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A weak dynamic programming principle for zero-sum stochastic differential games

Pedro Miguel Almeida Serra Costa Vitória

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Presidente: Professora Doutora Ana Bela Cruzeiro
Orientador: Professor Doutor Diogo Aguiar Gomes
Vogal: Professor Doutor António Atalaia Serra
Vogal: Doutor Gabriele Terrone

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"I was born not knowing and have had only
a little time to change that here and there."
Richard Feynman

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Resumo

Neste trabalho extendemos uma versão fraca do princípio de programação dinâmica, provado pela primeira vez em [1] para problemas de controlo optimal, a problemas de jogos diferenciais estocásticos de soma nula. Deste modo, conseguimos derivar a equação de Hamilton-Jacobi-Bellman-Isaacs quando um dos jogadores pode utilizar estratégias com valores num conjunto ilimitado.

Palavras-chave: jogos diferenciais estocásticos, função valor, princípio de programação dinâmica, soluções de viscosidade.

Abstract

We extend a weak version of the dynamic programming principle, first proven in [1] for stochastic control problems, to the context of zero-sum stochastic differential games. By doing so we are able to derive the Hamilton-Jacobi-Bellman-Isaacs equation when one of the players is allowed to use strategies taking values in an unbounded set.

Keywords: stochastic differential games, value function, dynamic programming principle, viscosity solutions.

Contents

Preface

In [1], Bouchard and Touzi propose a weak version of the dynamic programming principle in the context of stochastic optimal control. Their objective was to avoid technical difficulties related to the measurable selection argument. By doing this they were able to derive the dynamic programming equation without requiring the value function to be measurable. One question that arises naturally is how to extend this approach to stochastic differential games.

Zero-sum stochastic differential games were studied rigorously for the first time by Fleming and Souganidis in [2]. These problems are usually studied in a setting where the strong assumptions imply that the value function is continuous, hence measurable. Considering a weak version of the dynamic programming principle gives us the opportunity of studying these problems in a more general setting where the value function does not have *a priori* much regularity.

This thesis tackles the problem of extending the weak dynamic programming principle of [1] to the context of stochastic differential games. This is done in Chapter 4. For convenience of the reader we develop in the first Chapters small introductions to the theories of deterministic optimal control, stochastic optimal control and deterministic two-person zero-sum differential games. In addition, 2 appendices in the end briefly recall and develop some basic notions and results of second-order viscosity solutions and stochastic calculus.

Chapter 1

Deterministic optimal control

In this Chapter we consider the *deterministic optimal control* problem in the finite horizon setting. Our objective is to establish the dynamic programming principle and derive from it the Hamilton-Jacobi-Bellman equation.

We aim to stress the main ideas and arguments. Thus, we consider the Mayer problem (no running cost) with compact valued controls and bounded value function. For a detailed exposition we refer the reader to [3].

1.1 Introduction

We start by outlining in a brief and informal way the main ideas of this Chapter. Consider a state variable, X, driven by a control ν through the following nonlinear system:

$$dX(s) = \mu\left(s, X(s); \nu(s)\right) ds,$$

with initial condition $X(t) = x$. Since this variable X depends on t, x, ν, we use the notation $X := X^{\nu}_{t,x}$.

We are interested in the terminal value of this variable, $X^{\nu}_{t,x}(T)$. More precisely, we are interested in the quantity

$$J(t, x; \nu) := f\left(X^{\nu}_{t,x}(T)\right),$$

which we want to maximize over all admissible controls. Thus we want to determine the value function

$$v(t, x) := \sup_{\nu} J(t, x; \nu).$$

We will use a dynamic programming approach to this optimization problem.

To establish heuristically the dynamic programming principle we assume that there is an optimal control ν^*, i.e., there exists ν^* such that, for all (t, x),

$$v(t, x) = J(t, x; \nu^*).$$

Then, given $\tau \geq t$, we have, by the flow property for ordinary differential equations, that

$$\begin{aligned} v(t, x) &= f\left(X^{\nu^*}_{t,x}(T)\right) \\ &= f\left(X^{\nu^*}_{\tau, X^{\nu^*}_{t,x}(\tau)}(T)\right) \\ &= J\left(\tau, X^{\nu^*}_{t,x}(\tau); \nu^*\right) \\ &= v\left(\tau, X^{\nu^*}_{t,x}(\tau)\right). \end{aligned}$$

If the existence of admissible controls is not assumed then we should expect the previous equality to be replaced by the so-called dynamic programming principle:

$$v(t,x) = \sup_{\nu} v\left(\tau, X^{\nu}_{t,x}(\tau)\right).$$

The dynamic programming principle gives us important information on the local behavior of the value function. Letting $\tau \to t$ we are able to derive an infinitesimal version of it, the Hamilton-Jacobi-Bellman equation. Indeed, if we assume that v is C^1, then we have, by the chain rule, that

$$0 = \sup_{\nu} \frac{v\left(\tau, X^{\nu}_{t,x}(\tau)\right) - v(t,x)}{\tau - t} \underset{\tau \to t}{\longrightarrow} \sup_{u} \left(\partial_t v(t,x) + \langle \mu(t,x;u), Dv(t,x)\rangle\right)$$

Thus, we should expect v to be a solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\left(-\partial_t v + \inf_{u} -\langle \mu(.;u), Dv\rangle\right)(t,x) = 0.$$

Since in many applications the value function lacks the regularity to be a classical solution of the previous equation, we need a weak definition of solution for such an equation. As it turns out, the theory of viscosity solutions provides the adequate framework for this derivation. In fact, as we will see, the value function is a viscosity solution of the HJB equation.

1.2 The controlled dynamical system

We consider a nonlinear system in $\mathbb{R}^d$, written in differential form as

$$\begin{cases} dX^{\nu}_{t,x}(s) &= \mu\left(s, X^{\nu}_{t,x}(s); \nu(s)\right) ds \\ X^{\nu}_{t,x}(t) &= x, \end{cases} \tag{1.1}$$

where ν is a control function in a set $\mathcal{U}$ to be specified later, (t,x) are the initial conditions and $\mu : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ is a continuous function which is Lipschitz continuous in the space variable, that is

$$|\mu(t,x;u) - \mu(t,y;u)| \leq K|x-y|,$$

for some constant K.

We call $[0,T] \times \mathbb{R}^d$ the *state space* and we denote it by $\mathbb{S}$.

The space of *admissible controls*, $\mathcal{U}$, is defined as

$$\mathcal{U} := \{\psi : [0,T] \to U : \psi \text{ measurable}\},$$

where $U \subset \mathbb{R}^M$ is a compact set.

Under the assumptions made on μ, the existence and uniqueness of a solution $X^{\nu}_{t,x}$ for (1.1) follows from the standard theory of ordinary differential equations, for each ν. Furthermore there is continuity with respect to the initial conditions (t,x) uniformly in t and ν. More precisely, for each x, there is a constant, C_x, depending only on K, T, x such that for all $\nu \in \mathcal{U}$, $t \in [0,T]$, $t' \geq t$, $s \geq t'$, $x' \in B_1(x)$,

$$\left|X^{\nu}_{t',x'}(s) - X^{\nu}_{t,x}(s)\right| \leq C_x \left(|x - x'| + |t - t'|^{\frac{1}{2}}\right). \tag{1.2}$$

This estimate is a particular case of Lemma 148, when there is no diffusion.

1.3 Value function

In this Section we characterize our problem. We consider a *terminal reward* which we want to maximize,

$$J(t,x;\nu) := f\left(X^{\nu}_{t,x}(T)\right).$$

The *payoff function* f is assumed to be bounded and continuous.

The problem we consider is to determine the *value function* given by

$$v(t,x) := \sup_{\nu \in \mathcal{U}} J(t,x;\nu).$$

The following regularity result for v is easy to obtain.

Proposition 1. *v is bounded and continuous in* $[0,T] \times \mathbb{R}^d$.

Proof. The boundedness of v follows directly from the boundedness of f.

To prove continuity we consider ν^ε such that

$$v(t,x) \leq J\left(t,x;\nu^\varepsilon\right) + \varepsilon.$$

Then

$$\begin{aligned} v(t,x) - v(t',x') &\leq J(t,x;\nu^\varepsilon) - J(t',x';\nu^\varepsilon) + \varepsilon \\ &= f\left(X^{\nu^\varepsilon}_{t,x}(T)\right) - f\left(X^{\nu^\varepsilon}_{t',x'}(T)\right) + \varepsilon \\ &\to \varepsilon, \end{aligned}$$

where the convergence, as $(t',x') \to (t,x)$, follows from the continuity of f and from (1.2).

Thus, for (t',x') sufficiently close to (t,x), we have

$$v(t,x) - v(t',x') \leq 2\varepsilon.$$

The inequality

$$v(t,x) - v(t',x') \geq -2\varepsilon,$$

is obtained in an analogous way. □

1.4 Dynamic programming principle

In this Section we establish the *dynamic programming principle* for the value function v.

The essential ingredient is the following property:

$$J(t,x;\nu_1 \oplus_\tau \nu_2) = J\left(\tau, X^{\nu_1}_{t,x}(\tau);\nu_2\right), \tag{1.3}$$

where

$$(\nu_1 \oplus_\tau \nu_2)(s) := \nu_1(s)\mathbf{1}_{[t,\tau]}(s) + \nu_2(s)\mathbf{1}_{(\tau,T]}(s)$$

denotes the *concatenation* of controls ν_1, ν_2 at time $\tau \geq t$.

This property follows directly from the *flow property* for dynamical systems and a simple computation:

$$\begin{aligned} J(t,x;\nu_1 \oplus_\tau \nu_2) &= f\left(X^{\nu_1 \oplus_\tau \nu_2}_{t,x}(T)\right) \\ &= f\left(X^{\nu_2}_{\tau, X^{\nu_1}_{t,x}(\tau)}(T)\right) \\ &= J\left(\tau, X^{\nu_1}_{t,x}(\tau);\nu_2\right) \end{aligned}$$

Theorem 2 (Dynamic programming principle)**.** *For all* $x \in \mathbb{R}^d$, $t \in [0,T]$ *and* $\tau \in [t,T]$ *the following holds:*

$$v(t,x) = \sup_{\nu \in \mathcal{U}} v\left(\tau, X^\nu_{t,x}(\tau)\right)$$

Proof. We prove the two inequalities separately.

Step 1: $v(t,x) \le \sup_{\nu\in\mathcal{U}} v\left(\tau, X^{\nu}_{t,x}(\tau)\right)$

By (1.3) we have

$$\begin{aligned} J(t,x;\nu) &= J\left(\tau, X^{\nu}_{t,x}(\tau);\nu\right) \\ &\le \sup_{\nu_2\in\mathcal{U}} J\left(\tau, X^{\nu}_{t,x}(\tau);\nu_2\right) \\ &= v\left(\tau, X^{\nu}_{t,x}(\tau)\right). \end{aligned}$$

Taking the supremum on both sides of the inequality yields

$$v(t,x) \le \sup_{\nu\in\mathcal{U}} v\left(\tau, X^{\nu}_{t,x}(\tau)\right).$$

Step 2: $v(t,x) \ge \sup_{\nu\in\mathcal{U}} v\left(\tau, X^{\nu}_{t,x}(\tau)\right)$

Fix $\varepsilon > 0$ and consider $\nu_1, \nu_2 \in \mathcal{U}$ such that

$$\begin{aligned} \sup_{\nu\in\mathcal{U}} v\left(\tau, X^{\nu}_{t,x}(\tau)\right) &\le v\left(\tau, X^{\nu_1}_{t,x}(\tau)\right) + \varepsilon \\ v\left(\tau, X^{\nu_1}_{t,x}(\tau)\right) &\le J\left(\tau, X^{\nu_1}_{t,x}(\tau);\nu_2\right) + \varepsilon. \end{aligned}$$

Then

$$\begin{aligned} \sup_{\nu\in\mathcal{U}} v\left(\tau, X^{\nu}_{t,x}(\tau)\right) &\le v\left(\tau, X^{\nu_1}_{t,x}(\tau)\right) + \varepsilon \\ &\le J\left(\tau, X^{\nu_1}_{t,x}(\tau);\nu_2\right) + 2\varepsilon \\ &= J(t,x;\nu_1 \oplus_\tau \nu_2) + 2\varepsilon \\ &\le v(t,x) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary we conclude that

$$\sup_{\nu\in\mathcal{U}} v\left(\tau, X^{\nu}_{t,x}(\tau)\right) \le v(t,x).$$

□

1.5 Hamilton-Jacobi-Bellman equation

In this Section we prove that the dynamic programming principle implies that v is a viscosity solution of the *Hamilton-Jacobi-Bellman* equation.

We introduce the Hamiltonian

$$H(t,x,p) := \inf_{u\in U} H^u(t,x,p),$$

where

$$H^u(t,x,p) := -\langle p, \mu(t,x;u)\rangle.$$

Notice that $H^u(t,x,p)$ is continuous in t,x,p,u. Thus, due to the compactness of U, H is also continuous.

Theorem 3. *The value function v is a viscosity solution of*

$$(-\partial_t v + H(.,Dv))(t,x) = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}^d. \tag{1.4}$$

Proof. We begin by proving that v is a viscosity supersolution of (1.4). Consider $\phi \in C^1(\mathbb{S})$ and $(\hat{t}, \hat{x}) \in \operatorname{argmin}(v - \phi)$.

By Remark 101, in the Appendix, we can suppose that $(\hat{t}, \hat{x})$ is a strict global minimizer of $v - \phi$. Thus, for all $(t, x) \in \mathbb{S}$,

$$(v - \phi)(t, x) \geq (v - \phi)(\hat{t}, \hat{x}),$$

that is

$$\phi(\hat{t}, \hat{x}) - \phi(t, x) \geq v(\hat{t}, \hat{x}) - v(t, x).$$

Fix $u \in U$ and consider the constant control $\nu(s) := u$. Then, for $\tau \geq \hat{t}$,

$$\phi(\hat{t}, \hat{x}) - \phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \geq v(\hat{t}, \hat{x}) - v\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right).$$

By the dynamic programming principle we know that

$$v(\hat{t}, \hat{x}) \geq v\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right),$$

hence

$$\phi(\hat{t}, \hat{x}) - \phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \geq 0. \tag{1.5}$$

Dividing both sides of (1.5) by $\tau - \hat{t}$ and letting $\tau \to \hat{t}$ we conclude that

$$\left(-\partial_t \phi - \langle D\phi, \mu(., u)\rangle\right)(\hat{t}, \hat{x}) = \left(-\partial_t \phi + H^u(., D\phi)\right)(\hat{t}, \hat{x}) \geq 0,$$

where we have used the differentiability of ϕ. Since $u \in U$ is arbitrary we conclude that

$$\left(-\partial_t \phi + H(., D\phi)\right)(\hat{t}, \hat{x}) \geq 0.$$

Thus v is a viscosity supersolution of (1.4).

To check that v is a subsolution we consider $\phi \in C^1(\mathbb{S})$ and $(\hat{t}, \hat{x}) \in \operatorname{argmax}(v - \phi)$. By Remark 101, in the Appendix, we can suppose that $(\hat{t}, \hat{x})$ is a strict global maximizer of $v - \phi$. Thus, for all $(t, x) \in \mathbb{S}$,

$$(v - \phi)(t, x) \leq (v - \phi)(\hat{t}, \hat{x}),$$

that is

$$\phi(\hat{t}, \hat{x}) - \phi(t, x) \leq v(\hat{t}, \hat{x}) - v(t, x). \tag{1.6}$$

We suppose by contradiction that

$$\left(-\partial_t \phi + H(., D\phi)\right)(\hat{t}, \hat{x}) \geq 2\delta,$$

for some $\delta > 0$. Then, for all $u \in U$,

$$\left(-\partial_t \phi + H^u(., D\phi)\right)(\hat{t}, \hat{x}) \geq 2\delta.$$

Since $H^u(t, x, p)$ is continuous with respect to u, t, x, p and U is compact we deduce that there exists $R > 0$ such that, for all $u \in U$,

$$\left(-\partial_t \phi + H^u(., D\phi)\right)(t, x) \geq \delta, \text{ for all } (t, x) \in B_R(\hat{t}, \hat{x}).$$

By (1.2), for τ sufficiently close to $\hat{t}$, we have

$$\left|\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) - (\hat{t}, \hat{x})\right| < R,$$

for all $\nu \in \mathcal{U}$ and all $s \in [\hat{t}, \tau]$.

Since $\phi \in C^1(\mathbb{S})$, we have

$$\begin{aligned} \phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) - \phi(\hat{t}, \hat{x}) &= \int_{\hat{t}}^{\tau} \left(\partial_t \phi - H^{\nu(s)}(., D\phi)\right)\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) ds \\ &\leq -(\tau - \hat{t})\delta. \end{aligned}$$

By (1.6) we then conclude that

$$\begin{aligned} v(\hat{t}, \hat{x}) - v\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) &\geq \phi(\hat{t}, \hat{x}) - \phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \\ &\geq (\tau - \hat{t})\delta. \end{aligned}$$

Taking the supremum in ν we conclude that

$$v(\hat{t}, \hat{x}) \geq \sup_{\nu \in \mathcal{U}} v\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) + (\tau - \hat{t})\delta,$$

which contradicts the dynamic programming principle. □

We remark that v also satisfies the terminal condition

$$v(T, x) = f(x).$$

This provides a characterization of the value function as a solution of a partial differential equation. For the characterization to be complete, uniqueness of solution for the HJB equation must be established. It is, indeed, possible to prove uniqueness by the standard techniques of first order viscosity solutions. We will not do this study here. Instead we refer the reader to [3].

Chapter 2

Stochastic optimal control

In this Chapter we turn our attention to *stochastic optimal control* problems. The results are analogous to those of the previous Chapter. A modern reference on the subject is the well known book by Fleming and Soner, [4].

2.1 Introduction

Stochastic optimal control is, in many points, similar to its deterministic counterpart. Thus the procedure we will use to study this problem will be analogous to the previous one. In this introduction we outline the main differences.

In the stochastic scenario we consider a state variable, $X^\nu_{t,x}$, which is a stochastic process satisfying a stochastic differential equation,

$$dX^\nu_{t,x}(s) \quad = \quad \mu\left(s, X^\nu_{t,x}(s); \nu_s\right) ds + \sigma\left(s, X^\nu_{t,x}(s); \nu_s\right) dW_s,$$

with initial condition $X^\nu_{t,x}(t) = x$. This state variable is driven by a control ν, which must be a progressively measurable stochastic process.

As in the deterministic case, we have a terminal payoff, $f\left(X^\nu_{t,x}(T)\right)$, which we want to maximize. One difference is that, in the stochastic setting, when the choice of control is to be made, at time t, we can not predict the terminal payoff. Thus we maximize instead its conditional expectation,

$$J(t,x;\nu) := \mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right) \Big| \mathcal{F}_t\right].$$

We call this random variable the terminal reward. The stochastic optimal control problem is then to determine the value function:

$$V(t,x) := \operatorname*{esssup}_{\nu} J(t,x;\nu).$$

Even though, *a priori*, $V(t,x)$ is a random variable, we will see that it is, in fact, a constant random variable. Thus we can think of V as a function $V : [0,T] \times \mathbb{R}^d \to \mathbb{R}$.

As in the deterministic scenario, it is easy to derive heuristically the dynamic programming principle for this problem. It takes the following form:

$$V(t,x) = \operatorname*{esssup}_{\nu} \mathbb{E}\left[V\left(\tau, X^\nu_{t,x}(\tau)\right) \Big| \mathcal{F}_t\right].$$

Though the heuristic derivation is simple and the analogies with the deterministic case are many, the stochastic version of the dynamic programming principle encloses subtle technical difficulties related to measurability issues. Indeed, as we can see, for the statement of the principle to make sense, we must prove first that V is measurable. At this point, either we work under a restricted setting where strong assumptions imply easily that V is measurable (typically that it is even continuous), or we have to use deep measurable selection arguments to prove it. In this Chapter we will take the first route. For the reader interested in following the second, we refer to [5], where that approach is used in the discrete time scenario.

Recently, Bouchard and Touzi, proposed in [1] an alternative approach, where a weak version of the dynamic programming principle is used. Following this approach we avoid the measurability issues and we are still able to derive the Hamilton-Jacobi-Bellman equation which is the utmost objective of the whole dynamic programming approach. In Chapter 4 we will use this approach in the context of stochastic differential games.

With the dynamic programming principle we are able to derive the Hamilton-Jacobi-Bellman equation (HJB). Again we must take a limit $\tau \to t$, in the context of the dominated convergence Theorem, and use the stochastic counterpart of the chain rule, Itô's formula. Indeed, under the assumption that V is $C^{1,2}$, we have by Itô's formula that

$$\begin{aligned} 0 &= \operatorname*{esssup}_{\nu} \mathbb{E}\left[V\left(\tau, X^{\nu}_{t,x}(\tau)\right) - V(t,x)\Big|\mathcal{F}_t\right] \\ &= \operatorname*{esssup}_{\nu} \mathbb{E}\left[\int_t^{\tau}\left(\partial_t V + \langle \mu(.;\nu_s), DV\rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(,;\nu_s)\, D^2V\right)\right)\left(s, X^{\nu}_{t,x}(s)\right) ds\Big|\mathcal{F}_t\right] + \\ &\quad + \mathbb{E}\left[\int_t^{\tau}\left(DV\,\sigma(.;\nu_s)\right)\left(s, X^{\nu}_{t,x}(s)\right) dW_s\Big|\mathcal{F}_t\right]. \end{aligned} \tag{2.1}$$

If $X^{\nu}_{t,x}(s)$ remains bounded for $s \in [t,\tau]$, we then have, by the martingale properties of the stochastic integral, that

$$\mathbb{E}\left[\int_t^{\tau}\left(DV\,\sigma(.;\nu)\right)\left(s, X^{\nu}_{t,x}(s)\right) dW_s\Big|\mathcal{F}_t\right] = 0.$$

In that case, we can divide both sides of (2.1) by $\tau - t$ and take the limit as $\tau \to t$ to conclude that

$$\sup_u \left(\partial_t V + \langle \mu(.;u), DV\rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(,;u)\, D^2V\right)\right)(t,x) = 0.$$

Thus we should expect V to be a solution of the HJB equation:

$$\left(-\partial_t V + \inf_u\left(-\langle \mu(.;u), DV\rangle - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(,;u)\, D^2V\right)\right)\right)(t,x) = 0.$$

Notice that this is a parabolic second order partial differential equation. Like in the first order case, the theory of viscosity solutions provides the adequate framework for the study of such equations.

The HJB equation provides us also with a procedure to find optimal strategies. This procedure is explored in Section 2.6 and applied in Section 2.7.

2.2 The controlled Markov diffusion

We fix a time horizon, T, and consider the *classical Wiener space*, $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the *standard $N-$dimensional Brownian motion*, W, and the natural *filtration induced* by W, $\mathbb{F} = \{\mathcal{F}_s : 0 \le s \le T\}$. For further details see Section 4.2.

As in the previous Chapter we denote by $\mathbb{S} := [0,T] \times \mathbb{R}^d$ the *state space*. We consider a control space, $\mathcal{U}$, consisting of the progressively measurable processes with values in U, where $U \subset \mathbb{R}^M$ is a compact set.

Let $\mu : \mathbb{S} \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{S} \times U \to \mathbb{R}^{d\times N}$ be continuous functions such that

$$\begin{aligned} |\mu(t,x;u) - \mu(t,y;u)| + |\sigma(t,x;u) - \sigma(t,y;u)| &\le K|x-y|, \\ |\mu(t,x;u)| + |\sigma(t,x;u)| &\le K(1+|x|). \end{aligned}$$

The dynamics of the state variable is given by a *stochastic differential equation*,

$$\begin{cases} dX^{\nu}_{t,x}(s) &= \mu\left(s, X^{\nu}_{t,x}(s); \nu_s\right) ds + \sigma\left(s, X^{\nu}_{t,x}(s); \nu_s\right) dW_s \\ X^{\nu}_{t,x}(t) &= x, \end{cases} \tag{2.2}$$

where (t,x) are the initial conditions and $\nu \in \mathcal{U}$.

Under the assumptions made on μ, σ, we know by Theorem 147 that, for each ν, there exists a unique strong solution, $X^{\nu}_{t,x}$, of (2.2). Furthermore, by Lemma 148, there is continuity with respect to the initial conditions, uniformly in t and ν, in the sense that, for each x and $p \geq 2$, there is a constant, C_x, depending only on K, T, x, p such that for all $\nu \in \mathcal{U}$, $t \in [0,T]$, $t' \geq t$, $s \geq t'$, $x' \in B_1(x)$,

$$\mathbb{E}\left[\left|X^{\nu}_{t',x'}(s) - X^{\nu}_{t,x}(s)\right|^p \middle| \mathcal{F}_t\right] \leq C_x\left(|x-x'|^p + |t-t'|^{\frac{p}{2}}\right). \tag{2.3}$$

2.3 Value function

We consider a *terminal reward* which we want to maximize,

$$J(t,x;\nu) := \mathbb{E}\left[f\left(X^{\nu}_{t,x}(T)\right) \middle| \mathcal{F}_t\right],$$

where the *payoff function* f is assumed to be bounded and globally Lipschitz.

Notice that $J(t,x;\nu)$ is a random variable and may depend on the past.

The problem we consider is to determine the *value function* given by

$$V(t,x) := \operatorname*{esssup}_{\nu \in \mathcal{U}} J(t,x;\nu).$$

For the notion of *essential supremum*, see Definition 39.

Even though, *a priori*, $V(t,x)$ is a random variable, it turns out that it is, in fact, a constant random variable. For a proof of this result we refer the reader to Section 4.4.1, where an analogous result is proved in the context of stochastic differential games. Thus, we may think of $V(t,x)$ as a function $V : \mathbb{S} \to \mathbb{R}$.

It is possible to find controls which are uniformly $\varepsilon-$optimal. More precisely we have:

Lemma 4. *Fix $\varepsilon > 0$. Then there exists $\nu^{\varepsilon} \in \mathcal{U}$ such that*

$$V(t,x) \leq J(t,x;\nu^{\varepsilon}) + \varepsilon.$$

Proof. By Theorem 40 there exists a countable collection $\{\nu_i\} \subset \mathcal{U}$ such that

$$V(t,x) = \sup_i J(t,x;\nu_i).$$

Consider $\tilde{\Lambda}_i := \{V(t,x) \leq J(t,x;\nu_i) + \varepsilon\} \in \mathcal{F}_t$. We define $\Lambda_1 := \tilde{\Lambda}_1$, $\Lambda_{i+1} = \tilde{\Lambda}_{i+1} \setminus \bigcup_{k=1}^{i} \tilde{\Lambda}_k$. Then $\{\Lambda_i\} \subset \mathcal{F}_t$ forms a countable partition of Ω, modulo null sets.

We now define

$$\nu^{\varepsilon} := \sum_i \nu_i \mathbf{1}_{\Lambda_i}.$$

Then, since U is compact, $\nu^{\varepsilon} \in \mathcal{U}$ and

$$\begin{aligned} J(t,x;\nu^{\varepsilon}) &= \sum_i J(t,x;\nu_i)\mathbf{1}_{\Lambda_i} \\ &\geq V(t,x) - \varepsilon, \end{aligned}$$

where we used in the first equality a property of J, to be proved in Lemma 35, that implies:

$$J\left(t,x;\sum_i \mathbf{1}_{\Lambda_i}\nu_i\right) = \sum_i \mathbf{1}_{\Lambda_i} J(t,x;\nu_i).$$

We call this property independence of irrelevant alternatives. □

Remark 5. *Even though, in the proof of the previous Lemma, we use the compactness of U, the same result still holds if U is not compact. This will be a consequence of Proposition 68, and it will be explored in Section 4.5.1.*

By the previous result and by the non-randomness of V we get the following alternative definition for the value function:

Proposition 6. *The following holds:*

$$V(t,x) = \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[J(t,x;\nu)\right].$$

Proof. Since V is a constant random-variable we have

$$V(t,x) = \mathbb{E}[V(t,x)].$$

Thus, on one hand,

$$\begin{aligned} V(t,x) &= \mathbb{E}[\operatorname*{esssup}_{\nu \in \mathcal{U}} J(t,x;\nu)] \\ &\geq \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[J(t,x;\nu)\right]. \end{aligned}$$

On the other hand, by the previous Lemma,

$$\begin{aligned} V(t,x) &= \mathbb{E}[V(t,x)] \\ &\leq \mathbb{E}[J(t,x;\nu^{\varepsilon})] + \varepsilon \\ &\leq \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[J(t,x;\nu)\right] + \varepsilon, \end{aligned}$$

and since ε is arbitrary we deduce that

$$V(t,x) \leq \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[J(t,x;\nu)\right].$$

□

Using Proposition 6, the next regularity result for V is easy to obtain.

Proposition 7. *The function $V(t,x)$ is bounded and continuous. Furthermore $V(t,.)$ is Lipschitz continuous and $V(.,x)$ is $\frac{1}{2}-$Hölder continuous.*

Proof. The boundedness of V follows directly from the boundedness of f.

By Proposition 6, we have for all $x' \in B_1(x)$:

$$\begin{aligned} V(t',x') - V(t,x) &= \sup_{\nu} \mathbb{E}[J(t',x';\nu)] - \sup_{\nu} \mathbb{E}[J(t,x;\nu)] \\ &\leq \sup_{\nu} \mathbb{E}\left[f\left(X^{\nu}_{t',x'}(T)\right) - f\left(X^{\nu}_{t,x}(T)\right)\right] \\ &\leq \sup_{\nu} K\mathbb{E}\left[\left|X^{\nu}_{t',x'}(T) - X^{\nu}_{t,x}(T)\right|\right] \\ &\leq \sup_{\nu} K\mathbb{E}\left[\left|X^{\nu}_{t',x'}(T) - X^{\nu}_{t,x}(T)\right|^2\right]^{\frac{1}{2}} \\ &\leq C_x K \left(|x-x'|^2 + |t-t'|\right)^{\frac{1}{2}}, \end{aligned}$$

where the last inequality follows from (2.3). Changing the roles of (t',x') and (t,x) we get the desired continuity result on V. □

Typically, the $\frac{1}{2}$−Hölder continuity of V, in the time variable, is proved using the dynamic programming principle. by a slightly different argument. Because we state the dynamic programming principle with stopping times we need some *a priori* regularity of V in the time variable. This is the reason why we proved at this point this regularity result.

The fact that $V(t,x)$ is continuous is essential to simplify the subsequent exposition. Indeed, due to this fact, we conclude that $V\left(\tau, X^{\nu}_{t,x}(\tau)\right)$ is measurable, for any stopping time τ. As we will see in the next Section, this is crucial for the statement of the dynamic programming principle. In [1], Bouchard and Touzi establish a weak version of the dynamic programming principle where such a regularity of the value function is not required.

2.4 Dynamic programming principle

In this Section we establish the *dynamic programming principle* for the value function V.

The essential ingredient is a result analogous to (1.3):

$$J(t,x;\nu_1 \oplus_\tau \nu_2) = \mathbb{E}\left[J\left(\tau, X^{\nu_1}_{t,x}(\tau);\nu_2\right)\Big|\mathcal{F}_t\right], \tag{2.4}$$

where

$$(\nu_1 \oplus_\tau \nu_2)(s) := \nu_1(s)\mathbf{1}_{[t,\tau]}(s) + \nu_2(s)\mathbf{1}_{(\tau,T]}(s)$$

denotes the *concatenation* of controls ν_1, ν_2 at the stopping time $\tau \geq t$.

This property follows directly from the *flow property* for solutions of stochastic differential equations and the tower property of conditional expectations:

$$\begin{aligned} J(t,x;\nu_1 \oplus_\tau \nu_2) &= \mathbb{E}\left[f\left(X^{\nu_1\oplus_\tau \nu_2}_{t,x}(T)\right)|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f\left(X^{\nu_2}_{\tau, X^{\nu_1}_{t,x}(\tau)}(T)\right)\Big|\mathcal{F}_\tau\right]\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[J\left(\tau, X^{\nu_1}_{t,x}(\tau);\nu_2\right)\Big|\mathcal{F}_t\right]. \end{aligned}$$

Theorem 8 (Dynamic programming principle)**.** *For all $x \in \mathbb{R}^d$, $t \in [0,T]$ and stopping time $\tau \in [t,T]$ the following holds:*

$$V(t,x) = \operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V\left(\tau, X^{\nu}_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right].$$

Proof. We prove the two inequalities separately.

Step 1: $V(t,x) \leq \operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V(\tau, X^{\nu}_{t,x}(\tau))|\mathcal{F}_t\right]$

By (2.4) we have

$$\begin{aligned} J(t,x;\nu) &= \mathbb{E}\left[J\left(\tau, X^{\nu}_{t,x}(\tau);\nu\right)\Big|\mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\operatorname*{esssup}_{\nu_2\in\mathcal{U}} J\left(\tau, X^{\nu}_{t,x}(\tau);\nu_2\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[V\left(\tau, X^{\nu}_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right]. \end{aligned}$$

Taking the supremum on both sides of the inequality yields

$$V(t,x) \leq \operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V\left(\tau, X^{\nu}_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right].$$

Step 2: $V(t,x) \geq \operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V\left(\tau, X^{\nu}_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right]$

This inequality requires more work. Fix $\varepsilon > 0$ and consider $\nu^\varepsilon \in \mathcal{U}$ such that

$$\operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V\left(\tau, X^\nu_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right] \quad\leq\quad \mathbb{E}\left[V\left(\tau, X^{\nu^\varepsilon}_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right] + \varepsilon.$$

For each $(s,y) \in \mathbb{S}$ consider $\nu_{(s,y)} \in \mathcal{U}$ such that

$$V(s,y) \quad\leq\quad J(s,y;\nu_{(s,y)}) + \varepsilon.$$

We know, by continuity of V and by (2.3), that, for each (s,y) there is also $r_{(s,y)}$ such that

$$\begin{aligned} |V(s',y') - V(s,y)| &\quad\leq\quad \varepsilon, \\ |J(s',y';\nu) - \mathbb{E}[J(s,y;\nu)|\mathcal{F}_{s'}]| &\quad\leq\quad \varepsilon, \end{aligned}$$

for all $\nu \in \mathcal{U}$ and for all $(s',y') \in B(s,y;r_{(s,y)})$, where $B(s,y;r) := [s-r,s]\times B_r(y)$. Since $\{B(s,y;r) : (s,y) \in \mathbb{S},\ 0 < r \leq r_{(s,y)}\}$ forms a Vitali covering of $\mathbb{S}$, we can find a countable sequence (s_i,y_i,r_i) such that $\{B(s_i,y_i;r_i)\}_i$ forms a partition of $\mathbb{S}$, modulo null sets, and $0 < r_i \leq r_{(s_i,y_i)}$. For the notion of Vitali covering we refer the reader to [6, p. 158].

Define

$$\begin{aligned} \Lambda_i &:= \left\{\left(\tau, X^{\nu^\varepsilon}_{t,x}(\tau)\right) \in B(s_i,y_i;r_i)\right\} \in \mathcal{F}_\tau \cap \mathcal{F}_{s_i}, \\ \nu_i &:= \nu^\varepsilon \oplus_{\tau\vee s_i} \nu_{(s_i,y_i)}, \\ \nu &:= \sum_i \mathbf{1}_{\Lambda_i}\nu_i = \sum_i \mathbf{1}_{\Lambda_i}\nu^\varepsilon \oplus_{s_i} \nu_{(s_i,y_i)}. \end{aligned}$$

Then $\nu \in \mathcal{U}$ and

$$\begin{aligned} V\left(\tau, X^{\nu^\varepsilon}_{t,x}(\tau)\right)\mathbf{1}_{\Lambda_i} &\leq (V(s_i,y_i)+\varepsilon)\mathbf{1}_{\Lambda_i} \\ &= \left(\mathbb{E}\left[V(s_i,y_i)|\mathcal{F}_\tau\right]+\varepsilon\right)\mathbf{1}_{\Lambda_i} \\ &\leq (\mathbb{E}[J(s_i,y_i;\nu_{(s_i,y_i)}))|\mathcal{F}_\tau] + 2\varepsilon)\mathbf{1}_{\Lambda_i} \\ &= (\mathbb{E}[J(s_i,y_i;\nu_i)|\mathcal{F}_\tau] + 2\varepsilon)\mathbf{1}_{\Lambda_i} \\ &\leq \left(J\left(\tau, X^{\nu^\varepsilon}_{t,x}(\tau);\nu_i\right) + 3\varepsilon\right)\mathbf{1}_{\Lambda_i} \\ &= \left(J\left(\tau, X^{\nu}_{t,x}(\tau);\nu_i\right) + 3\varepsilon\right)\mathbf{1}_{\Lambda_i}. \end{aligned}$$

Thus,

$$\begin{aligned} \operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V\left(\tau, X^\nu_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right] &\leq \mathbb{E}\left[V\left(\tau, X^{\nu^\varepsilon}_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right] + \varepsilon \\ &= \mathbb{E}\left[\sum_i V\left(\tau, X^{\nu^\varepsilon}_{t,x}(\tau)\right)\mathbf{1}_{\Lambda_i}\Big|\mathcal{F}_t\right] + \varepsilon \\ &\leq \mathbb{E}\left[\sum_i J\left(\tau, X^{\nu}_{t,x}(\tau);\nu_i\right)\mathbf{1}_{\Lambda_i}\Big|\mathcal{F}_t\right] + 4\varepsilon \\ &= \mathbb{E}\left[J\left(\tau, X^{\nu}_{t,x}(\tau);\nu\right)\Big|\mathcal{F}_t\right] + 4\varepsilon \\ &= J(t,x;\nu) + 4\varepsilon \\ &\leq V(t,x) + 4\varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude that

$$\operatorname*{esssup}_{\nu\in\mathcal{U}} \mathbb{E}\left[V\left(\tau, X^\nu_{t,x}(\tau)\right)\Big|\mathcal{F}_t\right] \leq V(t,x).$$

□

Remark 9. *In our setting the value function is continuous, thus implying that* $V\left(\tau, X^{\nu}_{t,x}(\tau)\right)$ *is measurable. If this is not the case then more complicated arguments must be used. Examples of this can be seen in [5], where a deep measurable selection argument is used, and [7], where a compactification of the space of controls is carried through. As an alternative to this, in [1], a weak formulation of the dynamic programming principle is proposed.*

2.5 Hamilton-Jacobi-Bellman equation

Now that we established the dynamic programming principle and proved that V is a continuous function we can prove that it is a viscosity solution of the *Hamilton-Jacobi-Bellman* equation. In the stochastic setting the Hamilton-Jacobi-Bellman equation is a parabolic partial differential equation of second order.

The Hamiltonian we consider in this case is

$$H(t,x,p,X) := \inf_{u\in U} H^u(t,x,p,X),$$

where

$$H^u(t,x,p,X) := -\langle p, \mu(t,x;u)\rangle - \frac{1}{2}\mathrm{Tr}\left((\sigma\sigma^T)(t,x;u)X\right).$$

Notice that $H^u(t,x,p,X)$ is continuous in t,x,p,u,X. Thus, due to the compactness of U, H is also continuous.

Theorem 10. *The value function* V *is a viscosity solution of*

$$\left(-\partial_t V + H(.,DV,D^2V)\right)(t,x) = 0, \qquad (t,x)\in]0,T[\times\mathbb{R}^d. \tag{2.5}$$

Proof. We begin by proving that V is a viscosity supersolution of (2.5). Consider $\phi\in C^{1,2}(\mathbb{S})$ and $(\hat t,\hat x)\in \mathrm{argmin}(V-\phi)$. By Remark 101, in the Appendix, we can suppose that $(\hat t,\hat x)$ is a strict global minimizer of $V-\phi$. Thus, for all $(t,x)\in\mathbb{S}$,

$$(V-\phi)(t,x) \geq (V-\phi)(\hat t,\hat x),$$

that is

$$\phi(\hat t,\hat x) - \phi(t,x) \geq V(\hat t,\hat x) - V(t,x).$$

Fix $u\in U$ and consider the constant control $\nu_s := u$. For each $r>0$, consider the stopping time $\tau_r := \inf\left\{s\geq \hat t : X^{\nu}_{\hat t,\hat x}(s)\notin B_r(\hat x)\right\}$. Then,

$$\phi(\hat t,\hat x) - \phi\left(\tau_r, X^{\nu}_{\hat t,\hat x}(\tau_r)\right) \geq V(\hat t,\hat x) - V\left(\tau_r, X^{\nu}_{\hat t,\hat x}(\tau_r)\right).$$

By the dynamic programming principle we know that

$$V(\hat t,\hat x) \geq \mathbb{E}\left[V\left(\tau_r, X^{\nu}_{\hat t,\hat x}(\tau_r)\right)\Big|\mathcal{F}_{\hat t}\right],$$

hence

$$\phi(\hat t,\hat x) - \mathbb{E}\left[\phi\left(\tau_r, X^{\nu}_{\hat t,\hat x}(\tau_r)\right)\Big|\mathcal{F}_{\hat t}\right] \geq 0. \tag{2.6}$$

By Itô's formula we have

$$\begin{aligned}\phi\left(\tau_r, X^{\nu}_{\hat t,\hat x}(\tau_r)\right) - \phi(\hat t,\hat x) &= \int_{\hat t}^{\tau_r}\left(\partial_t\phi - H^{\nu_s}\left(.;D\phi,D^2\phi\right)\right)\left(s, X^{\nu}_{\hat t,\hat x}(s)\right)ds + \\ &+ \int_{\hat t}^{\tau_r}\left(D\phi\,\sigma(.;\nu_s)\right)\left(s, X^{\nu}_{\hat t,\hat x}(s)\right)dW_s.\end{aligned} \tag{2.7}$$

Since τ_r is an exit time and $\sigma, D\phi$ are continuous we have that $(D\phi\,\sigma(.;\nu_s))\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right)$ remains bounded for $s \in [\hat{t}, \tau]$, hence

$$\mathbb{E}\left[\int_{\hat{t}}^{\tau_r} (D\phi\,\sigma(.;\nu_s))\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) dW_s \Big| \mathcal{F}_{\hat{t}}\right] = 0.$$

Taking expectations on both sides of (2.7) and using (2.6) we get

$$\mathbb{E}\left[\int_{\hat{t}}^{\tau_r} \left(\partial_t\phi - H^{\nu_s}\left(.; D\phi, D^2\phi\right)\right)\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) ds \Big| \mathcal{F}_{\hat{t}}\right] \le 0$$

Dividing the previous inequality by $\tau_r - \hat{t}$ and letting $r \to 0$ we have, by dominated convergence,

$$\left(\partial_t\phi - H^{u}\left(.; D\phi, D^2\phi\right)\right)(\hat{t}, \hat{x}) \le 0.$$

Thus V is a supersolution of (2.5).

To check that V is a subsolution we consider $\phi \in C^{1,2}(\mathbb{S})$ and $(\hat{t}, \hat{x}) \in \operatorname{argmax}(V - \phi)$. By Remark 101 we can suppose that $(\hat{t}, \hat{x})$ is a strict global maximizer of $V - \phi$. Thus, for all $(t, x) \in \mathbb{S}$,

$$(V - \phi)(t, x) \le (V - \phi)(\hat{t}, \hat{x}),$$

that is

$$\phi(\hat{t}, \hat{x}) - \phi(t, x) \le V(\hat{t}, \hat{x}) - V(t, x). \tag{2.8}$$

We suppose by contradiction that

$$\left(-\partial_t\phi + H(., D\phi, D^2\phi)\right)(\hat{t}, \hat{x}) \ge 2\delta,$$

for some $\delta > 0$.

Consider $\varphi(t, x) := \phi(t, x) + |t - \hat{t}|^2 + |x - \hat{x}|^4$. Then, for all $u \in U$,

$$\left(-\partial_t\varphi + H^u(., D\varphi, D^2\varphi)\right)(\hat{t}, \hat{x}) \ge 2\delta.$$

Since $H^u(t, x, p, X)$ is continuous with respect to u, t, x, p, X and U is compact, we deduce that there exists $R > 0$ such that, for all $u \in U$,

$$\left(-\partial_t\varphi + H^u(., D\varphi, D^2\varphi)\right)(t, x) \ge \delta, \text{ for all } (t, x) \in B_R(\hat{t}, \hat{x}).$$

We define

$$\eta := \min_{\partial B_R(\hat{t}, \hat{x})} (\varphi - \phi) > 0.$$

Let ν be such that

$$V(\hat{t}, \hat{x}) \le J(\hat{t}, \hat{x}; \nu) + \frac{\eta}{2}, \tag{2.9}$$

and consider the stopping time $\tau := \inf\left\{s \ge \hat{t} : \left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) \notin B_R(\hat{t}, \hat{x})\right\}$.

Since $\varphi \in C^{1,2}(\mathbb{S})$, we have by Itô's formula that

$$\begin{aligned}\varphi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) - \varphi(\hat{t}, \hat{x}) &= \int_{\hat{t}}^{\tau} \left(\partial_t\varphi - H^{\nu_s}(., D\varphi, D^2\varphi)\right)\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) ds + \\ &\quad + \int_{\hat{t}}^{\tau} (D\varphi\,\sigma(.;\nu_s))\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) dW_s \\ &\le -(\tau - \hat{t})\delta + \int_{\hat{t}}^{\tau} (D\varphi\,\sigma(.;\nu_s))(s, X^{\nu}_{\hat{t},\hat{x}}(s))dW_s. \end{aligned} \tag{2.10}$$

Since τ is an exit time and $\sigma, D\varphi$ are continuous we have that $(D\varphi\,\sigma(.;\nu_s))\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right)$ remains bounded for $s \in [\hat{t}, \tau]$, hence

$$\mathbb{E}\left[\int_{\hat{t}}^{\tau} (D\varphi\,\sigma(.;\nu_s))\left(s, X^{\nu}_{\hat{t},\hat{x}}(s)\right) dW_s \Big| \mathcal{F}_{\hat{t}}\right] = 0.$$

Thus, taking expectations on (2.10) we get

$$\mathbb{E}\left[\varphi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \Big| \mathcal{F}_{\hat{t}}\right] - \varphi(\hat{t}, \hat{x}) \leq 0.$$

By definition of η and since $\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \in \partial B_R(\hat{t}, \hat{x})$, we have

$$\varphi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \geq \phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) + \eta,$$

and hence

$$\mathbb{E}\left[\phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \Big| \mathcal{F}_{\hat{t}}\right] - \phi(\hat{t}, \hat{x}) \leq -\eta.$$

By (2.8) we then conclude that

$$\begin{aligned} V(\hat{t}, \hat{x}) - \mathbb{E}\left[V\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \Big| \mathcal{F}_{\hat{t}}\right] &\geq \phi(\hat{t}, \hat{x}) - \mathbb{E}\left[\phi\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \Big| \mathcal{F}_{\hat{t}}\right] \\ &\geq \eta. \end{aligned}$$

Thus

$$V(\hat{t}, \hat{x}) \geq \mathbb{E}\left[V\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \Big| \mathcal{F}_{\hat{t}}\right] + \eta.$$

On the other hand, we have by (2.9) and (2.4) that

$$\begin{aligned} V(\hat{t}, \hat{x}) &\leq J(\hat{t}, \hat{x}; \nu) + \frac{\eta}{2} \\ &= \mathbb{E}\left[J\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau); \nu\right) \Big| \mathcal{F}_{\hat{t}}\right] + \frac{\eta}{2} \\ &\leq \mathbb{E}\left[V\left(\tau, X^{\nu}_{\hat{t},\hat{x}}(\tau)\right) \Big| \mathcal{F}_{\hat{t}}\right] + \frac{\eta}{2}. \end{aligned}$$

We have thus reached a contradiction. □

Like in the deterministic case, the value function V also satisfies the terminal condition

$$V(T, x) = f(x).$$

Regarding uniqueness of solution, we refer the reader to Section 4.6.1. There we discuss uniqueness of solution for the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, which is a second-order partial differential equation similar to the HJB equation. The same arguments can be adapted easily to prove uniqueness of solution for the HJB equation in this case. In fact, it is much easier, since in this Chapter we only consider the case of bounded value function.

2.6 Verification Theorem

In this section we establish a verification result useful for the synthesis of optimal controls.

Theorem 11 (Verification Theorem)**.** *Let $v \in C^{1,2}([0,T)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ be a classical solution of (2.5) such that $v(T,x)=f(x)$, and v has polynomial growth.*

Suppose that there exists a measurable function $u^* : \mathbb{S}\to U$ such that

$$H(.;Dv,D^2v)(t,x) = H^{u^*(.)}(.;Dv,D^2v)(t,x).$$

Then

$$V(t,x) = v(t,x),$$

and $\nu^*_s := u^*\left(s, X_{t,x}(s)\right)$ is an optimal control.

Proof. Since u^* is measurable then $\nu^*_s := u^*\left(s, X_{t,x}(s)\right) \in \mathcal{U}$. Let $X := X^{\nu^*}_{t,x}$ be the unique strong solution of (2.2) and consider the stopping time $\tau_n := T\wedge \inf\{s\geq t : X_s \notin [-n,n]\}$.

Applying Itô's formula we have

$$\begin{aligned} v(\tau_n, X(\tau_n)) - v(t,x) &= \int_t^{\tau_n}\left(\partial_t v - H^{u^*(.)}(.;Dv,D^2v)\right)(s,X(s))ds + \\ &\quad + \int_t^{\tau_n}(Dv\,\sigma(.;u^*(.)))(s,X(s))dW_s \\ &= \int_t^{\tau_n}(Dv\,\sigma(.;u^*(.)))(s,X(s))dW_s. \end{aligned}$$

Since τ_n is an exit time, we have

$$\mathbb{E}\left[\int_t^{\tau_n}(Dv\,\sigma(.;u^*(.)))(s,X(s))dW_s\Big|\mathcal{F}_t\right] = 0,$$

hence

$$v(t,x) = \mathbb{E}\left[v(\tau_n, X(\tau_n))|\mathcal{F}_t\right].$$

Since v has polynomial growth we can take $n\to\infty$ to conclude, by the dominated convergence Theorem, that

$$\begin{aligned} v(t,x) &= \mathbb{E}\left[v(T,X(T))|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f(X(T))|\mathcal{F}_t\right] \\ &= J(t,x;\nu^*). \end{aligned}$$

On the other hand, if we consider an arbitrary control $\nu\in\mathcal{U}$ and X^ν, the unique strong solution of (2.2), then we remark that

$$\left(H^{u^*(.)}(.;Dv,D^2v)\right)(s,X^\nu(s)) \leq \left(H^{\nu_s}(.;Dv,D^2v)\right)(s,X^\nu(s)),$$

hence, by an analogous argument to the one used before, we deduce that

$$\mathbb{E}\left[v(\tau_n, X^\nu(\tau_n))|\mathcal{F}_t\right] - v(t,x) \leq \mathbb{E}\left[\int_t^{\tau_n}(Dv\,\sigma(.;\nu_s))(s,X^\nu(s))dW_s\Big|\mathcal{F}_t\right] = 0.$$

Letting $n\to\infty$ we deduce that

$$\begin{aligned} v(t,x) &\geq \mathbb{E}\left[v(T,X^\nu(T))|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f(X^\nu(T))|\mathcal{F}_t\right] \\ &= J(t,x;\nu). \end{aligned}$$

Since ν is arbitrary and $v(t,x) = J(t,x;\nu^*)$ we conclude that

$$\begin{aligned} V(t,x) &\leq v(t,x) \\ &= J(t,x;\nu^*) \\ &\leq V(t,x). \end{aligned}$$

Thus ν^* is an optimal control. □

2.7 Merton's optimal portfolio

We now give an example of application of the previous results to the well known *Merton's optimal portfolio problem*.

The setting for Merton's problem is that of a market with a risky asset S_t and a non-risky asset S_t^0 which follow the following stochastic differential equation

$$\begin{aligned} dS_t &= S_t(\mu_t dt + \sigma_t dW_t), \\ dS_t^0 &= S_t^0 r_t dt. \end{aligned}$$

In this setting we consider a self-financed portfolio which value, X_t, we want to maximize. The value of the portfolio is to be maximized over all admissible strategies. An admissible strategy, $\pi \in \mathcal{U}$, indicates the ratio, $\pi_t \in [\pi_0, \pi_1]$, of the value of the portfolio to invest at time t in the risky asset. The *self-financing conditions* are then:

$$\begin{aligned} dX_t &= \frac{X_t \pi_t}{S_t} dS_t + \frac{X_t(1-\pi_t)}{S_t^0} dS_t^0 \\ &= X_t \left((\pi_t \mu_t + (1-\pi_t) r_t) dt + \pi_t \sigma_t dW_t \right). \end{aligned}$$

By choosing only self-financed strategies we are not allowing money to be added or taken from the portfolio. Furthermore it is easy to see that the value of the portfolio never becomes negative. Indeed, if at some point the value reaches 0 the we have $dX_t = 0$ so that it remains 0.

Given a terminal time T, Merton's optimal portfolio problem consists in optimizing, over all admissible strategies $\pi \in \mathcal{U}$, the utility of the terminal value of the portfolio, that is, determine

$$V(t,x) = \sup_{\pi \in \mathcal{U}} \mathbb{E}\left[f\left(X_{t,x}^\pi(T) \right) \middle| \mathcal{F}_t \right],$$

where $\mathcal{U}$ is the set of progressively measurable processes taking values in $[\pi_0, \pi_1]$, f is an *utility function*, $X_{t,x}^\pi(s)$ is the value of the portfolio at time s, with strategy π and initial data (t,x).

We consider in the following a power utility function, $f(x) = x^p$, for $p \in (0,1)$, and deterministic parameters, $\mu_t = \mu, \sigma_t = \sigma$. In this case we have $f\left(\mathbb{R}_0^+\right) = \mathbb{R}_0^+$, thus $V(t,x) \geq 0$.

Because the dynamics is homogeneous, that is, $X_{t,\lambda x}^\pi = \lambda X_{t,x}^\pi$, the power utility function is particularly easy to deal with. This can be seen in the following Lemma:

Lemma 12. *If $f(x)$ is multiplicative, i.e. $f(xy) = f(x)f(y)$, then*

$$V(t,x) = f(x)V(t,1)$$

Proof. This is a consequence of the following computation:

$$\begin{aligned} J(t,x;\pi) &= \mathbb{E}\left[f\left(X_{t,x}^\pi(T) \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}\left[f\left(\frac{X_{t,x}^\pi(T)}{x} x \right) \middle| \mathcal{F}_t \right] \\ &= f(x)\mathbb{E}\left[f\left(X_{t,1}^\pi(T) \right) \middle| \mathcal{F}_t \right] \\ &= f(x)J(t,1;\pi). \end{aligned}$$

□

Thus we conclude that V is differentiable in the space variable and

$$\begin{aligned} DV(t,x) &= f'(x)V(t,1) = px^{p-1}V(t,1) > 0, \\ D^2V(t,x) &= f''(x)V(t,1) = -p(1-p)x^{p-2}V(t,1) < 0, \end{aligned}$$

for $x > 0$. Taking these facts in account and denoting $h(t) := V(t,1)$, we now need to determine π^* such that

$$H(t,x;px^{p-1}h(t),-p(1-p)x^{p-2}h(t)) = H^{\pi^*(t,x)}(t,x;px^{p-1}h(t),-p(1-p)x^{p-2}h(t)).$$

Thus

$$\begin{aligned}\pi^*(t,x) &= \underset{\pi\in[\pi_0,\pi_1]}{\text{argmin}} \left(-(\pi(\mu-r)+r)px^p h(t) + \frac{1}{2}\sigma^2\pi^2 p(1-p)x^p h(t)\right)\\ &= \underset{\pi\in[\pi_0,\pi_1]}{\text{argmin}} \left(-(\pi(\mu-r)+r) + \frac{1}{2}\sigma^2\pi^2(1-p)\right)\\ &= \min\left(\pi_1, \frac{\mu-r}{\sigma^2(1-p)} \vee \pi_0\right).\end{aligned}$$

It is then easy to deduce that

$$v(t,x) = x^p e^{p(T-t)\left(r+\pi^*\left((\mu-r)-\frac{1-p}{2}\pi^*\sigma^2\right)\right)}$$

is a $C^{1,2}$ solution of (2.5) such that

$$H(.;Dv,D^2v)(t,x) = H^{\pi^*(.)}(.;Dv,D^2v)(t,x).$$

Thus we conclude, by the verification Theorem, that $V(t,x) = v(t,x)$ and π^* is an optimal control.

Later, in Section 4.7, we will perform a worst-case approach to this problem, when the parameters μ, σ are considered to be stochastic.

Chapter 3

Deterministic differential games

In this Chapter we consider two-person zero-sum differential games. Analogously to Chapter 1 our objective is to establish the dynamic programming principle and use it to characterize the value functions as viscosity solutions of the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. The exposition is similar to the one in [3]. For more classical approaches (without viscosity solutions) we refer the reader to [8, 9].

3.1 Introduction

The scenario for a two-person differential game is similar to that of optimal control. One main difference is that now the state variable is controlled by two controls, $a \in \mathcal{A}$ and $b \in \mathcal{B}$:

$$dX^{a,b}_{t,x}(s) \quad = \quad \mu\left(s, X^{a,b}_{t,x}(s); a(s), b(s)\right) ds.$$

The controls are adjusted by two players, who have antagonic objectives. We assume that player one wants to maximize a terminal reward, while the other wants to minimize it. The terminal reward is the quantity

$$J(t, x; a, b) := f\left(X^{a,b}_{t,x}(T)\right).$$

In this scenario we will give advantage to one of the players by allowing him to use strategies. A strategy for player 1, $\alpha := \alpha[b]$, is a mapping $\alpha : \mathcal{B} \to \mathcal{A}$. Analogously we define strategy for player 2. We then consider the optimization problems of determining the value functions

$$\begin{aligned} v(t,x) &:= \inf_{\beta} \sup_{a} J(t, x; a, \beta[a]), \\ u(t,x) &:= \sup_{\alpha} \inf_{b} J(t, x; \alpha[b], b). \end{aligned}$$

These value functions, v, u, are called, respectively, lower and upper value of the game. The names are justified by the inequality,

$$v \leq u,$$

which is a consequence of the fact that on v player two is given advantage, while the opposite happens on u. In fact, not to give too much advantage to one of the players, and to consider a problem that can be tackled via a dynamic programming approach, we need to consider only strategies that do not foresee the future of the opponent's control. This is made precise by introducing the notion of non-anticipativity, which is a property of interest in many applications.

As the reader may predict, each of the value functions will verify a dynamic programming principle that will allows us to prove that they are solutions to a partial differential equation. This is, indeed, true. The dynamic programming principle now takes the form:

$$\begin{aligned} v(t,x) &= \inf_{\beta} \sup_{a} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right), \\ u(t,x) &= \sup_{\alpha} \inf_{b} u\left(\tau, X_{t,x}^{\alpha[b],b}(\tau)\right). \end{aligned}$$

The equations that the value functions satisfy are the so-called Hamilton-Jacobi-Bellman-Isaacs equations:

$$\left(-\partial_t v + \inf_a \sup_b -\langle \mu(.;a,b), Dv\rangle\right)(t,x) = 0,$$
$$\left(-\partial_t u + \sup_b \inf_a -\langle \mu(.;a,b), Du\rangle\right)(t,x) = 0.$$

After establishing uniqueness for the above equations we can deduce that if Isaac's condition holds, that is, if, for all (t,x) and p,

$$\inf_a \sup_b -\langle \mu(t,x;a,b), p\rangle = \sup_b \inf_a -\langle \mu(t,x;a,b), p\rangle$$

holds, then the game has a value, i.e., $v = u$.

3.2 The controlled dynamical system

In two-person differential games we must consider two control spaces $\mathcal{A}, \mathcal{B}$ consisting of the measurable functions taking values in compact sets $A \subset \mathbb{R}^{d_a}, B \subset \mathbb{R}^{d_b}$, respectively. We say that $\mathcal{A}$, $\mathcal{B}$ are the control spaces of players one and two respectively.

The nonlinear system in $\mathbb{R}^d$ that characterizes the state variable is similar to the one considered in optimal control,

$$\begin{cases} dX_{t,x}^{a,b}(s) &= \mu\left(s, X_{t,x}^{a,b}(s); a(s), b(s)\right) ds \\ X_{t,x}^{a,b}(t) &= x, \end{cases} \tag{3.1}$$

where $a \in \mathcal{A}, b \in \mathcal{B}$, (t,x) are the initial conditions and $\mu : [0,T] \times \mathbb{R}^d \times A \times B \to \mathbb{R}^d$ is a continuous function which is Lipschitz continuous in the space variable, that is

$$|\mu(t,x;a,b) - \mu(t,y;a,b)| \le K|x-y|,$$

for some constant K.

The space $\mathbb{S} := [0,T] \times \mathbb{R}^d$ is again denoted by *state space*.

Under the assumptions made on μ, the existence and uniqueness of a solution $X_{t,x}^{a,b}$ for (1.1) follows from the standard theory of ordinary differential equations, for each (a,b). Furthermore there is continuity with respect to the initial conditions (t,x) uniformly in t and a,b. More precisely, for each x, there is a constant, C_x, depending only on K,T,x such that for all $a \in \mathcal{A}, b \in \mathcal{B}$, $t \in [0,T]$, $t' \ge t$, $s \ge t'$, $x' \in B_1(x)$,

$$\left|X_{t',x'}^{a,b}(s) - X_{t,x}^{a,b}(s)\right| \le C_x \left(|x-x'| + |t-t'|^{\frac{1}{2}}\right). \tag{3.2}$$

This estimate is a particular case of Lemma 148, when there is no diffusion.

3.3 Lower and upper values

We consider a *terminal reward*, which player one wants to maximize and player two wants to minimize,

$$J(t,x;a,b) := f\left(X_{t,x}^{a,b}(T)\right).$$

The *payoff function* f is assumed to be bounded and continuous. We can think of this payoff as something that player two will have to pay to player one in the terminal time T. This would explain their antagonic interests.

In this scenario it is natural to define the two following *value functions*:

$$v_s(t,x) := \sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} J(t,x;a,b),$$
$$u_s(t,x) := \inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}} J(t,x;a,b).$$

We call v_s the *lower static value function* and u_s the *upper static value function*.

Notice that, on the lower value function, advantage is given to player 2, who is allowed to make his choice with the information of the whole future response of the first player. Similarly, on the upper value, advantage is given to the first player.

In a dynamic formulation of the game we want to give advantage to one of the players but without letting him foresee the future. This is achieved by introducing the notion of non-anticipating strategies:

Definition 13. *A* strategy *for player* 2 *is a mapping* $\beta : \mathcal{A} \to \mathcal{B}$.

A strategy, β, is said to verify the non-anticipativity property *if for every* $a_1, a_2 \in \mathcal{A}$, $\tau \in [t,T]$ *we have:*

$$\left(\forall_{s\in[t,\tau]}\ a_1(s) = a_2(s)\right) \Rightarrow \left(\forall_{s\in[t,\tau]}\ \beta[a_1](s) = \beta[a_2](s)\right).$$

The space of non-anticipative strategies for player 2 *is denoted by* Δ.

The definition of strategy for player 1 *is analogous. The space of non-anticipative strategies for player* 1 *is denoted by* Γ.

We are now able to define the value functions which will be studied in the subsequent sections.

Definition 14. *The* lower value *of a differential game is defined as*

$$v(t,x) = \inf_{\beta\in\Delta} \sup_{a\in\mathcal{A}} J(t,x;a,\beta[a]).$$

Similarly, the upper value *of a differential game is*

$$u(t,x) = \sup_{\alpha\in\Gamma} \inf_{b\in\mathcal{B}} J(t,x;\alpha[b],b).$$

The names upper and lower value are justified by the following inequality:

$$v \leq u. \tag{3.3}$$

This intuitive inequality needs to be proved. We will do that, in an indirect way, using the HJBI equations associated with the game.

Using the same arguments as in the deterministic optimal control problem we deduce the following result on the regularity of v, u:

Proposition 15. *The lower and upper value functions, v, u, are bounded and continuous in* $\mathbb{S}$.

3.4 Dynamic programming principle

In this Section we establish the *dynamic programming principle* for the value functions v, u. Like in the case of optimal control, property (1.3) will be essential.

Theorem 16 (Dynamic programming principle)**.** *For all* $x \in \mathbb{R}^d$, $t \in [0,T]$ *and* $\tau \in [t,T]$ *the following holds:*

$$\begin{aligned} v(t,x) &= \inf_{\beta\in\Delta} \sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right), \\ u(t,x) &= \sup_{\alpha\in\Gamma} \inf_{b\in\mathcal{B}} u\left(\tau, X_{t,x}^{\alpha[b],b}(\tau)\right). \end{aligned}$$

Proof. We give only the proof for the lower value function, since the other is similar. We prove the two inequalities separately.

Step 1: $v(t,x) \leq \inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right)$.

Let β_1^ε be such that

$$\inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right) \geq \sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta_1^\varepsilon[a]}(\tau)\right) - \varepsilon.$$

For each y, let β_y^ε be such that

$$v(\tau,y) \geq \sup_{a\in\mathcal{A}} J\left(\tau, y; a, \beta_y^\varepsilon[a]\right) - \varepsilon.$$

We now define β^ε by

$$\beta^\varepsilon[a] := \beta_1^\varepsilon[a] \oplus_\tau \beta^\varepsilon_{X_{t,x}^{a,\beta_1^\varepsilon[a]}(\tau)}[a].$$

It is straightforward to check that $\beta^\varepsilon \in \Delta$. Furthermore, for any $a \in \mathcal{A}$,

$$\begin{aligned}
J(t,x;a,\beta^\varepsilon[a]) &= J\left(\tau, X_{t,x}^{a,\beta^\varepsilon[a]}(\tau); a, \beta^\varepsilon[a]\right) \\
&= J\left(\tau, X_{t,x}^{a,\beta_1^\varepsilon[a]}(\tau); a, \beta^\varepsilon_{X_{t,x}^{a,\beta_1^\varepsilon[a]}(\tau)}[a]\right) \\
&\leq v\left(\tau, X_{t,x}^{a,\beta_1^\varepsilon[a]}(\tau)\right) + \varepsilon \\
&\leq \sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta_1^\varepsilon[a]}(\tau)\right) + \varepsilon \\
&\leq \inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right) + 2\varepsilon.
\end{aligned}$$

Since $a \in \mathcal{A}$ is arbitrary, we deduce that

$$\sup_{a\in\mathcal{A}} J(t,x;a,\beta^\varepsilon[a]) \leq \inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right) + 2\varepsilon.$$

Thus,

$$v(t,x) \quad\leq\quad \inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right) + 2\varepsilon.$$

Step 2: $v(t,x) \geq \inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right)$.

Let β^ε be such that

$$v(t,x) \geq \sup_{a\in\mathcal{A}} J(t,x;a,\beta^\varepsilon[a]) - \varepsilon,$$

and a_1^ε be such that

$$\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta^\varepsilon[a]}(\tau)\right) \leq v\left(\tau, X_{t,x}^{a_1^\varepsilon,\beta^\varepsilon[a_1^\varepsilon]}(\tau)\right) + \varepsilon.$$

Finally, define $\tilde\beta^\varepsilon$ by

$$\tilde\beta^\varepsilon[a] := \beta^\varepsilon[a_1^\varepsilon \oplus_\tau a],$$

and let a_2^ε be such that

$$\sup_a J\left(\tau, X_{t,x}^{a_1^\varepsilon,\beta^\varepsilon[a_1^\varepsilon]}(\tau); a, \tilde{\beta}^\varepsilon[a]\right) \le J\left(\tau, X_{t,x}^{a_1^\varepsilon,\beta^\varepsilon[a_1^\varepsilon]}(\tau); a_2^\varepsilon, \tilde{\beta}^\varepsilon[a_2^\varepsilon]\right) + \varepsilon.$$

Let $a^\varepsilon := a_1^\varepsilon \oplus_\tau a_2^\varepsilon$. Then, by the non-anticipativity property of β^ε and by definition of $\tilde{\beta}^\varepsilon$, we have

$$\beta^\varepsilon[a^\varepsilon] = \beta^\varepsilon[a_1^\varepsilon] \oplus_\tau \tilde{\beta}^\varepsilon[a_2^\varepsilon].$$

Thus,

$$\begin{aligned}
v(t,x) &\ge \sup_{a\in\mathcal{A}} J(t,x;a,\beta^\varepsilon[a]) - \varepsilon \\
&\ge J(t,x;a^\varepsilon,\beta^\varepsilon[a^\varepsilon]) - \varepsilon \\
&= J\left(\tau, X_{t,x}^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}(\tau); a^\varepsilon, \beta^\varepsilon[a^\varepsilon]\right) - \varepsilon \\
&= J\left(\tau, X_{t,x}^{a_1^\varepsilon,\beta^\varepsilon[a_1^\varepsilon]}(\tau); a_2^\varepsilon, \tilde{\beta}^\varepsilon[a_2^\varepsilon]\right) - \varepsilon \\
&\ge \sup_a J\left(\tau, X_{t,x}^{a_1^\varepsilon,\beta^\varepsilon[a_1^\varepsilon]}(\tau); a, \tilde{\beta}^\varepsilon[a]\right) - 2\varepsilon \\
&\ge v\left(\tau, X_{t,x}^{a_1^\varepsilon,\beta^\varepsilon[a_1^\varepsilon]}(\tau)\right) - 2\varepsilon \\
&\ge \sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta^\varepsilon[a]}(\tau)\right) - 3\varepsilon \\
&\ge \inf_{\beta\in\Delta}\sup_{a\in\mathcal{A}} v\left(\tau, X_{t,x}^{a,\beta[a]}(\tau)\right) - 3\varepsilon.
\end{aligned}$$

□

3.5 Hamilton-Jacobi-Bellman-Isaacs equation

In this Section we use the dynamic programming principle to prove that the value functions are viscosity solutions of the associated *Hamilton-Jacobi-Bellman-Isaacs* equations.

Consider the Hamiltonians

$$\begin{aligned}
H^-(t,x,p) &:= \inf_{a\in A}\sup_{b\in B} H^{a,b}(t,x,p), \\
H^+(t,x,p) &:= \sup_{b\in B}\inf_{a\in A} H^{a,b}(t,x,p),
\end{aligned}$$

where

$$H^{a,b}(t,x,p) := -\langle p, \mu(t,x;a,b)\rangle.$$

Notice that $H^{a,b}(t,x,p)$ is continuous in t,x,p,a,b. Thus, due to the compactness of A and B, H is also continuous.

Theorem 17. *The lower value function v is a viscosity solution of*

$$(-\partial_t v + H^-(.,Dv))(t,x) = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}^d. \tag{3.4}$$

The upper value function u is a viscosity solution of

$$(-\partial_t u + H^+(.,Du))(t,x) = 0, \qquad (t,x) \in]0,T[\times\mathbb{R}^d. \tag{3.5}$$

Proof. We only make the proof for the lower value, since the other is analogous. We start with the supersolution property, arguing by contradiction. With that purpose in mind consider $\phi \in C^1(\mathbb{S})$, $(\hat{t},\hat{x}) \in \operatorname{argmin}(v-\phi)$ and suppose that

$$(-\partial_t\phi + H^-(.,D\phi))(\hat{t},\hat{x}) \le -3\delta, \tag{3.6}$$

where $\delta > 0$.

By Remark 101, in the Appendix, we can suppose that $(\hat{t}, \hat{x})$ is a strict global minimizer. Thus, for all $(t, x) \in \mathbb{S}$,

$$(v - \phi)(t, x) \geq (v - \phi)(\hat{t}, \hat{x}),$$

that is

$$\phi(\hat{t}, \hat{x}) - \phi(t, x) \geq v(\hat{t}, \hat{x}) - v(t, x). \tag{3.7}$$

By (3.6), there is $a^* \in A$ such that, for all $b \in B$,

$$\left(-\partial_t \phi + H^{a^*,b}(., D\phi)\right)(\hat{t}, \hat{x}) \leq -2\delta.$$

Because B is compact, then $H^{a,b}(t, x)$ is continuous in (t, x) uniformly with respect to b. Thus there is R such that, for all $b \in B$,

$$\left(-\partial_t \phi + H^{a^*,b}(., D\phi)\right)(t, x) \leq -\delta, \text{ for all } (t, x) \in B_R(\hat{t}, \hat{x}).$$

Let τ be sufficiently close to $\hat{t}$ so that, for all $a \in A, b \in B$,

$$\left(s, X^{a,b}_{\hat{t},\hat{x}}(s)\right) \in B_R(\hat{t}, \hat{x}), \text{ for all } s \in [\hat{t}, \tau].$$

Consider $\beta \in \Delta$ arbitrary. Then, because ϕ is C^1, we have

$$\begin{aligned} \phi\left(\tau, X^{a^*,\beta[a^*]}_{\hat{t},\hat{x}}(\tau)\right) - \phi(\hat{t}, \hat{x}) &= \int_{\hat{t}}^{\tau} \left(\partial_t \phi - H^{a^*,\beta[a^*]}(., D\phi)\right)\left(s, X^{a^*,\beta[a^*]}_{\hat{t},\hat{x}}(s)\right) ds \\ &\geq (\tau - \hat{t})\delta. \end{aligned}$$

Thus, by (3.7),

$$\begin{aligned} v\left(\tau, X^{a^*,\beta[a^*]}_{\hat{t},\hat{x}}(\tau)\right) - v(\hat{t}, \hat{x}) &\geq \phi\left(\tau, X^{a^*,\beta[a^*]}_{\hat{t},\hat{x}}(\tau)\right) - \phi(\hat{t}, \hat{x}) \\ &\geq (\tau - \hat{t})\delta. \end{aligned}$$

Hence

$$\sup_{a \in \mathcal{A}} v\left(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)\right) \geq v(\hat{t}, \hat{x}) + (\tau - \hat{t})\delta,$$

and, since β is arbitrary,

$$\inf_{\beta \in \Delta} \sup_{a \in \mathcal{A}} v\left(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)\right) \geq v(\hat{t}, \hat{x}) + (\tau - \hat{t})\delta,$$

which contradicts the dynamic programming principle.

To prove that v is a subsolution of (3.4) we also proceed by contradiction. Consider $\phi \in C^1(\mathbb{S})$, $(\hat{t}, \hat{x}) \in \operatorname{argmax}(v - \phi)$ and suppose that

$$(-\partial_t \phi + H^-(., D\phi))(\hat{t}, \hat{x}) \geq 4\delta. \tag{3.8}$$

By Remark 101, in the Appendix, we can suppose that $(\hat{t}, \hat{x})$ is a strict global maximizer. Thus, for all $(t, x) \in \mathbb{S}$,

$$(v - \phi)(t, x) \leq (v - \phi)(\hat{t}, \hat{x}),$$

that is

$$\phi(\hat{t}, \hat{x}) - \phi(t, x) \leq v(\hat{t}, \hat{x}) - v(t, x). \tag{3.9}$$

By (3.8), for each $a \in A$ there is $b_a \in B$ such that

$$\left(-\partial_t \phi + H^{a,b_a}(., D\phi)\right)(\hat{t}, \hat{x}) \geq 3\delta.$$

Furthermore, by continuity of $H^{a,b}$ with respect to a, for each a there is r_a such that, for all $\tilde{a} \in B_{r_a}(a)$,

$$\left(-\partial_t \phi + H^{\tilde{a},b_a}(., D\phi)\right)(\hat{t}, \hat{x}) \geq 2\delta.$$

Since A is compact and $\{B_{r_a}(a) : a \in A\}$ is an open covering of A, there is a finite collection $\{a_i, r_i\}$ such that $\{B_{r_i}(a_i)\}$ covers A. We then define $\Lambda_1 := B_{r_1}(a_1)$, $\Lambda_{i+1} := B_{r_{i+1}}(a_{i+1}) \setminus \bigcup_{k=1}^{i} \Lambda_k$. Now define $\psi : A \to B$ by

$$\psi(a) := \sum_i \mathbf{1}_{\Lambda_i}(a) b_{a_i},$$

and $\beta \in \Delta$ by $\beta[a]_s := \psi(a_s)$. Then, for all $a \in A$,

$$\left(-\partial_t \phi + H^{a,\psi(a)}(., D\phi)\right)(\hat{t}, \hat{x}) \geq 2\delta.$$

Because A, B are compact then $H^{a,b}(t,x)$ is continuous in (t,x) uniformly with respect to (a,b). Thus, there is R such that, for all $a \in A$,

$$\left(-\partial_t \phi + H^{a,\psi(a)}(., D\phi)\right)(t, x) \geq \delta, \text{ for all } (t,x) \in B_R(\hat{t}, \hat{x}).$$

Let τ be sufficiently close to $\hat{t}$ so that, for all $a \in A, b \in B$,

$$\left(s, X^{a,b}_{\hat{t},\hat{x}}(s)\right) \in B_R(\hat{t}, \hat{x}), \text{ for all } s \in [\hat{t}, \tau].$$

Then, because ϕ is C^1, we have, for an arbitrary $a \in \mathcal{A}$,

$$\begin{aligned} \phi\left(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)\right) - \phi(\hat{t}, \hat{x}) &= \int_{\hat{t}}^{\tau} \left(\partial_t \phi - H^{a_s,\psi(a_s)}(., D\phi)\right)\left(s, X^{a,\beta[a]}_{\hat{t},\hat{x}}(s)\right) ds \\ &\leq -(\tau - \hat{t})\delta. \end{aligned}$$

Thus, by (3.9),

$$\begin{aligned} v\left(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)\right) - v(\hat{t}, \hat{x}) &\leq \phi\left(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)\right) - \phi(\hat{t}, \hat{x}) \\ &\leq -(\tau - \hat{t})\delta. \end{aligned}$$

Taking the supremum in a we get

$$\sup_{a \in \mathcal{A}} v\left(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)\right) \leq v(\hat{t}, \hat{x}) - (\tau - \hat{t})\delta.$$

Thus

$$\inf_{\beta \in \Delta} \sup_{a \in \mathcal{A}} v(\tau, X^{a,\beta[a]}_{\hat{t},\hat{x}}(\tau)) \leq v(\hat{t}, \hat{x}) - (\tau - \hat{t})\delta,$$

which contradicts the dynamic programming principle. □

It remains to establish uniqueness of solution for the HJBI equation.

Theorem 18. *The lower value v is the minimal supersolution and maximal subsolution of* (3.4) *in the class of bounded functions. In particular, it is the unique viscosity solution of* (3.4) *in that class. An analogous result holds for the upper value, u.*

For the proof of the previous Theorem we refer the reader to [3, p. 442]. We will establish later uniqueness of solution for the analogous second order HJBI equation appearing in the stochastic scenario, see Section 4.6.1.

We can finally prove (3.3).

Corollary 19. *Let v, u be the lower and upper values of the game, respectively. Then*

$$v \leq u.$$

Proof. Notice that $H^- \geq H^+$. This implies that v is a subsolution of (3.5). Since, by the previous Theorem, u is the maximal subsolution of (3.5) we conclude that

$$v \leq u.$$

□

Using the same arguments we can also establish a criterion for the game to have a *value*:

Corollary 20 (Isaac's condition)**.** *If $H^- = H^+$ then the game has a value, that is,*

$$v = u.$$

Proof. The proof follows directly from the fact that in this case equations (3.4) and (3.5) are the same, hence, by uniqueness of solution, we have $v = u$. □

Chapter 4

Stochastic differential games

4.1 Introduction

Consider a scenario where two players compete in such a way that at some terminal time T the second player pays to the first one a certain payoff. This payoff is determined by the terminal value of a *state variable*, $X^{a,b}(T)$, depending on parameters a, b which are controlled, respectively, by players 1 and 2. More precisely, the *payoff* is $f\left(X^{a,b}_{t,x}(T)\right)$, where f is a measurable function.

We consider the case where the state variable is random with dynamics given by the following *stochastic differential equation*:

$$\begin{cases} dX^{a,b}_{t,x}(s) &= \mu\left(s, X^{a,b}_{t,x}(s); a_s, b_s\right) ds + \sigma\left(s, X^{a,b}_{t,x}(s); a_s, b_s\right) dW_s \\ X^{a,b}_{t,x}(t) &= x, \end{cases}$$

where t is the time when the players start competing and x is the initial value of the state variable.

By adjusting the *controls*, both players can drive the state variable, hence changing the terminal payoff. Obviously, if player 1 is rational then he is interested in maximizing the payoff. Similarly, player 2 is interested in minimizing the payoff.

At the initial time t, when both players choose their controls, they can not predict the payoff, hence they are not able to optimize it. Instead they can optimize the expected value of the payoff:

$$J(t, x; a, b) := \mathbb{E}\left[f\left(X^{a,b}_{t,x}(T)\right)\middle|\mathcal{F}_t\right].$$

This quantity is also a random variable, but is $\mathcal{F}_t-$measurable, hence its value is accessible to the players at the time of the start of the game. We shall call J the *reward function*. Both players will try to choose a, b in order to optimize J according to their antagonic objectives.

If the players are rational then they are interested in choosing controls a, b in such a way that maximizes or minimizes $J(t, x; a, b)$. Naively we could consider the problems

$$V_s(t, x) := \sup_a \inf_b J(t, x; a, b),$$
$$U_s(t, x) := \inf_b \sup_a J(t, x; a, b).$$

These two functions are of interest but they will not be the scope of this work. They are called, respectively, the *lower static value* and *upper static value* of the game. Clearly on the lower static value, player 2 is given an advantage over player 1 because he is able to look at the other player's control before choosing his own. Analogously, on the upper static value it is player 1 who is in advantage.

These values are called static, because on each of them, one of the players is given advantage by being allowed to know the control of the other player for the entire duration of the game, that is, in $[t, T]$. In this case we see that some information about the future is being revealed to one of the players.

In a dynamic version of the same game we will still give advantage to one of the players, but in such a way that no information about the future is revealed. In this case information to one player about the other player's control is revealed, in a dynamic way, as time goes by. This justifies the names static and dynamic given to these two different versions of the same game. More precisely, we will consider

$$V(t, x) := \inf_\beta \sup_a J(t, x; a, \beta[a]),$$
$$U(t, x) := \sup_\alpha \inf_b J(t, x; \alpha[b], b),$$

where $\alpha := \alpha[b]$ and $\beta := \beta[b]$ are, respectively, strategies for players 1 and 2. A *strategy* for player 1, $\alpha[b]$, can be seen as the reply of player 1 when player 2 chooses to use the control b. If no restrictions are made on the admissible strategies then we get the static version of the game. Since we are interested in not revealing information about the future, we require the admissible strategies to be non-anticipating, in a sense to be specified in the next paragraph.

We say that a strategy for player 1, α, is *non-anticipating* if, given any two controls of player 2, b and b', which are equal up to some time s, $b_r = b'_r$ for all $r \in [t, s]$, the replies of player 1 using strategy α, $\alpha[b]$ and $\alpha[b']$, are also equal up to time s, $\alpha[b]_r = \alpha[b']_r$ for all $r \in [t, s]$.

The scope of this chapter is to study and characterize the lower value V. The approach for the upper value is analogous and will be omitted. The main results are:

- $V(t, x)$ is a constant random variable, that is, $V(t, x) = \mathbb{E}[V(t, x)]$;
- V is the *viscosity solution* of the *Hamilton-Jacobi-Bellman-Isaacs* (HJBI) equation associated with this game.

To derive the HJBI equation we will use a dynamic programming approach different from the one existent in the literature. Instead of the *dynamic programming principle* we will follow [1] and use a weak version of this principle. With this approach we avoid measurability problems and the need of continuity of the value function which will allow us to study V in a more general setting than the usual one.

Thus, we emphasize the two main contributions of this thesis, which are discussed in this chapter:

- Extending the weak dynamic programming principle to the context of stochastic differential games;
- Considering stochastic differential games in a more general setting, the setting where strategies are allowed to take values in an unbounded set and where f is locally Lipschitz instead of globally Lipschitz.

Two main references for zero-sum stochastic differential games are [2], [10]. In their pioneering work, [2], Fleming and Souganidis studied rigorously for the first time zero-sum stochastic differential games. There they considered only controls which are independent of the past, thus getting deterministic value functions. In [10], Buckdahn and Li take a more modern approach to the same problem, using backward stochastic differential equations (BSDEs). Our approach will follow more closely this last reference.

The structure of this Chapter is the following:

- Section 4.2 concerns some notation and preliminary definitions and estimates;
- In Section 4.3 we consider zero-sum stochastic differential games in the Markovian setting. We discuss general properties of the dynamics, the space of controls, the non-anticipative strategies, the reward function, and the upper and lower values. A detailed discussion of strategies' properties, essential in the subsequent sections, ends this Section;
- In the following Section we prove that the value function is a constant random variable and can thus be seen as a deterministic function, $V(t, x) = \mathbb{E}[V(t, x)]$. It is also proved in this section that the value function has polynomial growth. This result is essential in the discussion of uniqueness of the HJBI equation;
- Section 4.5 is the main section of the Chapter. There we prove a version of the weak dynamic programming principle for stochastic differential games;
- In Section 4.6 we use the weak dynamic programming principle to prove that the value function is a viscosity solution of the HJBI equation. It is also proved in this section that when A, B are both compact sets then the HJBI equation admits a unique solution on a class of functions that contains all the functions of polynomial growth;
- In Section 4.7 we discuss an application of zero-sum stochastic differential games to Merton's problem. There we use a worst-case approach to deal with stochastic volatilities;
- Finally, in Section 4.8, we gather the results and assumptions from this Chapter to extract conclusions and some directions where further research can be carried.

For convenience of the reader we review in the end of the Chapter the notation used throughout this work.

4.2 Preliminaries

In this Chapter we consider zero-sum stochastic differential games on a time interval $[0, T]$.

Therefore we consider the *classical Wiener space* $(\Omega, \mathcal{F}, \mathbb{P})$, on this finite horizon T. More precisely, Ω is the space of continuous functions $\omega : [0, T] \to \mathbb{R}^N$ such that $\omega(0) = 0$, and $\mathcal{F}$ is the Borel $\sigma-$algebra in Ω completed with respect to the *Wiener measure* $\mathbb{P}$. We denote by W the *standard $N-$dimensional Brownian motion* corresponding to the coordinate process, $W_s(\omega) = \omega_s$.

We consider in $(\Omega, \mathcal{F}, \mathbb{P})$ the natural *filtration induced* by W augmented with the $\mathbb{P}-$null sets, $\mathbb{F} = \{\mathcal{F}_s, 0 \le s \le T\}$, where

$$\mathcal{F}_s = \sigma\{W_r : r \le s\} \vee \mathcal{N}_{\mathbb{P}}.$$

We denote by $\mathcal{T}$ the collection of all stopping times in $\mathbb{F}$. Given τ_1, τ_2 such that $\tau_1 \le \tau_2$, $\mathcal{T}_{[\tau_1,\tau_2]}$ denotes the collection of all $\tau \in \mathcal{T}$ such that $\tau_1 \le \tau \le \tau_2$. When $\tau_1 = 0$ we simply write $\mathcal{T}_{\tau_2}$.

The essential extrema of a random variable is defined as

Definition 21. *Let X be a random variable. Then $M \in \mathbb{R} \cup \{+\infty\}$ is said to be the* essential supremum *of X, $M = \operatorname{esssup} X$, if:*

- $X \le M \ \ \mathbb{P} - \text{a.s.}$*;*
- *If there exists $\tilde{M} \in \mathbb{R}$ such that $X \le \tilde{M}$, $\mathbb{P} - \text{a.s.}$, then $M \le \tilde{M}$.*

The essential infimum *of X is defined as*

$$\operatorname{essinf} X := -\operatorname{esssup} - X.$$

We consider processes in the following space

$$\mathbb{H}^p(t, T; A) := \Big\{ (\psi_s)_{s\in[t,T]},\ \mathbb{F} - \text{progressively measurable with values in } A \text{ s.t.:} \\ \mathbb{E}\left[\int_t^T |\psi_s|^p ds\right] < \infty \Big\}.$$

In the above space we consider a norm $\|.\|_{\mathbb{H}^p}$, for $p \ge 1$, defined by

$$\|\psi\|_{\mathbb{H}^p} := \left(\mathbb{E}\left[\int_t^T |\psi_s|^p ds\right]\right)^{\frac{1}{p}}.$$

Notice that the above space is a $\mathbb{L}^p$ space. Indeed,

$$\mathbb{H}^p(t, T; A) = \mathbb{L}^p([t, T] \times \Omega, \mathcal{B}([t, T]) \otimes \mathcal{F}, m \otimes \mathbb{P}),$$

where $\mathcal{B}([t, T])$ denotes the Borel $\sigma-$algebra of $[t, T]$ and m is the Lebesgue measure in $[t, T]$.

We also consider the subspace of $\mathbb{H}^p(t, T; A)$ consisting of the processes which have essentially bounded integral:

$$\mathbb{H}^{p,\infty}(t, T; A) := \left\{ \psi \in \mathbb{H}^p(t, T; A) : \operatorname{esssup} \int_t^T |\psi_s|^p ds < \infty \right\}.$$

In this space we consider the following two uniform norms:

$$\|\psi\|_{\mathbb{H}_t^{p,\infty}} := \operatorname{esssup} \left(\mathbb{E}\left[\int_t^T |\psi_s|^p ds \Big| \mathcal{F}_t\right]\right)^{\frac{1}{p}},$$

$$\|\psi\|_{\mathbb{H}^{p,\infty}} := \operatorname{esssup} \left(\int_t^T |\psi_s|^p ds\right)^{\frac{1}{p}}.$$

Notice that $\|\psi\|_{\mathbb{H}^{p,\infty}_t} \le \|\psi\|_{\mathbb{H}^{p,\infty}}$.

Finally, we consider the subspace of $\mathbb{H}^{p,\infty}(t,T;A)$ consisting of the processes which are essentially bounded:

$$\mathbb{H}^{\infty}(t,T;A) := \left\{ \psi \in \mathbb{H}^{p,\infty}(t,T;A) : \operatorname{esssup} \sup_{s\in[t,T]} |\psi_s| < \infty \right\}.$$

In this space we also consider two uniform norms:

$$\begin{aligned} \|\psi\|_{\mathbb{H}^{\infty}_{t,p}} &:= \operatorname{esssup} \left(\mathbb{E}\left[\sup_{s\in[t,T]} |\psi_s|^p \Big| \mathcal{F}_t \right] \right)^{\frac{1}{p}}, \\ \|\psi\|_{\mathbb{H}^{\infty}} &:= \operatorname{esssup} \sup_{s\in[t,T]} |\psi_s|. \end{aligned}$$

In the following Lemma we collect some inequalities about the above norms:

Lemma 22. *Let $\psi \in \mathbb{H}^{\infty}(t,T;A)$ and let $q \le p$. Then the following hold:*

(i) $\|\psi\|_{\mathbb{H}^{\infty}_{t,q}} \le \|\psi\|_{\mathbb{H}^{\infty}_{t,p}} \le \|\psi\|_{\mathbb{H}^{\infty}}$;

(ii) $\|\psi\|_{\mathbb{H}^{p,\infty}_t} \le (T-t)^{\frac{1}{p}} \|\psi\|_{\mathbb{H}^{\infty}_{t,p}}$;

(iii) $\|\psi\|_{\mathbb{H}^{p,\infty}} \le (T-t)^{\frac{1}{p}} \|\psi\|_{\mathbb{H}^{\infty}}$.

Remark 23. *The space $\mathbb{H}^{\infty}(t,T;A)$, with the norm of $\mathbb{H}^p$, is a dense subspace of $\mathbb{H}^p(t,T;A)$.*

$\mathbb{H}^p_{rcll}(t,T;A)$ denotes the processes in $\mathbb{H}^p(t,T;A)$ which are right-continuous and have finite left limits, also called *càdlàg*.

Unless otherwise stated, equalities and inequalities between random variables are to be understood in the $\mathbb{P}-$a.s. sense.

4.3 The Markovian scenario

4.3.1 State dynamics

We consider *zero-sum stochastic differential games* between two players. There is a controlled state process that determines the reward of each of them. This process is a mapping taking values in $\mathbb{R}^d$,

$$(t,x;a,b) \in \mathbb{S} \times \mathcal{A}_t \times \mathcal{B}_t \mapsto X_{t,x}^{a,b} \in \mathbb{H}_{rcll}^0(t,T;\mathbb{R}^d),$$

where $\mathbb{S} := [0,T] \times \mathbb{R}^d$ is the *state space* and $\mathcal{A}_t \subset \mathbb{H}^0(t,T;\mathbb{R}^{d_a})$, $\mathcal{B}_t \subset \mathbb{H}^0(t,T;\mathbb{R}^{d_b})$ are the collections of admissible controls at time t for players 1 and 2 respectively. Here d_a, d_b represent the dimensions of the space of controls for players 1 and 2, respectively.

In this work we consider the case where the controlled state process is a diffusion:

$$\begin{cases} dX_{t,x}^{a,b}(s) &= \mu\left(s, X_{t,x}^{a,b}(s); a_s, b_s\right) ds + \sigma\left(s, X_{t,x}^{a,b}(s); a_s, b_s\right) dW_s \\ X_{t,x}^{a,b}(t) &= x, \end{cases} \tag{4.1}$$

In the above equation μ, σ are functions such that the above SDE has an unique solution:

$$\mu : \mathbb{S} \times A \times B \to \mathbb{R}^d,$$
$$\sigma : \mathbb{S} \times A \times B \to \mathbb{R}^{d\times N},$$

where $A \subset \mathbb{R}^{d_a}, B \subset \mathbb{R}^{d_b}$, and we think of $\mathbb{R}^{d\times N}$ as the space of $d \times N$ matrices with the *Fröbenius norm*, i.e., $|\sigma| = \sqrt{\mathrm{Tr}(\sigma\sigma^T)}$.

We start by making the typical global Lipschitz and linear growth conditions that assure existence and uniqueness of a strong solution for (4.1). That is, there is K such that

$$\begin{cases} |\mu(t,x;a,b) - \mu(t,y;a,b)| + |\sigma(t,x;a,b) - \sigma(t,y;a,b)| \le K|x-y| \\ |\mu(t,x;a,b)| + |\sigma(t,x;a,b)| \le K(1 + |x| + |a| + |b|). \end{cases} \tag{4.2}$$

In addition, as we proceed, further assumptions on the controls will be considered in order to establish our main results.

Example 24. *We now give a few examples of state dynamics:*

- *In a pursuit-evasion game there is a pursuer P and an evader E. In this case we can think of the state variable as being the relative position between P and E. If the pursuer and evader can control their velocities a, b, respectively, then we consider the dynamics of the state variable to be*

  $$dX_{t,x}^{a,b}(s) = (a_s - b_s)ds + \sigma dW_s,$$

  where σ is a volatility factor that controls the Brownian noise on the pursuit.

  Clearly in this case conditions (4.2) are satisfied.

- *We can also think of a game where there is a particle with a velocity controlled by players 1 and 2: the first player controls the direction while the second controls the intensity. In this case the state variable is the position of the particle. In dimension 2 with a 2−dimensional Brownian noise in the dynamics we get*

  $$dX_{t,x}^{a,b}(s) = b_s \cos(a_s)ds + \sigma_{1,1}dW_s^1 + \sigma_{1,2}dW_s^2$$
  $$dY_{t,x}^{a,b}(s) = b_s \sin(a_s)ds + \sigma_{2,1}dW_s^1 + \sigma_{2,2}dW_s^2,$$

  where X, Y represent, respectively, the x and y coordinates of the particle's position.

  Again, the dynamics satisfies (4.2).

- *We can do a worst-case approach to Merton's optimal portfolio problem by considering a differential game where a fictitious player controls the volatility. In this case we can take the value of the portfolio to be the state variable, which has dynamics*

$$dX_{t,x}^{a,b}(s) = (a_s\mu + (1-a_s)r)X_{t,x}^{a,b}(s)ds + a_s b_s X_{t,x}^{a,b}(s)dW_s,$$

  where μ, r are, respectively, the growth rate of the risky asset and the interest rate, a_s is the ratio of the portfolio's value invested in the risky asset at time s, and b_s is the volatility of the risky asset at time s.

  In this example, if we consider compact-valued controls, that is, if a, b take values in a compact set, then conditions (4.2) *are satisfied.*

If conditions (4.2) are satisfied we know, by Theorem 147, that for each $p \geq 2$ and each pair of controls $a \in \mathcal{A}_t \cap \mathbb{H}^p(t,T;A), b \in \mathcal{B}_t \cap \mathbb{H}^p(t,T;B)$, there is a unique strong solution $X_{t,x}^{a,b} \in \mathbb{H}^p(t,T;\mathbb{R}^d)$ to (4.1), which verifies additionally

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X_{t,x}^{a,b}(s)\right|^p\right] \leq C(1+|x|^p+\|a\|_{\mathbb{H}^p}^p+\|b\|_{\mathbb{H}^p}^p), \tag{4.3}$$

where C is a constant depending only on K, T, p. We are interested in considering conditional expectations of the state process. Hence we consider a stronger version of (4.3):

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X_{t,x}^{a,b}(s)\right|^p\middle|\mathcal{F}_t\right] \leq C\left(1+|x|^p+\mathbb{E}\left[\int_t^T(|a_s|^p+|b_s|^p)ds\middle|\mathcal{F}_t\right]\right).$$

If we know additionally that $a, b \in \mathbb{H}^{p,\infty}$ then we deduce from the previous estimate that

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X_{t,x}^{a,b}(s)\right|^p\middle|\mathcal{F}_t\right] \leq C\left(1+|x|^p+\|a\|_{\mathbb{H}_t^{p,\infty}}^p+\|b\|_{\mathbb{H}_t^{p,\infty}}^p\right). \tag{4.4}$$

This estimate has the advantage of being uniform with respect to ω.

Remark 25. *If we consider controls a, b in bounded sets $\mathcal{A}, \mathcal{B}$ of $\mathbb{H}^p$ then estimate* (4.3) *is uniform with respect to a, b. More precisely, if $\mathcal{A} \subset \mathcal{A}_t$, $\mathcal{B} \subset \mathcal{B}_t$ are bounded sets, in the norm of $\mathbb{H}^p$, then there is C depending only on K, T, $\sup\{\|a\|_{\mathbb{H}^p} : a \in \mathcal{A}\}$, and $\sup\{\|b\|_{\mathbb{H}^p} : b \in \mathcal{B}\}$ such that, for all $a \in \mathcal{A}, b \in \mathcal{B}$,*

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X_{t,x}^{a,b}(s)\right|^p\right] \leq C(1+|x|^p). \tag{4.5}$$

Similarly, if we consider controls a, b in sets $\mathcal{A}, \mathcal{B}$ bounded in the norm $\mathbb{H}_t^{p,\infty}$ then estimate (4.4) *is uniform with respect to a, b, in the sense that, for all $a \in \mathcal{A}, b \in \mathcal{B}$,*

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|X_{t,x}^{a,b}(s)\right|^p\middle|\mathcal{F}_t\right] \leq C\left(1+|x|^p\right). \tag{4.6}$$

4.3.2 Admissible controls

In order to be able to use estimate (4.4) we should consider

$$\mathcal{A}_t \subset \mathbb{H}^{p,\infty}(t,T;A),\ \ \mathcal{B}_t \subset \mathbb{H}^{p,\infty}(t,T;B).$$

Since we also want to use Lemma 148[1] we consider the following spaces of *admissible controls* for players 1 and 2:

$$\mathcal{A}_t := \mathbb{H}^{\infty}(t,T;A),\ \ \mathcal{B}_t := \mathbb{H}^{\infty}(t,T;B).$$

[1] In fact, to use Lemma 148, we could have just considered

$$\mathcal{A}_t := \mathbb{H}^{p,\infty}(t,T;A),\ \ \mathcal{B}_t := \mathbb{H}^{p,\infty}(t,T;B),$$

for some $p > 2$.

In this case, estimates (4.3) and (4.4) are valid for all $p \geq 2$.

Remark 26. *If A, B are compact then*

$$\mathcal{A}_t = \mathbb{H}^0(t, T; A), \ \ \mathcal{B}_t = \mathbb{H}^0(t, T; B).$$

This is the case considered in [10].

In the following we establish basic properties concerning the controls for player 1. Obviously the same results apply to the controls of player 2.

Note that by considering restrictions of controls we have the inclusion $\mathcal{A}_t \subset \mathcal{A}_s$, whenever $t \leq s$. More precisely

Lemma 27. *Let $t \leq s$. If $a \in \mathcal{A}_t$ then $a_{|[s,T]} \in \mathcal{A}_s$.*

Thus, given $a \in \mathcal{A}_t$, when there is no ambiguity we will write $a \in \mathcal{A}_s$ instead of $a_{|[s,T]} \in \mathcal{A}_s$.

The following definition is useful to compare stochastic processes:

Definition 28. *Given stochastic processes X, Y, stopping times τ_1, τ_2 and a measurable set $\Lambda \in \mathcal{F}$, we write*

$$X \equiv_\Lambda Y \text{ on } [\tau_1, \tau_2] \tag{4.7}$$

to mean

$$\mathbb{P}(\{\mathbf{1}_\Lambda X_s = \mathbf{1}_\Lambda Y_s; s \ a.e. \ on \ [\tau_1, \tau_2]\}) = 1.$$

If $\Lambda = \Omega$ we write (4.7) as

$$X \equiv Y \text{ on } [\tau_1, \tau_2].$$

Remark 29. *Notice that if X, Y are stochastic processes with enough regularity then we may lift the 'a.e.' in the definition of $\equiv$. More precisely, if X, Y are right continuous, then*

$$X \equiv_\Lambda Y \text{ on } [\tau_1, \tau_2] \Rightarrow \mathbb{P}(\{\mathbf{1}_\Lambda X_s = \mathbf{1}_\Lambda Y_s; \forall s \in [\tau_1, \tau_2)\}) = 1,$$

and if X, Y are left continuous, then

$$X \equiv_\Lambda Y \text{ on } [\tau_1, \tau_2] \Rightarrow \mathbb{P}(\{\mathbf{1}_\Lambda X_s = \mathbf{1}_\Lambda Y_s; \forall s \in (\tau_1, \tau_2]\}) = 1.$$

We would like to identify controls that have an equivalent effect on the dynamics. Therefore we make the following definition:

Definition 30. *Let $a_1, a_2 \in \mathcal{A}_t$, $\tau_1 \in \mathcal{T}_{[t,T]}$, $\tau_2 \in \mathcal{T}_{[\tau_1,T]}$, $\Lambda \in \mathcal{F}$.*

- *We say that a_1 equals a_2 on $[\tau_1, \tau_2]$ for all events in Λ if $a_1 \equiv_\Lambda a_2$ on $[\tau_1, \tau_2]$.*
- *If $a_1 \equiv a_2$ on $[t, T]$ we simply write $a_1 \equiv a_2$ and say that a_1, a_2 are* equivalent controls.

There are two very natural and useful ways of constructing controls out of existing ones. One is to *concatenate* two different controls. For this operation we use the following notation:

Definition 31. *For $t \leq s \leq T$ let $a_1 \in \mathcal{A}_t$, $a_2 \in \mathcal{A}_s$ and $\theta \in \mathcal{T}_{[s,T]}$. Then we define $a_1 \oplus_\theta a_2 \in \mathcal{A}_t$ by*

$$(a_1 \oplus_\theta a_2)_s(\omega) := (a_1)_s(\omega)\mathbf{1}_{[t,\theta(\omega)]}(s) + (a_2)_s(\omega)\mathbf{1}_{(\theta(\omega),T]}(s).$$

The other one is to *patch* two controls in a suitable way by using one of the controls in a subset of Ω and the other in the complement. The next Lemma gives conditions under which these operations yield admissible controls.

Lemma 32. *The following hold:*

- *Let $s \geq t$ and $\theta \in \mathcal{T}_{[s,T]}$. If $a_1 \in \mathcal{A}_t$ and $a_2 \in \mathcal{A}_s$ then*

$$a_1 \oplus_\theta a_2 \in \mathcal{A}_t.$$

- *Let $\theta \in \mathcal{T}_{[t,T]}$ and $a_i \in \mathcal{A}_t$ be such that $a_i \equiv a_j$ on $[t,\theta]$ and $\sup_i \|a_i\|_{\mathbb{H}^\infty} < \infty$. Suppose $\{\Lambda_i\}_{i\geq 1} \subset \mathcal{F}_\theta$ forms a partition of Ω. Then*

$$\sum_{i\geq 1} \mathbf{1}_{\Lambda_i} a_i \in \mathcal{A}_t.$$

Example 33. *One example of admissible controls are the so called* Markov control policies. *Given a measurable function $\psi : \mathbb{S} \to A$, a Markov control policy is the control defined by*

$$a_s := \psi(s, X_s).$$

More precisely, if the control policy above is used then $X_{t,x}^{a,b}$ is the solution to

$$\begin{cases} dX(s) &= \mu(s, X(s); \psi(s, X(s)), b_s)ds + \sigma(s, X(s); \psi(s, X(s)), b_s)dW_s \\ X(t) &= x. \end{cases}$$

Obviously, further conditions on ψ may be required to obtain indeed an admissible control (for example boundedness of ψ). In the case of A compact then no other conditions are required.

4.3.3 Terminal reward

We consider zero-sum games, meaning that the rewards of both players add to zero. The *terminal reward* of player 1 is the random variable

$$J(t, x; a, b) := \mathbb{E}\left[f\left(X_{t,x}^{a,b}(T)\right) \Big| \mathcal{F}_t \right],$$

where f is a *payoff function* such that the above definition makes sense, i.e., f is measurable and for all $a \in \mathcal{A}_t, b \in \mathcal{B}_t$:

$$\mathbb{E}\left[f\left(X_{t,x}^{a,b}(T)\right)\right] < +\infty.$$

For example we can take f to be a bounded measurable function. More generally we will assume that f has polynomial growth, i.e., there are p, C such that

$$|f(x)| \leq C(1 + |x|^p). \tag{4.8}$$

Then by (4.3) we conclude that J is well defined.

Due to the zero-sum condition, the reward of player 2 is $-J(t, x; a, b)$.

Remark 34. *For fixed t, x, a, b, $(J(s, x; a, b))_{s\in[t,T]}$ is a martingale. Hence, by the martingale representation Theorem (Theorem 140), there exists $\psi(., x; a, b)$ such that*

$$J(s, x; a, b) = J(t, x; a, b) + \int_t^s \psi(r, x; a, b)dW_r.$$

Furthermore, since by (4.3) and (4.8), $J(T, x; a, b) = f\left(X_{t,x}^{a,b}(T)\right)$ is square-integrable, then

$$\psi(., x; a, b) \in \mathbb{H}^2(t, T; \mathbb{R}^N)$$

is unique. Thus,

$$(J(s, x; a, b), \psi(s, x; a, b))_{s\in[t,T]}$$

is the unique solution of the BSDE

$$\begin{cases} dJ(s, x; a, b) &= \psi(s, x; a, b)dW_s \\ J(T, x; a, b) &= f\left(X_{t,x}^{a,b}(T)\right). \end{cases}$$

In [10] the reward function is seen as a solution of a similar BSDE.

By the definition of J it follows that $J(t, x; a, b)$ as function of a, b verifies a property of *independence of irrelevant alternatives*[2] in the following sense:

Lemma 35. *Let $\Lambda \in \mathcal{F}_t$ and $a_1, a_2 \in \mathcal{A}_t$, $b_1, b_2 \in \mathcal{B}_t$ such that*

$$(a_1, b_1) \equiv_\Lambda (a_2, b_2) \text{ on } [t, T].$$

Then

$$J(t, x; a_1, b_1) =_\Lambda J(t, x; a_2, b_2).$$

Proof. We have

$$(a_1, b_1) \equiv_\Lambda (a_2, b_2) \text{ on } [t, T] \Rightarrow X_{t,x}^{a_1,b_1} \equiv_\Lambda X_{t,x}^{a_2,b_2} \text{ on } [t, T],$$

and, by continuity of $X_{t,x}^{a_1,b_1}, X_{t,x}^{a_2,b_2}$, we conclude that

$$X_{t,x}^{a_1,b_1}(T) =_\Lambda X_{t,x}^{a_2,b_2}(T).$$

Thus, since $\Lambda \in \mathcal{F}_t$,

$$J(t, x; a_1, b_1) =_\Lambda J(t, x; a_2, b_2).$$

□

Remark 36. *We can allow a* running cost, *l, by considering an extra state variable $Y^{a,b}$ that satisfies*

$$\begin{cases} dY^{a,b}(s) &= l\left(s, X_{t,x}^{a,b}(s); a_s, b_s\right) ds \\ Y^{a,b}(t) &= 0, \end{cases}$$

and the payoff function $g(x, y) = f(x) + y$.

Then

$$g\left(X_{t,x}^{a,b}(T), Y^{a,b}(T)\right) = f\left(X_{t,x}^{a,b}(T)\right) + \int_t^T l\left(s, X_{t,x}^{a,b}(s); a_s, b_s\right) ds$$

is a terminal reward that has incorporated the running cost. Therefore the fact that we are dealing with terminal payoffs is not as restrictive as it may seem.

4.3.4 Strategies

Given a control $a \in \mathcal{A}_t$ for player 1, the second player needs to choose a control $b \in \mathcal{B}_t$. In general, however, the second player at each time t only knows a_s for $s \leq t$. The choice of controls of the second player must take into account his lack of information about the future. This is guaranteed by imposing a condition of non-anticipativity which is made precise by the next Definition.

Definition 37. *A* strategy *for player 2 is a function $\beta : \mathcal{A}_t \to \mathcal{B}_t$ that maps equivalent controls to equivalent controls.*

A strategy, β, is said to verify the non-anticipativity property *if for every $a_1, a_2 \in \mathcal{A}_t$, $\tau \in \mathcal{T}_{[t,T]}$ we have:*

$$a_1 \equiv a_2 \text{ on } [t, \tau] \Rightarrow \beta[a_1] \equiv \beta[a_2] \text{ on } [t, \tau].$$

The space of non-anticipative strategies for player 2 is denoted by $\Delta(t)$.

The definition of strategy for player 1 is analogous. The space of non-anticipative strategies for player 1 is denoted by $\Gamma(t)$.

[2] Independence of irrelevant alternatives (IIA) is a term for an axiom of decision theory.

This is the stochastic analogue to strategies in deterministic differential games. We allow player 2 to use all information he can access up to the present time without foreseeing the future. This information includes the strategy of player 1 and the state of the world.

Of course the definition would be useless if in the end $\Delta(t) = \emptyset$. Thus we give now some examples of strategies.

Example 38. *1. If β is the constant strategy, $\beta[a] = b \in \mathcal{B}_t$, then $\beta \in \Delta(t)$.*

2. If $\psi : A \to B$ is measurable and bounded then the strategy β defined by

$$\beta[a]_s(\omega) := \psi(a_s(\omega))$$

is in $\Delta(t)$.

In this example we can replace the condition of boundedness by linear growth or continuity.

3. Generalizing the previous example and in analogy with the Markov control policies we can consider $\beta \in \Delta(t)$ defined by

$$\beta[a]_s := \psi\left(s, X_{t,x}^{a,\beta[a]}(s); a_s\right),$$

where $\psi : \mathbb{S} \times A \to B$ is a measurable bounded function.

More precisely, in this case $X_{t,x}^{a,\beta[a]}$ is the solution to

$$\begin{cases} dX_s &= \mu(s, X_s; a_s, \psi(s, X_s; a_s))ds + \sigma(s, X_s; a_s, \psi(s, X_s; a_s))dW_s \\ X_t &= x. \end{cases}$$

These are the Markov control policies *for the player* 2 *when he is allowed to use strategies.*

For properties of strategies we refer the reader to the end of the Section.

4.3.5 Lower and upper values

Since the reward functions of both players are symmetric, then it should be clear that if players 1 and 2 are rational then they should be interested in maximizing and minimizing, respectively, the reward function of player 1. Thus we will take the supremum and infimum of the reward function J. Actually, because J is a random variable, we will need to use instead the essential versions of the supremum and infimum which we recall here.

Definition 39. *Given a family of indexed real-valued random variables, X^ν, $\nu \in \mathcal{U}$, a random variable X is said to be the* essential supremum *of X^ν with respect to $\nu \in \mathcal{U}$, $X = \mathrm{esssup}_{\nu\in\mathcal{U}} X^\nu$, if*

1. *$X \geq X^\nu$, $\mathbb{P} -$ a.s., for all $\nu \in \mathcal{U}$;*
2. *If there is another random variable $\tilde{X}$ such that $\tilde{X} \geq X^\nu$, $\mathbb{P} -$ a.s. for all $\nu \in \mathcal{U}$, then $\tilde{X} \geq X$, $\mathbb{P} -$ a.s..*

The essential infimum *of X^ν with respect to $\nu \in \mathcal{U}$, $X = \mathrm{essinf}_{\nu\in\mathcal{U}} X^\nu$ is defined as*

$$\operatorname*{essinf}_{\nu\in\mathcal{U}} X^\nu = -\operatorname*{esssup}_{\nu\in\mathcal{U}} (-X^\nu).$$

Since a probability space is $\sigma-$finite we have the following important property of essential extrema, [6, p. 71]:

Theorem 40. *Let $\{X^\nu\}_{\nu\in\mathcal{U}}$ be a collection of measurable random variables. Then the essential supremum, $\mathrm{esssup}_{\nu\in\mathcal{U}} X^\nu$, exists, is unique up to null sets and there is a suitable countable sequence $\{X_{\nu_n}\}$ such that*

$$\operatorname*{esssup}_{\nu} X^\nu = \sup_n X^{\nu_n}.$$

We can now define the upper and lower values of a stochastic differential game. In the lower value we allow player 2 to use strategies. This gives him an advantage over player 1. In the upper value we have the opposite situation. More precisely, we have the following:

Definition 41. *The* lower value *of a stochastic differential game is defined as*

$$V(t,x) = \operatorname*{essinf}_{\beta\in\Delta(t)} \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a]).$$

Similarly, the upper value *of a stochastic differential game is*

$$U(t,x) = \operatorname*{esssup}_{\alpha\in\Gamma(t)} \operatorname*{essinf}_{b\in\mathcal{B}_t} J(t,x;\alpha[b],b).$$

The name upper and lower values is justified by the following inequality, which is intuitive but needs to be proved,

$$V(t,x) \leq U(t,x).$$

This comparison is discussed in Corollary 96. If the lower and upper values are equal then we say that the game has a *value*.

Remark 42. *For these values to exist we need further assumptions. For example, any of the following guarantees this:*

- *f is bounded;*
- *A, B are compact sets. Indeed, by (4.6) and by the polynomial growth of f we conclude that if A, B are compact then $J(t,x;.)$ is uniformly bounded.*
- *The following holds:*

$$\exists_{a^*\in\mathcal{A}_t} \lim_{\|b\|_{\mathbb{H}^\infty}\to\infty} J(t,x;a^*,b) = +\infty,$$
$$\exists_{b^*\in\mathcal{B}_t} \lim_{\|a\|_{\mathbb{H}^\infty}\to\infty} J(t,x;a,b^*) = -\infty,$$

  where the limits are taken uniformly in ω.

  Indeed, if this is the case then there is M sufficiently large such that

$$V(t,x) \leq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,b^*) = \operatorname*{esssup}_{a\in\mathcal{A}_t;\|a\|_{\mathbb{H}^\infty}\leq M} J(t,x;a,b^*) < +\infty$$
$$V(t,x) \geq \operatorname*{essinf}_{\beta\in\Delta(t)} J(t,x;a^*,\beta[a^*]) \geq \operatorname*{essinf}_{\beta\in\Delta(t);\|\beta[a^*]\|_{\mathbb{H}^\infty}\leq M} J(t,x;a^*,\beta[a*]) > -\infty$$

- *The following holds:*

$$\exists_{a\in\mathcal{A}_t}\forall_{b\in\mathcal{B}_t}\, J(.;a,b) \text{ is bounded from below,}$$
$$\exists_{b\in\mathcal{B}_t}\forall_{a\in\mathcal{A}_t}\, J(.;a,b) \text{ is bounded from above.}$$

Notice that, *a priori*, for each (t,x), $V(t,x)$ and $U(t,x)$ are random variables. Still, because the dynamics is Markovian we see that if we fix the history up to time t then there is no point for both players to use controls which are not independent of $\mathcal{F}_t$ and for these controls we have that $J(t,x;a,b)$ is deterministic. Thus we should expect V, U to be deterministic in the sense that

$$V(t,x) = \mathbb{E}[V(t,x)],$$
$$U(t,x) = \mathbb{E}[U(t,x)].$$

Our objective is to prove a weak version of the dynamic programming principle for $V(t,x)$ which will allow us to prove that it is a viscosity solution of the HJBI equation. Using the same arguments, analogous results can be obtained for the upper value.

4.3.6 Properties of strategies

In this Section we study basic properties of strategies. These will be required when proving the dynamic programming principle. We restrict ourselves to strategies of player 2 but obviously analogous results hold for strategies of player 1. We start by considering the non-anticipativity property.

Proposition 43. *Consider a strategy β. Then $\beta \in \Delta(t)$ iff for every $a_1, a_2 \in \mathcal{A}_t$, $\tau \in \mathcal{T}_{[t,T]}$*

$$\beta[a_1 \oplus_\tau a_2] \equiv \beta[a_1] \oplus_\tau \beta[a_1 \oplus_\tau a_2]. \tag{4.9}$$

Proof. Consider $a_1, a_2 \in \mathcal{A}_t$, $\tau \in \mathcal{T}_{[t,T]}$. On one hand, if $\beta \in \Delta(t)$ then

$$a_1 \equiv a_1 \oplus_\tau a_2 \text{ on } [t, \tau] \Rightarrow \beta[a_1] \equiv \beta[a_1 \oplus_\tau a_2] \text{ on } [t, \tau],$$

hence we conclude that β satisfies (4.9).

On the other hand, suppose $a_1 \equiv a_2$ on $[t, \tau]$. Then $a_1 \oplus_\tau a_2 \equiv a_2$ so if β is a strategy we have that

$$\beta[a_1 \oplus_\tau a_2] \equiv \beta[a_2]. \tag{4.10}$$

Thus, if β additionally satisfies (4.9) we conclude that

$$\beta[a_1] \oplus_\tau \beta[a_1 \oplus_\tau a_2] \equiv \beta[a_1 \oplus_\tau a_2] \equiv \beta[a_2],$$

by (4.10). This implies that $\beta[a_1] \equiv \beta[a_2]$ on $[t, \tau]$, i.e., β is a non-anticipative strategy. □

Due to the arbitrariness of the stopping time in the definition of strategy we can deduce that a strategy verifies the, apparently stronger, property stated in the next Proposition.

Proposition 44. *Let $\beta \in \Delta(t)$, $a_1, a_2 \in \mathcal{A}_t$. Then for every sequence of stopping times $\tau_0 := t \le \tau_1 \le ... \le \tau_n$, and every sequence of events $\Lambda_1 \supseteq \Lambda_2 ... \supseteq \Lambda_n$, $\Lambda_i \in \mathcal{F}_{\tau_{i-1}}$, we have:*

$$\textit{if } \forall_i \ (a_1 \equiv_{\Lambda_i} a_2 \text{ on } [\tau_{i-1}, \tau_i]) \ \textit{ then } \forall_i \ (\beta[a_1] \equiv_{\Lambda_i} \beta[a_2] \text{ on } [\tau_{i-1}, \tau_i]) \,.$$

Proof. Consider a sequence of stopping times $\tau_0 := t \le \tau_1 \le ... \le \tau_n$ and a sequence of events $\Lambda_1 \supseteq \Lambda_2 ... \supseteq \Lambda_n$, $\Lambda_i \in \mathcal{F}_{\tau_{i-1}}$, and define

$$\tilde{\tau} := \sum_{i=0}^{n} \mathbf{1}_{\Lambda_i} \mathbf{1}_{\Lambda_{i+1}^c} \tau_i,$$

where $\Lambda_0 := \Omega, \Lambda_{n+1} := \emptyset$. Then $\tilde{\tau}$ is a stopping time because

$$\{\tilde{\tau} \le s\} = \bigcup_{i=0}^{n} \{\tau_i \le s\} \cap \Lambda_i \cap \Lambda_{i+1}^c,$$

and $\Lambda_i \cap \Lambda_{i+1}^c \in \mathcal{F}_{\tau_i} \Rightarrow \{\tau_i \le s\} \cap \Lambda_i \cap \Lambda_{i+1}^c \in \mathcal{F}_s$.

Now notice that

$$\forall_i \ (a_1 \equiv_{\Lambda_i} a_2 \text{ on } [\tau_{i-1}, \tau_i])$$

is equivalent to

$$\forall_i \ (a_1 \equiv_{\Lambda_i} a_2 \text{ on } [t, \tau_i]) \,.$$

The result then follows easily by the non-anticipativity of β. Indeed,

$$\begin{aligned} \forall_i \ (a_1 \equiv_{\Lambda_i} a_2 \text{ on } [t, \tau_i]) \ &\Rightarrow \ a_1 \equiv a_2 \text{ on } [t, \tilde{\tau}] \\ &\Rightarrow \ \beta[a_1] \equiv \beta[a_2] \text{ on } [t, \tilde{\tau}] \\ &\Rightarrow \ \forall_i \ (\beta[a_1] \equiv_{\Lambda_i} \beta[a_2] \text{ on } [\tau_{i-1}, \tau_i]) \,, \end{aligned}$$

where the first and third implications follow, respectively, from

$$\begin{aligned}\tilde{\tau}\mathbf{1}_{\Lambda_{i+1}^c} &\leq \tau_i\mathbf{1}_{\Lambda_{i+1}^c},\\ \tilde{\tau}\mathbf{1}_{\Lambda_i} &\geq \tau_i\mathbf{1}_{\Lambda_i}.\end{aligned}$$

□

The previous property tells us two important facts:

- What player 2 does up to time s depends only on what player 1 does up to time s;
- If at time s the control of player 1 is fragmented across Ω in sets of events in $\mathcal{F}_s$ then the reply of player 2 should be fragmented across the same sets as well.

In particular, a non-anticipative strategy verifies the property of *independence of irrelevant alternatives*:

Corollary 45. *Let $\beta \in \Delta(t)$. Then β is independent of irrelevant alternatives, that is, given a family of controls $a_i \in \mathcal{A}_t$ and a partition $\{\Lambda_i\} \subset \mathcal{F}_t$ of Ω such that $\sum_i \mathbf{1}_{\Lambda_i} a_i \in \mathcal{A}_t$ we have:*

$$\beta\left[\sum_i \mathbf{1}_{\Lambda_i} a_i\right] \equiv \sum_i \mathbf{1}_{\Lambda_i}\beta\left[a_i\right].$$

As a consequence of the independence of irrelevant alternatives we have the following interesting and useful property:

Proposition 46. *Consider a set $\mathcal{A} \subset \mathcal{A}_t$ bounded in the norm $\|.\|_{\mathbb{H}^\infty}$. Then, for each $p \geq 1$, $\beta[\mathcal{A}]$ is bounded in the norm $\|.\|_{\mathbb{H}^\infty_{t,p}}$.*

Similarly, if $\mathcal{A}$ is bounded in the norm $\|.\|_{\mathbb{H}^{p,\infty}}$ then $\beta[\mathcal{A}]$ is bounded in the norm $\|.\|_{\mathbb{H}^{p,\infty}_t}$.

Proof. We suppose, by contradiction, that $\beta[\mathcal{A}]$ is unbounded in the norm $\|.\|_{\mathbb{H}^\infty_{t,p}}$. Then there is a sequence $(a_i) \subset \mathcal{A}$ such that $\|\beta[a_i]\|_{\mathbb{H}^\infty_{t,p}}$ is unbounded. We suppose, without loss of generality, that $\|\beta[a_i]\|_{\mathbb{H}^\infty_{t,p}} \geq i$.

We define

$$\begin{aligned}\tilde{\Lambda}_i &:= \left\{\mathbb{E}\left[\sup_{s\in[t,T]} |\beta[a_i]_s|^p \Big| \mathcal{F}_t\right] \geq i\right\},\\ \tilde{\lambda}_i &:= \mathbb{P}(\tilde{\Lambda}_i).\end{aligned}$$

Since $\|\beta[a_i]\|_{\mathbb{H}^\infty_{t,p}} \geq i$ then $\tilde{\lambda}_i > 0$. We now define λ_i by

$$\begin{aligned}\lambda_1 &:= \tilde{\lambda}_1,\\ \lambda_{i+1} &:= \frac{\lambda_i \wedge \tilde{\lambda}_i}{3}.\end{aligned}$$

By Lemma 122, for each i, there is a $\mathcal{F}_t-$measurable set $\Lambda_i \subset \tilde{\Lambda}_i$ such that

$$\mathbb{P}(\Lambda_i) = \lambda_i.$$

We now remark that

$$\begin{aligned}\mathbb{P}\left(\bigcup_{k>i}\Lambda_k\right) &\leq \sum_{k>i}\lambda_k\\ &\leq \lambda_i\sum_{k>i}\frac{1}{3^{k-i}}\\ &= \frac{\lambda_i}{2}.\end{aligned}$$

Let $\Gamma_i := \Lambda_i \setminus \bigcup_{k>i} \Lambda_k$. Then $\{\Gamma_i\}$ is a family of disjoint sets and

$$\mathbb{P}(\Gamma_i) \geq \frac{\lambda_i}{2} > 0.$$

By Lemma 32 and because $\mathcal{A}$ is bounded, we have

$$a := a_1 \mathbf{1}_{\left(\bigcup_{i\geq 1} \Gamma_i\right)^c} + \sum_{i\geq 1} a_i \mathbf{1}_{\Gamma_i} \in \mathcal{A}_t.$$

Furthermore, by the independence of irrelevant alternatives, we have

$$\beta[a] = \beta[a_1]\mathbf{1}_{\left(\bigcup_{i\geq 1} \Gamma_i\right)^c} + \sum_{i\geq 1} \beta[a_i]\mathbf{1}_{\Gamma_i}.$$

Thus

$$\begin{aligned}
\|\beta[a]\|_{\mathbb{H}^\infty_{t,p}} &\geq \operatorname{esssup}\left(\sum_{i\geq 1}\mathbb{E}\left[\sup_{s\in[t,T]} |\beta[a_i]_s|^p \Big| \mathcal{F}_t\right]\mathbf{1}_{\Gamma_i}\right)\\
&\geq \operatorname{esssup}\left(\sum_{i\geq 1} i\mathbf{1}_{\Gamma_i}\right)\\
&= +\infty.
\end{aligned}$$

Since $\beta[a] \in \mathcal{B}_t$ and $\|\beta[a]\|_{\mathbb{H}^\infty} \geq \|\beta[a]\|_{\mathbb{H}^\infty_{t,p}}$ we have a contradiction.

The case when $\mathcal{A}$ is bounded in the norm $\|.\|_{\mathbb{H}^{p,\infty}}$ is analogous. □

The previous property tells us that a non-anticipating strategy is in some sense locally bounded with respect to the control of the other player.

Remark 47. *Using the previous Proposition we conclude that if A is compact, then $\beta[\mathcal{A}_t]$ is bounded with the norm $\|.\|_{\mathbb{H}^\infty_{t,p}}$. Thus, in this case, we can define*

$$\|\beta\|_p := \sup_{a\in\mathcal{A}_t} \|\beta[a]\|_{\mathbb{H}^\infty_{t,p}}. \tag{4.11}$$

It is immediate that, if $\Delta(t)$ is a vector space then (4.11) defines a norm and that, in general,

$$d_p(\beta_1, \beta_2) := \sup_{a\in\mathcal{A}_t} \|\beta_1[a] - \beta_2[a]\|_{\mathbb{H}^\infty_{t,p}}$$

defines a metric in $\Delta(t)$.

Notice that we can see a strategy as a controlled process. In this context, the property of non-anticipativity should say that the state of the controlled process in the present does not depend on the future of the control, which is a natural condition to require. Thus we make the next definition, where we denote by $\mathcal{U}$ a suitable control space. For our purposes, $\mathcal{U} = \mathcal{A}_t, \mathcal{B}_t$ or $\mathcal{A}_t \times \mathcal{B}_t$.

Definition 48. *A controlled process, $\mathcal{U} \ni \nu \mapsto X^\nu$, is said to verify the* non-anticipativity property *if for every $\nu_1, \nu_2 \in \mathcal{U}$, $\tau \in \mathcal{T}_{[t,T]}$ we have*

$$\nu_1 \equiv \nu_2 \text{ on } [t,\tau] \Rightarrow X^{\nu_1} \equiv X^{\nu_2} \text{ on } [t,\tau].$$

Example 49. *The solution to (4.1), $X^{a,b}_{t,x}$, is a non-anticipative controlled process.*

This is a consequence of the fact that $X^{a,b}_{t,x}(s) = \psi((W_r; a_r, b_r)_{t\leq r\leq s})$ for some function ψ.

The notion of non-anticipativity is preserved by algebraic operations, that is:

Proposition 50. *Let X^a, Y^a be non-anticipative controlled processes and $\lambda \in \mathbb{R}$. Then the following are non-anticipative controlled processes as well: λX^a, $X^a + Y^a$, $X^a Y^a$, X^a / Y^a.*

In the remainder of this section we will study natural ways of constructing strategies. We start by the next two Propositions, which are the analogue of Lemma 32 in the context of strategies.

Proposition 51. *Let $t \leq s \leq T$, $\beta_1 \in \Delta(t)$, $\beta_2 \in \Delta(s)$ and $\theta \in \mathcal{T}_{[s,T]}$. Then*

$$\beta_1 \oplus_\theta \beta_2 := \beta_1 \mathbf{1}_{[t,\theta]} + \beta_2 \mathbf{1}_{(\theta,T]} \in \Delta(t).$$

Proposition 52. *Let $\theta \in \mathcal{T}_{[t,T]}$ and $\beta_i \in \Delta(t)$ be such that $\forall_{a \in \mathcal{A}_t} \beta_i[a] \equiv \beta_j[a]$ on $[t, \theta]$. If $\{\Lambda_i\}_{i=1}^n \subset \mathcal{F}_\theta$ forms a partition of Ω then*

$$\sum_{i=1}^n \mathbf{1}_{\Lambda_i} \beta_i \in \Delta(t).$$

Because in strategies there is a dependence in a there is a natural extension to the previous constructions where we consider a stopping time dependent of a, θ^a. But to respect the flow of information the choice of θ^a must have some restrictions, which intuitively will impose that it must not look in the future of the controls. These are made in the following definition.

Definition 53. *A* non-anticipative controlled stopping time*, θ^ν, is a mapping*

$$\mathcal{U} \ni \nu \mapsto \theta^\nu \in \mathcal{T}_{[t,T]}$$

such that, for all $\nu_1, \nu_2 \in \mathcal{U}$, $\tau \in \mathcal{T}_{[t,T]}$, $f : \mathbb{R} \to \mathbb{R}$, we have

$$\nu_1 \equiv \nu_2 \text{ on } [t, \tau] \Rightarrow f(\theta^{\nu_1}) \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} = f(\theta^{\nu_2}) \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}.$$

Remark 54. *Notice that in the previous Definition*

$$f(\theta^{\nu_1}) \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} = f(\theta^{\nu_2}) \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}$$

is equivalent to

$$\begin{aligned} \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} &= \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}, \\ \theta^{\nu_1} \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} &= \theta^{\nu_2} \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}. \end{aligned}$$

Notice as well that we considered $\mathbf{1}_{\{\theta^{\nu_1} < \tau\}}$ instead of $\mathbf{1}_{\{\theta^{\nu_1} \leq \tau\}}$. This is important because, due to Remark 29, given a controlled stochastic process X^ν, the condition $X^{\nu_1} \equiv X^{\nu_2}$ on $[t, \tau]$ does not compare $X^{\nu_1}(\tau)$ with $X^{\nu_2}(\tau)$, unless X^{ν_1}, X^{ν_2} are know to be left-continuous. However we will need to consider controlled stopping times associated with right continuous processes, as in the next example.

Example 55. *Consider a controlled stochastic process, X^ν, and suppose that it is either right or left continuous. Let S be such that $X^\nu(t) \in S$. Then $\theta^\nu := \inf\{s > t : X^\nu(s) \notin S\}$ is a non-anticipative controlled stopping time.*

Indeed, if we consider $\nu_1, \nu_2 \in \mathcal{U}$, $\tau \in \mathcal{T}_{[t,T]}$ such that $\nu_1 \equiv \nu_2$ on $[t, \tau]$, then $X^{\nu_1} \equiv X^{\nu_2}$ on $[t, \tau]$, hence by the assumption of continuity and Remark 29, we conclude that

$$\mathbb{P}(\{X^{\nu_1}(s) = X^{\nu_2}(s); \forall s \in (t, \tau)\}) = 1.$$

Thus,

$$\begin{aligned} \mathbf{1}_{\{\theta^{\nu_1} \geq \tau\}} &= \mathbf{1}_{\{X^{\nu_1}(s) \in S,\ \forall s \in (t,\tau)\}} \\ &= \mathbf{1}_{\{X^{\nu_2}(s) \in S,\ \forall s \in (t,\tau)\}} \\ &= \mathbf{1}_{\{\theta^{\nu_2} \geq \tau\}}. \end{aligned}$$

Besides

$$\begin{aligned} \theta^{\nu_1} \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} &= \inf\{s : \tau > s > t,\ X^{\nu_1}(s) \notin S\} \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} \\ &= \inf\{s : \tau > s > t,\ X^{\nu_2}(s) \notin S\} \mathbf{1}_{\{\theta^{\nu_2} < \tau\}} \\ &= \theta^{\nu_2} \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}. \end{aligned}$$

Proposition 56. *Let θ_1^ν, θ_2^ν be non-anticipative controlled stopping times. Then $\theta_1^\nu \wedge \theta_2^\nu$, $\theta_1^\nu \vee \theta_2^\nu$ are non-anticipative controlled stopping times.*

Proof. Let $\nu_1, \nu_2 \in \mathcal{U}$ and $\tau \in \mathcal{T}_{[t,T]}$ be such that

$$\nu_1 \equiv \nu_2 \text{ on } [t, \tau].$$

Then

$$\begin{aligned} f(\theta_1^{\nu_1} \vee \theta_2^{\nu_1})\mathbf{1}_{\theta_1^{\nu_1} \vee \theta_2^{\nu_1} < \tau} &= f(\theta_1^{\nu_1} \vee \theta_2^{\nu_1})\mathbf{1}_{\theta_1^{\nu_1} < \tau}\mathbf{1}_{\theta_2^{\nu_1} < \tau} \\ &= f(\theta_1^{\nu_2} \vee \theta_2^{\nu_2})\mathbf{1}_{\theta_1^{\nu_2} < \tau}\mathbf{1}_{\theta_2^{\nu_2} < \tau} \\ &= f(\theta_1^{\nu_2} \vee \theta_2^{\nu_2})\mathbf{1}_{\theta_1^{\nu_2} \vee \theta_2^{\nu_2} < \tau}, \end{aligned}$$

and

$$\begin{aligned} f(\theta_1^{\nu_1} \wedge \theta_2^{\nu_1})\mathbf{1}_{\theta_1^{\nu_1} \wedge \theta_2^{\nu_1} < \tau} &= f(\theta_1^{\nu_1})\mathbf{1}_{\theta_1^{\nu_1} < \tau}\mathbf{1}_{\theta_2^{\nu_1} \geq \tau} + f(\theta_1^{\nu_1} \wedge \theta_2^{\nu_1})\mathbf{1}_{\theta_1^{\nu_1} < \tau}\mathbf{1}_{\theta_2^{\nu_1} < \tau} + f(\theta_2^{\nu_1})\mathbf{1}_{\theta_2^{\nu_1} < \tau}\mathbf{1}_{\theta_1^{\nu_1} \geq \tau} \\ &= f(\theta_1^{\nu_2})\mathbf{1}_{\theta_1^{\nu_2} < \tau}\mathbf{1}_{\theta_2^{\nu_2} \geq \tau} + f(\theta_1^{\nu_2} \wedge \theta_2^{\nu_2})\mathbf{1}_{\theta_1^{\nu_2} < \tau}\mathbf{1}_{\theta_2^{\nu_2} < \tau} + f(\theta_2^{\nu_2})\mathbf{1}_{\theta_2^{\nu_2} < \tau}\mathbf{1}_{\theta_1^{\nu_2} \geq \tau} \\ &= f(\theta_1^{\nu_2} \wedge \theta_2^{\nu_2})\mathbf{1}_{\theta_1^{\nu_2} \wedge \theta_2^{\nu_2} < \tau}. \end{aligned}$$

Thus, $\theta_1^\nu \wedge \theta_2^\nu$, $\theta_1^\nu \vee \theta_2^\nu$ are non-anticipative controlled stopping times. □

With the notion of non-anticipative controlled stopping time the extension of Proposition 51 is now natural.

Proposition 57. *Let $t \leq s \leq T$, $\beta_1 \in \Delta(t)$, $\beta_2 \in \Delta(s)$ and θ^a a non-anticipative controlled stopping time. Then β, defined by*

$$\beta[a] := \beta_1[a] \oplus_{\theta^a \vee s} \beta_2[a],$$

is in $\Delta(t)$.

Proof. For $a \in \mathcal{A}_t$, $\beta[a]$ is well defined because $\mathcal{A}_t \subset \mathcal{A}_s$ and hence $\beta_2[a]$ is well defined. Besides, by Lemma 32, we have that $\beta[a] \in \mathcal{B}_t$.

Now we need to verify the non-anticipativity property. For that, let $a_1, a_2 \in \mathcal{A}_t$, $\tau \in \mathcal{T}_{[t,T]}$ be such that

$$a_1 \equiv a_2 \text{ on } [t, \tau].$$

Then we have by the non-anticipativity of $\beta_1 \in \Delta(t)$ and $\beta_2 \in \Delta(s)$

$$\begin{cases} \beta_1[a_1] \equiv \beta_1[a_2] \text{ on } [t, \tau], \\ \beta_2[a_1] \equiv \beta_2[a_2] \text{ on } [s, s \vee \tau]. \end{cases} \tag{4.12}$$

Since θ^a is non-anticipative then so is $\tilde{\theta}^a := \theta^a \vee s$. Thus

$$\begin{cases} \mathbf{1}_{\{\tilde{\theta}^{a_1} < \tau\}}\mathbf{1}_{[t,\tilde{\theta}^{a_1}]} = \mathbf{1}_{\{\tilde{\theta}^{a_2} < \tau\}}\mathbf{1}_{[t,\tilde{\theta}^{a_2}]}, \\ \mathbf{1}_{\{\tilde{\theta}^{a_1} < \tau\}}\mathbf{1}_{(\tilde{\theta}^{a_1},T]} = \mathbf{1}_{\{\tilde{\theta}^{a_2} < \tau\}}\mathbf{1}_{(\tilde{\theta}^{a_2},T]}. \end{cases} \tag{4.13}$$

Furthermore,

$$\begin{aligned} \beta[a_1] &= \mathbf{1}_{[t,\tilde{\theta}^{a_1}]}\beta_1[a_1] + \mathbf{1}_{(\tilde{\theta}^{a_1},T]}\beta_2[a_1] \\ &= \mathbf{1}_{\{\tilde{\theta}^{a_1} \geq \tau\}}\left(\mathbf{1}_{[t,\tilde{\theta}^{a_1}]}\beta_1[a_1] + \mathbf{1}_{(\tilde{\theta}^{a_1},T]}\beta_2[a_1]\right) + \\ &\quad + \mathbf{1}_{\{\tilde{\theta}^{a_1} < \tau\}}\left(\mathbf{1}_{[t,\tilde{\theta}^{a_1}]}\beta_1[a_1] + \mathbf{1}_{(\tilde{\theta}^{a_1},T]}\beta_2[a_1]\right), \end{aligned} \tag{4.14}$$

and

$$\mathbf{1}_{\{\tilde{\theta}^{a_1} \geq \tau\}}\left(\mathbf{1}_{[t,\tilde{\theta}^{a_1}]}\beta_1[a_1] + \mathbf{1}_{(\tilde{\theta}^{a_1},T]}\beta_2[a_1]\right) \equiv \mathbf{1}_{\{\tilde{\theta}^{a_1} \geq \tau\}}\beta_1[a_1] \text{ on } [t, \tau]. \tag{4.15}$$

Combining (4.12), (4.13) and (4.15) in (4.14) we conclude that

$$\beta[a_1] \equiv \beta[a_2] \text{ on } [t, \tau].$$

□

To extend Proposition 52 we need an additional definition that has the following motivation. In many applications we are interested in observing certain quantities at a stopping time. More precisely, given a stopping time θ we are interested in considering a $\mathcal{F}_\theta$−measurable random variable, X^θ. In the case of a non-anticipative controlled stopping time, θ^ν, we are interested in considering a family of observations, X^{θ^ν}, which are also non-anticipative. Intuitively this should mean that if ν_1 and ν_2 are equal up to time s and $\theta^{\nu_1} < s$ then $\theta^{\nu_1} = \theta^{\nu_2}$ and the observations for these controls are equal, $X^{\theta^{\nu_1}} = X^{\theta^{\nu_2}}$. Rigorously we have:

Definition 58. *Given a non-anticipative controlled stopping time θ^ν, a family of random variables $\{X^\nu\}_{\nu \in \mathcal{U}}$ is called a* controlled observation *associated with θ^ν if $X^\nu \in \mathcal{F}_{\theta^\nu}$, and, for every $\nu_1, \nu_2 \in \mathcal{U}$, $\tau \in \mathcal{T}_{[t,T]}$ such that*

$$\nu_1 \equiv \nu_2 \text{ on } [t, \tau],$$

we have

$$X^{\nu_1} \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} \quad = \quad X^{\nu_2} \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}.$$

Example 59. *Let θ^ν be a non-anticipative controlled stopping time, $\nu \in \mathcal{U}$. The following are controlled observations associated with θ^ν:*

- *θ^ν;*
- *$\psi(X^\nu)$, where X^ν is any controlled observation associated with θ^ν and ψ is measurable;*
- *$X^\nu(\theta^\nu)$, where X^ν is a right-continuous non-anticipative controlled process. Indeed, if*

  $$\nu_1 \equiv \nu_2 \text{ on } [t, \tau],$$

  then we have, by the non-anticipativity of X^ν and by Remark 29, that

  $$\mathbb{P}(\{X^{\nu_1}(s) = X^{\nu_2}(s) : \forall s \in [t, \tau)\}) = 1.$$

  Thus

  $$X^{\nu_1}(\theta^{\nu_1}) \mathbf{1}_{\{\theta^{\nu_1} < \tau\}} = X^{\nu_2}(\theta^{\nu_2}) \mathbf{1}_{\{\theta^{\nu_2} < \tau\}}.$$

- *$X^\nu(\theta^\nu)$, where X^ν is a left-continuous non-anticipative controlled process such that $X^{\nu_1}(t) = X^{\nu_2}(t)$, for all $\nu_1, \nu_2 \in \mathcal{U}$. Like the previous example, this is also a consequence of Remark 29.*

We can now extend Proposition 52 in the following way:

Proposition 60. *Let θ^a be a non-anticipative controlled stopping time and $\{\beta_i\}_{i \geq 1} \subset \Delta(t)$ be such that $\forall_{a \in \mathcal{A}_t} \beta_i[a] \equiv \beta_j[a]$ on $[t, \theta^a]$. If for each $a \in \mathcal{A}_t$, $\{\Lambda_i^a\}_{i=1}^n \subset \mathcal{F}_{\theta^a}$ forms a partition of Ω, and for each $1 \leq i \leq n$, $\{\mathbf{1}_{\Lambda_i^a}\}_{a \in \mathcal{A}_t}$ is a controlled observation associated with θ^a, then β, defined by*

$$\beta[a] := \sum_{i=1}^{n} \mathbf{1}_{\Lambda_i^a} \beta_i[a],$$

is in $\Delta(t)$.

Proof. Given $a \in \mathcal{A}_t$, $\beta[a]$ is in $\mathcal{B}_t$ by Lemma 32.

We just need to prove the non-anticipativity property. As usual, we start by considering $a_1, a_2 \in \mathcal{A}_t$, $\tau \in \mathcal{T}_{[t,T]}$ such that

$$a_1 \equiv a_2 \text{ on } [t, \tau].$$

Then, for each i, we have

$$\beta_i[a_1] \equiv \beta_i[a_2] \text{ on } [t, \tau].$$

Thus, for any i,

$$\beta[a_1] \equiv_{\{\theta^{a_1} \geq \tau\}} \beta_i[a_1] \equiv_{\{\theta^{a_1} \geq \tau\}} \beta_i[a_2] \equiv_{\{\theta^{a_1} \geq \tau\} \cap \{\theta^{a_2} \geq \tau\}} \beta[a_2] \text{ on } [t, \tau].$$

Because θ^a is a non-anticipative controlled stopping time,

$$\mathbf{1}_{\{\theta^{a_1} \geq \tau\} \cap \{\theta^{a_2} \geq \tau\}} = \mathbf{1}_{\{\theta^{a_1} \geq \tau\}} = \mathbf{1}_{\{\theta^{a_2} \geq \tau\}}, \ \mathbb{P} - \text{a.s.}.$$

Thus we conclude that

$$\beta[a_1] \equiv_{\{\theta^{a_1} \geq \tau\}} \beta[a_2] \text{ on } [t, \tau]. \tag{4.16}$$

On the other hand, because $\mathbf{1}_{\Lambda_i^a}$ is a controlled observation,

$$\beta_i[a_1] \mathbf{1}_{\Lambda_i^{a_1}} \mathbf{1}_{\{\theta^{a_1} < \tau\}} \equiv \beta_i[a_2] \mathbf{1}_{\Lambda_i^{a_2}} \mathbf{1}_{\{\theta^{a_2} < \tau\}} \text{ on } [t, \tau],$$

hence, by definition of β, we get

$$\beta[a_1] \equiv_{\{\theta^{a_1} < \tau\}} \beta[a_2] \text{ on } [t, \tau]. \tag{4.17}$$

Combining (4.16) and (4.17) we conclude that

$$\beta[a_1] \equiv \beta[a_2] \text{ on } [t, \tau].$$

□

The next Proposition gives a natural construction for strategies in
$Delta(s)$ from strategies in $\Delta(t)$, for $s \geq t$.

Proposition 61. *For $t \leq s \leq T$ and $\theta \in \mathcal{T}_{[s,T]}$ let $\tilde{\beta} \in \Delta(t)$, $\tilde{a} \in \mathcal{A}_t$ be fixed. Define β by*

$$\beta[a] := \tilde{\beta}[\tilde{a} \oplus_\theta a],$$

for all $a \in \mathcal{A}_s$. Then $\beta \in \Delta(s)$.

Proof. β is well defined because if $a \in \mathcal{A}_s$ then, by Lemma 32, $\tilde{a} \oplus_\theta a \in \mathcal{A}_t$, and because $\mathcal{B}_t \subset \mathcal{B}_s$.

We need to prove the non-anticipativity property. As usual, we consider $a_1, a_2 \in \mathcal{A}_s$, $\tau \in \mathcal{T}_{[t,T]}$ such that

$$a_1 \equiv a_2 \text{ on } [s, \tau].$$

Then

$$\tilde{a} \oplus_\theta a_1 \quad \equiv \quad \tilde{a} \oplus_\theta a_2 \text{ on } [t, \tau].$$

From the non-anticipativity of $\tilde{\beta}$ we conclude that

$$\tilde{\beta}[\tilde{a} \oplus_\theta a_1] \quad \equiv \quad \tilde{\beta}[\tilde{a} \oplus_\theta a_2] \text{ on } [t, \tau].$$

Thus,

$$\beta[a_1] \quad \equiv \quad \beta[a_2] \text{ on } [s, \tau].$$

□

4.4 Properties of the value function

4.4.1 Non-randomness

Recall the definitions of lower and upper values:

$$V(t,x) = \operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\beta[a]),$$
$$U(t,x) = \operatorname*{esssup}_{\alpha \in \Gamma(t)} \operatorname*{essinf}_{b \in \mathcal{B}_t} J(t,x;\alpha[b],b).$$

In this Section we prove that these value functions, which are *a priori* random variables, are in fact deterministic. More precisely, we show that

$$\begin{cases} V(t,x) = \mathbb{E}[V(t,x)], \\ U(t,x) = \mathbb{E}[U(t,x)]. \end{cases} \tag{4.18}$$

This result was first proven in [10] and follows from the invariance of the value function with respect to the Girsanov transformation:

$$\rho_h \omega(.) = \omega(.) + \int_0^{\cdot} h(s)ds,$$

where $h \in \mathbb{L}^2([0,T];\mathbb{R}^N)$ is arbitrary.

We start by deducing the law of ρ_h.

Lemma 62. *Define*

$$Z_h := \exp\left(\int_0^T h(s)dW_s - \frac{1}{2}\int_0^T |h(s)|^2 ds\right).$$

Then the law of ρ_h is given by $\mathbb{P} \circ (\rho_h)^{-1} = Z_h.\mathbb{P}$. More precisely, for any random variable X, we have

$$\mathbb{E}\left[X \circ \rho_h\right] = \mathbb{E}[X Z_h].$$

Proof. Let $\tilde{W} := W \circ \rho_h$. Then $\tilde{W}_t = W_t + \int_0^t h(s)ds$, hence by Girsanov's Theorem (Theorem 141), $\tilde{W}$ is a $\mathbb{Q}-$Brownian motion, where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T h(s)dW_s - \frac{1}{2}\int_0^T |h(s)|^2 ds\right).$$

Thus

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{W \circ \rho_h \in \Lambda\}}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\mathbf{1}_{\{\tilde{W} \in \Lambda\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T h(s)dW_s + \frac{1}{2}\int_0^T |h(s)|^2 ds\right)\mathbf{1}_{\{\tilde{W} \in \Lambda\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T h(s)d\tilde{W}_s - \frac{1}{2}\int_0^T |h(s)|^2 ds\right)\mathbf{1}_{\{\tilde{W} \in \Lambda\}}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^T h(s)dW_s - \frac{1}{2}\int_0^T |h(s)|^2 ds\right)\mathbf{1}_{\{W \in \Lambda\}}\right] \\
&= \mathbb{E}^{\mathbb{P}}[Z_h \mathbf{1}_{\{W \in \Lambda\}}].
\end{aligned}$$

□

As a consequence of the previous Lemma we have the following Corollary:

Corollary 63. *Let X be a random variable. Then*

$$\mathbb{E}[X \circ \rho_h|\mathcal{F}_t] = \mathbb{E}[Z_{-h}|\mathcal{F}_t](\mathbb{E}[XZ_h|\mathcal{F}_t] \circ \rho_h). \tag{4.19}$$

Furthermore, if Z_h is $\mathcal{F}_t-$measurable then

$$\mathbb{E}[X \circ \rho_h|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] \circ \rho_h. \tag{4.20}$$

Proof. We just need to check that for any $\Lambda \in \mathcal{F}_t$

$$\mathbb{E}[X \circ \rho_h \mathbf{1}_\Lambda] = \mathbb{E}\left[\mathbb{E}[Z_{-h}|\mathcal{F}_t](\mathbb{E}\left[XZ_h|\mathcal{F}_t\right] \circ \rho_h)\mathbf{1}_\Lambda\right].$$

This is just a simple computation:

$$\begin{aligned}
\mathbb{E}[X \circ \rho_h \mathbf{1}_\Lambda] &= \mathbb{E}[\big(X\mathbf{1}_{\rho_h(\Lambda)}\big) \circ \rho_h] \\
&= \mathbb{E}[Z_h X \mathbf{1}_{\rho_h(\Lambda)}] \\
&= \mathbb{E}[\mathbb{E}[Z_h X|\mathcal{F}_t]\mathbf{1}_{\rho_h(\Lambda)}] \\
&= \mathbb{E}[(\mathbb{E}[Z_h X|\mathcal{F}_t] \circ \rho_h \mathbf{1}_\Lambda) \circ \rho_{-h}] \\
&= \mathbb{E}[Z_{-h}(\mathbb{E}[Z_h X|\mathcal{F}_t] \circ \rho_h)\mathbf{1}_\Lambda] \\
&= \mathbb{E}[\mathbb{E}[Z_{-h}|\mathcal{F}_t](\mathbb{E}[Z_h X|\mathcal{F}_t] \circ \rho_h)\mathbf{1}_\Lambda].
\end{aligned}$$

To obtain (4.20) we first notice that

$$\begin{aligned}
Z_h \circ \rho_h &= \exp\left(\int_0^T h(s)d\tilde{W}_s - \frac{1}{2}\int_0^T |h(s)|^2 ds\right) \\
&= \exp\left(\int_0^T h(s)dW_s + \frac{1}{2}\int_0^T |h(s)|^2 ds\right) \\
&= (Z_{-h})^{-1}.
\end{aligned}$$

Thus, if Z_h is $\mathcal{F}_t$ measurable,

$$\begin{aligned}
\mathbb{E}[X \circ \rho_h|\mathcal{F}_t] &= \mathbb{E}[Z_{-h}|\mathcal{F}_t](\mathbb{E}[XZ_h|\mathcal{F}_t] \circ \rho_h) \\
&= Z_{-h}(Z_h \circ \rho_h)(\mathbb{E}[X|\mathcal{F}_t] \circ \rho_h) \\
&= \mathbb{E}[X|\mathcal{F}_t] \circ \rho_h.
\end{aligned}$$

□

We can now proceed to the proof of the main result of this section:

Theorem 64. *The random variables $V(t,x)$ and $U(t,x)$ are constant, i.e.,*

$$V(t,x) = \mathbb{E}[V(t,x)],$$
$$U(t,x) = \mathbb{E}[U(t,x)].$$

Proof. The result for the lower value is proved in detail and is a consequence of the next two Propositions. The proof for the upper value is analogous. □

Proposition 65. *$V(t,x)$ is invariant by ρ_h, that is*

$$V(t,x) \circ \rho_h = V(t,x).$$

Proof. We consider a fixed $(t,x) \in \mathbb{S}$ and for each h we define $h^t := \mathbf{1}_{[0,t]}h$. The proof is now divided into several steps.

Step 1: $J(t,x;a,b)(\rho_h) = J(t,x;a(\rho_{h^t}), b(\rho_{h^t}))$.

First we notice that

$$\left(\int_t^s \psi_r dW_r\right)(\rho_{h^t}) = \int_t^s \psi_r(\rho_{h^t}) dW_r,$$

because $h^t(r) = 0$, for $r \in [t,T]$, and so $dW_r(\rho_{h^t}) = dW_r$ for all $r \in [t,T]$.

Thus,

$$\begin{aligned} X_{t,x}^{a,b}(s)(\rho_{h^t}) &= x + \left(\int_t^s \mu\left(r, X_{t,x}^{a,b}(r); a_r, b_r\right) dr\right)(\rho_{h^t}) + \left(\int_t^s \sigma\left(r, X_{t,x}^{a,b}(r); a_r, b_r\right) dW_r\right)(\rho_{h^t}) \\ &= x + \int_t^s \mu\left(r, X_{t,x}^{a,b}(r)(\rho_{h^t}); a_r(\rho_{h^t}), b_r(\rho_{h^t})\right) dr + \\ &\quad + \int_t^s \sigma\left(r, X_{t,x}^{a,b}(r)(\rho_{h^t}); a_r(\rho_{h^t}), b_r(\rho_{h^t})\right) dW_r, \end{aligned}$$

which implies that $X_{t,x}^{a,b}(\rho_{h^t})$ is a solution of (4.1) with a, b replaced by $a(\rho_{h^t}), b(\rho_{h^t})$. By uniqueness of solution, we conclude that

$$X_{t,x}^{a,b}(\rho_{h^t}) = X_{t,x}^{a(\rho_{h^t}), b(\rho_{h^t})}.$$

Since Z_{h^t} is $\mathcal{F}_t$−measurable, we can use (4.20) to get

$$\begin{aligned} J(t,x;a(\rho_{h^t}), b(\rho_{h^t})) &= \mathbb{E}\left[f\left(X_{t,x}^{a(\rho_{h^t}),b(\rho_{h^t})}(T)\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f\left(X_{t,x}^{a,b}(T)\right) \circ \rho_{h^t}\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f\left(X_{t,x}^{a,b}(T)\right)\Big|\mathcal{F}_t\right] \circ \rho_{h^t} \\ &= J(t,x;a,b)(\rho_{h^t}). \end{aligned}$$

Furthermore, since $J(t,x;a,b)$ is $\mathcal{F}_t$−measurable it will only depend on ω through $\omega_{|[0,t]}$, thus

$$J(t,x;a,b)(\rho_{h^t}) = J(t,x;a,b)(\rho_h).$$

Step 2: Given $\beta \in \Delta(t)$, we define β^h by $\beta^h[a] := \beta[a(\rho_{-h})](\rho_h)$. Then we claim that $\beta^h \in \Delta(t)$.

Note that β^h is well defined and $\beta^h[a] \in \mathcal{B}_t$. We just need to verify the non-anticipativity property. If $a_1, a_2 \in \mathcal{A}_t$ and $\tau \in \mathcal{T}_{[t,T]}$ are such that $a_1 \equiv a_2$ on $[t,\tau]$, then we have

$$a_1(\rho_{-h}) \equiv a_2(\rho_{-h}) \text{ on } [t, \tau(\rho_{-h})].$$

Since $\tau(\rho_{-h})$ is a stopping time we get by the non-anticipativity of β that

$$\beta[a_1(\rho_{-h})] \equiv \beta[a_2(\rho_{-h})] \text{ on } [t, \tau(\rho_{-h})],$$

hence

$$\beta[a_1(\rho_{-h})](\rho_h) \equiv \beta[a_2(\rho_{-h})](\rho_h) \text{ on } [t,\tau].$$

Step 3: $\left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a])\right)(\rho_h) = \operatorname*{esssup}_{a\in\mathcal{A}_t}\big(J(t,x;a,\beta[a])(\rho_h)\big)$.

We have, for all $a \in \mathcal{A}_t$,

$$\operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \geq J(t, x; a, \beta[a]),$$

hence

$$\left(\operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \right) (\rho_h) \geq J(t, x; a, \beta[a])(\rho_h),$$

and since a is arbitrary we conclude that

$$\left(\operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \right) (\rho_h) \geq \operatorname*{esssup}_{a \in \mathcal{A}_t} \big(J(t, x; a, \beta[a])(\rho_h) \big).$$

Similarly, we deduce that

$$\left(\operatorname*{esssup}_{a \in \mathcal{A}_t} \big(J(t, x; a, \beta[a])(\rho_h) \big) \right) (\rho_{-h}) \geq \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]),$$

that is,

$$\operatorname*{esssup}_{a \in \mathcal{A}_t} \big(J(t, x; a, \beta[a])(\rho_h) \big) \geq \left(\operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \right) (\rho_h).$$

Thus

$$\left(\operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \right) (\rho_h) = \operatorname*{esssup}_{a \in \mathcal{A}_t} \big(J(t, x; a, \beta[a])(\rho_h) \big).$$

Note that repeating the same argument yields

$$\left(\operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \right) (\rho_h) = \operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} \big(J(t, x; a, \beta[a])(\rho_h) \big).$$

Step 4: $V(t, x)(\rho_h) = V(t, x)$.

Using the previous properties we compute:

$$\begin{aligned}
V(t, x)(\rho_h) &= \left(\operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \right) (\rho_h) \\
&= \operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} \ (J(t, x; a, \beta[a])(\rho_h)) \\
&= \operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} J \left(t, x; a(\rho_{h^t}), \beta^{h^t}[a(\rho_{h^t})] \right) \\
&= \operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]) \\
&= V(t, x).
\end{aligned}$$

□

Proposition 66. *Let X be a random variable such that $X(\rho_h) = X$, for any $h \in \mathbb{L}^2([0, T]; \mathbb{R}^N)$. Then $X = \mathbb{E}[X]$.*

Proof. For any Borel set $O \subset \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\{X \in O\}}] &= \mathbb{E}\left[\mathbf{1}_{\rho_h \{X \in O\}} \circ \rho_h\right] \\
&= \mathbb{E}\left[Z_h \mathbf{1}_{\rho_h \{X \in O\}}\right] \\
&= \mathbb{E}\left[Z_h \mathbf{1}_{\{X(\rho_{-h}) \in O\}}\right] \\
&= \mathbb{E}\left[Z_h \mathbf{1}_{\{X \in O\}}\right].
\end{aligned}$$

Thus, by definition of Z_h, we deduce that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^T h(s)dW_s\right)\mathbf{1}_{X\in O}\right] &= \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |h(s)|^2 ds\right)\right]\mathbb{E}[\mathbf{1}_{X\in O}] \\ &= \mathbb{E}\left[\exp\left(\int_0^T h(s)dW_s\right)\right]\mathbb{E}[\mathbf{1}_{X\in O}],\end{aligned}$$

where the second equality follows from $\mathbb{E}[Z_h] = 1$.

Since O and h are arbitrary we conclude that $X \perp\!\!\!\perp W$ and hence $X \perp\!\!\!\perp \mathcal{F}_T$, which is only possible if X is constant. □

By the previous result and from this point onwards, $V(t,x)$ can denote both a random variable or a function of (t,x), without any possible ambiguity. Obviously the same applies to the upper value function, U.

The previous result reveals a connection between our value functions and the ones defined by Fleming and Souganidis in their original article [2]. This connection is explored in the next Proposition.

Proposition 67. *Suppose that the essential infimum and essential supremum in the definition of V are achieved in an uniform way. More precisely assume that:*

$$\forall_{\varepsilon>0}\,\exists_{\beta^\varepsilon\in\Delta(t)}\;\; V(t,x) \geq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^\varepsilon[a]) - \varepsilon,$$
$$\forall_{\beta\in\Delta(t)}\,\forall_{\varepsilon>0}\,\exists_{a^{\beta,\varepsilon}\in\mathcal{A}_t}\;\; \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a]) \leq J(t,x;a^{\beta,\varepsilon},\beta\left[a^{\beta,\varepsilon}\right]) + \varepsilon.$$

Then

$$V(t,x) = \inf_{\beta\in\Delta(t)}\sup_{a\in\mathcal{A}_t}\mathbb{E}\left[f\left(X_{t,x}^{a,\beta[a]}(T)\right)\right]. \tag{4.21}$$

Proof. On one hand we have

$$\begin{aligned}V(t,x) &= \mathbb{E}[V(t,x)] \\ &= \mathbb{E}\left[\operatorname*{essinf}_{\beta\in\Delta(t)}\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a])\right] \\ &\leq \inf_{\beta\in\Delta(t)}\mathbb{E}\left[\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a])\right] \\ &\leq \inf_{\beta\in\Delta(t)}\mathbb{E}\left[J\left(t,x;a^{\beta,\varepsilon},\beta\left[a^{\beta,\varepsilon}\right]\right)\right] + \varepsilon \\ &\leq \inf_{\beta\in\Delta(t)}\sup_{a\in\mathcal{A}_t}\mathbb{E}\left[J(t,x;a,\beta[a])\right] + \varepsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}V(t,x) &= \mathbb{E}[V(t,x)] \\ &= \mathbb{E}\left[\operatorname*{essinf}_{\beta\in\Delta(t)}\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a])\right] \\ &\geq \mathbb{E}\left[\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^\varepsilon[a])\right] - \varepsilon \\ &\geq \sup_{a\in\mathcal{A}_t}\mathbb{E}\left[J(t,x;a,\beta^\varepsilon[a])\right] - \varepsilon \\ &\geq \inf_{\beta\in\Delta(t)}\sup_{a\in\mathcal{A}_t}\mathbb{E}\left[J(t,x;a,\beta[a])\right] - \varepsilon.\end{aligned}$$

Thus,

$$\inf_{\beta\in\Delta(t)}\sup_{a\in\mathcal{A}_t}\mathbb{E}\left[J(t,x;a,\beta[a])\right] - \varepsilon \leq V(t,x) \leq \inf_{\beta\in\Delta(t)}\sup_{a\in\mathcal{A}_t}\mathbb{E}\left[J(t,x;a,\beta[a])\right] + \varepsilon,$$

and, since ε is arbitrary, we conclude that

$$V(t,x) = \inf_{\beta\in\Delta(t)} \sup_{a\in\mathcal{A}_t} \mathbb{E}\left[f\left(X_{t,x}^{a,\beta[a]}(T)\right)\right].$$

□

The difference to [2] is that in (4.21) we allow controls and strategies which are dependent on the past, that is, we allow controls and strategies which are not independent of $\mathcal{F}_t$.

On the previous Proposition we assume the existence of strategies that achieve the infimum in the definition of V in an uniform way. On the next Proposition we give sufficient conditions for such strategies to exist.

Proposition 68. *Let $\Lambda \in \mathcal{F}_t$, with $\mathbb{P}(\Lambda) > 0$, and $\beta \in \Delta(t)$ be such that*

$$\mathbf{1}_\Lambda V(t,x) \geq \mathbf{1}_\Lambda \left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a]) - \varepsilon\right). \tag{4.22}$$

Suppose that one of the following holds:

(i) For each $h \in \mathbb{L}^2$, $\beta[a\circ\rho_{h^t}] = \beta[a]\circ\rho_{h^t}$;

(ii) $\sup_{a\in\mathcal{A}_t} \|\beta[a]\|_{\mathbb{H}^\infty} < \infty$.

Then there exists $\beta^\varepsilon \in \Delta(t)$ such that

$$V(t,x) \geq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^\varepsilon[a]) - 2\varepsilon.$$

Proof. Suppose that (i) holds. Then

$$\begin{aligned}
\left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a])\right)\circ\rho_h &= \operatorname*{esssup}_{a\in\mathcal{A}_t} \left(J(t,x;a,\beta[a])\circ\rho_h\right)\\
&= \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a\circ\rho_{h^t},\beta[a]\circ\rho_{h^t})\\
&= \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a\circ\rho_{h^t},\beta[a\circ\rho_{h^t}])\\
&= \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a]).
\end{aligned}$$

Thus, by Proposition 66, $\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a])$ is a constant random variable. Since $V(t,x)$ is also constant we deduce from (4.22) that

$$V(t,x) \geq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a]) - \varepsilon.$$

Thus we can take $\beta^\varepsilon := \beta$.

Now suppose that (ii) holds. From (4.22) it follows that

$$\begin{aligned}
\mathbf{1}_{\rho_{-h}(\Lambda)} V(t,x) &= (\mathbf{1}_\Lambda V(t,x))\circ\rho_h\\
&\geq \left(\mathbf{1}_\Lambda\left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta[a]) - \varepsilon\right)\right)\circ\rho_h\\
&= \mathbf{1}_{\rho_{-h}(\Lambda)}\left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a\circ\rho_{h^t},\beta[a]\circ\rho_{h^t}) - \varepsilon\right)\\
&= \mathbf{1}_{\rho_{-h}(\Lambda)}\left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J\left(t,x;a\circ\rho_{h^t},\beta^{h^t}[a\circ\rho_{h^t}]\right) - \varepsilon\right)\\
&= \mathbf{1}_{\rho_{-h}(\Lambda)}\left(\operatorname*{esssup}_{a\in\mathcal{A}_t} J\left(t,x;a,\beta^{h^t}[a]\right) - \varepsilon\right)\\
&\geq \mathbf{1}_{\rho_{-h}(\Lambda)}\left(\operatorname*{essinf}_{\tilde\beta\in\tilde{\mathcal{B}}} \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\tilde\beta[a]) - \varepsilon\right),
\end{aligned}$$

where

$$\tilde{\mathcal{B}} := \left\{ \beta^{h^t} : h \in \mathbb{L}^2 \right\}.$$

Since $h \in \mathbb{L}^2$ is arbitrary we deduce that

$$\mathbf{1}_{\tilde{\Lambda}} V(t,x) \geq \mathbf{1}_{\tilde{\Lambda}} \left(\operatorname*{essinf}_{\tilde{\beta} \in \tilde{\mathcal{B}}} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}[a]) - \varepsilon \right), \tag{4.23}$$

where

$$\tilde{\Lambda} = \bigcup_{h \in \mathbb{L}^2} \rho_{-h}(\Lambda).$$

Now we remark that

$$\mathbf{1}_{\tilde{\Lambda}} \circ \rho_h = \mathbf{1}_{\rho_{-h}(\tilde{\Lambda})} = \mathbf{1}_{\tilde{\Lambda}}.$$

Thus, by Proposition 66,

$$\mathbf{1}_{\tilde{\Lambda}} = \mathbb{E}\left[\mathbf{1}_{\tilde{\Lambda}}\right] = \mathbb{P}(\tilde{\Lambda}) > 0,$$

which implies that $\mathbf{1}_{\tilde{\Lambda}} = 1$. From (4.23) we then deduce that

$$V(t,x) \geq \operatorname*{essinf}_{\tilde{\beta} \in \tilde{\mathcal{B}}} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}[a]) - \varepsilon,$$

By Theorem 40, there is a sequence $\{\tilde{\beta}_i := \beta^{h_i^t}\} \subset \tilde{\mathcal{B}}$ such that

$$\operatorname*{essinf}_{\tilde{\beta} \in \tilde{\mathcal{B}}} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}[a]) = \inf_i \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}_i[a]).$$

Let

$$\tilde{\Lambda}_i := \left\{ \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}_i[a]) \leq \operatorname*{essinf}_{\tilde{\beta} \in \tilde{\mathcal{B}}} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}[a]) + \varepsilon \right\} \in \mathcal{F}_t,$$

and define $\Lambda_1 := \tilde{\Lambda}_1$, $\Lambda_{i+1} := \tilde{\Lambda}_{i+1} \setminus \bigcup_{k=1}^{i} \tilde{\Lambda}_k$. Then $\{\Lambda_i\} \subset \mathcal{F}_t$ forms a partition, modulo null sets, of Ω.

We now remark that, for each $a \in \mathcal{A}_t$,

$$\|\tilde{\beta}_i[a]\|_{\mathbb{H}^\infty} = \|\beta^{h_i^t}[a]\|_{\mathbb{H}^\infty} = \|\beta[a \circ \rho_{-h_i^t}] \circ \rho_{h_i^t}\|_{\mathbb{H}^\infty} = \|\beta[a \circ \rho_{-h_i^t}]\|_{\mathbb{H}^\infty} \leq \sup_{a \in \mathcal{A}_t} \|\beta[a]\|_{\mathbb{H}^\infty},$$

and hence

$$\sup_i \|\tilde{\beta}_i[a]\|_{\mathbb{H}^\infty} < \infty.$$

Thus, we can apply Lemma 32 to conclude that β^ε, defined by

$$\beta^\varepsilon[a] := \sum_i \mathbf{1}_{\Lambda_i} \tilde{\beta}_i[a],$$

is a well defined strategy. Furthermore,

$$\begin{aligned}
\operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\beta^\varepsilon[a]) &= \sum_i \mathbf{1}_{\Lambda_i} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}_i[a]) \\
&\leq \operatorname*{essinf}_{\tilde{\beta} \in \tilde{\mathcal{B}}} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t,x;a,\tilde{\beta}[a]) + \varepsilon \\
&\leq V(t,x) + 2\varepsilon.
\end{aligned}$$

□

Remark 69. *Notice that, on the previous Proposition, if β is a Markov control policy as in Example 38, then condition (i) is satisfied.*

4.4.2 Growth rate

In this section we prove that the value function has polynomial growth. This will be important in the discussion of uniqueness of solution of the HJBI equation.

Recall Remark 47, where $\|\beta\|_p$ is defined by (4.11) when A is compact.

Proposition 70. *Let A be compact and suppose that for each (t,x) and for each ε there exists $\beta^{\varepsilon}_{(t,x)} \in \Delta(t)$ such that*

$$V(t,x) \geq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^{\varepsilon}_{(t,x)}[a]) - \varepsilon, \ \ \mathbb{P}-\text{a.s.},$$

and

$$\|\beta^{\varepsilon}_{(t,x)}\|_p \leq C(1+|x|^m), \tag{4.24}$$

for some C, m. Here p is the growth power of f, as in (4.8).

Then V has polynomial growth, i.e.,

$$|V(t,x)| \leq C(1+|x|^p+|x|^{pm}), \tag{4.25}$$

for some constant C.

Proof. Consider the collection $\beta^{\varepsilon}_{(t,x)}$ in the hypothesis of the Proposition. Then

$$V(t,x) \geq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^{\varepsilon}_{(t,x)}[a]) - \varepsilon, \ \ \mathbb{P}-\text{a.s.}.$$

Now we estimate J:

$$\begin{aligned}
|J(t,x;a,\beta^{\varepsilon}_{(t,x)}[a])| &\leq \mathbb{E}\left[\left|f\left(X^{a,\beta^{\varepsilon}_{(t,x)}[a]}_{t,x}(T)\right)\right|\Big|\mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[C\left(1+\left|X^{a,\beta^{\varepsilon}_{(t,x)}[a]}_{t,x}(T)\right|^p\right)\Big|\mathcal{F}_t\right] \\
&\leq C\left(1+|x|^p+\|\beta^{\varepsilon}_{(t,x)}[a]\|^p_{\mathbb{H}^{p,\infty}_t}\right) \\
&\leq C\left(1+|x|^p+(T-t)\|\beta^{\varepsilon}_{(t,x)}[a]\|^p_{\mathbb{H}^{\infty}_{t,p}}\right) \\
&\leq C\left(1+|x|^p+(T-t)|x|^{pm}\right),
\end{aligned}$$

where the second inequality follows by (4.8), the third inequality follows by (4.4) and the fact that A is compact, in the fourth inequality we used inequality (ii) of Lemma 22, and the last inequality follows from (4.24).

Thus (4.25) holds, for some constant C. □

Remark 71. *If B is compact then we may use $m=0$ on the previous Proposition, thus getting that V has at most the same growth as f.*

4.4.3 Continuity in the space variable

In this Section we show that if f is globally Lipschitz then the value function is also Lipschitz in the space variable.

Proposition 72. *Suppose f is Lipschitz. Then*

$$|V(t,x)-V(t,x')| \leq K|x-x'|$$

Proof. We start with the following estimate:

$$\begin{aligned}
(J(t,x;a,b)-J(t,x';a,b))^2 &\leq \mathbb{E}\left[\left(f\left(X^{a,b}_{t,x}(T)\right)-f\left(X^{a,b}_{t,x'}(T)\right)\right)^2 \Big| \mathcal{F}_t\right] \\
&\leq K^2\mathbb{E}\left[\left|X^{a,b}_{t,x}(T)-X^{a,b}_{t,x'}(T)\right|^2 \Big| \mathcal{F}_t\right] \\
&\leq CK^2|x-x'|^2,
\end{aligned}$$

where the last inequality follows from Lemma 148.

The Lipschitz continuity of V now follows easily from the relations

$$\operatorname*{essinf}_{a,\beta}(J-J') \leq \operatorname*{essinf}_{\beta}\ \operatorname*{esssup}_{a} J - \operatorname*{essinf}_{\beta}\ \operatorname*{esssup}_{a} J' \leq \operatorname*{esssup}_{a,\beta}(J-J'),$$

and the fact that C does not depend on a, b. □

If A, B are compact and f is locally Lipschitz we get that for all t, $V(t, .)$ is continuous. The proof of this fact uses an estimate for $|J(t,x;a,b)-J(t,x';a,b)|$ that is analogous to the one used in the proof of Proposition 85. Thus we omit the proof of this fact and, instead, we refer the reader to Remark 87.

4.5 Weak dynamic programming principle

This Section is the core of this Chapter. We prove a weak version of the dynamic programming principle (DPP) that will be used later to derive the HJBI equation. The original version of the weak DPP is due to Touzi and Bouchard and was proved in [1] in the context of stochastic control.

This weak version of the DPP avoids measurability problems by considering weaker inequalities involving test functions for the value function. This is enough, however, to prove that the value function is a viscosity solution of the HJBI equation. Indeed, when working in the viscosity sense, the value function is replaced by a smooth test function. Therefore it is enough to prove a dynamic programming principle that gives information on these functions.

There are two main differences between the weak and the traditional versions of the dynamic programming principle. One, already mentioned, is that in the weak version we use a test function more regular than the value function. The other is that in this version we will consider an intermediate time that is a stopping time, $\theta \in \mathcal{T}_{[t,T]}$, instead of a deterministic time, $t+\delta$. Considering stopping times will be useful in the proof of the HJBI equation because there we will be interested in bounding the state process to use the martingale properties of the stochastic integral.

There will be two main assumptions in the proof of the weak DPP. One has to do with the continuity of the reward function while the other is technical. We will try to motivate both before stating and proving the Theorem.

When proving the traditional DPP we follow an optimal strategy up to a fixed time $t+\delta$ and then we concatenate it with a strategy which is optimal from that time onwards. In the stochastic setting at time $t+\delta$ the state process will end up randomly in an uncountable number of positions. We can not consider an uncountable number of strategies to patch while preserving for instance measurability. Therefore we must make a small error by choosing only one fixed strategy for each given neighborhood, instead of for each point. To control the error made by choosing a single strategy for an entire neighborhood we must have some continuity in the space variable of the reward function, that is, require continuity of $J(t+\delta, x; a, b)$ in x.

In this version of the DPP we use stopping times, hence, we will switch strategies at a random time and, again, we will have to make an error. To control this error in time we will need continuity of the reward function in the time variable. Thus we must show that $J(t, x; a, b)$ is continuous in (t, x). Furthermore, because we have 2 players and 2 controls, we do not want to allow one player to ruin the other player's patching of strategies by augmenting too much the error of the patching. Thus, we will have to require continuity of $J(.; a, \beta[a])$ uniformly in a.

One question that arises naturally in this context is how we should interpret the continuity of $J(.; a, b)$. Indeed, J is a random variable, and even worse, $J(t, .; a, b)$ is $\mathcal{F}_t-$measurable while $J(t+\delta, .; a, b)$ is $\mathcal{F}_{t+\delta}-$measurable. It should be clear that we can not ask for continuity $J(.; a, b)$ uniformly in ω. Indeed, even if δ is very small, there will certainly exist histories ω for which the state process changes drastically from t to $t+\delta$ thus changing the reward function as well. The solution to this problem is averaging: by averaging through all the histories from t to $t+\delta$ we can obtain continuity. More precisely:

Definition 73. *We say that $J(t, x; a, b)$ is* continuous in x and from the left in t, uniformly with respect to ω *if for each $\varepsilon > 0$ and $(t, x) \in \mathbb{S}$ there is $r^{a,b}_{(t,x)} > 0$ such that*

$$|\mathbb{E}[J(t, x; a, b)|\mathcal{F}_{t'}] - J(t', x'; a, b)| < \varepsilon, \qquad \mathbb{P}-\text{a.s.},$$

for all $(t', x') \in B\left(t, x; r^{a,b}_{(t,x)}\right)$, where $B(t, x; r) := [t-r, t] \times B_r(x)$.

If we can find a radius $r^{b}_{(t,x)}$ that is valid for all $a \in \mathcal{A}$ then we say that the continuity is uniform with respect to $a \in \mathcal{A}$. Analogously, we can define uniform continuity with respect to t, x, b.

Usually, to simplify the text, we will just say that $J(.; a, b)$ is left-continuous uniformly with respect to ω.

To obtain continuity uniformly with respect to a and to avoid other technicalities we will assume that A is compact.

The other assumption that will be made is technical in nature and has to do with the existence of a sequence of strategies that approximate the essential infimum of the reward function in an uniform way. More precisely, we make the following definition:

Definition 74. $\beta^{\varepsilon}_{(t,x)} \in \Delta(t)$ *is an* uniformly $\varepsilon-$optimal strategy *for* $V(t,x)$ *if*

$$V(t,x) \geq \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^{\varepsilon}_{(t,x)}[a]) - \varepsilon, \qquad \mathbb{P}-\text{a.s.}.$$

The definition of uniformly $\varepsilon-$optimal strategy for $U(t,x)$ is analogous.

We will assume that for each (t,x) and each ε there is an uniformly $\varepsilon-$optimal strategy $\beta^{\varepsilon}_{(t,x)} \in \Delta(t)$. Furthermore, to be able to use the uniform continuity of the reward function, we will need to bound these optimal strategies by requiring $\|\beta^{\varepsilon}_{(t,x)}\|_{p+\delta}$ to be locally bounded as a function of (t,x), for some $\delta > 0$, where p it the growth power of f. Here we recall that, since A is compact, $\|\beta\|_{p+\delta} := \sup_{a\in\mathcal{A}_t} \|\beta[a]\|_{\mathbb{H}^{\infty}_{t,p+\delta}}$ is well defined for all $\beta \in \Delta(t)$.

We are now ready to state and prove the *weak dynamic programming principle*.

Theorem 75 (Weak dynamic programming principle)**.** *Let p be the power growth of f, as in* (4.8)*, and consider $\delta > 0$ such that $p+\delta \geq 2$.*

Suppose that A is compact and for every M, $J(t,x;a,b)$ is left-continuous in (t,x) uniformly with respect to $t \in [0,T], a \in \mathcal{A}_t, b \in \{b \in \mathcal{B}_t : \|b\|_{\mathbb{H}^{\infty}_{t,p+\delta}} \leq M\}, \omega \in \Omega$.

Suppose also that, for each (t,x) and each ε, there exists an uniformly $\varepsilon-$optimal strategy for $V(t,x)$, $\beta^{\varepsilon}_{(t,x)} \in \Delta(t)$, such that $\|\beta^{\varepsilon}_{(t,x)}\|_{p+\delta}$ is locally bounded as a function of (t,x).

Let $\phi : \mathbb{S} \to \mathbb{R}$ be a continuous function[3] *and $\theta^{a,b}$ be a non-anticipative controlled stopping time such that*

$$W(t,x;a,b) := \mathbb{E}\left[\phi\left(\theta^{a,b}, X^{a,b}_{t,x}(\theta^{a,b})\right)\Big|\mathcal{F}_t\right]$$

makes sense for every $a \in \mathcal{A}_t, b \in \mathcal{B}_t$.

It follows that:

1. *If $\phi \geq V$ and $\phi(t,x) \geq -C(1+|x|^m)$ for some C, m, then*

$$V(t,x) \leq \operatorname*{essinf}_{\beta\in\Delta(t)} \operatorname*{esssup}_{a\in\mathcal{A}_t} W(t,x;a,\beta[a]). \tag{4.26}$$

2. *If $\phi \leq V$ then*

$$V(t,x) \geq \operatorname*{essinf}_{\beta\in\Delta(t)} \operatorname*{esssup}_{a\in\mathcal{A}_t} W(t,x;a,\beta[a]). \tag{4.27}$$

Proof. We start by defining

$$\begin{aligned} W(t,x;\beta) &:= \operatorname*{esssup}_{a\in\mathcal{A}_t} W(t,x;a,\beta[a]), \\ W(t,x) &:= \operatorname*{essinf}_{\beta\in\Delta(t)} W(t,x;\beta), \end{aligned}$$

and fixing (t,x).

Notice that due to the fact that $\theta^{a,b}$ is a non-anticipative controlled stopping time, $W(t,x;a,b)$ satisfies the independence of irrelevant alternatives as $J(t,x;a,b)$, see Lemma 35.

The proof of the Theorem is divided into Lemmas 76, 78, which establish (4.26) and (4.27), respectively. □

Lemma 76. *In the conditions of Theorem 75,* (4.26) *is valid.*

[3] In practice, later we will use the weak dynamic programming principle on functions $\phi \in C^{1,2}(\mathbb{S})$.

Proof. Let $\varepsilon > 0$. We will proceed in several steps:

Step 1: Find $\beta^{\varepsilon,m} \in \Delta(t)$ and $\mathcal{F}_t \ni \Lambda^m \nearrow \Omega$ such that $\mathbf{1}_{\Lambda^m} W(t,x) \geq \mathbf{1}_{\Lambda^m}(W(t,x;\beta^{\varepsilon,m}) - \varepsilon)$.

There is a countable sequence $(\beta_i) \subset \Delta(t)$ such that

$$W(t,x) = \inf_{i\geq 1} W(t,x;\beta_i), \;\; \mathbb{P} - \text{a.s.}$$

We consider $\tilde{\Lambda}_i := \{W(t,x) - W(t,x;\beta_i) \geq -\varepsilon\} \in \mathcal{F}_t$ and define $\Lambda_1 := \tilde{\Lambda}_1$, $\Lambda_{i+1} := \tilde{\Lambda}_{i+1} \setminus \bigcup_{j=1}^{i} \Lambda_i$.

Then we note that $\mathbb{P}\left(\bigcup_{i\geq 1} \Lambda_i\right) = 1$ and $\Lambda_i \cap \Lambda_j = \emptyset$ thus, by Proposition 52,

$$\beta^{\varepsilon,m} := \sum_{i=1}^{m} \mathbf{1}_{\Lambda_i}\beta_i + \mathbf{1}_{(\Lambda^m)^c}\beta_1 \in \Delta(t),$$

where $\Lambda^m = \bigcup_{i=1}^{m} \Lambda_i$. Besides, due to the independence of irrelevant alternatives of W we have[4]

$$\begin{aligned}
\mathbf{1}_{\Lambda^m} W(t,x;\beta^{\varepsilon,m}) &= \mathbf{1}_{\Lambda^m} \operatorname*{esssup}_{a\in\mathcal{A}_t} W\left(t,x;a,\sum_{i=1}^{m} \mathbf{1}_{\Lambda_i}\beta_i[a] + \mathbf{1}_{(\Lambda^m)^c}\beta_1[a]\right)\\
&= \operatorname*{esssup}_{a\in\mathcal{A}_t} \sum_{i=1}^{m} \mathbf{1}_{\Lambda_i} W\left(t,x;a,\beta_i[a]\right)\\
&= \sum_{i=1}^{m} \mathbf{1}_{\Lambda_i} W(t,x;\beta_i)\\
&\leq \mathbf{1}_{\Lambda^m}\left(W(t,x) + \varepsilon\right),
\end{aligned}$$

that is,

$$\mathbf{1}_{\Lambda^m} W(t,x) \geq \mathbf{1}_{\Lambda^m}(W(t,x;\beta^{\varepsilon,m}) - \varepsilon), \qquad , \mathbb{P} - \text{a.s.} \tag{4.28}$$

Step 2: Fix m and define $\beta^\varepsilon := \beta^{\varepsilon,m}$. Find $\varepsilon-$optimal strategies for V, $\tilde{\beta}_i^\varepsilon$, to patch with β^ε after the stopping time.

By hypothesis, there is a family $\left(\tilde{\beta}^\varepsilon_{(s,y)}\right)_{(s,y)}$ such that $\tilde{\beta}^\varepsilon_{(s,y)} \in \Delta(s)$ and

$$\begin{aligned}
\phi(s,y) \geq V(s,y) &\geq J\left(s,y;\tilde{\beta}^\varepsilon_{(s,y)}\right) - \varepsilon\\
&\geq J\left(s,y;a,\tilde{\beta}^\varepsilon_{(s,y)}[a]\right) - \varepsilon, \qquad \mathbb{P} - \text{a.s.},
\end{aligned} \tag{4.29}$$

for any $a \in \mathcal{A}_s$.

Furthermore, $\|\tilde{\beta}^\varepsilon_{(s,y)}\|_{p+\delta}$ is locally bounded. We define

$$\beta^\varepsilon_{(s,y)} := \beta^\varepsilon \oplus_s \tilde{\beta}^\varepsilon_{(s,y)} \in \Delta(t),$$

and remark that $\|\beta^\varepsilon_{(s,y)}\|_{p+\delta}$ is also locally bounded. Hence, for each n, there is M_n such that $\forall_{y\in \bar{B}_n(0)}\; \|\beta^\varepsilon_{(s,y)}\|_{p+\delta} \leq M_n$.

Since ϕ is continuous and, for each n, $\bar{B}_n(0)$ is compact and $J(t,x;a,b)$ is left-continuous in (t,x), uniformly with respect to $t \in [0,T]$, $a \in \mathcal{A}_t$ and $b \in \{b \in \mathcal{B}_t : \|b\|_{\mathbb{H}^\infty_{t,p+\delta}} \leq M_n\}$, there is also a family (r_n) such that

$$\phi(s,y) - \phi(s',y') \leq \varepsilon, \tag{4.30}$$

$$\mathbb{E}\left[J\left(s,y;a,\beta^\varepsilon_{(s,y)}[a]\right)\Big|\mathcal{F}_{s'}\right] - J\left(s',y';a,\beta^\varepsilon_{(s,y)}[a]\right) \geq -\varepsilon, \tag{4.31}$$

[4] We use the fact that if $\{\Lambda_i\}_i$ is a collection of disjoint sets then $\operatorname{esssup}_a \sum_i \mathbf{1}_{\Lambda_i} X^a = \sum_i \mathbf{1}_{\Lambda_i} \operatorname{esssup}_a X^a$.

for all $a \in \mathcal{A}_t$, $(s,y) \in [t,T] \times \bar{B}_n(0)$, $(s',y') \in B(s,y;r_n) \cap [t,T] \times \mathbb{R}^d$.

Since $\{B_{r_n}(y) : y \in \bar{B}_n(0)\}$ is an open covering of $\bar{B}_n(0)$, then, for each n, we can find a finite collection (s_i, y_i) such that $\{\tilde{O}_i^n := B(s_i, y_i; r_n)\}$ is a covering of $[t,T] \times \bar{B}_n(0)$. We define $O_1^n := \tilde{O}_1^n$, $O_{i+1}^n := \tilde{O}_{i+1}^n \setminus \bigcup_{k=1}^i \tilde{O}_k^n$ and $O^n = \bigcup_{i\geq 1} O_i^n$.

Combining (4.29), (4.30), (4.31) we conclude that if $(s',y') \in O_i^n$, $a \in \mathcal{A}_t$ then

$$J(s',y';a,\beta_i^\varepsilon[a]) \leq \phi(s',y') + 3\varepsilon, \ \ \mathbb{P}-\text{a.s.} \tag{4.32}$$

where $\beta_i^\varepsilon := \beta_{(s_i,y_i)}^\varepsilon \in \Delta(s_i)$. Indeed,

$$\begin{aligned} J(s',y';a,\beta_i^\varepsilon[a]) &\leq \mathbb{E}\left[J\left(s_i,y_i;a,\beta^\varepsilon[a] \oplus_{s_i} \tilde{\beta}_{(s_i,y_i)}^\varepsilon[a]\right)\Big|\mathcal{F}_{s'}\right] + \varepsilon \\ &= \mathbb{E}\left[J\left(s_i,y_i;a,\tilde{\beta}_{(s_i,y_i)}^\varepsilon[a]\right)\Big|\mathcal{F}_{s'}\right] + \varepsilon \\ &\leq \phi(s_i,y_i) + 2\varepsilon, \qquad \text{using (4.29)}, \\ &\leq \phi(s',y') + 3\varepsilon. \end{aligned}$$

Step 3: We will now patch the strategies together to get a sequence of almost optimal strategies for player 2, β^n.

Given $a \in \mathcal{A}_t$, we then define $\beta^n[a]$ by

$$\beta^n[a] := \sum_{i\geq 1} \mathbf{1}_{\Pi_i^{a,n}} \beta_i^\varepsilon[a] + \mathbf{1}_{(\Pi^{a,n})^c} \beta^\varepsilon[a], \tag{4.33}$$

where $\Pi_i^{a,n} = \left\{\left(\theta^a, X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a)\right) \in O_i^n\right\} \in \mathcal{F}_{\theta^a}$, $\Pi^{a,n} = \cup_{i\geq 1}\Pi_i^{a,n} \in \mathcal{F}_{\theta^a}$ and $\theta^a := \theta^{a,\beta^\varepsilon[a]}$. The proof that $\beta^n \in \Delta(t)$ is done in Lemma 77.

By definition of β^n, we have that

$$\begin{cases} \beta^n[a] \equiv \beta^\varepsilon[a] & \text{on } [t,\theta^a] \\ \beta^n[a] \equiv_{\Pi_i^{a,n}} \beta_i^\varepsilon[a] & \text{on } [t,T]. \end{cases} \tag{4.34}$$

Then we notice that for all $a \in \mathcal{A}_t$ and for almost all $\omega \in \Pi_i^{a,n}$,

$$\begin{aligned} \mathbb{E}\left[f\left(X_{t,x}^{a,\beta^n[a]}(T)\right)\Big|\mathcal{F}_{\theta^a}\right](\omega) &= \mathbb{E}\left[f\left(X^{a,\beta_i^\varepsilon[a]}_{\theta^a(\omega),X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a(\omega))}(T)\right)\Big|\mathcal{F}_{\theta^a(\omega)}\right](\omega) \\ &= J\left(\theta^a(\omega), X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a(\omega)); a, \beta_i^\varepsilon[a]\right)(\omega) \\ &\leq \phi\left(\theta^a(\omega), X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a(\omega))\right) + 3\varepsilon, \end{aligned}$$

where the first equality follows from the *flow property* for solutions of SDE's[5] and (4.34), and the inequality follows from (4.32). Thus,

$$\mathbb{E}\left[f\left(X_{t,x}^{a,\beta^n[a]}(T)\right)\Big|\mathcal{F}_{\theta^a}\right]\mathbf{1}_{\Pi^{a,n}} \leq \left(\phi\left(\theta^a, X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a)\right) + 3\varepsilon\right)\mathbf{1}_{\Pi^{a,n}}. \tag{4.35}$$

Step 4: Finally, we take the limit as $n \to \infty$.

By the tower property of expectations and (4.35), we deduce that,

$$\begin{aligned} J(t,x;a,\beta^n[a]) &= \mathbb{E}\left[f\left(X_{t,x}^{a,\beta^n[a]}(T)\right)\mathbf{1}_{\Pi^{a,n}}\Big|\mathcal{F}_t\right] + \mathbb{E}\left[f\left(X_{t,x}^{a,\beta^n[a]}(T)\right)\mathbf{1}_{(\Pi^{a,n})^c}\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f\left(X_{t,x}^{a,\beta^n[a]}(T)\right)|\mathcal{F}_{\theta^a}\right]\mathbf{1}_{\Pi^{a,n}}\Big|\mathcal{F}_t\right] + \mathbb{E}\left[f\left(X_{t,x}^{a,\beta^\varepsilon[a]}(T)\right)\mathbf{1}_{(\Pi^{a,n})^c}\Big|\mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\left(\phi\left(\theta^a, X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a)\right) + 3\varepsilon\right)\mathbf{1}_{\Pi^{a,n}}\Big|\mathcal{F}_t\right] + \mathbb{E}\left[f\left(X_{t,x}^{a,\beta^\varepsilon[a]}(T)\right)\mathbf{1}_{(\Pi^{a,n})^c}\Big|\mathcal{F}_t\right] \end{aligned}$$

[5] Here we refer to the following property: $X_{t,x}(T) = X_{s,X_{t,x}(s)}(T)$, where $t \leq s \leq T$.

In the following we simplify the notation and let $X_s := X_{t,x}^{a,\beta^\varepsilon[a]}(s)$. We recall that, by hypothesis, $\phi(x) + C(1 + |x|^m) \geq 0$. Hence,

$$\begin{aligned}
\mathbb{E}\left[\phi(\theta^a, X_{\theta^a})\mathbf{1}_{\Pi^{a,n}}|\mathcal{F}_t\right] &= \mathbb{E}\left[\left(\phi(\theta^a, X_{\theta^a}) + C\left(1 + |X_{\theta^a}|^m\right)\right)\mathbf{1}_{\Pi^{a,n}}|\mathcal{F}_t\right] - \mathbb{E}\left[C\left(1 + |X_{\theta^a}|^m\right)\mathbf{1}_{\Pi^{a,n}}|\mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\left(\phi(\theta^a, X_{\theta^a}) + C\left(1 + |X_{\theta^a}|^m\right)\right)|\mathcal{F}_t\right] - \mathbb{E}\left[C\left(1 + |X_{\theta^a}|^m\right)\mathbf{1}_{\Pi^{a,n}}|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\phi(\theta^a, X_{\theta^a}) + C\left(1 + |X_{\theta^a}|^m\right)\mathbf{1}_{(\Pi^{a,n})^c}|\mathcal{F}_t\right] \\
&\to \mathbb{E}\left[\phi(\theta^a, X_{\theta^a})|\mathcal{F}_t\right],
\end{aligned}$$

where the convergence as $n \to \infty$ is uniform with respect to a and ω, and follows from dominated convergence and the following facts:

- By (4.6), $\mathbb{E}\left[\sup_{s\in[t,T]}\left|X_{t,x}^{a,\beta^\varepsilon[a]}(s)\right|^m\middle|\mathcal{F}_t\right]$ is bounded uniformly in a;
- $\mathbf{1}_{(\Pi^{a,n})^c} \leq \mathbf{1}_{\left\{\sup_{s\in[t,T]}\left|X_{t,x}^{a,\beta^n[a]}(s)\right|>n\right\}} \to 0$ uniformly with respect to a.

Using the same convergence arguments and the fact that f has polynomial growth, we deduce that

$$\mathbb{E}\left[f\left(X_{t,x}^{a,\beta^\varepsilon[a]}(T)\right)\mathbf{1}_{(\Pi^{a,n})^c}\middle|\mathcal{F}_t\right] \to 0,$$

uniformly in a.

Thus, for n sufficiently large, the following holds, for all $a \in \mathcal{A}_t$,

$$J(t,x;a,\beta^n[a]) \leq W(t,x;a,\beta^\varepsilon[a]) + 4\varepsilon \tag{4.36}$$

Now we recall that, by (4.28) and the definition of $W(t,x;\beta^\varepsilon)$,

$$\mathbf{1}_{\Lambda^m}W(t,x) \geq \mathbf{1}_{\Lambda^m}(W(t,x;\beta^\varepsilon) - \varepsilon) \geq \mathbf{1}_{\Lambda^m}(W(t,x;a,\beta^\varepsilon[a]) - \varepsilon). \tag{4.37}$$

Combining (4.36) and (4.37) we get

$$\mathbf{1}_{\Lambda^m}J(t,x;a,\beta^n[a]) \leq \mathbf{1}_{\Lambda^m}(W(t,x) + 5\varepsilon),$$

and since a, ε, m are arbitrary we conclude that

$$V(t,x) \leq W(t,x).$$

□

Lemma 77. *Let β^n be defined by (4.33). Then $\beta^n \in \Delta(t)$.*

Proof. Notice that we have

$$\beta^n[a]_s = \sum_{i\geq 1}\bar{\beta}^{i,\varepsilon}[a]_s\mathbf{1}_{\Pi_i^{a,n}} + \mathbf{1}_{(\Pi^{a,n})^c}\beta^\varepsilon[a]_s,$$

where

$$\bar{\beta}^{i,\varepsilon} := \beta^\varepsilon \oplus_{s_i\vee\theta^a} \tilde{\beta}^\varepsilon_{(s_i,y_i)}.$$

Since $\theta^{a,b}$ is a non-anticipative stopping time and β^ε is non-anticipative we conclude that $\theta^a := \theta^{a,\beta^\varepsilon[a]}$ is also a non-anticipative stopping time and thus by Proposition 57 we conclude that $\bar{\beta}^{i,\varepsilon} \in \Delta(t)$.

On the other hand

$$\bar{\beta}^{i,\varepsilon}[a] \equiv \bar{\beta}^{j,\varepsilon}[a] \equiv \beta^\varepsilon[a] \text{ on } [t,\theta^a],$$

and

$$\mathbf{1}_{\Pi_i^{a,n}} = \mathbf{1}_{[s_i - r_n, s_i]}(\theta^a)\mathbf{1}_{B_{r_n}(y_i)}\left(X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a)\right).$$

Because $X_{t,x}^{a,b}$ is a non-anticipative controlled process then so is $X_{t,x}^{a,\beta^\varepsilon[a]}$. Thus, by Example 59, $X_{t,x}^{a,\beta^\varepsilon[a]}(\theta^a)$ is a non-anticipative controlled observation associated with θ^a.

We conclude that $\mathbf{1}_{\Pi_i^{a,n}}$ is a non-anticipative controlled observation associated with θ^a, hence, by Proposition 60 $\beta^n \in \Delta(t)$. □

Lemma 78. *In the conditions of Theorem 75,* (4.27) *is valid.*

Proof. This proof is similar to the one of Lemma 76, hence we will only outline the main points and the ones that are different. Once again we proceed by steps:

Step 1: Find the strategy for player 2, β^ε, and the initial reply of player 1, a^ε.

By hypothesis there is $\beta^\varepsilon \in \Delta(t)$ such that

$$V(t,x) \geq J(t,x;\beta^\varepsilon) - \varepsilon, \qquad \mathbb{P}-\text{a.s.}. \tag{4.38}$$

On the other hand there is a family $(a_i) \subset \mathcal{A}_t$ such that

$$W(t,x) \leq W(t,x;\beta^\varepsilon) = \sup_{i\geq 1} W(t,x;a_i,\beta^\varepsilon[a_i]), \qquad \mathbb{P}-\text{a.s.}.$$

Consider $\tilde{\Lambda}_i := \{W(t,x;a_i,\beta^\varepsilon[a_i]) - W(t,x;\beta^\varepsilon) \geq -\varepsilon\} \in \mathcal{F}_t$ and define $\Lambda_1 = \tilde{\Lambda}_1$, $\Lambda_{i+1} = \tilde{\Lambda}_{i+1} \setminus \bigcup_{k=1}^i \Lambda_k$. Then $\{\Lambda_i\}_i$ is a partition of Ω, modulo null sets, thus, by Lemma 32 and because A is compact,

$$a^\varepsilon := \sum_{i\geq 1} \mathbf{1}_{\Lambda_i} a_i \in \mathcal{A}_t. \tag{4.39}$$

Moreover, by the independence of irrelevant alternatives for W and β^ε, we have

$$\begin{aligned}
W(t,x;a^\varepsilon,\beta^\varepsilon[a^\varepsilon]) &= W\left(t,x;\sum_{i\geq 1}\mathbf{1}_{\Lambda_i}a_i,\beta^\varepsilon\left[\sum_{i\geq 1}\mathbf{1}_{\Lambda_i}a_i\right]\right)\\
&= W\left(t,x;\sum_{i\geq 1}\mathbf{1}_{\Lambda_i}a_i,\sum_{i\geq 1}\mathbf{1}_{\Lambda_i}\beta^\varepsilon\left[a_i\right]\right)\\
&= \sum_{i\geq 1}\mathbf{1}_{\Lambda_i}W\left(t,x;a_i,\beta^\varepsilon\left[a_i\right]\right)\\
&\geq W(t,x)-\varepsilon,
\end{aligned}$$

Thus, for a^ε given by (4.39), we have

$$W(t,x) \leq W(t,x;a^\varepsilon,\beta^\varepsilon[a^\varepsilon]) + \varepsilon, \;\; \mathbb{P}-\text{a.s.}. \tag{4.40}$$

Step 2: Find $\varepsilon-$optimal controls, $a^{i,\varepsilon}$, for player 1 to continue the game after the stopping time and the neighborhoods, A_i, where they shall be used.

Define, for each $s \geq t$, $\tilde{\beta}_s^\varepsilon$ by

$$\tilde{\beta}_s^\varepsilon[a] := \beta^\varepsilon[a^\varepsilon \oplus_s a].$$

We have $\tilde{\beta}_s^\varepsilon \in \Delta(s)$ by Proposition 61. In addition, we have that $\tilde{\beta}_s^\varepsilon \in \Delta(r)$ for each $r \in [t,s]$, because $\beta^\varepsilon \in \Delta(t)$. Notice that, by Proposition 43,

$$\tilde{\beta}_s^\varepsilon[a] = \beta^\varepsilon[a^\varepsilon] \oplus_s \tilde{\beta}_s^\varepsilon[a].$$

As in step 1, for each (s,y) we can find $a_{(s,y)}^\varepsilon \in \mathcal{A}_s$ such that

$$\phi(s,y) \leq V(s,y) \leq J\left(s,y;a_{(s,y)}^\varepsilon,\tilde{\beta}_s^\varepsilon[a_{(s,y)}^\varepsilon]\right) + \varepsilon, \qquad ,\mathbb{P}-\text{a.s.} \tag{4.41}$$

Because $J(.;a,b)(\omega)$ is left-continuous and ϕ is continuous there is also a family $(r_{(s,y)})_{(s,y)}$ such that

$$\begin{aligned} \phi(s,y)-\phi(s',y') &\geq -\varepsilon, && (4.42)\\ \mathbb{E}\left[J\left(s,y;\tilde{a}^\varepsilon_{(s,y)},\tilde{\beta}^\varepsilon_s[\tilde{a}^\varepsilon_{(s,y)}]\right)\Big|\mathcal{F}_{s'}\right]-J\left(s',y';\tilde{a}^\varepsilon_{(s,y)},\tilde{\beta}^\varepsilon_s[\tilde{a}^\varepsilon_{(s,y)}]\right) &\leq \varepsilon, && (4.43)\end{aligned}$$

for all $(s',y')\in B(s,y;r_{(s,y)})$, where $\tilde{a}^\varepsilon_{(s,y)}:=a^\varepsilon\oplus_s a^\varepsilon_{(s,y)}\in\mathcal{A}_t$.

Since $\{B(s,y;r):(s,y)\in S,\ 0<r\leq r_{(s,y)}\}$ forms a Vitali covering of $\mathbb{S}$, we can find a countable sequence (s_i,y_i,r_i) such that $\{B(s_i,y_i;r_i)\}_i$ forms a partition of $\mathbb{S}$ and $0<r_i\leq r_{(s_i,y_i)}$. For the notion of Vitali covering we refer the reader to [6, p. 158].

Combining (4.41), (4.42), (4.43) we conclude that if $(s',y')\in B(s_i,y_i;r_i)$, then

$$J\left(s',y';\tilde{a}^\varepsilon_i,\tilde{\beta}^\varepsilon_{s_i}[\tilde{a}^\varepsilon_i]\right)\geq\phi(s',y')-3\varepsilon,\ \ \mathbb{P}-\text{a.s.} \tag{4.44}$$

where $\tilde{a}^\varepsilon_i:=\tilde{a}^\varepsilon_{(s_i,y_i)}$. Indeed,

$$\begin{aligned} J\left(s',y';\tilde{a}^\varepsilon_i,\tilde{\beta}^\varepsilon_{s_i}[\tilde{a}^\varepsilon_i]\right) &\geq \mathbb{E}\left[J\left(s_i,y_i;a^\varepsilon\oplus_{s_i}a^\varepsilon_i,\tilde{\beta}^\varepsilon_{s_i}[a^\varepsilon\oplus_{s_i}a^\varepsilon_i]\right)\Big|\mathcal{F}_{s'}\right]-\varepsilon\\ &= \mathbb{E}\left[J\left(s_i,y_i;a^\varepsilon_i,\tilde{\beta}^\varepsilon_{s_i}[a^\varepsilon_i]\right)\Big|\mathcal{F}_{s'}\right]-\varepsilon\\ &\geq \phi(s_i,y_i)-2\varepsilon, \qquad \text{using (4.41)},\\ &\geq \phi(s',y')-3\varepsilon.\end{aligned}$$

Step 3: Patching the controls together to get the reply of player 1, a, to β^ε.

We define $a\in\mathcal{A}_t$ by

$$a:=\sum_{i\geq1}\tilde{a}^\varepsilon_i\mathbf{1}_{\Pi_i}=\sum_{i\geq1}a^\varepsilon\oplus_{s_i}a^\varepsilon_i\mathbf{1}_{\Pi_i},$$

where $\Pi_i=\left\{\left(\theta,X^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}_{t,x}(\theta)\right)\in B(s_i,y_i;r_i)\right\}\in\mathcal{F}_\theta$ and $\theta:=\theta^{a^\varepsilon,\beta^\varepsilon}$. Then, by Corollary 45 and Proposition 43, we have

$$\begin{aligned}\beta^\varepsilon[a] &= \sum_{i\geq1}\mathbf{1}_{\Pi_i}\beta^\varepsilon[a^\varepsilon]\oplus_{s_i}\beta^\varepsilon[a^\varepsilon\oplus_{s_i}a^\varepsilon_i]\\ &= \sum_{i\geq1}\mathbf{1}_{\Pi_i}\beta^\varepsilon[a^\varepsilon]\oplus_{s_i}\tilde{\beta}^\varepsilon_{s_i}[\tilde{a}^\varepsilon_i]\\ &= \sum_{i\geq1}\mathbf{1}_{\Pi_i}\tilde{\beta}^\varepsilon_{s_i}[\tilde{a}^\varepsilon_i].\end{aligned}$$

Thus,

$$\begin{cases} a\equiv_{\Pi_i}\tilde{a}^\varepsilon_i\\ \beta^\varepsilon[a]\equiv_{\Pi_i}\tilde{\beta}^\varepsilon_{s_i}[\tilde{a}^\varepsilon_i]\end{cases} \tag{4.45}$$

By (4.44), (4.45), we have for almost all $\omega\in\Pi_i$

$$\begin{aligned}\mathbb{E}\left[f\left(X^{a,\beta^\varepsilon[a]}_{t,x}(T)\right)\Big|\mathcal{F}_\theta\right](\omega) &= \mathbb{E}\left[f\left(X^{a,\beta^\varepsilon[a]}_{\theta(\omega),X^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}_{t,x}(\theta(\omega))}(T)\right)\Big|\mathcal{F}_{\theta(\omega)}\right](\omega)\\ &= J\left(\theta(\omega),X^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}_{t,x}(\theta(\omega));\tilde{a}^\varepsilon_i,\tilde{\beta}^\varepsilon_{s_i}[\tilde{a}^\varepsilon_i]\right)(\omega)\\ &\geq \phi\left(\theta(\omega),X^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}_{t,x}(\theta(\omega))\right)-3\varepsilon.\end{aligned}$$

Thus,

$$\mathbb{E}\left[f\left(X^{a,\beta^\varepsilon[a]}_{t,x}(T)\right)\Big|\mathcal{F}_\theta\right]\mathbf{1}_{\Pi_i}\geq\left(\phi\left(\theta,X^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}_{t,x}(\theta)\right)-3\varepsilon\right)\mathbf{1}_{\Pi_i},\ \ \mathbb{P}-\text{a.s.}. \tag{4.46}$$

Step 4: Combining a and β^ε.

By (4.38) and the definition of $J(t,x;\beta^\varepsilon)$, we have that

$$V(t,x) \geq J(t,x;\beta^\varepsilon) - \varepsilon \geq J(t,x;a,\beta^\varepsilon[a]) - \varepsilon, \qquad \mathbb{P}-\text{a.s.}. \tag{4.47}$$

Also, we have, by the tower property of conditional expectation and (4.46), that

$$\begin{aligned} J(t,x;a,\beta^\varepsilon[a]) &= \sum_{i\geq 1} \mathbb{E}\left[f\left(X_{t,x}^{a,\beta^\varepsilon[a]}(T)\right) \mathbf{1}_{\Pi_i} \middle| \mathcal{F}_t \right] \\ &= \sum_{i\geq 1} \mathbb{E}\left[\mathbb{E}\left[f\left(X_{t,x}^{a,\beta^\varepsilon[a]}(T)\right) | \mathcal{F}_\theta \right] \mathbf{1}_{\Pi_i} \middle| \mathcal{F}_t \right] \\ &\geq \sum_{i\geq 1} \mathbb{E}\left[\left(\phi\left(\theta, X_{t,x}^{a^\varepsilon,\beta^\varepsilon[a^\varepsilon]}(\theta)\right) - 3\varepsilon \right) \mathbf{1}_{\Pi_i} \middle| \mathcal{F}_t \right] \\ &= W(t,x;a^\varepsilon,\beta^\varepsilon[a^\varepsilon]) - 3\varepsilon. \end{aligned} \tag{4.48}$$

By (4.47), (4.48), (4.40), we deduce that

$$V(t,x) \geq J(t,x;a,\beta^\varepsilon[a]) - \varepsilon \geq W(t,x;a^\varepsilon,\beta^\varepsilon[a^\varepsilon]) - 4\varepsilon \geq W(t,x) - 5\varepsilon,$$

and, since ε is arbitrary, we conclude that

$$V(t,x) \geq W(t,x).$$

□

Remark 79. *Since $V(t,x)$ is deterministic, and if we assume additionally that it is continuous and of polynomial growth, then we can take $\phi(t,x) = V(t,x)$ in both inequalities, thus getting a version of the traditional dynamic programming principle with controlled stopping times.*

Remark 80. *If B is bounded then the assumption on the existence of uniformly $\varepsilon-$optimal strategies, $\beta^\varepsilon_{(t,x)}$, is satisfied. Indeed, we know that there is a sequence (β_i) such that*

$$V(t,x) = \inf_i \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta_i[a]).$$

We can now consider $\tilde{\Lambda}_i = \{V(t,x) - \operatorname{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta_i[a]) \geq -\varepsilon\} \in \mathcal{F}_t$ and define $\Lambda_1 := \tilde{\Lambda}_1$, $\Lambda_{i+1} := \tilde{\Lambda}_{i+1} \backslash \bigcup_{k=1}^i \tilde{\Lambda}_k$. Then $\{\Lambda_i\}$ is a partition of Ω and because B is bounded we have by Proposition 52 that

$$\beta^\varepsilon := \sum_{i\geq 1} \mathbf{1}_{\Lambda_i}\beta_i \in \Delta(t).$$

Moreover, by the property of independence of irrelevant alternatives of J, we have

$$\begin{aligned} \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta^\varepsilon[a]) &= \sum_{i\geq 1} \mathbf{1}_{\Lambda_i} \operatorname*{esssup}_{a\in\mathcal{A}_t} J(t,x;a,\beta_i[a]) \\ &\leq V(t,x) + \varepsilon. \end{aligned}$$

Furthermore, the condition of local boundedness of $\|\beta^\varepsilon_{(t,x)}\|_{p+\delta}$ with respect to (t,x) is immediate since B is bounded.

If B is not bounded and the essential infimum is not achieved uniformly with respect to ω then we run into problems in Lemma 76 when searching for the countable sequence of strategies β^ε_i to patch. This is due to the fact that in this case we are only able to find strategies $\beta^\varepsilon_{(s,y)}$ which approximate $V(s,y)$ in $\Omega \setminus \Lambda_{(s,y)}$. When we consider all the points (s,y) we may miss an important part of Ω.

Remark 81. *We now make a remark on the need of uniform continuity for J in the proof of the weak DPP.*

First we notice that for the second inequality we only need continuity for each a, b fixed instead of uniformly in a, b. The reason is that for the second inequality we need to construct carefully the control for the first player in order to approximate the value function and this is done after player 2 chooses his strategy, hence the second player can not perturb the approximation made by player 1.

On the other hand, on the proof of the first inequality, uniform continuity seems to be fundamental. The reason is that for this inequality we are constructing the strategy for the second player before player 1 chooses his control. Thus we must make a choice that ensures that our approximation of the value function is good enough independently of the first player's control.

We end this Section by stating the corresponding Theorem for the upper value function. The proof is completely analogous.

Theorem 82. *Let p be the power growth of f, as in (4.8), and consider $\delta > 0$ such that $p + \delta \geq 2$.*

Suppose that B is compact and for every M, $J(t, x; a, b)$ is left-continuous in (t, x) uniformly with respect to $t \in [0, T], a \in \{a \in \mathcal{A}_t : \|a\|_{\mathbb{H}^\infty_{t,p+\delta}} \leq M\}, b \in \mathcal{B}_t, \omega \in \Omega$.

Suppose also that, for each (t, x) and each ε, there exists an uniformly $\varepsilon-$optimal strategy for $U(t, x)$, $\alpha^\varepsilon_{(t,x)} \in \Gamma(t)$, such that $\|\alpha^\varepsilon_{(t,x)}\|_{p+\delta}$ is locally bounded as a function of (t, x).

Let $\phi : \mathbb{S} \to \mathbb{R}$ be a continuous function and $\theta^{a,b}$ be a non-anticipative controlled stopping time such that

$$W(t, x; a, b) := \mathbb{E}\left[\phi\left(\theta^{a,b}, X^{a,b}_{t,x}(\theta^{a,b})\right) \Big| \mathcal{F}_t\right]$$

makes sense for every $a \in \mathcal{A}_t, b \in \mathcal{B}_t$.

It follows that:

1. *If $\phi \leq U$ and $\phi(t, x) \leq C(1 + |x|^m)$ for some C, m, then*

$$U(t, x) \geq \operatorname*{esssup}_{\alpha \in \Gamma(t)} \operatorname*{essinf}_{b \in \mathcal{B}_t} W(t, x; \alpha[b], b).$$

2. *If $\phi \geq U$ then*

$$U(t, x) \leq \operatorname*{esssup}_{\alpha \in \Gamma(t)} \operatorname*{essinf}_{b \in \mathcal{B}_t} W(t, x; \alpha[b], b).$$

4.5.1 Optimal stochastic control as a particular case

We now consider the particular case of Theorem 75 in the context of stochastic control. For that we set $A = \{a\}$ which is obviously compact. In this case player 1 has no choices, hence we have only one player, player two, like in stochastic control.

In this situation we have

$$V(t, x) = \operatorname*{essinf}_{b \in \mathcal{B}_t} J(t, x; a, b),$$

and Theorem 75 reads as

Theorem 83 (Weak DPP for stochastic control 1)**.** *Let p be the power growth of f, as in (4.8), and consider $\delta > 0$ such that $p + \delta \geq 2$.*

Suppose that for each (t, x) and each ε there exists $b^\varepsilon_{(t,x)} \in \mathcal{B}_t$ such that

$$V(t, x) \geq J\left(t, x; a, b^\varepsilon_{(t,x)}\right) - \varepsilon, \qquad \mathbb{P} - \text{a.s.},$$

and $\|b^\varepsilon_{(t,x)}\|_{\mathbb{H}^\infty_{t,p+\delta}}$ is locally bounded as a function of (t, x).

Let $\phi : \mathbb{S} \to \mathbb{R}$ be a continuous function and θ^b be a non-anticipative controlled stopping time such that

$$W(t,x;b) := \mathbb{E}\left[\phi\left(\theta^b, X^{a,b}_{t,x}(\theta^b)\right)\Big|\mathcal{F}_t\right]$$

makes sense for every $b \in \mathcal{B}_t$.

It follows that:

1. *If $\phi \geq V$ and $\phi(t,x) \geq -C(1+|x|^m)$ for some C, m, then*

$$V(t,x) \leq \operatorname*{essinf}_{b\in\mathcal{B}_t} W(t,x;b). \tag{4.49}$$

2. *If $\phi \leq V$ then*

$$V(t,x) \geq \operatorname*{essinf}_{b\in\mathcal{B}_t} W(t,x;b). \tag{4.50}$$

In the previous version of the Theorem we have already lifted most of the assumptions that are no longer needed because we no longer need uniform continuity with respect to a. Yet we can still adjust a few details namely:

- Since, in this case, the condition (ii) of Proposition 68 is immediately satisfied, we conclude that for each (t,x) there exists $b^\varepsilon_{(t,x)} \in \mathcal{B}_t$ such that

$$V(t,x) \geq J\left(t,x;a,b^\varepsilon_{(t,x)}\right) - \varepsilon, \qquad \mathbb{P}-\text{a.s.}.$$

- The local boundedness of $\|b^\varepsilon_{(t,x)}\|_{\mathbb{H}^\infty_{t,p+\delta}}$ and the condition $\phi(t,x) \geq -C(1+|x|^m)$ were there to ensure uniform convergence with respect to a. Thus, they are no longer required;

- Since now we do not need strategies then there is no need to require θ^b to be non-anticipative;

- In this context the second inequality is easier to obtain and does not require much regularity of ϕ. In fact we only need ϕ to be measurable. Indeed:

$$\begin{aligned} V(t,x) &= \operatorname*{essinf}_{b\in\mathcal{B}_t} \mathbb{E}\left[f\left(X^{a,b}_{t,x}(T)\right)\Big|\mathcal{F}_t\right] \\ &= \operatorname*{essinf}_{b\in\mathcal{B}_t} \mathbb{E}\left[\mathbb{E}\left[f\left(X^{a,b}_{t,x}(T)\right)\Big|\mathcal{F}_{\theta^b}\right]\Big|\mathcal{F}_t\right] \\ &= \operatorname*{essinf}_{b\in\mathcal{B}_t} \mathbb{E}\left[\mathbb{E}\left[f\left(X^{a,b}_{\theta^b, X^{a,b}_{t,x}(\theta^b)}(T)\right)\Big|\mathcal{F}_{\theta^b}\right]\Big|\mathcal{F}_t\right], \end{aligned}$$

  and

$$\begin{aligned} \mathbb{E}\left[f\left(X^{a,b}_{\theta^b, X^{a,b}_{t,x}(\theta^b)}(T)\right)\Big|\mathcal{F}_{\theta^b}\right](\omega) &\geq \operatorname*{essinf}_{\tilde{b}\in\mathcal{B}_{\theta^b(\omega)}} \mathbb{E}\left[f\left(X^{a,\tilde{b}}_{\theta^b, X^{a,b}_{t,x}(\theta^b)}(T)(\omega)\right)\Big|\mathcal{F}_{\theta^b}\right](\omega) \\ &= V\left(\theta^b(\omega), X^{a,b}_{t,x}(\theta^b)(\omega)\right) \\ &\geq \phi\left(\theta^b(\omega), X^{a,b}_{t,x}(\theta^b)(\omega)\right). \end{aligned}$$

  Thus,

$$V(t,x) \geq \operatorname*{essinf}_{b\in\mathcal{B}_t} \phi\left(\theta^b, X^{a,b}_{t,x}(\theta^b)\right).$$

- Due to the previous observation, the regularity of ϕ now is only needed to prove the first inequality of the weak dynamic programming principle. In this case ϕ needs not be continuous but just lower semi-continuous.

By the previous fourth remark we can, in particular, consider the lower semi-continuous envelope of v, v_*, as ϕ in the second inequality. Indeed, v_* is lower semi-continuous, hence measurable, and $v_* \leq v$. For the definition of semi-continuous envelope we refer the reader to Definition 113, in the Appendix.

Taking in account all the previous remarks we get the following version of the dynamic programming principle for stochastic control problems:

Theorem 84 (Weak DPP for stochastic control 2)**.** *Let $\phi : \mathbb{S} \to \mathbb{R}$ and θ^b be a family of stopping times such that*

$$W(t,x;b) := \mathbb{E}\left[\phi\left(\theta^b, X^{a,b}_{t,x}(\theta^b)\right)\Big|\mathcal{F}_t\right]$$

makes sense for every $b \in \mathcal{B}_t$.

It follows that:

1. *If $\phi \in \mathrm{LSC}(\mathbb{S})$, $\phi \geq V$, then*

$$V(t,x) \leq \operatorname*{essinf}_{b\in\mathcal{B}_t} W(t,x;b).$$

2.

$$V(t,x) \geq \operatorname*{essinf}_{b\in\mathcal{B}_t} \mathbb{E}\left[v_*\left(\theta^b, X^{a,b}_{t,x}(\theta^b)\right)\Big|\mathcal{F}_t\right].$$

This version of the DPP is analogous to the one in [1] for diffusion processes, when controls can depend on the past.

4.5.2 Continuity of the reward function

We will now study the continuity of the reward function J in the context of Definition 73. More precisely we will find conditions under which we can deduce that J has the uniform continuity required for the proof of the weak dynamic programming principle.

Proposition 85. *Suppose that A is compact and f is locally Lipschitz. Then $J(t,x;a,b)$ is left-continuous uniformly with respect to $t \in [0,T], a \in \mathcal{A}_t, b \in \{b \in \mathcal{B}_t : \|b\|_{\mathbb{H}^\infty_{t,p+\delta}} \leq M\}$, and ω. Here p is the growth power of f as in* (4.8)*, and $\delta > 0$ is such that $p+\delta \geq 2$.*

Proof. Consider $x, x' \in \mathbb{R}^d$ and $t, t' \in [0,T]$ such that $t' \leq t$. Let $a \in \mathcal{A}_{t'}$ and $b \in \{b \in \mathcal{B}_t : \|b\|_{\mathbb{H}^\infty_{t,p+\delta}} \leq M\}$ be arbitrary. We use the following notation:

$$X_s := X^{a,b}_{t,x}(s), \qquad X'_s := X^{a,b}_{t',x'}(s).$$

Since f is locally Lipschitz there is K_n such that:

$$|f(x) - f(x')| \quad \leq \quad K_n|x - x'|, \qquad \text{if } |x|, |x'| \leq n.$$

Using Hölder's inequality we have

$$\mathbb{E}\left[|f(X_T) - f(X'_T)|\,\mathbf{1}_{\{|X_T|>n\}}|\mathcal{F}_{t'}\right] \leq \left[|f(X_T) - f(X'_T)|^{\frac{p+\delta}{p}}\Big|\mathcal{F}_{t'}\right]^{\frac{p}{p+\delta}} \mathbb{E}\left[\mathbf{1}_{\{|X_T|>n\}}|\mathcal{F}_{t'}\right]^{\frac{\delta}{p+\delta}}.$$

On one hand we have, by (4.8) and by Jensen's inequality, that

$$|f(X_T) - f(X'_T)|^{\frac{p+\delta}{p}} \leq C\left(1 + |X_T|^{p+\delta} + |X'_T|^{p+\delta}\right),$$

and hence, by (4.4) we conclude that

$$\left[|f(X_T) - f(X'_T)|^{\frac{p+\delta}{p}}\Big|\mathcal{F}_{t'}\right] \leq C\left(1 + |x|^{p+\delta} + |x'|^{p+\delta} + M^{p+\delta}\right).$$

On the other hand, we have by Chebyshev's inequality that

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{|X_T|>n\}}|\mathcal{F}_{t'}\right] &\leq \frac{1}{n^2}\mathbb{E}\left[|X_T|^2|\mathcal{F}_{t'}\right]\\ &\leq \frac{C}{n^2}\left(1+|x|^2+M^2\right)\\ &\leq \frac{C}{n^2}\left(1+|x|^{p+\delta}+|x'|^{p+\delta}+M^{p+\delta}\right)^{\frac{2}{p+\delta}}\\ &\leq \frac{C}{n^2}\left(1+|x|^{p+\delta}+|x'|^{p+\delta}+M^{p+\delta}\right).\end{aligned}$$

Thus we have

$$\mathbb{E}\left[|f(X_T)-f(X'_T)|\,\mathbf{1}_{\{|X_T|>n\}}|\mathcal{F}_{t'}\right] \leq \frac{C}{n^{\frac{2\delta}{p+\delta}}}\left(1+|x|^{p+\delta}+|x'|^{p+\delta}+M^{p+\delta}\right),$$

and we have an analogous inequality for $\mathbb{E}\left[|f(X_T)-f(X'_T)|\,\mathbf{1}_{\{|X'_T|>n\}}|\mathcal{F}_{t'}\right]$.

Now recall Definition 73. We then have

$$\begin{aligned}|\mathbb{E}[J(t,x;a,b)|\mathcal{F}_{t'}]-J(t',x';a,b)| &= |\mathbb{E}\left[f(X_T)-f(X'_T)|\mathcal{F}_{t'}\right]|\\ &\leq \mathbb{E}\left[|f(X_T)-f(X'_T)|\,|\mathcal{F}_{t'}\right]\\ &= \mathbb{E}\left[|f(X_T)-f(X'_T)|\,\mathbf{1}_{\{|X_T|,|X'_T|\leq n\}}|\mathcal{F}_{t'}\right]+\\ &\quad +\mathbb{E}\left[|f(X_T)-f(X'_T)|\,\mathbf{1}_{\{|X_T|,|X'_T|\leq n\}^c}|\mathcal{F}_{t'}\right]\\ &\leq K_n\mathbb{E}\left[|X_T-X'_T|\,|\mathcal{F}_{t'}\right]+\\ &\quad +\frac{C}{n^{\frac{2\delta}{p+\delta}}}\left(1+|x|^{p+\delta}+|x'|^{p+\delta}+M^{p+\delta}\right)\\ &\leq K_n\left(\mathbb{E}\left[|X_T-X'_T|^2\,\middle|\mathcal{F}_{t'}\right]\right)^{\frac{1}{2}}+\frac{\varepsilon}{2},\end{aligned}$$

for a fixed n large enough, depending only on $K,T,M,|x|,|x'|$.

By Lemma 148, there is r, such that

$$\forall_{(t',x')\in B(t,x;r_{|x|})}\ K_n\mathbb{E}\left[|X_T-X'_T|^2\,\middle|\mathcal{F}_{t'}\right]^{\frac{1}{2}} \leq \frac{\varepsilon}{2}.$$

Since n depends only on $K,T,M,|x|,|x'|$ we conclude that r depends only on $K,T,M,|x|$. Thus, $J(t,x;a,b)$ is left-continuous uniformly with respect to $t\in[0,T]$, $a\in\mathcal{A}_t$, $b\in\{b\in\mathcal{B}_t:\|b\|_{\mathbb{H}^\infty_{t,p+\delta}}\leq M\}$, and $\omega\in\Omega$. ☐

Remark 86. *We remark that on the previous Proposition even if $t'=t$ the rate of convergence*

$$\mathbb{E}\left[|f(X_T)-f(X'_T)|\,\mathbf{1}_{\{|X_T|,|X'_T|\leq n\}^c}|\mathcal{F}_{t'}\right]\to 0$$

may depend on M. Thus if f is not globally Lipschitz, i.e. if $K_n\to\infty$, the rate of convergence of

$$|\mathbb{E}[J(t,x;a,b)|\mathcal{F}_{t'}]-J(t',x';a,b)|\to 0$$

will in general depend on M as well.

If this did not happen, then the inequality

$$|J(t,x;a,b)-J(t,x';a,b)|\leq\varepsilon$$

would be true for all $a\in\mathcal{A}_t, b\in\mathcal{B}_t$. Thus, as a consequence of the inequality

$$\operatorname*{essinf}_{a,\beta}(J-J')\leq\operatorname*{essinf}_{\beta}\operatorname*{esssup}_{a}J-\operatorname*{essinf}_{\beta}\operatorname*{esssup}_{a}J'\leq\operatorname*{esssup}_{a,\beta}(J-J'),$$

we would get that $V(t,.)$ would be continuous.

Remark 87. *If B is compact then we can forget the dependence on M on Proposition 85. Thus, by the previous Remark, we conclude that if A,B are compact and f is locally Lipschitz then $V(t,.)$ is continuous.*

4.6 Hamilton-Jacobi-Bellman-Isaacs equation

We know from Theorem 64 that $V(t,x)$ is deterministic, that is, we can think of it as a function $V:\mathbb{S}\to\mathbb{R}$. In this section we give a PDE characterization for V by means of a HJBI equation.

We introduce the Hamiltonians $H^{\pm}:\mathbb{S}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\to\mathbb{R}$ by

$$\begin{aligned} H^-(t,x,p,X) &= \inf_{a\in A}\sup_{b\in B} H^{a,b}(t,x,p,X),\\ H^+(t,x,p,X) &= \sup_{b\in B}\inf_{a\in A} H^{a,b}(t,x,p,X), \end{aligned}$$

where

$$H^{a,b}(t,x,p,X) = -\langle \mu(t,x;a,b),p\rangle - \frac{1}{2}\mathrm{Tr}\left(\left(\sigma\sigma^T\right)(t,x;a,b)X\right).$$

Throughout this section we assume that both μ,σ are continuous in all variables. This implies that $H^{a,b}(t,x,p,X)$ is continuous in a,b,t,x,p,X.

Remark 88. *If A,B are compact then we know that H^+,H^- are continuous functions. However if either A or B fails to be compact then both the Hamiltonians can be discontinuous. In that case we can only ensure semi-continuity. Indeed, if A is compact then H^- is lower semi-continuous (see [5, p. 148]) and H^+ is also lower semi-continuous because it is the supremum of continuous functions. Analogously, if B is compact then H^- and H^+ are upper semi-continuous.*

The main result of this section is the next Theorem which states that V is a discontinuous viscosity solution of the following *Hamilton-Jacobi-Bellman-Isaacs equation*:

$$\begin{cases} -\partial_t V + H^-(.,DV,D^2V) = 0, & \text{on} \quad (0,T)\times\mathbb{R}^d\\ -V = -f & \text{on} \quad \{T\}\times\mathbb{R}^d. \end{cases} \tag{4.51}$$

We need to consider the notion of discontinuous viscosity solutions because we did not prove yet that V is a continuous function. For the notion of discontinuous viscosity solutions we refer the reader to the Appendix, section A.3. This notion makes use of the concepts of semi-continuous envelopes, defined in Definition 113 of the Appendix.

Theorem 89. *Suppose A is compact. Then*

1. *V^* is a viscosity subsolution of*

$$\begin{cases} -\partial_t V^* + H^-(.,DV^*,D^2V^*) \le 0, & \textit{on} \quad (0,T)\times\mathbb{R}^d\\ -V^* \le -f & \textit{on} \quad \{T\}\times\mathbb{R}^d; \end{cases}$$

2. *V_* is a viscosity supersolution of*

$$\begin{cases} -\partial_t V_* + (H^-)^*(.,DV_*,D^2V_*) \ge 0, & \textit{on} \quad (0,T)\times\mathbb{R}^d\\ -V_* \ge -f & \textit{on} \quad \{T\}\times\mathbb{R}^d. \end{cases}$$

Proof of 1. We start with the subsolution property. For the boundary condition we just notice that

$$V^*(T,x) \ge V(T,x) = f(x).$$

In the interior of the domain we proceed by contradiction, that is, we suppose that there is (t_0,x_0) and ϕ smooth satisfying

$$(V^*-\phi)(t,x) < (V^*-\phi)(t_0,x_0) = 0, \text{ for all } (t,x)\in(0,T)\times\mathbb{R}^d, (t,x)\neq(t_0,x_0),$$

such that

$$\left(-\partial_t\phi + H^-(.,D\phi,D^2\phi)\right)(t_0,x_0) \ge 3\delta,$$

for some $\delta > 0$.

Step 1: Find a strategy β that contradicts the weak DPP.

We consider $\varphi(t,x) = \phi(t,x) + |t-t_0|^2 + |x-x_0|^4$ and notice that

$$\left(-\partial_t\varphi + H^-(., D\varphi, D^2\varphi)\right)(t_0,x_0) \geq 3\delta.$$

Thus, for each $a \in A$, there is $b_a \in B$ such that

$$\left(-\partial_t\varphi + H^{a,b_a}(., D\varphi, D^2\varphi)\right)(t_0,x_0) \geq 2\delta,$$

and since $H^{\cdot,b}(t,x)$ is continuous, for each a, there is a r_a such that, for all $\tilde{a} \in B_{r_a}(a)$,

$$\left(-\partial_t\varphi + H^{\tilde{a},b_a}(., D\varphi, D^2\varphi)\right)(t_0,x_0) \geq \delta.$$

Since $\{B_{r_a}(a) : a \in A\}$ is an open covering of A, we can find a finite family $\{a_i\}$ such that

$$\bigcup_{i\geq 1} B_{r_i}(a_i) = A,$$

where $r_i := r_{a_i}$.

Let $\Lambda_1 := B_{r_1}(a_1)$, $\Lambda_{i+1} := B_{r_{i+1}}(a_{i+1}) \setminus \bigcup_{k=1}^{i} \Lambda_k$, and define

$$\psi(a) := \sum_{i\geq 1} b_i \mathbf{1}_{\Lambda_i}(a),$$

where $b_i := b_{a_i}$.

Then $\psi : A \to B$ is measurable, bounded and, for all $a \in A$,

$$\Big(-\partial_t\varphi + H^{a,\psi(a)}(., D\varphi, D^2\varphi)\Big)(t_0,x_0) \geq \delta. \tag{4.52}$$

With ψ we can define a strategy for player 2 by $\beta[a]_s := \psi(a_s)$. We have seen in Example 38 that $\beta \in \Delta(t)$.

Step 2: Find a non-anticipative controlled stopping time, θ^a.

Due to (4.52), and since A and $\psi(A)$ are compact, we conclude, by the continuity of $H^{a,b}(.)$, that there is R such that, for all $a \in A$,

$$\Big(-\partial_t\varphi + H^{a,\psi(a)}(., D\varphi, D^2\varphi)\Big)(t,x) \geq 0, \text{ for all } (t,x) \in B_R(t_0,x_0). \tag{4.53}$$

Let (t_n,x_n) be a sequence in $B_R(t_0,x_0)$ converging to (t_0,x_0) such that $V(t_n,x_n) \to V^*(t_0,x_0)$ and let $X^{a,n}_\cdot := X^{a,\beta[a]}_{t_n,x_n}(.)$. For each n, we consider the family of stopping times

$$\theta^a_n := \inf\{s \geq t_n : (s, X^{a,n}_s) \notin B_R(t_0,x_0)\} < +\infty.$$

Then, for each n, by Example 55, $\{\theta^a_n\}_{a\in\mathcal{A}_{t_n}}$ is a non-anticipative controlled stopping time.

Step 3: Getting the contradiction.

By Itô's formula, we have

$$\varphi\left(\theta^a_n, X^{a,n}_{\theta^a_n}\right) - \varphi(t_n,x_n) = \int_{t_n}^{\theta^a_n} \Big(\partial_t\varphi - H^{a_s,\beta[a]_s}(., D\varphi, D^2\varphi)\Big)(s, X^{a,n}_s)\, ds + \\ + \int_{t_n}^{\theta^a_n} \left(D\varphi\, \sigma(.; a_s, \beta[a]_s)\right)(s, X^{a,n}_s)\, dW_s.$$

Since $\sigma, D\varphi$ are continuous and $(s, X_s^{a,n})$ is bounded for $s \in [t_n, \theta_n^a]$, we have that

$$\mathbb{E}\left[\int_{t_n}^{\theta_n^a} (D\varphi\, \sigma(.; a_s, \beta[a]_s))\, (s, X_s^{a,n})\, dW_s \Big| \mathcal{F}_{t_n}\right] = 0.$$

On the other hand, $\beta[a]_s = \psi(a_s)$, hence, by taking expectations, we get

$$\begin{aligned} \varphi(t_n, x_n) &= \mathbb{E}\left[\varphi\left(\theta_n^a, X_{\theta_n^a}^{a,n}\right) - \int_{t_n}^{\theta_n^a} \left(\partial_t \varphi - H^{a_s, \psi(a_s)}(., D\varphi, D^2\varphi)\right)(s, X_s^{a,n}) ds \Big| \mathcal{F}_{t_n}\right] \\ &\geq \mathbb{E}\left[\varphi\left(\theta_n^a, X_{\theta_n^a}^{a,n}\right) \Big| \mathcal{F}_{t_n}\right], \end{aligned} \tag{4.54}$$

where the inequality follows from (4.53).

Now we notice that $X_{\theta_n^a}^{a,n} \in \partial B_R(t_0, x_0)$, hence there is $\eta > 0$ such that

$$\varphi\left(\theta_n^a, X_{\theta_n^a}^{a,n}\right) \geq \phi\left(\theta_n^a, X_{\theta_n^a}^{a,n}\right) + 2\eta. \tag{4.55}$$

Since $\varphi(t_n, x_n) \to \varphi(t_0, x_0) = V^*(t_0, x_0)$ and $V(t_n, x_n) \to V^*(t_0, x_0)$ we conclude that, for n sufficiently large,

$$V(t_n, x_n) \geq \varphi(t_n, x_n) - \eta. \tag{4.56}$$

Combining (4.54), (4.55) and (4.56), we conclude that

$$V(t_n, x_n) \geq \mathbb{E}\left[\phi\left(\theta_n^a, X_{\theta_n^a}^{a,n}\right) \Big| \mathcal{F}_{t_n}\right] + \eta.$$

Since $a \in \mathcal{A}_{t_n}$ is arbitrary we deduce that

$$V(t_n, x_n) \geq \operatorname*{esssup}_{a \in \mathcal{A}_{t_n}} \mathbb{E}\left[\phi\left(\theta_n^a, X_{\theta_n^a}^{a,}\right) \Big| \mathcal{F}_{t_n}\right] + \eta,$$

hence

$$V(t_n, x_n) \geq \operatorname*{essinf}_{\beta \in \Delta(t_n)} \operatorname*{esssup}_{a \in \mathcal{A}_{t_n}} \mathbb{E}\left[\phi\left(\theta_n^a, X_{t_n, x_n}^{a, \beta[a]}(\theta_n^a)\right) \Big| \mathcal{F}_{t_n}\right] + \eta,$$

which contradicts the inequality (4.26) of the weak DPP. □

Proof of 2. We now give the proof of the supersolution property. The boundary condition is completely analogous to that of the subsolution. In the interior of the domain we proceed again by contradiction, that is, we suppose that there is (t_0, x_0) and ϕ smooth satisfying

$$(V_* - \phi)(t, x) > (V_* - \phi)(t_0, x_0) = 0, \text{ for all } (t, x) \in [0, T) \times \mathbb{R}^d, (t, x) \neq (t_0, x_0),$$

such that

$$\left(-\partial_t \phi + (H^-)^*(., D\phi, D^2\phi)\right)(t_0, x_0) \leq -3\delta,$$

for some $\delta > 0$.

Step 1: Find a control a that contradicts the weak DPP.

Consider $\varphi(t, x) = \phi(t, x) - |t - t_0|^2 - |x - x_0|^4$. Then

$$\left(-\partial_t \varphi + (H^-)^*(., D\varphi, D^2\varphi)\right)(t_0, x_0) \leq -3\delta.$$

Since $(H^-)^*$ is upper semi-continuous, there is R such that

$$\left(-\partial_t \varphi + (H^-)^*(., D\varphi, D^2\varphi)\right)(t, x) \leq -2\delta, \text{ for all } (t, x) \in \overline{B_R(t_0, x_0)}.$$

By $(H^-)^* \geq H^-$, we then conclude that

$$\left(-\partial_t \varphi + H^-(., D\varphi, D^2\varphi)\right)(t,x) \leq -2\delta, \text{ for all } (t,x) \in \overline{B_R(t_0,x_0)}.$$

We now remark that $\sup_{b\in B} H^{a,b}(., D\varphi, D^2\varphi)(t,x)$ is lower semi-continuous in a and (t,x) because it is the supremum of continuous functions. Taking in account this fact we deduce that

$$F := \left\{(t,x;a) \in \overline{B_R(t_0,x_0)} \times A : \left(-\partial_t\varphi + \sup_{b\in B} H^{a,b}(., D\varphi, D^2\varphi)\right)(t,x) \leq -\delta\right\}$$

is a closed set and, by the previous inequality, $\pi_{\overline{B_R(t_0,x_0)}}(F) = \overline{B_R(t_0,x_0)}$.

Thus, by Proposition 151, there is a measurable function $a^* : \overline{B_R(t_0,x_0)} \to A$ such that, for all $b \in B$,

$$\left(-\partial_t\varphi + H^{a^*(t,x),b}(., D\varphi, D^2\varphi)\right)(t,x) \leq -\delta \leq 0, \text{ for all } (t,x) \in \overline{B_R(t_0,x_0)}. \tag{4.57}$$

We then have by Example 33 that $a_s := a^*(s, X_s)$ defines an admissible control.

Step 2: Find a non-anticipative controlled stopping time, θ^b.

Consider

$$\eta := \frac{1}{2} \min_{(t,x)\in\partial B_R(t_0,x_0)} (\phi(t,x) - \varphi(t,x)) > 0.$$

Let (t_n, x_n) be a sequence in $B_R(t_0,x_0)$ converging to (t_0,x_0) such that $V(t_n,x_n) \to V_*(t_0,x_0)$. Then $\varphi(t_n,x_n) \to \varphi(t_0,x_0) = V_*(t_0,x_0)$, hence, for a fixed n sufficiently large,

$$V(t_n,x_n) \leq \varphi(t_n,x_n) + \eta. \tag{4.58}$$

We now define $(t,x) := (t_n,x_n)$, $X^b_. := X^{a,b}_{t,x}(.)$ and

$$\theta^b := \inf\{s \geq t : (s, X^b_s) \notin B_R(t_0,x_0)\} < +\infty.$$

Then, by Example 55, $\{\theta^b\}_{b\in\mathcal{B}_t}$ is a non-anticipative controlled stopping time.

Step 3: Getting the contradiction.

Let $\beta \in \Delta(t)$. By Itô's formula, we have

$$\varphi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right) - \varphi(t,x) = \int_t^{\theta^{\beta[a]}} \left(\partial_t\varphi - H^{a_s,\beta[a]_s}(., D\varphi, D^2\varphi)\right)\left(s, X^{\beta[a]}_s\right) ds + \\ + \int_t^{\theta^{\beta[a]}} (D\varphi\, \sigma(.; a_s, \beta[a]_s))\left(s, X^{\beta[a]}_s\right) dW_s,$$

Since $\sigma, D\varphi$ are continuous and $\left(s, X^{\beta[a]}_s\right)$ is bounded for $s \in \left[t, \theta^{\beta[a]}\right]$, we have that

$$\mathbb{E}\left[\int_t^{\theta^{\beta[a]}} (D\varphi\, \sigma(.; a_s, \beta[a]_s))\left(s, X^{\beta[a]}_s\right) dW_s \Big| \mathcal{F}_t\right] = 0.$$

On the other hand $a_s = a^*$, hence, by taking expectations we get

$$\begin{aligned} \varphi(t,x) &= \mathbb{E}\left[\varphi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right) - \int_t^{\theta^{\beta[a]}} \left(\partial_t\varphi - H^{a^*(.),\beta[a]_s}(., D\varphi, D^2\varphi)\right)\left(s, X^{\beta[a]}_s\right) ds \Big| \mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\varphi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right)\Big|\mathcal{F}_t\right], \end{aligned} \tag{4.59}$$

where the inequality follows from (4.57).

Since $X^{\beta[a]}_{\theta^{\beta[a]}} \in \partial B_R(t_0, x_0)$ we have, by definition of η, that

$$\varphi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right) \leq \phi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right) - 2\eta. \tag{4.60}$$

Combining (4.58), (4.59) and (4.60), we conclude that

$$V(t,x) \leq \mathbb{E}\left[\phi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right)\Big|\mathcal{F}_t\right] - \eta \leq \operatorname*{esssup}_{a\in\mathcal{A}_t} \mathbb{E}\left[\phi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right)\Big|\mathcal{F}_t\right] - \eta,$$

and, since $\beta \in \Delta(t_n)$ is arbitrary, we deduce that

$$V(t,x) \leq \operatorname*{essinf}_{\beta\in\Delta(t)} \operatorname*{esssup}_{a\in\mathcal{A}_t} \mathbb{E}\left[\phi\left(\theta^{\beta[a]}, X^{\beta[a]}_{\theta^{\beta[a]}}\right), \Big|\mathcal{F}_t\right] - \eta.$$

which contradicts the inequality (4.27) of the weak DPP. □

Analogously we obtain a PDE characterization for the upper value:

Theorem 90. *Suppose B is compact. Then*

1. *U^* is a viscosity subsolution of*

$$\begin{cases} -\partial_t U^* + H^+_*(., DU^*, D^2U^*) \leq 0, & on \quad (0,T)\times\mathbb{R}^d \\ -U^* \leq -f & on \quad \{T\}\times\mathbb{R}^d; \end{cases}$$

2. *U_* is a viscosity supersolution of*

$$\begin{cases} -\partial_t U_* + H^+(., DU_*, D^2U_*) \geq 0, & on \quad (0,T)\times\mathbb{R}^d \\ -U_* \geq -f & on \quad \{T\}\times\mathbb{R}^d. \end{cases}$$

4.6.1 Uniqueness

In this Section we prove uniqueness of solution for (4.51) in the case where both A, B are compact. The procedure will be similar to that of the Laplace equation: we will prove that the difference of solutions is a subsolution of an equation for which there is a maximum principle. From that we extract that the supremum is attained at the boundary and hence is zero.

We assume throughout this Section that A, B are compact. In that case we know that H^- is continuous, hence $H^-_* = (H^-)^* = H^-$.

Lemma 91. *Suppose that A, B are compact. If v_1, v_2 are respectively a viscosity subsolution and supersolution of (4.51) then $w = v_1 - v_2$ is a viscosity subsolution of*

$$\begin{cases} -\partial_t w + \inf_{a\in A,\ b\in B} H^{a,b}(.; Dw, D^2w) \leq 0, & on \quad (0,T)\times\mathbb{R}^d \\ w \leq 0 & on \quad \{T\}\times\mathbb{R}^d \end{cases} \tag{4.61}$$

Proof. Let $\phi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and fix $(t_0, x_0) \in \operatorname{argmax}(w - \phi)$. By Remark 101 we can suppose that (t_0, x_0) is actually a strict maximum, that is:

$$(w-\phi)(t,x) < (w-\phi)(t_0,x_0) = 0, \text{ for all } (t,x) \in \overline{B_R(t_0,x_0)}, (t,x) \neq (t_0,x_0).$$

We then consider the function

$$\psi_\varepsilon(t,x,y) = v_1(t,x) - v_2(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \phi(t,x),$$

and the points (depending on ε)

$$(\bar{t},\bar{x},\bar{y}) \in \operatorname*{argmax}_{[t_0-R,t_0+R]\times\overline{B_R(x_0)}\times\overline{B_R(x_0)}} \psi_\varepsilon(t,x,y).$$

Since (t_0, x_0) is the global maximum of $w - \phi$ we know, by Lemma 109, that

- $(\bar{t}, \bar{x}, \bar{y}) \to (t_0, x_0, x_0)$ as $\varepsilon \to 0$;
- $\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}$ is bounded and converges to 0 as $\varepsilon \to 0$.

It follows that for ε sufficiently small, $(\bar{t}, \bar{x}, \bar{y}) \in (t_0-R, t_0+R)\times B_R(x_0)\times B_R(x_0)$ is a local maximum of ψ_ε, hence by the parabolic version of Ishii's Lemma, Lemma 119, there exist $\bar{q}_1, \bar{q}_2 \in \mathbb{R}$, and $X, Y \in \mathcal{S}^d$ such that

$$\begin{aligned}
(\bar{q}_1 + \partial_t\phi(\bar{t},\bar{x}), \bar{p} + D\phi(\bar{t},\bar{x}), X) &\in \bar{D}^{(2,1),+}v_1(\bar{t},\bar{x}),\\
(\bar{q}_2, \bar{p}, Y) &\in \bar{D}^{(2,1),-}v_2(\bar{t},\bar{y}),\\
q_1 - q_2 &= 0,\\
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} &+ \begin{pmatrix} D^2\phi(\bar{t},\bar{x}) & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{4.62}$$

where

$$\bar{p} = \frac{2(\bar{x}-\bar{y})}{\varepsilon^2}$$

Applying vectors of the form (x, x) to the quadratic forms of inequality (4.62) we conclude that

$$X \le Y + D^2\phi(\bar{t},\bar{x}).$$

Since v_1, v_2 are, respectively, a subsolution and a supersolution of (4.51), we have

$$\begin{aligned}
-\bar{q}_1 - \partial_t\phi(\bar{t},\bar{x}) + H^-(\bar{t},\bar{x},\bar{p}+D\phi(\bar{t},\bar{x}),X) &\le 0,\\
-\bar{q}_2 + H^-(\bar{t},\bar{y},\bar{p},Y) &\ge 0.
\end{aligned}$$

Subtracting the 2 inequalities we get

$$-\partial_t\phi(\bar{t},\bar{x}) + H^-(\bar{t},\bar{x},\bar{p}+D\phi(\bar{t},\bar{x}),X) - H^-(\bar{t},\bar{y},\bar{p},Y) \le 0. \tag{4.63}$$

Now we notice that $\inf_a \sup_b H^{a,b} - \inf_a \sup_b H'^{a,b} \ge \inf_{a,b}\left(H^{a,b} - H'^{a,b}\right)$. Indeed, choose a_ε such that $\inf_a \sup_b H^{a,b} \ge \sup_b H^{a_\varepsilon,b} - \varepsilon$, and choose b_ε such that $\sup_b H'^{a_\varepsilon,b} \le H'^{a_\varepsilon,b_\varepsilon} + \varepsilon$. Then:

$$\begin{aligned}
\inf_a \sup_b H^{a,b} - \inf_a \sup_b H'^{a,b} &\ge \sup_b H^{a_\varepsilon,b} - \varepsilon - \sup_b H'^{a_\varepsilon,b}\\
&\ge H^{a_\varepsilon,b_\varepsilon} - H'^{a_\varepsilon,b_\varepsilon} - 2\varepsilon\\
&\ge \inf_{a,b}\left(H^{a,b} - H'^{a,b}\right) - 2\varepsilon.
\end{aligned}$$

Applying this inequality on (4.63) we get

$$-\partial_t\phi(\bar{t},\bar{x}) + \inf_{a\in A, b\in B}\left(H^{a,b}(\bar{t},\bar{x},\bar{p}+D\phi(\bar{t},\bar{x}),X) - H^{a,b}(\bar{t},\bar{y},\bar{p},Y)\right) \le 0.$$

We now need to estimate the difference $H^{a,b}(\bar{t},\bar{x},\bar{p}+D\phi(\bar{t},\bar{x}),X) - H^{a,b}(\bar{t},\bar{y},\bar{p},Y)$. Following Example 112 we conclude that, since $\sigma(t,x;a,b), \mu(t,x;a,b)$ are Lipschitz continuous in x uniformly with respect to t, a, b, then

$$H^{a,b}(\bar{t},\bar{x},\bar{p}+D\phi(\bar{t},\bar{x}),X) - H^{a,b}(\bar{t},\bar{y},\bar{p},Y) \ge H^{a,b}(., D\phi, D^2\phi)(\bar{t},\bar{x}) - C\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2},$$

where C is a constant depending only on K.

Thus,

$$-\partial_t\phi(\bar{t},\bar{x}) + \inf_{a\in A, b\in B} H^{a,b}\left(\bar{t},\bar{x},D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right) - C\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \le 0.$$

Since A, B are compact then $\inf_{a,b} H^{a,b}(.)$ is continuous, hence we can let $\varepsilon \to 0$ to conclude that

$$-\partial_t\phi(t_0,x_0) + \inf_{a\in A, b\in B} H^{a,b}(t_0,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0)) \le 0.$$

□

Since the state space $\mathbb{S}$ is unbounded in the space variable, we need to impose growth conditions to prove uniqueness. Thus, we will prove uniqueness for (4.51) in the space of functions:

$$\Theta = \Big\{w : \mathbb{S} \to \mathbb{R} : \exists \tilde{C} > 0 \text{ such that } \lim_{|x|\to+\infty} w(t,x)\exp\Big(-\tilde{C}\Psi(x)\Big) = 0, \text{ uniformly in } t\Big\},$$

where

$$\Psi(x) := \big(\log(|x|^2+1)+1\big)^2.$$

We remark that this space of functions contains the functions with polynomial growth.

Before proving the maximum principle for (4.61) we need the following auxiliary Lemma:

Lemma 92. *For any $\tilde{C}$, there exists $C_1 > 0$ such that, for all t_2, the function*

$$\chi(t,x) := \exp\Big((C_1(t_2-t)+\tilde{C})\Psi(x)\Big)$$

satisfies

$$-\partial_t\chi + \inf_{a\in A, b\in B} H^{a,b}(., D\chi, D^2\chi) > 0 \text{ in } [t_1,t_2]\times\mathbb{R}^d,$$

where $t_1 := t_2 - \frac{\tilde{C}}{C_1}$.

Proof. By a direct calculation we have

$$\begin{aligned} D\Psi(x) &= 4\frac{\log(|x|^2+1)+1}{|x|^2+1}x^T, \\ D^2\Psi(x) &= -8\frac{\log(|x|^2+1)}{(|x|^2+1)^2}xx^T + 4\frac{\log(|x|^2+1)+1}{|x|^2+1}I. \end{aligned}$$

Thus,

$$|D\Psi(x)| < C\frac{\sqrt{\Psi(x)}}{1+|x|}, \qquad |D^2\Psi(x)| < C\frac{\sqrt{\Psi(x)}}{1+|x|^2}.$$

Let $C_1 > 0$ be arbitrary and $t_1 := t_2 - \frac{\tilde{C}}{C_1}$. Then, for $t \in [t_1, t_2]$,

$$\begin{aligned} |D\chi(t,x)| &= \chi(t,x)(C_1(t_2-t)+\tilde{C})|D\Psi(x)| \\ &\le 2\tilde{C}\chi(t,x)|D\Psi(x)| \\ &< C\chi(t,x)\frac{\sqrt{\Psi(x)}}{1+|x|}, \end{aligned}$$

and, similarly,

$$\begin{aligned} |D^2\chi(t,x)| &= \Big|(C_1(t_2-t)+\tilde{C})\left((D\Psi(x))^T D\chi(t,x) + \chi(t,x)D^2\Psi(x)\right)\Big| \\ &\le C\chi(t,x)\frac{\Psi(x)}{1+|x|^2}. \end{aligned}$$

Notice that the constant C never depends on C_1 by virtue of the choice of t_1. On the other hand, $\partial_t\chi$ depends on C_1 in a crucial way:

$$\partial_t\chi(t,x) = -C_1\Psi(x)\chi(t,x).$$

By the growth conditions on μ, σ, we then conclude that

$$\begin{aligned} -\partial_t\chi(t,x) + \inf_{a\in A, b\in B} H^{a,b}(t,x,D\chi,D^2\chi) &\ge (C_1 - C)\Psi(t,x)\chi(t,x) \\ &> 0, \end{aligned}$$

for $C_1 > C$ large enough. ∎

We can now prove the maximum principle for (4.61).

Theorem 93. *Suppose that A, B are compact and let $w \in \Theta$ be a continuous subsolution of* (4.61)*. Then*

$$\sup_{[0,T]\times\mathbb{R}^d} w = \sup_{\{T\}\times\mathbb{R}^d} w.$$

Proof. Since $w \in \Theta$ there is $\tilde{C}$ such that

$$\lim_{|x|\to\infty} w(t,x)\exp\left(-\tilde{C}\Psi(x)\right) = 0.$$

Let $t_2 := T$, and set t_1 and χ as in Lemma 92. Then, for all ε,

$$M_\varepsilon := \max_{[t_1,T]\times\mathbb{R}^d} w - \varepsilon\chi$$

is attained at some point $(t_\varepsilon, x_\varepsilon)$.

Suppose that there is ε such that $t_\varepsilon < T$. We will proceed towards a contradiction. By the definition of $(t_\varepsilon, x_\varepsilon)$ we have,

$$w(t,x) \leq \phi(t,x) := \varepsilon\chi(t,x) + (w-\varepsilon\chi)(t_\varepsilon, x_\varepsilon),$$

hence $(w-\phi)(t,x) \leq 0 = (w-\phi)(t_\varepsilon, x_\varepsilon)$ for all $t \in [t_1, T]$. Since w is a viscosity subsolution of (4.61) and $t_\varepsilon \in [t_1, T)$ we conclude that

$$\left(-\partial_t\phi + \inf_{a\in A, b\in B} H^{a,b}(., D\phi, D^2\phi)\right)(t_\varepsilon, x_\varepsilon) \leq 0.$$

Notice that to obtain the previous inequality we had to extend the viscosity property of subsolutions up to the initial time t_1. This is made possible by Proposition 117 since w is continuous.

But since $\partial_t\phi = \varepsilon\partial_t\chi$, $D\phi = \varepsilon D\chi$, $D^2\phi = \varepsilon D^2\chi$ and $t_\varepsilon \in [t_1, T]$ we have by Lemma 92 that

$$\left(-\partial_t\phi + \inf_{a\in A, b\in B} H^{a,b}(., D\phi, D^2\phi)\right)(t_\varepsilon, x_\varepsilon) = \varepsilon\left(-\partial_t\chi + \inf_{a\in A, b\in B} H^{a,b}(., D\chi, D^2\chi)\right)(t_\varepsilon, x_\varepsilon) > 0,$$

which is a contradiction.

We conclude that for all ε, $t_\varepsilon = T$ and hence

$$(w-\varepsilon\chi)(t,x) \leq (w-\varepsilon\chi)(t_\varepsilon, x_\varepsilon) \leq \sup_{\{T\}\times\mathbb{R}^d} w,$$

for all $t \in [t_1, T]$.

Thus, taking $\varepsilon \to 0$, we conclude that

$$\sup_{[t_1,T]\times\mathbb{R}^d} w \leq \sup_{\{T\}\times\mathbb{R}^d} w.$$

Repeating the same argument with $t_2 := t_1$ we conclude that

$$\sup_{\left[t_1-\frac{\tilde{C}}{C_1}, t_1\right]\times\mathbb{R}^d} w \leq \sup_{\{t_1\}\times\mathbb{R}^d} w \leq \sup_{[t_1,T]\times\mathbb{R}^d} w \leq \sup_{\{T\}\times\mathbb{R}^d} w,$$

and, by iteration, we get

$$\sup_{[0,T]\times\mathbb{R}^d} w \leq \sup_{\{T\}\times\mathbb{R}^d} w.$$

□

As an immediate Corollary of Lemma 91 and Theorem 93 we get the comparison principle for solutions of (4.51).

Corollary 94 (Comparison principle for the HJBI equation)**.** *Suppose A, B are compact. If $v_1, v_2 \in \Theta$ are continuous functions such that v_1 is a viscosity subsolution and v_2 is a viscosity supersolution of* (4.51) *then $v_1 \leq v_2$.*

From the comparison result we are able to prove that there is a unique viscosity solution in Θ to (4.51).

Theorem 95. *Suppose A, B are compact. Then there is an unique solution $v \in \Theta$ to* (4.51)*.*

Proof. Suppose there exist two solutions v_1, v_2 to (4.51). Then each one is continuous and both are viscosity subsolutions and supersolutions, hence

$$\begin{aligned} v_1 &\leq v_2, \\ v_2 &\leq v_1. \end{aligned}$$

□

Since, by Remark 71, the value function $V(t, x)$ has polynomial growth then $V \in \Theta$. Furthermore, in the case of A, B compact we have by Remark 87 that V is continuous. Thus, it is the unique solution to (4.51).

As a Corollary of the comparison principle we also conclude that $V \leq U$.

Corollary 96. *Suppose A, B are compact. Then*

$$V \leq U.$$

Proof. We have seen that V is a solution of (4.51). Since $H^- \geq H^+$, we conclude that V is a subsolution to

$$\partial_t V + H^+(., DV, D^2V) \leq 0.$$

Since U is a solution to the above equation, we conclude by the comparison principle that $V \leq U$. □

By uniqueness of solution we get, under the *Isaacs' condition*, the existence of value for the differential game.

Corollary 97 (Isaacs' condition)**.** *Suppose A, B are compact. If $H^+ = H^-$ then $V = U$, that is, the game has a value.*

4.7 Merton's optimal portfolio: worst-case approach

In this Section we consider a worst-case approach on Merton's optimal portfolio problem in the setting of stochastic parameters. We will do this by transforming this problem into a differential game.

Recall Merton's problem, studied in Section 2.7. To convert it into a differential game we consider two players: the first one controls π and wants to maximize the utility of the portfolio's terminal value while the second controls μ, σ and wants to minimize it. We suppose that the interest rate, r, is fixed. More precisely, our state variable has dynamics given by (4.1) with coefficients[6]

$$\begin{aligned} \mu(t, x; \pi, \mu, \sigma) &:= x(\pi\mu + (1-\pi)r), \\ \sigma(t, x; \pi, \mu, \sigma) &:= x\pi\sigma, \end{aligned}$$

and the control spaces are $\mathcal{A}_t = \mathbb{H}^\infty(t, T; [\pi_0, \pi_1])$, $\mathcal{B}_t = \mathbb{H}^\infty(t, T; B)$, with B a compact set to be specified later. The reward function is

$$J(t, x; \pi, \mu, \sigma) := \mathbb{E}\left[f\left(X_{t,x}^{\pi,(\mu,\sigma)}(T) \right) \Big| \mathcal{F}_t \right],$$

where $f(x) = x^p$ is the power utility function ($0 < p < 1$).

For the *worst-case approach* we consider the upper value of this stochastic differential game,

$$U(t, x) := \operatorname*{esssup}_{\pi(.)\in\Gamma(t)} \operatorname*{essinf}_{(\mu,\sigma)\in\mathcal{B}_t} J(t, x; \pi(\mu, \sigma), \mu, \sigma),$$

which we know already to be a deterministic function. Moreover, we know that U is the unique viscosity solution to

$$-\partial_t U + \sup_{(\mu,\sigma)\in B} \inf_{\pi\in[\pi_0,\pi_1]} H^{\pi,(\mu,\sigma)}(., DU, D^2U) = 0,$$

where

$$H^{\pi,(\mu,\sigma)}(., DU, D^2U) = -\mu(.; \pi, \mu, \sigma)DU - \frac{1}{2}\sigma(.; \pi, \mu, \sigma)^2 D^2U.$$

Using the same arguments as in Section 2.7, we conclude that $U(t, x) = f(x)U(t, 1)$ and

$$\inf_{\pi\in[\pi_0,\pi_1]} H^{\pi,(\mu,\sigma)}(., DU, D^2U) = H^{\pi^*(\mu,\sigma),(\mu,\sigma)}(., DU, D^2U),$$

where

$$\pi^*(\mu, \sigma) = \min\left(\pi_1, \frac{(\mu - r)}{(1-p)\sigma^2} \vee \pi_0 \right).$$

It remains to obtain $h(t) := U(t, 1)$ which, due to the previous considerations, is a solution to

$$-h' - p(r - \lambda^*)h = 0,$$

where

$$\lambda^* = \sup_{(\mu,\sigma)\in B} -(\mu - r)\pi^*(\mu, \sigma) + (1-p)\frac{\sigma^2}{2}(\pi^*(\mu, \sigma))^2.$$

Thus,

$$u(t, x) = x^p e^{p(T-t)(r-\lambda^*)}.$$

[6] To avoid the use of many letters we will use the same letter for the controls and the values that they take. For example we use π for an element of $\mathbb{H}^\infty(t, T; [\pi_0, \pi_1])$ and also for an element of $[\pi_0, \pi_1]$.

In the following we assume $\pi_0 \le 0 \le \pi_1$. To compute λ^* we notice that

$$\lambda^* = \max(\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

where

$$\begin{aligned}
\lambda_1 &= \sup_{(\mu,\sigma)\in B:\frac{(\mu-r)}{(1-p)\sigma^2}\le\pi_0} -(\mu-r)\pi_0 + (1-p)\frac{\sigma^2}{2}\pi_0^2 \\
&\le \sup_{(\mu,\sigma,r)\in B:\frac{(\mu-r)}{(1-p)\sigma^2}\le\pi_0} -\frac{(\mu-r)}{2}\pi_0 \le 0, \\
\lambda_2 &= \sup_{(\mu,\sigma)\in B:\pi_0\le\frac{(\mu-r)}{(1-p)\sigma^2}\le 0} -\frac{(\mu-r)^2}{2(1-p)\sigma^2} \le 0, \\
\lambda_3 &= \sup_{(\mu,\sigma)\in B:0\le\frac{(\mu-r)}{(1-p)\sigma^2}\le\pi_1} -\frac{(\mu-r)^2}{2(1-p)\sigma^2} \le 0, \\
\lambda_4 &= \sup_{(\mu,\sigma)\in B:\frac{(\mu-r)}{(1-p)\sigma^2}\ge\pi_1} -(\mu-r)\pi_1 + (1-p)\frac{\sigma^2}{2}\pi_1^2 \\
&\le \sup_{(\mu,\sigma)\in B:\frac{(\mu-r)}{(1-p)\sigma^2}\ge\pi_1} -\frac{(\mu-r)}{2}\pi_1 \le 0.
\end{aligned}$$

Thus $\lambda^* \le 0$. To proceed any further we need to consider a particular set B. We consider B to be a rectangle in the product space, that is $B = [\mu_0, \mu_1] \times [\sigma_0, \sigma_1]$. It is then easy to see that the optimal μ, σ must be such that $|\mu - r|$ is minimal and σ is maximal.

4.8 Conclusions and further research

We have extended the weak dynamic programming principle of [1] to the context of stochastic two-person zero-sum differential games. We used this principle to derive, under weaker assumptions than the ones existing in the literature, that both value functions are viscosity solutions of the associated Hamilton-Jacobi-Bellman-Isaacs equations.

In this Section we compare the assumptions made to deduce our main results, with the ones typically made in the literature, in particular those of the pioneering paper by Fleming and Souganidis, [2], and those of the more recent paper by Buckdahn and Li, [10]. We also compare this version of the weak dynamic programming principle with the original one, obtained for the stochastic optimal control problem. Our objective in this comparison is to stress the main differences and the difficulties encountered when extending the principle. Finally we indicate some directions where further research can be made in order to complete this work.

A more general scenario

There are two points where the setting considered in this thesis is more general than the one in [2, 10]: we consider a larger control space for one of the players and a larger set of payoff functions. Indeed, to derive that the lower value function is a viscosity solution of the HJBI equation, the set B is allowed to be unbounded and the payoff function f needs only to be locally Lipschitz with polynomial growth (as opposed to globally Lipschitz).

If this approach is to be applied to both the lower and upper values at the same time then stronger conditions must be considered. Indeed, for our results to apply to the lower value we need A compact while for the upper value we need B compact. Thus, to apply our results to the lower and upper values simultaneously, then both A, B need to be compact sets. However, there are situations where only one of the value functions is of interest, like for example in worst-case approaches, as the one studied in Section 4.7. In such a situation our approach provides a more general framework.

Comparison with stochastic optimal control

In Section 4.5.1 we see the stochastic optimal control problem as a particular case of stochastic differential games. There we see that many assumptions used to deduce the weak dynamic programming principle are no longer necessary.

In fact, differential games are much more delicate than optimal control problems. The main reason for this is the existence, in a differential game, of two optimization problems: a minimization and a maximization. Because of this, none of the inequalities on the dynamic programming principle is deduced “for free”. Furthermore, since the two optimization problems have opposite objectives, we need extra regularity on the reward function, in order to control the effect that the actions of one player might have upon the other player's choices.

Another factor that makes differential games more difficult than optimal control is the non-anticipativity property. Indeed, when proving the dynamic programming principle we have the need to construct strategies that are non-anticipative, which makes the proof more technical. For example, just to formulate correctly the weak dynamic programming principle in this new context, we had the need to introduce the notion of non-anticipative controlled stopping time. Introducing this, and other related notions, and studying some of their properties, took us a big part of Section 4.3.6.

Some directions of further research

We indicate four main directions where further research can be taken:

- Regarding the existence of uniformly $\varepsilon-$optimal strategies, $\beta^{\varepsilon}_{(t,x)}$ (recall Definition 74). One question that remains open is whether considering A compact implies automatically the existence of such strategies. Some steps we took in this direction can be found in Proposition 68, where sufficient conditions were deduced. One of this conditions implies, in particular, that if there are $\varepsilon-$optimal Markov control policies for the player allowed to use strategies, then there exist uniformly $\varepsilon-$optimal strategies.

Another question to be answered has to do with the norm of these strategies, $\|\beta^{\varepsilon}_{(t,x)}\|_p$. Indeed, in the proof of the weak dynamic programming principle we assume that $\|\beta^{\varepsilon}_{(t,x)}\|_p$ is locally bounded as a function of (t,x). Finding sufficient conditions for this property to hold is, thus, an important issue. In fact, we would like to find conditions for the stronger property

$$\exists_{C,m} \ \|\beta^{\varepsilon}_{(t,x)}\|_p \leq C(1+|x|^m)$$

to hold. We assumed this result to prove that the value function has polynomial growth.

- Extending the result to the case where the set A is unbounded. We assume that A is compact so as to ensure that the reward function has the required continuity, which must be uniform with respect to the control of the first player. We discuss the need for this condition in Remark 81. It would be interesting if we could lift this assumption so as to be able to consider both A, B to be unbounded. If this is not possible it would then be interesting to find counterexamples.

- Considering even larger control spaces. The space of admissible controls considered in this thesis is the space of essentially bounded controls, $\mathbb{H}^{\infty}$. We chose this space so as to be able to use Lemma 148. As stated, in a footnote, in the beginning of Section 4.3.2, we could have considered $\mathbb{H}^{p,\infty}$ for $p > 2$ (even though in that case we would have to restrict the set of considered payoff functions). A question that remains open is whether we can make $p = 2$ and consider $\mathbb{H}^{2,\infty}$.

  In the literature on stochastic optimal control a weaker condition on the integrability of the controls is often made. For example, in the reference for stochastic optimal control [4, p. 153], the admissible controls must be such that all its moments are finite.

- Proving uniqueness of solution for the HJBI equation, when A or B are not compact. Regarding uniqueness, in this thesis we only considered the case where both A, B are compact. If one of these sets fails to be compact then the Hamiltonian can be discontinuous and the arguments in the proof of uniqueness need to be modified.

  If a comparison result is proved for the HJBI equation with discontinuous Hamiltonian then we would have as a consequence

$$V(t,x) \leq V^*(t,x) \leq V_*(t,x) \leq V(t,x),$$

  which proves that V is continuous.

4.9 Notation

In this Section we review the main notation used throughout this Chapter.

Ω - Space of continuous functions from $[0, T]$ to $\mathbb{R}^N$ starting from 0;

ω - Denotes a generic element $\omega \in \Omega$;

$\mathcal{F}$ - Borel $\sigma-$algebra in Ω;

Λ, Π - Generic elements of $\mathcal{F}$;

$\mathbb{P}$ - Wiener measure in Ω;

$\mathcal{N}_{\mathbb{P}}$ - $\mathbb{P}-$null sets;

$W_.$ - $N-$dimensional Brownian motion in $(\Omega, \mathcal{F}, \mathbb{P})$ corresponding to the coordinate process, $W_s(\omega) = \omega_s$;

$\mathbb{F}$ - Natural filtration induced by $\{W_s\}$ augmented with the $\mathbb{P}-$null sets, $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$;

$\mathcal{T}$ - Collection of all stopping times in $\mathbb{F}$;

τ, θ - Generic elements of $\mathcal{T}$;

$\mathbb{S}$ - State space, $\mathbb{S} = [0, T] \times \mathbb{R}^d$;

$(t, x), (t', x'), (s, y)$ - Generic elements of $\mathbb{S}$;

A, B - Sets where the controls of players 1 and 2, respectively, take values;

$\mathcal{A}_t, \mathcal{B}_t$ - Space of controls, starting at time t, for players 1 and 2, respectively;

a, b - Denote, respectively, generic elements of both A, B and $\mathcal{A}_t, \mathcal{B}_t$;

$\Gamma(t), \Delta(t)$ - Space of strategies, starting at time t, for players 1 and 2, respectively;

α, β - Denote, respectively, generic elements of $\Gamma(t), \Delta(t)$;

$X_{t,x}^{a,b}(.)$ - Controlled state process taking values in $\mathbb{R}^d$;

$\mu(t, x; a, b)$ - Growth rate of the state process, $\mu : \mathbb{S} \times A \times B \to \mathbb{R}^d$;

$\sigma(t, x; a, b)$ - Volatility of the state process, $\sigma : \mathbb{S} \times A \times B \to \mathbb{R}^{d \times N}$;

K - Lipschitz constant;

C - Generic constant depending at most in K, T;

$f(x)$ - Payoff function, $f : \mathbb{R}^d \to \mathbb{R}$;

p - Growth power of f, i.e., p is such that $|f(x)| \leq C(1 + |x|^p)$.

$J(t, x; a, b)$ - Reward function, $J(t, x; a, b) := \mathbb{E}\left[f\left(X_{t,x}^{a,b}(T)\right) \middle| \mathcal{F}_t\right]$;

$V(t, x)$, $U(t, x)$ - Lower and upper values of the stochastic differential game:

$$\begin{aligned} V(t,x) &:= \operatorname*{essinf}_{\beta \in \Delta(t)} \operatorname*{esssup}_{a \in \mathcal{A}_t} J(t, x; a, \beta[a]), \\ U(t,x) &:= \operatorname*{esssup}_{\alpha \in \Gamma(t)} \operatorname*{essinf}_{b \in \mathcal{B}_t} J(t, x; \alpha[b], b). \end{aligned}$$

$\mathcal{U}$ - Generic space of controls. Typically, $\mathcal{U} = \mathcal{A}_t$, $\mathcal{U} = \mathcal{B}_t$ or $\mathcal{U} = \mathcal{A}_t \times \mathcal{B}_t$;

ν - Generic element $\nu \in \mathcal{U}$;

$B_r(x)$ - Ball of radius r centered at $x \in \mathbb{R}^d$;

$B(t, x; r) := [t-r, t] \times B_r(x)$;

ϕ, φ - $C^{1,2}$ test functions;

Θ - Space of functions where uniqueness for the HJBI equation will hold;

I - Identity matrix of dimensions $d \times d$;

$\mathrm{USC}(Q), \mathrm{LSC}(Q)$ - Spaces of functions which are, respectively, upper semi-continuous or lower semi-continuous in $Q \subset \mathbb{S}$.

$\mathrm{graph}(\psi)$ - Denotes the graph of a real-valued function $\psi : S \to \mathbb{R}$, that is,

$$\mathrm{graph}(\psi) := \{(x, \psi(x)) : x \in S\} \subset S \times \mathbb{R}.$$

$\pi_A(F)$ - Denotes the projection of $F \subset A \times B$ in A;

$t_1 \wedge t_2, t_1 \vee t_2$ - Denotes, respectively, $\min(t_1, t_2)$ and $\max(t_1, t_2)$;

$X =_\Lambda Y$ - Abbreviation for $\mathbf{1}_\Lambda X = \mathbf{1}_\Lambda Y$, where X, Y are random variables and $\Lambda \in \mathcal{F}$;

$X \equiv_\Lambda Y$ on $[\tau_1, \tau_2]$ - Abbreviation for $\mathbb{P}(\{\mathbf{1}_\Lambda X_s = \mathbf{1}_\Lambda Y_s; s \text{ a.e. on } [\tau_1, \tau_2]\}) = 1$, where X, Y are stochastic processes, $\tau_1, \tau_2 \in \mathcal{T}$, and $\Lambda \in \mathcal{F}$;

$X \equiv Y$ on $[\tau_1, \tau_2]$ - Abbreviation for $X \equiv_\Omega Y$ on $[\tau_1, \tau_2]$;

$\nu_1 \oplus_\theta \nu_2$ - Denotes the concatenation of controls $\nu_1, \nu_2 \in \mathcal{U}$ at a stopping time $\theta \in \mathcal{T}$, that is

$$(\nu_1 \oplus_\theta \nu_2)_s := (\nu_1)_s \mathbf{1}_{[t,\theta]}(s) + (\nu_2)_s \mathbf{1}_{(\theta,T]}(s).$$

Appendix A

Viscosity solutions

In this Appendix we briefly survey important definitions and some key results of the theory of viscosity solutions of second order partial differential equations. The exposition follows closely [11].

A.1 Notion of viscosity solution

The theory of viscosity solutions applies to second order partial differential equations of the type

$$H(x, v, Dv, D^2 v) = 0, \tag{A.1}$$

where $H(x, r, p, X)$ is a continuous function that satisfies the following monotonicity condition:

$$H(x, r, p, X) \leq H(x, s, p, Y), \text{ when } r \leq s, \ X \geq Y.$$

When this condition is satisfied we say that H is *proper*.

Example 98. *To illustrate the scope of the theory we present some examples:*

- *First order equations, $H(x, v, Dv) = 0$, where $H(x, r, p)$ is nondecreasing in $r \in \mathbb{R}$.*
- Hamilton-Jacobi-Bellman *equations of first and second order:*

$$H(x, r, p, X) = \sup_{\alpha} \left(-\langle \mu(x; \alpha), p \rangle - \operatorname{Tr}((\sigma\sigma^T)(x; \alpha)X) + c(x; \alpha)r \right),$$

  where $c(x; \alpha) \geq 0$.

Remark 99. *If for each t, $H(t, x, r, p, X)$ is proper then so is the associated* parabolic equation

$$-\partial_t v + H(.; v, Dv, D^2 v) = 0, \tag{A.2}$$

which we write as $F(t, x, v, \partial_t v, Dv, D^2 v) = 0$.

In general, possibly degenerate elliptic or parabolic equations such as (A.1) or (A.2) do not admit smooth solutions. For instance it is very easy to construct examples using the method of characteristics for which (A.1) does not admit classical solutions, see [12], for instance.

Through the use of test functions we will be able to consider merely continuous solutions (or even discontinuous) to these equations, in the following sense:

Definition 100. *Let $Q \subset \mathbb{R}^d$. Then v is a* viscosity subsolution *of* (A.1) *in Q if $v \in \mathrm{USC}(Q)$ and for each $\varphi \in C^2(Q)$,*

$$H(., v, D\varphi, D^2\varphi)(\hat{x}) \leq 0, \tag{A.3}$$

at every $\hat{x} \in Q$ such that $v - \varphi$ has a local maximum at $\hat{x}$ on Q.

Similarly, v is a viscosity supersolution *of* (A.1) *in Q if $v \in \mathrm{LSC}(Q)$ and for each $\varphi \in C^2(Q)$,*

$$H(., v, D\varphi, D^2\varphi)(\hat{x}) \geq 0, \tag{A.4}$$

at every $\hat{x} \in Q$ such that $v - \varphi$ has a local minimum at $\hat{x}$ on Q.

We say that v is a viscosity solution *of* (A.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

We now make a few remarks on the definition of subsolution. The same remarks also apply to supersolutions.

Remark 101. *Let φ be a test function, and $\hat{x} \in \operatorname{argmax}_Q(v - \varphi)$. Let R be such that*

$$(v - \varphi)(x) \leq (v - \varphi)(\hat{x}), \quad \text{for all } x \in B_R(\hat{x}).$$

We make some observations regarding the nature of this maximizer and the definition of subsolution:

- *For the definition of subsolution we can consider that $\hat{x}$ is a strict maximizer. Indeed, $v - \bar{\varphi}$ has a strict local maximum at $\hat{x}$, where $\bar{\varphi}(x) := \varphi(x) + |x - \hat{x}|^4$. Furthermore, since $D\varphi(\hat{x}) = D\bar{\varphi}(\hat{x})$ and $D^2\varphi(\hat{x}) = D^2\bar{\varphi}(\hat{x})$, then the viscosity property* (A.3) *is valid for φ iff it is valid for $\bar{\varphi}$.*
- *Similarly, we can consider that $v(\hat{x}) = \varphi(\hat{x})$. If not, we can take the test function to be $\bar{\varphi}(x) := \varphi(x) + v(\hat{x}) - \varphi(\hat{x})$.*
- *If v is bounded above by some positive function $f \in C^2(Q)$, we can consider that $\hat{x}$ is a global maximizer. Indeed, let η_{r_1,r_2} denote a C^2 cutoff function such that $\eta_{r_1,r_2}\big|_{B_{r_1}(\hat{x})} = 1$ and $\eta_{r_1,r_2}\big|_{(B_{r_2}(\hat{x}))^c} = 0$. Define*

$$\bar{\varphi}(x) := \eta_{\frac{R}{2},R}(x)\varphi(x) + \left(1 - \eta_{\frac{R}{2},R}(x)\right) f(x).$$

By the previous observation we can assume that $\varphi(\hat{x}) = v(\hat{x})$. Then, it is immediate that

$$(v - \bar{\varphi})(x) \leq 0 = (v - \bar{\varphi})(\hat{x}),$$

hence $\hat{x}$ is a global maximizer of $v - \bar{\varphi}$. Since the derivatives of $\varphi, \bar{\varphi}$ at $\hat{x}$ are equal, we conclude that φ satisfies the viscosity property (A.3) *iff $\bar{\varphi}$ does.*

Remark 102. *In Definition 100 we have taken the space of test functions to be $C^2(Q)$. Due to the continuity of H, instead of $C^2(Q)$ we can consider any dense subspace of $(C^2(Q), ||.||_\infty)$. In particular, we can consider $C^\infty(Q)$ to be the space of test functions.*

Indeed if $v - \varphi$ has a strict local maximum at $\hat{x}$ and $\varphi_n \to \varphi$ uniformly then there is a sequence $x_n \to \hat{x}$ such that $v(x_n) \to v(\hat{x})$ and $v - \varphi_n$ has a strict local maximum at x_n. Thus, by continuity of H, φ and upper semi-continuity of v, we get

$$H\left(\hat{x}, v(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})\right) = \lim H\left(x_n, v(x_n), D\varphi_n(x_n), D^2\varphi_n(x_n)\right) \leq 0$$

We want the definition of viscosity solution to be compatible with that of classical solution. More precisely, if $v \in C^2(Q)$ is a solution of (A.1) in the classical sense then it should also be a solution in the viscosity sense. If Q is an open set then this is indeed true.

Proposition 103. *Suppose Q is an open set. If $v \in C^2(Q)$ is a classical solution of* (A.1) *then it is also a viscosity solution.*

Proof. Let $v \in C^2(Q)$ be a classical solution of (A.1). We prove that v is viscosity subsolution of (A.1). Similarly, we can prove that v is also a viscosity supersolution.

Consider $\varphi \in C^2(Q)$ and $\hat{x} \in \operatorname{argmax}(v - \varphi)$. Then, since $v, \varphi \in C^2(Q)$ and Q is an open set, the following hold:

$$\begin{aligned} Dv(\hat{x}) &= D\varphi(\hat{x}), \\ D^2v(\hat{x}) &\leq D^2\varphi(\hat{x}). \end{aligned}$$

Since H is proper, we have, by (A.2), that

$$H(.,v,D\varphi,D^2\varphi)(\hat{x}) \leq H(.,v,Dv,D^2v)(\hat{x}) = 0.$$

Thus, v is a viscosity subsolution of (A.1). □

Using these test functions we can give a sense to (Dv, D^2v) for a non-differentiable function v. This goes as follows:

Definition 104. *Let v be upper semi-continuous in Q. The* second-order superdifferential *of v at x is the set*

$$D_Q^{2,+}v(x) := \left\{ \left(D\varphi(x), D^2\varphi(x)\right) : \varphi \in C^2(Q) \text{ such that } v-\varphi \text{ has a local maximum at } x \right\}.$$

Let v be lower semi-continuous in Q. The second-order subdifferential *of v at x is the set $D_Q^{2,-}v(x) := -D_Q^{2,+}(-v)(x)$.*

The definition of viscosity solutions can then be made in terms of the superdifferentials and subdifferentials.

Remark 105. *If x is an interior point of Q then $D_Q^{2,+}v(x)$ does not depend on Q. More precisely, if $x \in \operatorname{int} Q \cap \operatorname{int} Q'$ then $D_Q^{2,+}v(x) = D_{Q'}^{2,+}v(x)$. In this case we define $D^{2,+}v(x) := D_Q^{2,+}v(x)$.*

Remark 106. *If $v-\varphi$ has a local maximum at $\hat{x}$ then $v(x) \leq \varphi(x) + v(\hat{x}) - \varphi(\hat{x})$ and, by the Taylor expansion of φ, we get*

$$v(x) \leq v(\hat{x}) + \langle D\varphi(\hat{x}), x-\hat{x}\rangle + \frac{1}{2}\langle D^2\varphi(\hat{x})(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2).$$

Thus,

$$D_Q^{2,+}v(\hat{x}) \subset \left\{ (p,X) : \limsup_{Q \ni x \to \hat{x}} \frac{v(x) - v(\hat{x}) - \langle p, x-\hat{x}\rangle - \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle}{|x-\hat{x}|^2} \leq 0 \right\}.$$

In fact, the other inclusion is also true, hence we get an alternative definition for $D_Q^{2,+}v$.

Remark 107. *If $v \in C^2$ then $D^{2,+}v(x) = \{(Dv(x), D^2v(x) + X) : X \geq 0\}$.*

A.2 Uniqueness for the Dirichlet problem

On this Section we suppose that Q is a compact set. We will study uniqueness of solution of (A.1) in the case of a Dirichlet boundary condition on ∂Q.

We start by recalling the argument which establishes a *comparison principle* between classical subsolutions and supersolutions.

Proposition 108 (Classic comparison principle)**.** *Suppose that $H(x,r,p,A)$ is strictly increasing in r. Let u, v be, respectively, a subsolution and a supersolution of* (A.1) *such that $u_{|\partial Q} = v_{|\partial Q}$. Then*

$$u \leq v.$$

Proof. Let $w := u - v$. The result follows by the following *maximum principle*:

$$\max_Q w = \max_{\partial Q} w = 0. \tag{A.5}$$

To establish (A.5) let $\hat{x} \in \operatorname{argmax}_Q w$ and suppose, by contradiction, that $w(\hat{x}) > 0$. Then $\hat{x} \in \operatorname{int} Q$ and

$$\begin{aligned} D(u-v)(\hat{x}) &= 0, \\ D^2(u-v)(\hat{x}) &\leq 0. \end{aligned}$$

By (A.2), we then get

$$H(.,u,Dv,D^2v)(\hat{x}) \le H(.,u,Du,D^2u)(\hat{x}) \le 0 \le H(.,v,Dv,D^2v)(\hat{x}).$$

Since $H(x,r,p,A)$ is strictly increasing in r and we are assuming that $w(\hat{x}) = u(\hat{x}) - v(\hat{x}) > 0$ we also have

$$H(.,v,Dv,D^2v)(\hat{x}) < H(.,u,Dv,D^2v)(\hat{x}),$$

which is a contradiction.

Thus, we conclude that $\max_Q w \le 0 = \max_{\partial Q} w$. □

In the case where u, v are semi-continuous we can not replicate the argument of the previous Proposition because we can not differentiate w. To go around the problem we double the number of variables and penalize the doubling. More precisely, we consider

$$M_\alpha := \max_{Q\times Q} u(x) - v(y) - \frac{\alpha}{2}|x-y|^2, \tag{A.6}$$

which is attained in some point (x_α, y_α). By penalization we mean taking $\alpha \to \infty$ which implies that we approximate the maximization of $u - v$ in Q. More precisely, we have the following Lemma:

Lemma 109. *Suppose $u, -v$ are upper semi-continuous. Let $M_\alpha, x_\alpha, y_\alpha$ be defined as before, and suppose that $x_\alpha \to \hat{x}$ as $\alpha \to \infty$. Then, as $\alpha \to \infty$,*

$$\begin{aligned} \alpha|x_\alpha - y_\alpha|^2 &\to 0, \\ M_\alpha &\to \max_Q (u(x) - v(x)), \\ \hat{x} &\in \operatorname*{argmax}_Q (u(x) - v(x)). \end{aligned}$$

Proof. We start by noting that $M_\alpha \ge \max_Q u(x) - v(x)$. This implies that $\alpha|x_\alpha - y_\alpha|^2$ remains bounded. Hence we have

$$|x_\alpha - y_\alpha| \to 0.$$

Since $x_\alpha \to \hat{x}$ then we must also have $y_\alpha \to \hat{x}$.

Since $u(x) - v(y)$ is upper semi-continuous and $(x_\alpha, y_\alpha) \to (\hat{x}, \hat{x})$ we conclude that for α large enough

$$M_\alpha \le u(\hat{x}) - v(\hat{x}) + \varepsilon - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \le \max_Q (u(x) - v(x)) + \varepsilon.$$

Thus, due to the arbitrariness of ε, we conclude that

$$M_\alpha \to \max_Q (u(x) - v(x)).$$

Furthermore, for α large enough,

$$\begin{aligned} \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 &= u(x_\alpha) - v(y_\alpha) - M_\alpha \\ &\le \max_Q (u(x) - v(x)) + \varepsilon - M_\alpha \\ &\to \varepsilon. \end{aligned}$$

Thus, $\alpha|x_\alpha - y_\alpha|^2 \to 0$.

It remains to show that $\hat{x} \in \operatorname{argmax}_Q (u - v)$. For that we recall that, for α large enough,

$$u(\hat{x}) - v(\hat{x}) \ge M_\alpha - \varepsilon \to \max_Q (u(x) - v(x)) - \varepsilon.$$

Thus, $u(\hat{x}) - v(\hat{x}) = \max_Q (u(x) - v(x))$. □

If u, v were both C^2 then we could compare the derivatives of u, v and $\frac{\alpha}{2}|x-y|^2$ at (x_α, y_α) to get $Du(x_\alpha) = -Dv(y_\alpha) = \alpha(x_\alpha - y_\alpha)$ and

$$\begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where $X_\alpha = D^2u(x_\alpha)$, $Y_\alpha = D^2v(y_\alpha)$ and I stands for the identity matrix. The previous inequality implies in particular that $X_\alpha \leq Y_\alpha$.

For $u, -v$ upper semi-continuous then it is still possible to obtain a perturbed version of the previous inequality. This result is due to Ishii and is a key result in the proof of uniqueness of viscosity solutions.

Lemma 110 (Ishii's Lemma)**.** *Let Q be a locally compact set, $u, -v \in \mathrm{USC}(Q)$, φ be twice differentiable in a neighborhood of $Q \times Q$. Suppose that $u(x) - v(y) - \varphi(x, y)$ has a local maximum at $(\hat{x}, \hat{y})$. Then for each $\varepsilon > 0$ there exist X, Y such that*

$$(D_x\varphi(\hat{x}, \hat{y}), X) \in \bar{D}_Q^{2,+}u(\hat{x}),$$
$$(-D_y\varphi(\hat{x}, \hat{y}), Y) \in \bar{D}_Q^{2,-}v(\hat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2,$$

where $A := D^2\varphi(\hat{x}, \hat{y})$.

If in the previous Lemma we consider $\varphi(x, y) := \frac{\alpha}{2}|x-y|^2$, $(\hat{x}, \hat{y}) = (x_\alpha, y_\alpha)$ and $\varepsilon = \frac{1}{\alpha}$, we get

$$D_x\varphi(x_\alpha, y_\alpha) = -D_y\varphi(x_\alpha, y_\alpha) = \alpha(x_\alpha - y_\alpha),$$
$$\begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{A.7}$$

In particular, like in the case of u, v twice differentiable, we have $X_\alpha \leq Y_\alpha$.

We can now establish the *comparison principle for viscosity solutions*, under two additional assumptions.

Theorem 111 (Comparison principle for viscosity solutions)**.** *Let u, v be continuous functions and, respectively, a subsolution and a supersolution of* (A.1) *such that $u_{|\partial Q} = v_{|\partial Q}$. Let x_α, y_α be defined as in* (A.6)*. Consider the X_α, Y_α given by Ishii's lemma, satisfying* (A.7)*.*

Suppose that there exists $\gamma > 0$ and a function ω with $\omega(0^+) = 0$ such that

$$\begin{cases} H(x, r, p, A) - H(x, s, p, A) \geq \gamma(r - s), \\ H(y_\alpha, r, p_\alpha, Y_\alpha) - H(x_\alpha, r, p_\alpha, X_\alpha) \leq \omega(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|^2), \end{cases} \tag{A.8}$$

where $p_\alpha := \alpha(x_\alpha - y_\alpha)$.

Then

$$u \leq v.$$

Proof. Because Q is compact we can suppose that $x_\alpha \to \hat{x} \in \mathrm{argmax}_Q(u - v)$. We know by Ishii's lemma that

$$\begin{aligned} (p_\alpha, X_\alpha) &\in \bar{D}^{2,+}u(x_\alpha), \\ (p_\alpha, Y_\alpha) &\in \bar{D}^{2,-}v(y_\alpha). \end{aligned}$$

Thus, since u, v are, respectively, a subsolution and a supersolution of (A.1), we have

$$H(x_\alpha, u(x_\alpha), p_\alpha, X_\alpha) \leq 0 \leq H(y_\alpha, v(y_\alpha), p_\alpha, Y_\alpha).$$

We then get

$$\begin{aligned}
\gamma(u(x_\alpha) - v(y_\alpha)) &\leq H(x_\alpha, u(x_\alpha), p_\alpha, X_\alpha) - H(x_\alpha, v(y_\alpha), p_\alpha, X_\alpha) \\
&= H(x_\alpha, u(x_\alpha), p_\alpha, X_\alpha) - H(y_\alpha, v(y_\alpha), p_\alpha, Y_\alpha) + \\
&\quad + H(y_\alpha, v(y_\alpha), p_\alpha, Y_\alpha) - H(x_\alpha, v(y_\alpha), p_\alpha, X_\alpha) \\
&\leq H(y_\alpha, v(y_\alpha), p_\alpha, Y_\alpha) - H(x_\alpha, v(y_\alpha), p_\alpha, X_\alpha) \\
&\leq \omega(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|^2),
\end{aligned}$$

and, taking $\alpha \to \infty$, we conclude

$$u(\hat{x}) - v(\hat{x}) \leq 0.$$

Thus,

$$u(x) \leq v(x).$$

□

Now that we established a comparison principle for (A.1), the uniqueness of solution will follow. To show that conditions (A.8) are not too restrictive we now give examples where the second condition is satisfied. Regarding the first condition we will see in the next Section that it poses no problems in the parabolic case.

Example 112. *We see in this example that if*

$$H(x, r, p, X) = -\langle \mu(x), p \rangle - \mathrm{Tr}((\sigma\sigma^T)(x)X),$$

then the second condition in (A.8) *is satisfied. For that purpose we suppose that μ, σ are Lipschitz.*

Then we have

$$|\langle p, \mu(y_\alpha)\rangle - \langle p, \mu(x_\alpha)\rangle| \quad \leq \quad C|p||y_\alpha - x_\alpha|,$$

and, if X_α, Y_α satisfy (A.7),

$$-\mathrm{Tr}\left((\sigma\sigma^T)(y_\alpha)Y_\alpha\right) + \mathrm{Tr}\left((\sigma\sigma^T)(x_\alpha)X_\alpha\right) \quad \leq \quad C\alpha|y_\alpha - x_\alpha|^2,$$

where the second inequality follows from multiplying both sides of (A.7) *by the positive semi-definite matrix*

$$\begin{pmatrix} (\sigma\sigma^T)(x_\alpha) & \sigma(x_\alpha)\sigma^T(y_\alpha) \\ \sigma(y_\alpha)\sigma^T(x_\alpha) & (\sigma\sigma^T)(y_\alpha) \end{pmatrix},$$

and taking traces.

A.3 Discontinuous viscosity solutions

In the definition of viscosity solution we require the solution to be continuous. In addition, in the elliptic equation (A.1) we suppose that H is continuous.

In this Section we extend the definition of viscosity solutions so as to be able to consider discontinuous solutions of elliptic equations of the form (A.1) with H not necessarily continuous. For that purpose we need to introduce the notion of *semi-continuous envelope*:

Definition 113. *Given a function v, the* lower semi-continuous envelope *of v is the function v_* defined by*

$$v_*(x) := \liminf_{x' \to x} v(x').$$

The upper semi-continuous envelope *of v, v^*, is defined as $v^* := -(-v)_*$.*

The definition of discontinuous viscosity solution then is as follows:

Definition 114. *A function v is a* discontinuous viscosity subsolution *of* (A.1) *in Q if v^* is a viscosity subsolution of*

$$H_*(., \varphi, D\varphi, D^2\varphi)(x) \leq 0,$$

for all $x \in Q$.

Analogously, v is a discontinuous viscosity supersolution *of* (A.1) *in Q if v_* is a viscosity supersolution of*

$$H^*(., \varphi, D\varphi, D^2\varphi)(x) \leq 0,$$

for all $x \in Q$.

We say that v is a discontinuous viscosity solution *of* (A.1) *if it is both a discontinuous viscosity subsolution and a discontinuous viscosity supersolution.*

A.4 Parabolic equations

We now consider the particular case of *parabolic equations*:

$$F(t, x, v, \partial_t v, Dv, D^2v) := -\partial_t v + H(t, x, v, Dv, D^2v) = 0. \tag{A.9}$$

In this equations we have $N+1$ variables (t, x) but the unknown v is only differentiated once in the time variable. Therefore we do not have to require the test functions to have more than one derivative in time. We consider domains of the form $Q :=]t, T[\times O$.

Definition 115. *A function v is a* viscosity subsolution *of* (A.2) *in Q if $v \in \mathrm{USC}(Q)$ and, for each $\varphi \in C^{1,2}(Q)$,*

$$F(., v, \partial_t\varphi, D\varphi, D^2\varphi)(\hat{t}, \hat{x}) \leq 0,$$

at every $(\hat{t}, \hat{x}) \in Q$ such that $v - \varphi$ has a local maximum at $(\hat{t}, \hat{x})$ on Q.

The definitions of viscosity supersolution *and viscosity* solution *are then obvious.*

The definition of superdifferential is adapted as well:

Definition 116. *Let v be upper semi-continuous in Q. The* parabolic second-order superdifferential *of v at (t, x) is the set*

$$D_Q^{(1,2),+}v(t,x) := \Big\{(\partial_t\varphi, D\varphi, D^2\varphi)(t,x) : \varphi \in C^{1,2}(Q) \text{ such that } v - \varphi \text{ has a local maximum at } (t,x)\Big\}.$$

The parabolic second-order subdifferential *of v at (t, x) is then defined as*

$$D_Q^{(1,2),-}v(t,x) := -D_Q^{(1,2),+}(-v)(t,x)$$

We can extend the viscosity property to the initial time, in the following sense:

Proposition 117. *Suppose $v \in C([t, T[\times O)$ is a viscosity solution of* (A.2) *in $]t, T[\times O$. Then v is a viscosity solution of* (A.2) *in $[t, T[\times O$.*

Proof. Let $\varphi \in C^{1,2}([t, T[\times O)$ and let $(t, x_0) \in \mathrm{argmax}(v - \varphi)$. We assume, without loss of generality, that (t, x_0) is a strict maximizer. For some fixed r, consider

$$(t_\varepsilon, x_\varepsilon) \in \underset{(s,x)\in(t,t+r]\times\overline{B_r(x_0)}}{\mathrm{argmax}} \left(v(s,x) - \varphi(s,x) - \frac{\varepsilon}{s-t}\right).$$

Since $(t_\varepsilon, x_\varepsilon) \in [t, t+r] \times \overline{B_r(x_0)}$, we can suppose that $(t_\varepsilon, x_\varepsilon) \to (\hat{t}, \hat{x})$. By definition of $(t_\varepsilon, x_\varepsilon)$, we have that, for all $(s, x) \in (t, T) \times O$,

$$\begin{aligned} v(s,x) - \varphi(s,x) &\leq v(t_\varepsilon, x_\varepsilon) - \varphi(t_\varepsilon, x_\varepsilon) - \frac{\varepsilon}{t_\varepsilon - t} + \frac{\varepsilon}{s-t} \\ &\leq v(t_\varepsilon, x_\varepsilon) - \varphi(t_\varepsilon, x_\varepsilon) + \frac{\varepsilon}{s-t} \\ &\to v(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}), \end{aligned}$$

where the convergence, as $\varepsilon \to 0$, follows from the continuity of v and φ.

Because $v - \varphi$ is continuous in $[t, T[\times O$, we can extend the previous inequality to $(s, x) \in [t, T) \times O$ getting

$$v(s,x) - \varphi(s,x) \leq v(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}), \text{ for all } (s,x) \in [t,T) \times O.$$

Since (t, x_0) is a strict maximizer of $v - \varphi$ in $[t, T) \times O$ we then get that $(\hat{t}, \hat{x}) = (t, x_0)$. This implies in particular that, for ε small enough, $(t_\varepsilon, x_\varepsilon) \in (t, T) \times B_r(x_0)$.

Thus, since v is a viscosity subsolution to (A.2) in $(t, T) \times O$, we conclude that

$$\frac{\varepsilon}{(t_\varepsilon - t)^2} - \partial_t \varphi(t_\varepsilon, x_\varepsilon) + H(., v, D\varphi, D^2\varphi)(t_\varepsilon, x_\varepsilon) \leq 0.$$

The previous inequality implies that

$$-\partial_t \varphi(t_\varepsilon, x_\varepsilon) + H(., v, D\varphi, D^2\varphi)(t_\varepsilon, x_\varepsilon) \leq 0.$$

Taking $\varepsilon \to 0$, we conclude that

$$-\partial_t \varphi(t, x_0) + H(., v, D\varphi, D^2\varphi)(t, x_0) \leq 0.$$

Thus v is a viscosity subsolution of (A.2) in $[t, T) \times O$. The supersolution property is proved in a similar way. □

For parabolic equations, we may consider the following convenient particular case of Ishii's Lemma.

Lemma 118. *Let $Q =]t, T[\times O$ be a locally compact set, $u, -v \in \mathrm{USC}(Q)$, $\varphi(t, s, x, y)$ be twice differentiable in a neighborhood of $Q \times Q$ such that $\partial_t D_x \varphi = \partial_s D_x \varphi = \partial_t D_y \varphi = \partial_s D_y \varphi = 0$. Suppose that $u(t, x) - v(s, y) - \varphi(t, s, x, y)$ has a local maximum at $(\hat{t}, \hat{s}, \hat{x}, \hat{y})$. Then, for each $\varepsilon > 0$, there exist X, Y such that*

$$\begin{aligned} (\partial_t \varphi(\hat{t}, \hat{s}, \hat{x}, \hat{y}), D_x \varphi(\hat{t}, \hat{s}, \hat{x}, \hat{y}), X) &\in \bar{D}_Q^{(1,2),+} u(\hat{t}, \hat{x}) \\ (-\partial_s \varphi(\hat{t}, \hat{s}, \hat{x}, \hat{y}), -D_y \varphi(\hat{t}, \hat{s}, \hat{x}, \hat{y}), Y) &\in \bar{D}_Q^{(1,2),-} v(\hat{s}, \hat{y}) \end{aligned}$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2,$$

where $A := D^2_{x,y} \varphi(\hat{t}, \hat{s}, \hat{x}, \hat{y})$.

We can apply the previous Lemma together with a doubling of variables argument, doubling (t, x) into (t, x, s, y), to establish uniqueness. However, if we do this, then to verify the second condition of (A.8) we need stronger regularity of H in the time variable. To avoid this, we need a parabolic analogue to Ishii's Lemma. This parabolic analogue can be found in [11, p. 50] and is the following:

Lemma 119 (*Ishii's Lemma for parabolic equations*)**.** *Let $Q =]t, T[\times O$ be a locally compact set, $u, -v \in \mathrm{USC}(Q)$ and $\varphi(t, x, y)$ be a function defined in a neighborhood of $]t, T[\times O \times O$, once continuously differentiable in t and twice continuously differentiable in x, y.*

Suppose that $u(t,x) - v(t,y) - \varphi(t,x,y)$ has a local maximum at $(\hat{t},\hat{x},\hat{y})$. Assume, moreover, that there is $r > 0$ such that, for every $M > 0$, there is $C \in \mathbb{R}$ such that

$$\begin{cases} q_1 \geq C, & \text{whenever } (q_1,p_1,X) \in D_Q^{(1,2),+}u(t,x), |x-\hat{x}| + |t-\hat{t}| \leq r \text{ and} \\ & |u(t,x)| + |p_1| + |X| \leq M. \\ q_2 \leq C, & \text{whenever } (q_2,p_2,Y) \in D_Q^{(1,2),-}v(t,y), |y-\hat{y}| + |t-\hat{t}| \leq r \text{ and} \\ & |v(t,y)| + |p_2| + |X| \leq M. \end{cases} \tag{A.10}$$

Then, for each $\varepsilon > 0$, there exist q_1, q_2, X, Y such that

$$\begin{cases} (i) & (q_1, D_x\varphi(\hat{t},\hat{x},\hat{y}), X) \in \bar{D}_Q^{(1,2),+}u(\hat{t},\hat{x}), \\ & (q_2, -D_y\varphi(\hat{t},\hat{x},\hat{y}), Y) \in \bar{D}_Q^{(1,2),-}v(\hat{t},\hat{y}), \\ (ii) & q_1 - q_2 = \partial_t\varphi(\hat{t},\hat{x},\hat{y}), \\ (iii) & \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2, \end{cases}$$

where $A := D^2_{x,y}\varphi(\hat{t},\hat{x},\hat{y})$.

We remark that condition (A.10) is guaranteed by having u, v, respectively, a viscosity subsolution and a viscosity supersolution of (A.2).

We end this Section by noticing that in the parabolic case the first condition in (A.8) is not satisfied if $H(t,x,r,p,X) = H(t,x,p,X)$ does not depend on r (which is the case of the HJB equation). However, it is easy to work around this issue. Indeed, if $H(t,x,p,X)$ is homogeneous in (p,X), i.e. if $H(t,x,\lambda p,\lambda X) = \lambda H(t,x,p,X)$, and u is a subsolution to (A.2), then $\tilde{u}(t,x) := e^{-(T-t)}u(t,x)$ is a subsolution to

$$\tilde{F}(t,x,\tilde{u},\partial_t\tilde{u},D\tilde{u},D^2\tilde{u}) := \tilde{u} + F(t,x,\partial_t\tilde{u},D\tilde{u},D^2\tilde{u}) \leq 0,$$

and

$$\tilde{F}(t,x,r,q,p,X) - \tilde{F}(t,x,s,q,p,X) = r - s.$$

Thus, instead of applying the maximum principle to $u - v$, we should apply the maximum principle to $e^{-(T-t)}(u-v)$.

Appendix B

Stochastic calculus

In this Appendix we review some basic definitions and results from stochastic calculus. These results are standard in the literature and can be found for instance in [13, 14].

B.1 Preliminaries

B.1.1 Brownian motion and filtration

Consider a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is a $\sigma-$algebra on Ω and $\mathbb{P}$ is a probability measure, i.e., $\mathbb{P}(\Omega) = 1$.

Definition 120. *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, a stochastic process* W *is a* Brownian motion *if:*

- $W_0 = 0$ *and* $t \mapsto W_t$ *is a continuous function a.s.;*
- W_t *has independent increments, i.e.,*

$$0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \Rightarrow W_{t_4} - W_{t_3} \perp\!\!\!\perp W_{t_2} - W_{t_1};$$

- W_t *has increments which are normally distributed, i.e.,*

$$0 \leq t_1 \leq t_2 \Rightarrow W_{t_2} - W_{t_1} \sim \mathcal{N}(0, t_2 - t_1).$$

As stated in Section 4.2 the probability space we consider in this thesis is the classical Wiener space and in it we consider the standard $N-$dimensional Brownian motion corresponding to the coordinate process, $W_s(\omega) = \omega_s$.

Definition 121. *A* filtration *is an increasing collection of* $\sigma-$*algebras* $\{\mathcal{F}_t\}_{t\geq 0}$*, that is,* $s < t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$*.*

We consider in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a filtration $\mathbb{F} = \{\mathcal{F}_s : 0 \leq s \leq T\}$. The filtration we consider is the one *induced* naturally by W augmented with the $\mathbb{P}-$null sets, that is,

$$\mathcal{F}_s = \sigma\{W_r : r \leq s\} \vee \mathcal{N}_{\mathbb{P}}.$$

We prove a useful property of this probability space:

Lemma 122. *Let* $\Lambda \in \mathcal{F}_t$ *and* $0 < \varepsilon < \mathbb{P}(\Lambda)$*. Then there is* $\tilde{\Lambda} \subset \Lambda$ *such that* $\tilde{\Lambda} \in \mathcal{F}_t$ *and* $\mathbb{P}(\tilde{\Lambda}) = \varepsilon$*.*

Proof. This result is a consequence of the fact that we can partition Ω in $\mathcal{F}_t-$measurable sets which have arbitrarily small measure. Indeed, if we consider

$$\Gamma_n^\delta := \{|W_t| \in [n\delta, (n+1)\delta)\},$$

then $\Gamma_n^\delta \in \mathcal{F}_t$, $\{\Gamma_n^\delta\}_{n\geq 0}$ is a partition of Ω and

$$\mathbb{P}(\Gamma_n^\delta) \leq \mathbb{P}(\Gamma_0^\delta) \to_{\delta\to 0} 0.$$

Using these partitions and considering a sequence $\delta_i \to 0$, we can construct $\tilde{\Lambda}_i \subset \Lambda$ such that

$$\begin{aligned} \tilde{\Lambda}_i &\in \mathcal{F}_t, \\ \tilde{\Lambda}_i &\subset \tilde{\Lambda}_{i+1}, \\ \varepsilon - \delta_i \leq \mathbb{P}(\Lambda_i) &\leq \varepsilon. \end{aligned}$$

Thus $\tilde{\Lambda} = \bigcup_{i\geq 1} \tilde{\Lambda}_i$ is $\mathcal{F}_t-$measurable and satisfies $\mathbb{P}(\tilde{\Lambda}) = \varepsilon$. □

B.1.2 Stopping times and progressive measurability

The $\sigma-$algebra $\mathcal{F}_t$ of the considered filtration represents intuitively the information acquired up to time t. In this subsection we define useful objects that respect the information accumulated up to that time.

Definition 123. *A random variable $\tau \geq 0$ is a* stopping time *with respect to a filtration $\mathbb{F}$ if $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \geq 0$.*

Definition 124. *The $\sigma-$*algebra induced by a stopping time *is*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

Definition 125. *Consider a stochastic process X:*

- *X is $\mathbb{F}-$*adapted *if, for all t, X_t is $\mathcal{F}_t-$measurable.*
- *X is $\mathbb{F}-$*progressively measurable *if, for all t, the application*

$$[0, t] \times \Omega \ni (s, \omega) \mapsto X_s(\omega)$$

  is $\mathcal{B}([0, t]) \times \mathcal{F}-$measurable.

Clearly, a progressively measurable process is adapted. A sufficient condition for an adapted process to be progressively measurable is that the sample paths be right continuous (or left continuous), see for instance [13, p. 5].

B.1.3 Martingales and local martingales

Definition 126. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a probability space with a filtration, and X be a $\mathbb{F}-$adapted process such that $\mathbb{E}[|X_t|] < \infty$. Then:*

- *X_t is a* martingale *if $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$;*
- *X_t is a* supermartingale *if $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$;*
- *X_t is a* submartingale *if $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$.*

Using Jensen's inequality for conditional expectations, we get the following result:

Proposition 127. *Let X be a martingale and ψ a convex function such that $\mathbb{E}[|\psi(X_t)|] < \infty$. Then $\psi(X_.)$ is a submartingale.*

The previous Proposition implies, in particular, that, if X is a martingale, then $|X|$ is a submartingale.

Regarding convergence of submartingales we have the next Theorem. Obviously, an analogous result holds for supermartingales.

Theorem 128 (Convergence of submartingales)**.** *Let X be a right-continuous submartingale such that* $\sup_t \mathbb{E}[X_t] < \infty$*. Then $X_\infty = \lim_t X_t$ exists a.s. and $\mathbb{E}[|X_\infty|] < \infty$.*

The *optional sampling theorem* extends the inequalities in the definition of martingale to stopping times.

Theorem 129 (Optional sampling theorem)**.** *Let X be a right-continuous submartingale such that $X_\infty = \lim_t X_t$ exists a.s., $\mathbb{E}[|X_\infty|] < \infty$ and $\mathbb{E}[X_\infty|\mathcal{F}_t] \geq X_t$. Then, given two stopping times $\tau_1 \leq \tau_2$, we have*

$$\mathbb{E}[X_{\tau_2}|\mathcal{F}_{\tau_1}] \geq X_{\tau_1}.$$

We can deduce an analogous result for supermartingales and martingales.

To get estimates on martingales the following inequality is frequently used:

Theorem 130 (Doob's maximal inequality)**.** *Let $p > 1$ and $X \geq 0$ be a right-continuous submartingale such that $\mathbb{E}[|X_t|^p] < \infty$. Then*

$$\mathbb{E}\left[\left|\sup_{t_1 \leq t \leq t_2} X_t\right|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_{t_2}|^p].$$

We use in this work a conditional version of Doob's maximal inequality which we prove in the next Corollary.

Theorem 131 (Doob's maximal inequality for conditional expectations)**.** *Let $p > 1$ and $X \geq 0$ be a right-continuous submartingale such that $\mathbb{E}[|X_t|^p] < \infty$. Then*

$$\mathbb{E}\left[\left|\sup_{t_1 \leq t \leq t_2} X_t\right|^p \middle| \mathcal{F}_{t_1}\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_{t_2}|^p|\mathcal{F}_{t_1}].$$

Proof. We use Doob's maximal inequality to prove that, for each $\Lambda \in \mathcal{F}_{t_1}$,

$$\mathbb{E}\left[\mathbb{E}\left[\left|\sup_{t_1 \leq t \leq t_2} X_t\right|^p \middle| \mathcal{F}_{t_1}\right] \mathbf{1}_\Lambda\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[\mathbb{E}[|X_{t_2}|^p|\mathcal{F}_{t_1}]\mathbf{1}_\Lambda\right].$$

Since $\mathbb{E}\left[\left|\sup_{t_1 \leq t \leq t_2} X_t\right|^p |\mathcal{F}_{t_1}\right]$ and $\mathbb{E}[|X_{t_2}|^p|\mathcal{F}_{t_1}]$ are $\mathcal{F}_{t_1}-$measurable the result follows.

To proceed we notice that, if X is a non-negative right-continuous submartingale, then so is $\tilde{X} := X\mathbf{1}_\Lambda$. Furthermore,

$$\left(\sup_{t_1 \leq t \leq t_2} X_t\right) \mathbf{1}_\Lambda = \sup_{t_1 \leq t \leq t_2} \tilde{X}_t.$$

Thus, we can apply Doob's maximal inequality to get:

$$\begin{aligned} \mathbb{E}\left[\mathbb{E}\left[\left|\sup_{t_1 \leq t \leq t_2} X_t\right|^p \middle| \mathcal{F}_{t_1}\right] \mathbf{1}_\Lambda\right] &= \mathbb{E}\left[\left|\sup_{t_1 \leq t \leq t_2} \tilde{X}_t\right|^p\right] \\ &\leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|\tilde{X}_{t_2}|^p] \\ &= \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[\mathbb{E}[|X_{t_2}|^p|\mathcal{F}_{t_1}]\mathbf{1}_\Lambda\right]. \end{aligned}$$

□

Local martingales

Definition 132. *X is a* local martingale *if there is a sequence of stopping times $\tau_n \to \infty$ such that $X_{\cdot\wedge\tau_n}$ is a martingale.*

Using the dominated converge theorem we can easily give a condition for a local martingale to be a martingale.

Lemma 133. *Let M be a local martingale such that $\mathbb{E}[|M|_t^*] < \infty$, where $|M|_t^* := \sup_{s\in[0,t]} |M_s|$. Then M is a martingale.*

Proof. Indeed, we have

$$\mathbb{E}[M_{t\wedge\tau_n}|\mathcal{F}_s] = M_{s\wedge\tau_n} \to_{a.s.} M_s$$

On the other hand $|M_{t\wedge\tau_n}| \leq |M|_t^*$ and $|M|_t^*$ is integrable so, by the dominated convergence Theorem, we have

$$\mathbb{E}[M_{t\wedge\tau_n}|\mathcal{F}_s] \to_{a.s} \mathbb{E}[M_t|\mathcal{F}_s].$$

Thus, $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$. □

Similarly, we can use Fatou's lemma to give a condition for a local martingale to be a supermartingale.

Lemma 134. *Let M^l be a local martingale such that M^l is bounded by below by a martingale M. Then M^l is a supermartingale.*

Proof. We want to show that $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$. By the bounding condition we have $M_t^l - M_t \geq 0$.

Since M^l is a local martingale:

$$\mathbb{E}\left[M_{t\wedge\tau_n}^l - M_{t\wedge\tau_n}\middle|\mathcal{F}_s\right] = M_{s\wedge\tau_n}^l - M_{s\wedge\tau_n}. \to_{a.s.} M_s^l - M_s$$

On the other hand, by Fatou's lemma,

$$\mathbb{E}\left[\liminf_n \left(M_{t\wedge\tau_n}^l - M_{t\wedge\tau_n}\right)\middle|\mathcal{F}_s\right] \leq \liminf_n \mathbb{E}\left[M_{t\wedge\tau_n}^l - M_{t\wedge\tau_n}\middle|\mathcal{F}_s\right] = M_s^l - M_s,$$

and we know that

$$\liminf_n \left(M_{t\wedge\tau_n}^l - M_{t\wedge\tau_n}\right) = M_t^l - M_t.$$

Thus,

$$\mathbb{E}\left[M_t^l - M_t|\mathcal{F}_s\right] \leq M_s^l - M_s.$$

Because M is a martingale we conclude that $\mathbb{E}\left[M_t^l|\mathcal{F}_s\right] \leq M_s^l$. □

B.2 Stochastic integral and Itô's formula

The *stochastic integral* $\int_0^t \psi_s dW_s$ is defined for processes, $\psi : [0,T]\times\Omega \to \mathbb{R}^{d\times N}$, in

$$\mathbb{H}_{loc}^2 = \left\{\psi : \mathbb{F} - \text{adapted processes with } \int_0^T |\psi_s|^2 ds < \infty \ a.s.\right\}.$$

However, in the smaller space

$$\mathbb{H}^2 = \left\{\psi : \mathbb{F} - \text{adapted processes with } \mathbb{E}\left[\int_0^T |\psi_s|^2 ds\right] < \infty\right\},$$

we can prove additional results since this one is a Hilbert space when equipped with the norm

$$||\psi||_{\mathbb{H}^2} = \sqrt{\mathbb{E}\left[\int_0^T |\psi_s|^2 ds\right]}.$$

Remark 135. *We recall that W is a $N-$dimensional Brownian motion. Thus, $I := \int_0^T \psi_s dW_s$ is an abbreviation for the vector $(I_i)_{i=1,\dots,d}$ such that*

$$I_i := \sum_{j=1}^{N} \int_0^T \psi_s^{i,j} dW_s^j,$$

where ψ is a $(d \times N)-$dimensional process. The norm of ψ is the Fröbenius norm:

$$|\psi| = \sqrt{\mathrm{Tr}(\psi\psi^T)}.$$

In the following Proposition we list some of the properties of the stochastic integral:

Proposition 136 (Properties of the stochastic integral)**.** *Let $\psi \in \mathbb{H}^2_{loc}$ and $I_t := \int_0^t \psi_s dW_s$. Then:*

- *I_t has continuous sample paths and $I_0 = 0$;*
- *I_t is a local martingale. If $\psi \in \mathbb{H}^2$ then I_t is a martingale;*
- *If $\psi \in \mathbb{H}^2$ then we have the so called* Itô's isometry*:*

$$\mathbb{E}\left[(I_t - I_s)^2 | \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t |\psi_r|^2 dr \Big| \mathcal{F}_s\right].$$

B.2.1 Itô processes

Definition 137. *An* Itô process*, X, is a continuous-time process defined by*

$$X_t := X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \quad t \geq 0,$$

where μ, σ are $\mathbb{F}-$adapted processes satisfying $\int_0^t |\mu_s| + |\sigma_s|^2 ds < \infty$.

Itô processes are frequently written in *differential notation* as

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

We will use this notation often, specially in the context of stochastic differential equations.

B.2.2 Itô's formula

Itô's formula can be seen as the chain rule of stochastic calculus. It tells us how stochastic differentials change under composition.

Theorem 138 (Itô's formula)**.** *Consider a function $f \in C^{1,2}([0,T] \times \mathbb{R}^N)$ and an Itô process,*

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s.$$

Then, with probability 1*,*

$$f(t, X_t) = f(0,0) + \int_0^t \partial_t f(s, X_s) + \langle Df(s, X_s), \mu_s \rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma_s \sigma_s^T D^2 f(s, X_s)\right) ds + \\ + \int_0^t Df(s, W_s)\sigma_s dW_s,$$

for all $t \in [0,T]$.

In differential form Itô's formula is written as

$$df(t,X_t) = \left(\partial_t f(t,X_t) + \langle Df(t,X_t), \mu_t\rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma_t\sigma_t^T D^2 f(t,X_t)\right)\right)dt + Df(t,W_t)\sigma_t dW_t.$$

As an application of Itô's formula, we obtain the next martingale inequality, similar to Itô's isometry, that will be useful to obtain estimates in SDE's. We will obtain this inequality for conditional expectations. The same result for expected values can be found in [15, p. 80] or [13, p. 163].

Lemma 139. *Let $\psi \in \mathbb{H}^2_{loc}$ and $q \geq 1$. Then*

$$\mathbb{E}\left[\left|\int_s^t \psi_r dW_r\right|^{2q}\middle|\mathcal{F}_s\right] \leq (t-s)^{q-1}(2q-1)^q \mathbb{E}\left[\int_s^t |\psi_r|^{2q} dr\middle|\mathcal{F}_t\right].$$

Proof. Let $X_t := \int_s^t \psi_r dW_r$. We apply Itô's formula to $|X_t|^{2q}$ to obtain

$$|X_t|^{2q} \quad = \quad \int_s^t 2q(q-1)|\psi_r^T X_r|^2|X_r|^{2q-4} + q|\psi_r|^2|X_r|^{2q-2}dr + \int_s^t 2q(q-1)|X_r|^{2q-2}X_r^T dW_r.$$

Since the Fröbenius norm is consistent, we have

$$|\psi_r^T X_r|^2 \leq |\psi_r^T|^2|X_r|^2 = |\psi_r|^2|X_r|^2.$$

For now, we suppose that X_t is bounded for $t \in [s,T]$. Then, the conditional expectation of the stochastic integral is zero, and we get

$$\mathbb{E}\left[|X_t|^{2q}|\mathcal{F}_s\right] \quad \leq \quad \int_s^t q(2q-1)\mathbb{E}\left[|\psi_r|^2|X_r|^{2q-2}|\mathcal{F}_s\right]dr$$

Applying Hölder's inequality for conditional expectations, we conclude that

$$\mathbb{E}\left[|X_t|^{2q}|\mathcal{F}_s\right] \quad \leq \quad \int_s^t q(2q-1)\mathbb{E}\left[|\psi_r|^{2q}|\mathcal{F}_s\right]^{\frac{1}{q}}\mathbb{E}\left[|X_r|^{2q}|\mathcal{F}_s\right]^{1-\frac{1}{q}}dr$$

Thus, if we denote

$$\begin{aligned} g(t,\omega) &:= q(2q-1)\mathbb{E}\left[|\psi_t|^{2q}|\mathcal{F}_s\right]^{\frac{1}{q}}(\omega), \\ f(t,\omega) &:= \mathbb{E}\left[|X_t|^{2q}|\mathcal{F}_s\right](\omega), \end{aligned}$$

we have that, for almost all $\omega \in \Omega$, $f(.,\omega)$ verifies the integral inequality

$$f(t,\omega) \leq \int_s^t g(r,\omega)f(r,\omega)^{1-\frac{1}{q}}dr.$$

This implies that (see [16, p. 561])

$$f(t,\omega) \leq \left(\frac{1}{q}\int_s^t g(r,\omega)dr\right)^q.$$

Thus,

$$\mathbb{E}\left[|X_t|^{2q}|\mathcal{F}_s\right] \leq (2q-1)^q\left(\int_s^t \mathbb{E}\left[|\psi_r|^{2q}|\mathcal{F}_s\right]^{\frac{1}{q}}dr\right)^q.$$

Using Jensen's inequality, we then conclude that

$$\mathbb{E}\left[|X_t|^{2q}|\mathcal{F}_s\right] \leq (2q-1)^q(t-s)^{q-1}\int_s^t \mathbb{E}\left[|\psi_r|^{2q}|\mathcal{F}_s\right]dr.$$

If X_t is not bounded then we consider the exit time, τ_n, of X_t from $B_n(0)$. Then $\tilde{X}_t := X_{t\wedge\tau_n}$ is bounded and

$$\tilde{X}_t = \int_s^t \mathbf{1}_{\{\sup_{h\in[s,r]} X_h \in B_n(0)\}} \psi_r dr.$$

Thus, we can apply the inequality for bounded processes, to get

$$\mathbb{E}\left[|X_{t\wedge\tau_n}|^{2q}|\mathcal{F}_s\right] \le (2q-1)^q (t-s)^{q-1}\mathbb{E}\left[\int_s^{t\wedge\tau_n} |\psi_r|^{2q} dr \Big| \mathcal{F}_s\right].$$

Using Fatou's lemma on the left term, monotone convergence on the right term and the fact that $\tau_n \to \infty$, we conclude that

$$\mathbb{E}\left[|X_t|^{2q}|\mathcal{F}_s\right] \le (2q-1)^q (t-s)^{q-1}\mathbb{E}\left[\int_s^t |\psi_r|^{2q} dr \Big| \mathcal{F}_s\right].$$

□

B.3 Martingale representation

Theorem 140 (Martingale representation)**.** *If M is a local martingale then there exists an adapted process $H \in \mathbb{H}^2_{loc}$ such that*

$$M_t = M_0 + \int_0^t H_s dW_s, \quad \text{for all } t \in [0,T].$$

Furthermore, if M is square integrable, $H \in \mathbb{H}^2$ is unique.

B.4 Girsanov's Theorem

Now we recall *Girsanov's Theorem*, which tells us how the distribution of a Brownian motion changes when we change from measure $\mathbb{P}$ to $\mathbb{Q}$.

We specify the new measure $\mathbb{Q}$ by its Radon-Nikodym derivative with respect to $\mathbb{P}$.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T := e^{-\int_0^T \psi_s dW_s - \frac{1}{2}\int_0^T \psi_s^2 ds},$$

where $\psi \in \mathbb{H}^2$.

This defines a measure, but not necessarily a probability measure. Thus we must require that $\mathbb{E}[Z_T] = 1$.

Girsanov's Theorem then tells us that we can compensate the drift of a Brownian motion by changing the measure:

Theorem 141 (Girsanov)**.** *Suppose $\mathbb{E}[Z_T] = 1$. Then*

$$B_t := W_t + \int_0^t \psi_s ds$$

is a Brownian motion under $\mathbb{Q}$, where $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$.

A condition to ensure that $\mathbb{E}[Z_T] = 1$ is given by *Novikov's criterion*:

Theorem 142 (Novikov)**.** *If*

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \psi_s^2 ds}\right] < \infty,$$

then $\mathbb{E}[Z_T] = 1$.

B.5 Stochastic differential equations

In this Section we give a meaning to the *stochastic differential equation*

$$\begin{cases} dX_s &= \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \\ X_t &= x. \end{cases} \tag{SDE}$$

Notice that we only consider deterministic initial conditions.

Definition 143. *A* strong solution *to* (SDE) *is an adapted process X with continuous sample paths such that*

- $X_t = x$;
- $\int_t^T |\mu(s,X_s)| + |\sigma(s,X_s)|^2 ds < \infty$, $\mathbb{P}-\text{a.s.}$;
- $X_s = x + \int_t^s \mu(r,X_r)dr + \int_t^s \sigma(r,X_r)dW_r$, *for all* $s \in [t,T]$.

The next Theorem gives sufficient conditions for the existence and uniqueness of strong solutions for (SDE).

Theorem 144. *Suppose μ,σ satisfy the global Lipschitz and linear growth conditions*

$$\begin{aligned} |\mu(t,x)-\mu(t,y)| + |\sigma(t,x)-\sigma(t,y)| &\leq K|x-y|, \\ |\mu(t,x)| + |\sigma(t,x)| &\leq K(1+|x|). \end{aligned}$$

Then there exists a unique strong solution, $X \in \mathbb{H}^2$, to (SDE).

Moreover, for each $p \geq 2$, there exists a constant C depending only on K,T,p such that

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X_s|^p \Big| \mathcal{F}_t\right] \leq C(1+|x|^p).$$

In the next section we obtain the estimate of the previous Theorem in the context of controlled diffusions.

B.6 Controlled diffusions

We now consider controlled processes.

Definition 145. *Given a set of controls $\mathcal{U}$, a* controlled process *is a mapping*

$$(t,x,\nu) \in [0,T] \times \mathbb{R}^d \times \mathcal{U} \mapsto X^\nu_{t,x} \in \mathbb{H}^0_{rcll}(\mathbb{R}^d).$$

The controlled processes we consider are given as solutions of a *controlled diffusion*:

$$\begin{cases} dX^\nu_s &= \mu(s,X^\nu_s;\nu_s)ds + \sigma(s,X^\nu_s;\nu_s)dW_s \\ X^\nu_t &= x \end{cases}, \tag{CSDE}$$

where $\nu \in \mathcal{U}_t$ and $\mathcal{U}_t$ is a suitable space of controls taking values in a set U, such that $\mathcal{U}_t \subset \mathcal{U}_s$ whenever $t \leq s$. We assume that μ,σ satisfy the following global Lipschitz and linear growth conditions:

$$\begin{cases} |\mu(t,x;u)-\mu(t,y;u)| + |\sigma(t,x;u)-\sigma(t,y;u)| \leq K|x-y|, \\ |\mu(t,x;u)| + |\sigma(t,x;u)| \leq K(1+|x|+|u|). \end{cases} \tag{B.1}$$

Remark 146. *Notice that, by* (B.1) *and Jensen's inequality, we have, for $p \geq 1$:*

$$\begin{aligned} |\sigma(t,x;u)|^p &\leq K^p(1+|x|+|u|)^p \\ &\leq 3^{p-1}K^p(1+|x|^p+|u|^p) \\ &\leq (3K)^p(1+|x|^p+|u|^p). \end{aligned}$$

Thus we can restate the growth condition of (B.1) *as*

$$|\mu(t,x;u)|^p + |\sigma(t,x;u)|^p \leq K^p(1+|x|^p+|u|^p), \text{ for all } p \geq 1. \tag{B.2}$$

In this case we can still prove existence and uniqueness.

Theorem 147. *Let $p \geq 2$. For each $\nu \in \mathcal{U}_t \cap \mathbb{H}^p(t,T;U)$ there exists a unique strong solution, $X^\nu \in \mathbb{H}^p(t,T;\mathbb{R}^d)$, to* (CSDE).

Moreover, there exists a constant C depending only on K, T, p such that

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|X_s^\nu|^p\Big|\mathcal{F}_t\right] \leq C\left(1+|x|^p+\mathbb{E}\left[\int_t^T|\nu_s|^p ds\Big|\mathcal{F}_t\right]\right). \tag{B.3}$$

Proof. Here we just give the proof of the estimate. We abbreviate the notation, writing $\mu_s := \mu(s, X_s^\nu; \nu_s)$ and $\sigma_s := \sigma(s, X_s^\nu; \nu_s)$. Then

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leq r\leq s}|X_r^\nu|^p\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[\sup_{t\leq r\leq s}\left|x+\int_t^r \mu_u du+\int_t^r\sigma_u dW_u\right|^p\Big|\mathcal{F}_t\right]\\
&\leq 3^{p-1}\left(|x|^p+T^{p-1}\mathbb{E}\left[\int_t^s|\mu_u|^p du\Big|\mathcal{F}_t\right]+\mathbb{E}\left[\sup_{t\leq r\leq s}\left|\int_t^r\sigma_u dW_u\right|^p\Big|\mathcal{F}_t\right]\right)\\
&\leq 3^{p-1}\left(|x|^p+T^{p-1}\mathbb{E}\left[\int_t^s|\mu_u|^p du\Big|\mathcal{F}_t\right]+\right.\\
&\quad\left.+\left(\frac{p}{p-1}\right)^p T^{\frac{p}{2}-1}(p-1)^{\frac{p}{2}}\mathbb{E}\left[\int_t^s|\sigma_u|^p du\Big|\mathcal{F}_t\right]\right),
\end{aligned}$$

where the first inequality follows from Jensen's inequality and the second follows from Doob's maximal inequality and Lemma 139.

Thus, by (B.2), we conclude that there is C such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leq r\leq s}|X_r^\nu|^p\Big|\mathcal{F}_t\right] &\leq C\left(1+|x|^p+\mathbb{E}\left[\int_t^T|\nu_s|^p ds\Big|\mathcal{F}_t\right]+\int_t^s\mathbb{E}\left[|X_u^\nu|^p\Big|\mathcal{F}_t\right]du\right)\\
&\leq C\left(1+|x|^p+\mathbb{E}\left[\int_t^T|\nu_s|^p ds\Big|\mathcal{F}_t\right]+\int_t^s\mathbb{E}\left[\sup_{t\leq r\leq u}|X_r^\nu|^p\Big|\mathcal{F}_t\right]du\right)
\end{aligned}$$

Applying Gronwall's Lemma and considering $s = T$, we get the desired result. □

We also have continuity in the initial conditions in the following sense:

Lemma 148. *Consider $t' \leq t$, $\nu \in \mathcal{U}_{t'} \subset \mathcal{U}_t$ and let $X_{t,x}^\nu$ be the strong solution of* (CSDE). *Suppose that either:*

(i) $\|\nu\|_{\mathbb{H}^\infty_{t',p}} \leq M$ *for some $p \geq 2$, or*

(ii) $\|\nu\|_{\mathbb{H}^{p,\infty}_{t'}} \leq M$ *for some $p > 2$.*

Then, for all $x' \in B_1(x)$ and for all $s \in [t,T]$,

$$\mathbb{E}\left[\left|X_{t,x}^\nu(s)-X_{t',x'}^\nu(s)\right|^p\Big|\mathcal{F}_{t'}\right] \leq C_{x,M}(|x-x'|^p+|t-t'|^\gamma), \tag{B.4}$$

where $C_{x,M}$ is a constant depending only on K, T, p, x, M and

$$\gamma := \begin{cases} \frac{p}{2} & \text{,if (i) holds,}\\ \frac{p}{2}-1 & \text{,if (ii) holds.}\end{cases}$$

Proof. We use the following notation:

$$\begin{aligned}
X_s := X_{t,x}^\nu(s) \quad &, \quad X_s' := X_{t',x'}^\nu(s),\\
\mu_s := \mu(s,X_s;\nu_s) \quad &, \quad \mu_s' := \mu(s,X_s';\nu_s),\\
\sigma_s := \sigma(s,X_s;\nu_s) \quad &, \quad \sigma_s' := \sigma(s,X_s';\nu_s),\\
d_{x'} = |x'-x| \quad &, \quad d_{t'} = |t'-t|.
\end{aligned}$$

First we notice that, by (B.3), we have, for all $\in [t', T]$ and for all $x' \in B_1(x)$,

$$\begin{aligned}\mathbb{E}\left[\left|X^{\nu}_{t',x'}(r)\right|^p \Big| \mathcal{F}_{t'}\right] &\leq C_M(1+|x'|^p)\\ &\leq C_M(1+|x|^p),\end{aligned}$$

where C_M depends only on K, T, p, M.

Suppose that (i) in the hypothesis of the Lemma holds. Then

$$\begin{aligned}\mathbb{E}\left[|X_s - X'_s|^p|\mathcal{F}_{t'}\right] &= \mathbb{E}\left[\left|x + \int_t^s \mu_r dr + \int_t^s \sigma_r dW_r - x' - \int_{t'}^s \mu'_r dr - \int_{t'}^s \sigma'_r dW_r\right|^p \Big| \mathcal{F}_{t'}\right]\\ &\leq C\mathbb{E}\left[\left|\int_t^s \mu_r - \mu'_r dr\right|^p + \left|\int_{t'}^t \mu'_r dr\right|^p + \right.\\ &\quad \left. + \left|\int_t^s \sigma_r - \sigma'_r dW_r\right|^p + \left|\int_{t'}^t \sigma'_r dW_r\right|^p + d^p_{x'} \Big| \mathcal{F}_{t'}\right]\\ &\leq C\mathbb{E}\left[\int_t^s |\mu_r - \mu'_r|^p dr + d^{p-1}_{t'} \int_{t'}^t |\mu'_r|^p dr + \right.\\ &\quad \left. + \int_t^s |\sigma_r - \sigma'_r|^p dr + d^{\frac{p}{2}-1}_{t'} \int_{t'}^t |\sigma'_r|^p dr + d^p_{x'} \Big| \mathcal{F}_{t'}\right]\\ &\leq C\mathbb{E}\left[d^{\frac{p}{2}-1}_{t'} \int_{t'}^t (1 + |X'_r|^p + |\nu_r|^p) dr + d^p_{x'} + \int_t^s |X_r - X'_r|^p dr \Big| \mathcal{F}_{t'}\right]\\ &\leq \bar{C}_{(|x|,M)}(d_{t'}, d_{x'}) + C\int_t^s \mathbb{E}[|X_r - X'_r|^p|\mathcal{F}_{t'}] dr,\end{aligned}$$

where in the second inequality we used Lemma 139, and

$$\begin{aligned}\bar{C}_{x,M}(d_{t'}, d_{x'}) &:= C_M d^{\frac{p}{2}}_{t'}\Big(1 + |x|^p + M^p\Big) + C d^p_{x'}\\ &\leq C_{x,M}\left(d^{\frac{p}{2}}_{t'} + d^p_{x'}\right).\end{aligned}$$

Thus, we conclude by Gronwall's Lemma that

$$\begin{aligned}\mathbb{E}\left[|X_s - X'_s|^p|\mathcal{F}_{t'}\right] &\leq \bar{C}_{x,M}(d_{t'}, d_{x'}) e^{C(s-t)}\\ &\leq C_{x,M}\left(d^{\frac{p}{2}}_{t'} + d^p_{x'}\right) e^{CT}.\end{aligned}$$

If we assume that (ii) holds then we can define $\bar{C}$ by

$$\bar{C}_{x,M}(d_{t'}, d_{x'}) := C_M d^{\frac{p}{2}-1}_{t'}\Big(1 + |x|^p + M^p\Big) + C d^p_{x'}.$$

□

Remark 149. *In this thesis we are interested in the case where $U = A \times B$, $\mathcal{U}_t = \mathcal{A}_t \times \mathcal{B}_t$. In that case we remark that, for $p \geq 1$ and $(a, b) \in U$,*

$$|(a,b)|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

As a consequence we also have that, for $p \geq 1$ and $(a, b) \in \mathcal{U}$,

$$\begin{aligned}\|(a,b)\|_{\mathbb{H}^{\infty}_{t',p}} &\leq 2^{p-1}(\|a\|_{\mathbb{H}^{\infty}_{t',p}} + \|b\|_{\mathbb{H}^{\infty}_{t',p}}),\\ \|(a,b)\|_{\mathbb{H}^{p,\infty}_{t'}} &\leq 2^{p-1}(\|a\|_{\mathbb{H}^{p,\infty}_{t'}} + \|b\|_{\mathbb{H}^{p,\infty}_{t'}}).\end{aligned}$$

Appendix C

A measure result

In this Appendix we prove a measure result useful in the proof of existence of solution for the HJBI equation.

Lemma 150. *Consider $X \subset \mathbb{R}^n$, $A \subset \mathbb{R}$ and $F \subset X \times A$ such that A is compact, F is closed and*

$$\pi_X(F) = X.$$

Then there is a measurable function $\psi : X \to A$ such that $(x, \psi(x)) \in F$.

Proof. Consider $\psi(x) := \max\{a \in A : (x,a) \in F\}$. Since A is compact, $\psi : X \to A$ is a well defined function and $(x, \psi(x)) \in F$. It remains to see that ψ is measurable, which we will prove by approximating ψ by a sequence of increasing measurable functions f_n.

We can suppose, without loss of generality, that $A \subset [0,1]$. We then define

$$f_n(x) := \sum_{i=1}^{2^n-1} 2^{-n} \mathbf{1}_{\pi_X\left(F \cap X \times \left[\frac{i}{2^n},1\right]\right)}(x)$$

and notice that f_n is measurable because, for each i, $\pi_X\left(F \cap X \times \left[\frac{i}{2^n},1\right]\right)$ is a closed subset of X.

Furthermore, it is easy to see that f_n is an increasing sequence and that

$$f_n(x) = \sum_{i=0}^{2^n-1} \frac{i}{2^n} \mathbf{1}_{\left\{x : \frac{i}{2^n} \leq \psi(x) < \frac{i+1}{2^n}\right\}},$$

which implies that

$$f_n(x) \nearrow \psi(x).$$

Thus $\psi(x)$ is measurable. □

Using the previous Lemma, we get:

Proposition 151. *Consider $X \subset \mathbb{R}^n$, $A \subset \mathbb{R}^m$ and $F \subset X \times A$ such that A is compact, F is closed and*

$$\pi_X(F) = X.$$

Then there is a measurable function $\psi : X \to A$ such that $(x, \psi(x)) \in F$.

Proof. We will make the proof by induction. By the previous Lemma the result is valid for $m = 1$. Now suppose it is valid for $m - 1$.

We can suppose that $A = A_1 \times A_2$ where A_1, A_2 are compact and $A_2 \subset \mathbb{R}$. Thus, we will think of F as a subset of $X \times A_1 \times A_2$.

Consider $F_1 = \pi_{X\times A_1}(F)$ which is a closed subset of $X \times A_1$. Then $\pi_X(F_1) = X$, hence, by the induction hypothesis, there is a measurable function $\psi_1 : X \to A_1$ such that

$$(x, \psi_1(x)) \in F_1.$$

Now define the closed sets $\tilde{X} := \overline{\text{graph}(\psi_1)} \subset X \times A_1$ and $F_2 := \tilde{X} \times \mathbb{R} \cap F$. It is clear that $F_2 \subset \tilde{X} \times A_2$.

We will prove later that for each $(x, a_1) \in \tilde{X}$ there is $a_2 \in A_2$ such that $(x, a_1, a_2) \in F$. Assuming this claim for now, we get

$$\pi_{\tilde{X}}(F_2) = \tilde{X}.$$

We can then apply the previous Lemma to obtain a measurable function $\psi_2 : \tilde{X} \to A_2$ such that

$$(\tilde{x}, \psi_2(\tilde{x})) \in F_2 \subset F.$$

We now define $\psi : X \to A_1 \times A_2$ by

$$\psi(x) := (\psi_1(x), \psi_2(x, \psi_1(x))).$$

It is then obvious that $\text{graph}(\psi) \subset F$. Furthermore, ψ is measurable because it is the composition of measurable functions.

We finally prove the claim that for each $(x, a_1) \in \tilde{X}$ there is $a_2 \in A_2$ such that $(x, a_1, a_2) \in F$. Indeed, if $(x, a_1) \in \tilde{X}$ then there is a sequence $\{(x^n, a_1^n)\} \subset \text{graph}(\psi_1)$ such that $(x^n, a_1^n) \to (x, a_1)$. Since $\text{graph}(\psi_1) \subset F_1 = \pi_{X\times A_1}(F)$, then for each n there is also $a_2^n \in A_2$ such that $(x^n, a_1^n, a_2^n) \in F$. Since A_2 is compact there is $a_2 \in A_2$ and a sequence n_k such that $a_2^{n_k} \to a_2$. Then $(x^{n_k}, a_1^{n_k}, a_2^{n_k}) \to (x, a_1, a_2)$ and, since F is closed, we conclude that $(x, a_1, a_2) \in F$ as desired. □

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