = \{ e'_0, e'_1, e'_2, e'_3 \}.
\] (49)

\[
\partial = \frac{1}{c} e_0 \frac{\partial}{\partial t} + \nabla \mapsto \partial' = \frac{1}{c} e'_0 \frac{\partial}{\partial t} + \nabla'.
\] (50)

\[
\begin{align*}
F &\mapsto F' \\
G &\mapsto G' \\
J &\mapsto J'.
\end{align*}
\]

We have \( \partial' F' = \partial' \wedge F' + \partial' \cdot F' \) so, the new Maxwell equation is given by
\[
\partial' F' = \eta J'.
\] (52)

For the case where there is lack of sources of field \( \partial' F' = 0 \).

Considering the propagation of plane monochromatic waves like
\[
F' = \Re \{ F'_0 \exp \left[ -i \left( \mathbf{k} \cdot \mathbf{r} \right) \right] \}.
\] (53)

So the Maxwell equation can also be written in the form
\[
\begin{align*}
\mathbf{k}' \wedge F'_0 &= 0 \\
\mathbf{k}' \cdot G'_0 &= 0 \\
\mathbf{k}' \cdot \mathbf{F}'_0 &= 0
\end{align*}
\] (54)

where the wave vector is obtained by the transformation \( \mathbf{k}' = \exp \left( \frac{\xi}{2} \tau_0 \right) \mathbf{k} \). The Maxwell equation \( \mathbf{k}' F'_0 = 0 \) is just as in vacuum. Formally we have
\[
(\mathbf{k}')^2 F'_0 = 0
\] (55)

so the solution to this equation, without consider the null Faraday bivector, is that wave vector is null
\[
(\mathbf{k}')^2 = 0.
\] (56)

Finally we obtain one of the most important conclusions of this paper. This last equation means that, in the equivalent space time induced by the Vacuum Form Reduction the wave vector is null- just as in real vacuum.

We know that
\[
\mathbf{k}' = \exp(\xi/2) \mathbf{k} \Rightarrow \mathbf{k}' = \cosh(\xi/2) \mathbf{k} + \sinh(\xi/2) \mathbf{k}_s
\] (57)

so we have
\[
\mathbf{k}' = \frac{1}{2} \sqrt{n_0} [(n_0 + 1) \mathbf{k} + (n_0 - 1) \mathbf{k}_s].
\] (58)

where \( \mathbf{k}_s = \tau_0(\mathbf{k}) \). Write \( (\mathbf{k}')^2 = 0 \) is equal to have
\[
(\mathbf{k}')^2 = 0 \Rightarrow \mathbf{k}^2 + \frac{1}{c^2} (n_0^2 - 1) (\mathbf{u} \cdot \mathbf{k})^2 = 0.
\] (59)

Now, in lab frame we have
\[
\begin{align*}
\mathbf{u} &= \gamma c (\mathbf{e}_0 + \bar{\beta}) \Rightarrow \mathbf{k}^2 &= k_0^2 (1 - n^2) \\
\mathbf{k} &= k_0 (\mathbf{e}_0 + \bar{n}) \Rightarrow (\mathbf{u} \cdot \mathbf{k})^2 &= \gamma^2 c^2 k_0^2 \left(1 + \bar{\beta} \bar{n}\right)^2
\end{align*}
\]

where \( \theta = \alpha(\bar{\beta}, \bar{n}) \) and \( \Omega = \gamma^2 (n_0^2 - 1) \). From the eq. (59) we have
\[
(\beta^2 \cos^2 \theta - 1) n^2 - 2 \beta \Omega \cos \theta n + (1 + \Omega) = 0.
\] (60)

Solving this equation we have
\[
n(\theta) = \sqrt{1 + \Omega \left(1 - \beta^2 \cos^2 \theta\right) - \beta \Omega \cos \theta} \sqrt{1 - \beta^2 \Omega^2 \cos^2 \theta}.
\] (62)

Thus we chose the solution corresponding to \( n = n_0 \) for \( \beta = 0 \). The phase velocity corresponding to \( \bar{v}_p = (w/k) \bar{x}_0 = (c/n) \bar{x}_0 \) is then
\[
\frac{v_p(\theta)}{c} = \sqrt{1 + \Omega \left(1 - \beta^2 \cos^2 \theta\right) + \beta \Omega \cos \theta} \frac{1 + \Omega}{1 + \Omega}.
\] (63)

\section{6. CONCLUSION}

This work allows us to understand how the geometric algebra has become a stylish and effective
mathematical tool in the study of electromagnetism. It was concluded that this algebra contains a Euclidean metric and a non-Euclidean metric (Lorentz metric). In Euclidean metric, we analyzed the algebra of space, whereas in non-Euclidean metric, we analyzed the algebra of Minkowski spacetime. This last is seen, as an ideal tool to clarify the relationship between mechanics and electromagnetism.

The essential definition of these algebras is the geometric or Clifford product between vectors, which represents the sum of inner product with a new product, the exterior product. Unlike the cross product of Gibbs, the exterior product is definable in n dimensions, and generates new geometrical objects such as the bivector and trivector. Another important characteristic is the inherent geometric interpretation one has, which makes this algebra so intuitive to understand and use, hence attractive for beginners and motivating new users.

With Einstein’s special theory of relativity, the relation between mechanics and electromagnetism is clarified and, as result, Newtonian mechanics is replaced by its relativistic formulation. So, to study this relativistic formulation the geometric algebra of Clifford is the best mathematical system to do it. With spacetime algebra the use of the conventional tensor and dyadic analyses are useless.

This work’s foundations rely on not only in geometric algebra, but on special relativity and Lorentz transformations as well, so several conclusions shall be presented. Simultaneity is a crucial concept when dealing with these matters: observers on different frames interpret events, each on a particular way. The study of Doppler Effect with geometric algebra using boost and rotor is explained in an intuitive way.

It was concluded that the algebra of Minkowski spacetime serves as a unifying tool of mathematics, since it does not separate time from space. The geometric algebra made possible the writing of Maxwell’s equations in a more general and simplified. Addressing electromagnetism, we have to introduce some new concepts. \( \delta \mathbf{F} = \eta \mathbf{J} \) is the most condensed way to write Maxwell equations in spacetime algebra. It is possible to distinguish absolute \( (\mathbf{F}, \mathbf{G}) \) and relative vectors \( (\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}) \); the absolute vectors are don’t depend on the frame the observer is located; on the other hand relative vectors depend on the work frame.

As emphasized by David Hestenes, the spacetime algebra is the best framework to tackle electromagnetism in the context of special relativity, due to the inherent reduced complexity, as well as the coherent results obtained. In fact, the two Maxwell equations in spacetime algebra are reduced to a single one, as the vacuum situation. This was able by applying Vacuum Form Reduction to a plane wave propagation in moving isotropic media. From the point of view of the laboratory, it may be concluded that it is actually a nonreciprocal bianisotropic media, which means its constitutive expressions \( \mathbf{D} \) and \( \mathbf{H} \) depend on \( \mathbf{E} \) and \( \mathbf{B} \).

This paper seeks to show how the geometric algebra is a structure clear, intuitive and geometric, and it becomes a universal and unifying language for the study and application in different areas of physics and engineering.

The geometric algebra is taking important steps in the scientific community and has been applied to quantum mechanics, computing, signal processing, among others.

As prospects for future work, it is possible to mention some areas such as:

- Thomas rotation because it allows calculating the composition of speed when references are not collinear.
- Theory of general relativity.

**REFERENCES**


