Relativistic Electrodynamics with Minkowski Spacetime
Algebra

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Abstract. The aim of this work is to study several electrodynamic effects using spacetime algebra – the geometric algebra of spacetime, which is supported on Minkowski spacetime. The motivation for submitting onto this investigation relies on the need to explore new formalisms which allow attaining simpler derivations with rational results. The practical applications for the examined themes are uncountable and diverse, such as GPS devices, Doppler ultra-sound, color space conversion, aerospace industry etc. The effects which will be analyzed include time dilation, space contraction, relativistic velocity addition for collinear vectors, Doppler shift, moving media and vacuum form reduction, Lorentz force, energy-momentum operator and finally the twins paradox with Doppler shift. The difference between active and passive Lorentz transformation is also established. The first regards a vector transformation from one frame to another, while the passive transformation is related to user interpretation, therefore considered passive interpretation, since two observers watch the same event with a different point of view.

Regarding time dilation and length contraction in the theory of special relativity, one concludes that those parameters (time and length) are relativistic dimensions, since their interpretation depends on the frame where an observer is located. The concept of simultaneity being relative is also introduced – the same event is interpreted on different ways, depending on the frame of reference.

The relativistic velocity addition for collinear vectors is analyzed without resorting to Einstein’s second postulate, which states that the velocity of transmission of light in vacuum has to be considered equal to \( c \) for all inertial frames (non-accelerated), therefore the framework of special relativity does not depend on electromagnetism.

Moving media is a subject which is addressed on this work as well, using the vacuum form reduction. It is concluded that applying this method to plane wave propagation in moving isotropic media (on its own frame) is considered a bianisotropic media on the rest frame.

The famous twins paradox is approached using the Doppler shift – the observers, opposed to expected, actually agree with the time divergence between them. This supposed deviation in calculations is explained by time dilation, therefore does not sustains as a mystery or inconsistency.

Keywords
Geometric algebra, Minkowski spacetime, Maxwell equations and Lorentz force, Vacuum Form Reduction (VFR), Doppler shift, Energy-momentum operator
1. Introduction

The main objective of this work is to study several electrodynamic and relativistic effects, using the most appropriate algebra for that meaning – STA (spacetime algebra). In order to do so, one must first understand the basic concepts of Euclidean algebra, which is the foundation of STA. The difference between both relies on the fact that Euclidean algebra only uses vectors in ordinary three-dimensional space, while STA unites spatial and temporal vectors, resulting in a non-Euclidean algebra. STA is supported on Minkowski spacetime, which symmetry group is the Poincaré group. To understand the applications which will be presented, it is also necessary to introduce Maxwell’s equations, including Lorentz force, since they are the basis of electromagnetism. Other applications which will take part on this work relate to kinematics and Doppler effect, which will be subsequently used to solve the famous problem of the twins’ paradox. The moving media problem is also addressed, and instead of grasping with tensors, which has proven to be a rather complex task, the vacuum form reduction method will be applied.

During the investigation which lead to this dissertation, several concepts had not yet been written, whether as a master’s dissertation or as literature. Most concepts which appear on this work are already known, the difference is the used formalism – in this case STA is the chosen foundation. This is the first master’s dissertation which gives a global, yet, in-depth study of diverse applications that STA may help grasp. STA shows its enormous ability to solve problems, which formerly were very complicated to solve, since required much more extended calculations and major complexity level. On this thesis, VFR will be used to solve and analyze the STA constitutive relation on an isotropic media and presents a novelty on a master’s thesis. Another innovation on this thesis is the resolution of the relativistic velocity addition for collinear vectors, without Einstein’s second postulate, which states that the velocity of transmission of light in vacuum has to be considered equal to \( c \). The reader will have the opportunity to grasp the fundamentals of this algebra, such as bivectors and trivectors – and its geometric interpretation in the corresponding algebra space, \( C\ell_2 \) and \( C\ell_3 \). It is essential to understand these algebras, as they are the foundation of STA, since they led to the latter and more perfect form. The essential rules will be presented, as well as the application of rotations and contractions, which will be useful on subsequent chapters. The relation of GA (geometric algebra) with dot product and wedge product will be studied as well. Gibbs’ and Grassmann’s algebras are related as well, their relation and review will be given on this chapter.

Some applications shall be addressed, therefore an introduction to the framework will be made previously. The major strength of STA is the ability to treat boosts, since it is the basis of all presented applications. The difference of STA’s indefinite metric to the former algebras will be presented. That is the big advantage of STA, when compared to the Euclidean algebras – space and time are connected. The presented concepts on this chapter will allow the reader to understand and interpret the applications, such as time dilation and space contraction, as well as relativistic velocity addition for collinear vectors and Doppler shift as well. The interaction of spacetime algebra with electromagnetism will be presented as well. Maxwell equations will be the kernel of this chapter, since those equations give the best characterization of the electric and magnetic fields and relate them to their sources, charge density and current density. These equations are used to show that light is an electromagnetic wave. The four equations, together with the Lorentz force law are the complete set of laws of classical electromagnetism. The Lorentz force law itself was actually derived by Maxwell under the name of "Equation for Electromotive Force" and was one of an earlier set of eight equations by Maxwell.

2. Geometric Algebra

Considering a \( C\ell_n \) algebra with \( n = 2 \ (n = 3) \), which means it is a two (three)-dimensional space, all its vectors follow the rule

\[
e_i \cdot e_j = \delta_{ij}
\]

(1)

According to the Kronecker delta function, defined by

\[
\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}
\]

(2)

Those algebras are spanned by the basis sets \( B = \{1, e_1, e_2, e_3\} \) and \( B = \{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\} \). These algebras are Euclidean, since the vectors’ square is always equal to 1. As follows \( |e_1|^2 = |e_2|^2 = |e_3|^2 = |e_1e_2e_3|^2 = 1 \). The axioms which define the algebras at hand are the following

- **Associativity** \( a(bc) = (ab)c \)
- **Distributivity** \( a(b + c) = ab + ac \)
- **Anti-symmetry** \( ba = -ab \)
- **Contraction rule** \( aa = a^2 = |a|^2 \)
- **Positive** \( a^2 = |a|^2 > 0 \)
- **Scalar multiplication** \( \lambda a = a \lambda \)

It is possible to obtain a certain algebra’s dimension, which is given by \( \dim(C\ell_n) = 2^n \), therefore the dimensions are 4 and 8, which may be corroborated by the number of objects for a given algebra. \( C\ell_2 \) algebra is composed by one scalar, two vectors and one bivector, while \( C\ell_3 \) algebra has one scalar, three vectors, three bivectors and one trivector. For their corresponding algebras (\( C\ell_2 \) and \( C\ell_3 \)), bivectors and trivectors are also known as *pseudoscalars*, being the element with the highest grade.
Figure 1 – Bivector described by a frame

Figure 2 – Trivector described by oriented frames

Bivectors and trivectors may be considered as oriented planes, formed by the outer product of vectors. As for instance, a bivector is formed by the outer product of two vectors \( e_2 = e_1 \wedge e_2 = -e_1 \wedge e_2 \) and a trivector by three vectors, \( e_{123} = e_1 \wedge e_2 \wedge e_3 \). Being an oriented plane means that the figure representing the object doesn’t matter, just its orientation and area. The orientation is given by the vectors and area by a certain scalar. The outer product is an anti-commutative operation, opposite to the inner product. The geometric product will be introduced as the sum of inner and outer product of two vectors, as shown \( a \cdot b = a \cdot b + a \wedge b \). This product is anti-commutative due to the inner product, so \( b \cdot a = b \cdot a + b \wedge a = a \cdot b - a \wedge b \).

Summing both equations one obtains \( a \wedge b = \frac{1}{2}(a \cdot b - b \cdot a) \).

Now the aim is to study rotations in \( Cl_2 \), algebra, which result from applying a rotor to a certain vector. This will originate another vector rotated form the initial one, both forming a certain angle, which is given by the one defined on the rotor’s expression. Considering \( B = \beta e_{12} \), one obtains the rotor

\[
R = \exp(\beta e_{12}) = \cos(\beta) + \sin(\beta)e_{12}
\]  

(3)

Applying the rotor to a vector, it is possible to obtain the final and rotated vector, which will be a clock-wise rotation.

\[
v \rightarrow u = \exp(\beta e_{12})v
\]  

(4)

Figure 3 – Clock-wise rotation

It is also possible to attain a counter clock-wise rotation, given by the following rotor and its application to a vector

\[
\tilde{R} = \exp(\beta e_{12}) = \cos(\beta) - \sin(\beta)e_{12}
\]

(6)

\[
v \rightarrow u = \exp(-\beta e_{12})v = \exp(\beta e_{12})v
\]

(7)

Figure 4 – Counter clock-wise rotation

These rotations are similar to the complex numbers. Performing a rotation with rotors gives a different result when that rotor is on the left or right side of the vector to be rotated. On the other hand, with complex numbers it is the same to apply on the right or left side. Regarding \( Cl_3 \) algebra, one may highlight the property of duality, where

\[
a = -F e_{123}
\]

(8)

And

\[
F = a e_{123}
\]

(9)

It is possible to establish the connection between Gibbs’ cross product with \( Cl_3 \) GA. It is only possible to define the cross product in \( \mathbb{R}^3 \) space, opposed to Grassmann’s outer product, as follows

\[
c = a \times b = -(a \wedge b) e_{123}
\]

(10)

While Gibbs’ cross product depends on the metric and is neither associative nor invertible, Grassmann’s doesn’t depend on its metric, and besides that, it’s associative and invertible. \( Cl_3 \) algebra apprehends several algebraic structures within, it is possible to define the even part of \( Cl_3 \), denoted as \( Cl_3^e \) and the odd part \( Cl_3^o \) as those structures, which result from the geometric product of an
even and odd number of vectors, respectively of $\mathbb{R}^3$ space. Only the even part incorporates sub-algebra, since the odd part is not closed regarding the geometric product. On $Cl_i$ algebra it is also possible to apply rotations, which is given by

$$a \rightarrow a' = RaR$$

(11)

![Figure 5 – Rotation on space algebra](image)

It may be concluded that when performing a rotation, the parallel component of the vector is unchanged, while the perpendicular component remains unchanged. Another important operation which will be introduced is the contraction – it may be done to the left or to the right. The consequence of the left operation is to reduce the degree of a bivector, assigned by letter $B$ to a unitary degree, represented by $a$

$$a = \frac{1}{2}(aB - Ba)$$

(12)

$$B = \frac{1}{2}(Ba - aB)$$

It is an anti-symmetric operation, hence

$$a B = -B a = a_i B$$

(13)

### 3. Spacetime Algebra

STA is useful to handle certain kinds of problems which cannot be addressed by Euclidean algebras. So, this algebra differs from the latter ones on a basic principle: the square of the vectors is not always equal to 1, as follows $[e_i^2] = 1, [e_n^2] = e_n^2 = -1$. The algebra is spanned by the basis set $\mathcal{B} = \{1, e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}$. It is constituted by one scalar, four vectors, six bivectors, four trivectors and the pseudoscalar – quadrivector, hence the dimension is equal to 16. An event may be described by the equation

$$r(t) = (ct)e_0 + x(t)e_1 + y(t)e_2 + z(t)e_3 \in 1^{1/3}$$

(14)

It is possible to distinguish three kinds of vectors, directly related to the spacetime trajectory – lightlike, when the movement is made at light speed $c$ and $r^2 = 0$; spacelike (space predominates over time and $r^2 < 0$) and timelike (time predominates over space and $r^2 > 0$). $r^2 = (ct)^2 - \left(x^2 + y^2 + z^2\right) = (ct)^2 - x^2$  

(15)

It is possible to explain trajectories using the lightcone

![Figure 6 - Spacetime trajectories represented on a lightcone](image)

On the previous figure, the region known as elsewhere is not accessible since it is necessary to travel at a certain speed greater than $c$, which is not possible. An important rule in $Cl_{1,3}$ algebra is that of commutation: while vectors and trivectors anti-commute with the pseudoscalar $I$, bivectors commute with $I$. Just like the studied algebra, it’s possible to define Clifford’s dual for a generic multivector $u = a + b + F + aI + bI$, which results in vector $v = uI = u e_{0123}$. Then, the resulting vector is $v = -\beta - b + F I + a + aI$. In $Cl_{1,3}$ algebra bivectors may be classified as simple (hyperbolic), when $F = e_i \wedge e_j \rightarrow F^2 = 1$, simple (elliptical) $F = e_i \wedge e_j \rightarrow F^2 = -1$, simple (parabolic) $F = e_i \wedge e_j + e_i \wedge e_j \rightarrow F^2 = 0$ or non-simple, when $F = e_i \wedge e_j + e_i \wedge e_j \rightarrow F^2 = 2I$. The most important highlight is that elliptical bivectors originate rotations ($\exp(\theta e_{23}) = \cos \theta + e_{23} \sin \theta$), while hyperbolic bivectors originate boosts ($\cosh \zeta + e_{03} \sinh \zeta$). A boost is a similar transformation to rotation, although not as simple. $\zeta$ is the rapidity parameter, which controls the intensity of the boost, so the goal is to expose that tool. A boost is also known as active Lorentz transformation and performs a transformation of a certain frame defined by $(e_0, e_3)$ to
another defined by \((f_0, f_1)\). \(\gamma = \cosh \zeta = \frac{1}{\sqrt{1 - \beta^2}}\) is the correction factor, which depends directly on the velocity of a certain particle. \(\beta = \frac{v}{c} = \tanh \zeta\) is the normalized velocity hence \(\gamma \beta = \sinh \zeta\). When the rapidity tends to infinite, the normalized velocity tends to \(\tan \left(\frac{\pi}{4}\right) = 1\), which means that \(f_0 \rightarrow e_0 + e_1\) and \(f_1 \rightarrow e_0 + e_1\), therefore tending to the bisection, as represented next.

Figure 7 - Representation of a boost

Boosts result Minkowski diagrams, which will be the used tool to study relativistic effects. For a purpose of simplicity, one will use a \(Ct_{1,1}\) diagram instead of \(Ct_{1,3}\)

![Figure 8 - Minkowski diagram](image)

An event is described by a person standing on a frame, which can be represented on the Minkowski diagram according to the expression which links both time and space. An observer located on the rest frame \(S\) shall interpret an event \(\mathbf{r} = (ct)\mathbf{e}_n + x\mathbf{e}_1\), while another observer located on the \(\overline{S}\) will have another interpretation \(\mathbf{r} = (ct)\mathbf{f}_n + x\mathbf{f}_1\), which means that the same event has diverse point of view, therefore it is not possible to define an absolute time or space. Simultaneity is a relative concept, and the passive Lorentz transformation explains that. While the active transformation performs an actual transformation of vectors, the passive merely shows the diverse points of view for a particular event. Any two events \(A\) and \(B\) which occur on a line parallel (also known as line of simultaneity in \(\bar{S}\)) to axis \(\bar{x}\), defined as \(ct_{0}\), happen simultaneously regarding an observer on the \(\bar{S}\) frame. This is not valid for another observer who is placed on the \(S\) frame, which is shown on next figure. The difference in time events regarding \(S\) frame is given by \(\Delta ct_{ab} = ct_A - ct_B\), which is different from zero. Following the same thread of thought, Any two events \(A\) and \(B\) which occur on a line parallel to axis \(x\), defined as \(ct_{0}\), also known as line of simultaneity in \(S\) happen simultaneously regarding an observer on the \(S\) frame. This is not valid for another observer who is placed on the \(\bar{S}\) frame, which is shown on next figure. The difference in time events regarding \(\bar{S}\) frame is given by \(\overline{\Delta ct}_{ab} = \bar{ct}_A - \bar{ct}_B\), which is different from zero.

Several application shall be addressed on this paper, starting with time dilation and space contraction. Either effect may be performed considering the rest frame as \(S\) or \(\bar{S}\), as the result is the same. For a matter of convention, the first approach shall be taken.

![Figure 9 – Time dilation](image)

From the previous figure it can be inferred that vector \(\overline{AB}\) is equal to the sum of other two vectors, as follows:

\[
\overline{AB} = \overline{AC} + \overline{CB}
\]

This results on the following equation

\[
cT_{0}f_0 = cTe_0 + vTe_1
\]
And therefore
\[ f_0 e_n = \gamma e_0 + \gamma \beta e_n \] (18)

Given that \( f_0 e_0 = 0 \), \( e_0 e_n = 1 \) and \( e_n e_n = 0 \), yields
\[ T_0 \gamma = \frac{T_0}{\sqrt{1 - \beta^2}} = T \] (19)

Now, analyzing the space contraction, the geometric equation which translates the problem is given by
\[ AB = AC + CB \] (20)

Figure 10 – Length contraction

This is equal to
\[ Lf_1 = L_0 e_1 + \beta L e_0 \] (21)

Thus
\[ f_1 e_1 = \gamma \beta e_0 e_1 + \gamma e_1 e_1 \] (22)

Given that \( f_1 e_1 = 0 \), \( e_1 e_1 = 1 \) and \( e_1 e_1 = 0 \), yields
\[ \frac{L_0}{\gamma} = \frac{L_0}{\sqrt{1 - \beta^2}} = L \] (23)

It may be concluded that the correction factor is the parameter that allows to transform the time or length observed on a specific frame to another which may not be at rest.

The importance of using \( C\ell_{1,3} \) algebra instead of \( C\ell_4 \) may be demonstrated when determining the relativistic velocity addition, in this case, for collinear vectors. According to Newton, the velocity addition for two particles traveling at a speed equal to \( c \) is equal to \( 2c \), constituted by a simple addition. This is not possible, since no particles travel at a velocity greater than \( c \). This was contradicted by Maxwell, who stated there is the need to perform a correction to Newton’s equation. Using \( C\ell_4 \) to determine the velocity composition, one obtains equation
\[ v = \frac{v_1 + v_2}{1 - \frac{v_1 v_2}{c^2}} \] (24)

Simulations were made, and it was concluded that it is incorrect – adding two particles’ velocities greater when the condition \( \beta_1 > \frac{1}{\beta_2} \) is complied, results on a velocity inferior to 0, which is physically unacceptable. Besides that fact, the velocity addition tends to infinite when both particles’ velocities tend to \( c \), which is also incorrect.

Figure 11 - \( C\ell_4 \) velocity addition

Striving for the same objective, \( C\ell_{1,3} \) algebra, with its union between time and space, accomplished a clear result as crystal – not only the equation corroborates the physical effects, as well as the velocity addition tends to \( c \), as supposed to. The correct formula is represented next, as well as its simulation
\[ v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \] (25)

Figure 12 - \( C\ell_{1,3} \) velocity addition
Doppler shift is another application which deserves special attention. Its equation is given by $1 + z = \frac{\omega_1}{\omega_0}$, where $z$ is the Doppler magnitude of redshift, $\omega_1$ ($\omega_0$) is the frequency on the receiver (emmitter) side. The goal is to attain the relation among both latter parameters. In order to reach the relation, on other words, to determine the receiving photon on the receiver side. The goal is to attain the equation which gives the desired relation is presented next

$$\omega / \omega_0 = \frac{1 - \beta \cos \phi}{\sqrt{1 - \beta^2}}$$

(26)

It is the general formula for Doppler shift – when angle $\phi = 0$ ($\phi = \frac{\pi}{2}$) results on the longitudinal (transverse) Doppler effect. The first situation degenerates on $\omega / \omega_0 = \frac{(1 - \beta)}{\sqrt{(1 + \beta)}}$ thus on a redshift, as $\omega_1 < \omega_0$; the second situation results on $\omega / \omega_0 = \frac{1}{\sqrt{1 - \beta^2}}$ or a blueshift, which happens when $\omega_1 > \omega_0$. Doppler shift may be used to make a spectral analysis of the sun’s light, for instance. To say that an image is blueshifted (where a blueshift occurs) means an observer is looking at that part of the sun that is moving towards him, or the light is compressed to shorter wavelengths, so the frequency is increasing. Likewise, the opposite can be said about a red image, where the opposite takes place, also called as a redshift.

4. Electrodynamics and Relativistic Effects

As shown, GA possesses innumerous techniques for studying problems in electrodynamics and electromagnetism. This chapter proposes several applications regarding the matters referred above, which will emphasize the power of GA compared to traditional algebras. On a first approach, Maxwell equations shall be presented whether on regular three-dimensional space or spacetime. As one will see later on, this is a great quality of STA, since it will shed light on a somewhat dark matter. It will simplify the study of how electromagnetic fields appear to observers on different frames. Another big improvement of this treatment, is a more recent and compact formulation on Maxwell’s equations. Geometric product and derivative vector will permit to transform Maxwell’s four equations into an astonishing individual one. In order to study electromagnetism and electrodynamics, one must introduce a fundamental operator – del.

$$\nabla = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

(27)

It is possible to represent the Maxwell equations in CI, on a most known structure, where one may distinguish two groups of equations - Faraday group, which includes Maxwell-Faraday’s equation, also known as the Faraday’s law of induction and Gauss’s law for magnetism

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

(28)

$E \in \mathbb{R}^3$ represents the intensity of the electric field; it is defined as the electric force per unit charge. The direction of the field is taken to be the direction of the force it would exert on a positive test charge. The electric field is radially outward from a positive charge and radially inward to a negative point charge. $B \in \mathbb{R}^3$ represents the intensity of the magnetic field. These two fields are denominated as strength values, since characterize how strong the fields are. The second pair of equations is identified as the Maxwell group – both Ampère’s circuital law with Maxwell’s correction and Gauss’s law are present, as the following set of equations show

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$\nabla \cdot D = -$$

(29)

$D \in \mathbb{R}^3$ represents the electric displacement, while $H \in \mathbb{R}^3$ represents the magnetic field strength. These two vectors reflect a certain material’s excitation amount, consequently are known as excitation values. It is possible to describe $D$ and $H$ in order to $E$ and $B$ although using another two vectors as well: polarization $P \in \mathbb{R}^3$ and magnetization $M \in \mathbb{R}^3$. The indicated equations are represented next.

Figure 13 – Addressed problem on Doppler shift

Bivector $\vec{B}$ defines the relation between two frames and its basis sets – given by a boost, thus it is possible to establish $e$, in order to $e_0$. When a rotation occurs, its frame remains, while the obtained angle suffers an alteration as represented on Figure 13. The referred angle is given by $s \cdot s_0 = -\cos \phi$. When applying a rotation, just the vector’s angle changes, while the modulus remains unaltered. On the other hand, when a boost is applied, the photon’s frame is altered. On both applications, the photon’s velocity does not change, its is always equal to the speed of light. The expression which gives the desired relation is presented next

$$\omega / \omega_0 = \frac{1 - \beta \cos \phi}{\sqrt{1 - \beta^2}}$$

(26)

It is the general formula for Doppler shift – when angle $\phi = 0$ ($\phi = \frac{\pi}{2}$) results on the longitudinal (transverse) Doppler effect. The first situation degenerates on $\omega / \omega_0 = \frac{(1 - \beta)}{\sqrt{(1 + \beta)}}$ thus on a redshift, as $\omega_1 < \omega_0$; the second situation results on $\omega / \omega_0 = \frac{1}{\sqrt{1 - \beta^2}}$ or a blueshift, which happens when $\omega_1 > \omega_0$. Doppler shift may be used to make a spectral analysis of the sun’s light, for instance. To say that an image is blueshifted (where a blueshift occurs) means an observer is looking at that part of the sun that is moving towards him, or the light is compressed to shorter wavelengths, so the frequency is increasing. Likewise, the opposite can be said about a red image, where the opposite takes place, also called as a redshift.
\[ D = \varepsilon_0 E + P \]
\[ H = \frac{1}{\mu_0} B - M \]

(30)

The following relations are valid as well
\[ p = -\nabla \cdot P \]
\[ J_p = \frac{\partial P}{\partial t} \]
\[ J_m = \nabla \times M \]

(31)

Where the total electric charge density and current are given
\[ \rho = \rho_e + \rho_m \]
\[ J = J_e + J_p + J_m \]

(32)

Finally one may attain the light speed on vacuum
\[ c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}} \]

(33)

Opposite to Maxwell’s equations, the auxiliary fields \( D \) and \( H \) are not present. Therefore it is considered a fundamental and reductionist viewpoint, since one intends to explain the most using the fewest principles possible. This new formulation is very attractive since both curl and divergence of field quantities \( E \) and \( B \), respectively, are specified. According to the Helmholtz theorem, a field vector requires its curl and divergence to be given, in order to be fully specified. One may separate the four equations regarding magnetic flux conservation and charge-current conservation
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times B = \mu_0 J_m + \frac{1}{c^2} \frac{\partial E}{\partial t} \]
\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]

(34)

(35)

Using some mathematic principles, it is possible to reduce those equations to a more compressed set, as follows
\[ \frac{\partial}{\partial t} \left( \mu_0 e_1 J_m \right) + \nabla \times E = 0 \]
\[ \nabla \cdot B = 0 \]
\[ \frac{\partial D}{\partial t} + \nabla \times (\mu_0 e_1 J_m) = -J \]
\[ \nabla \cdot D = \frac{1}{\eta_0} \frac{\partial \eta_0}{\partial t} \]

(36)

(37)

(38)

Besides that, the Dirac operator and electric charge – current densities are also necessary
\[ J = e_0 + \frac{1}{c} J = e_0 + \frac{1}{c} \left( J_1 e_1 + J_2 e_2 + J_3 e_3 \right) \]
\[ \frac{\partial}{\partial t} \left( e_0 + \frac{1}{c} \frac{\partial e_0}{\partial t} + e_1 \frac{\partial e_1}{\partial x_1} + e_2 \frac{\partial e_2}{\partial x_2} + e_3 \frac{\partial e_3}{\partial x_3} \right) \]

(39)

Now it is possible to define the essential bivectors of the electromagnetic field, which are absolute vectors – Faraday and Maxwell bivectors
\[ F = \frac{1}{c} E + I B \]
\[ G = D + \frac{1}{c} I H \]

(40)

Now, STA Maxwell shall be presented
\[ \nabla \times E = \frac{\partial B}{\partial t} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times H = J + \frac{\partial B}{\partial t} \]
\[ \nabla \cdot B = 0 \]

(41)

(42)

Having made an introduction to Maxwell’s equations, one may differentiate absolute and relative vectors. Another goal which may be achieved is to reduce Maxwell’s equations to a single one, which is only possible using STA. In vacuum, the constitutive relation is given by
\[ G = \frac{1}{\eta_0} F \]

(43)

Knowing that \( \nabla \cdot G = J \) and using the constitutive relation in vacuum, one obtains
\[ \nabla \cdot F = \frac{1}{\eta_0} \left( e_0 + \frac{1}{c} \left( J_1 e_1 + J_2 e_2 + J_3 e_3 \right) \right) \]

(44)

If the considered media does not have sources, then the right-side of the previous equation is equal to 0. One can
distinguish relative and absolute vectors on Maxwell equations. Vectors $\mathbf{E}$ and $\mathbf{B}$ are both relative vectors, while $\mathbf{F}$ is not, even being composed by the two previous vectors. The same can be verified for $\mathbf{D}$ and $\mathbf{H}$, which are relative but $\mathbf{G}$ is an absolute vector. Consequently, vectors $\mathbf{F}$ and $\mathbf{G}$ don’t depend on any observer, so they represent the electromagnetic field and can be designated as covariant forms of the electromagnetic field. It is possible to reduce Maxwell’s equations to a sole one, since in vacuum the existence of vector $\mathbf{G}$ is not required.

$C\ell_{1,3}$ algebra is useful to study a great variety of problems. One of the classic electrodynamics problems is moving media. Considering a simple isotropic medium, in $C\ell_{1,3}$ algebra is defined by the following constitutive relations (using STA)

$$
\begin{align*}
\mathbf{D} &= \varepsilon\mathbf{E} \\
\mathbf{H} &= \frac{1}{\mu}\mathbf{B}
\end{align*}
$$

(45)

Then, one may determine vector $\mathbf{G}$

$$
\mathbf{G} = \mathbf{D} + \frac{1}{c} \mathbf{H} = \frac{\varepsilon}{c} \mathbf{E} + \alpha \mathbf{IB}
$$

(46)

As $\mathbf{v} = c\mathbf{e}_0$ represents the frame where the media is at rest, one obtains

$$
\mathbf{F}_v = -\frac{1}{c} \mathbf{E} + \mathbf{IB}
$$

(47)

This represents vector $\mathbf{F}$ seen by an observer on frame $\mathbf{e}_0$.

For a certain linear combination of $\mathbf{F}$ with $\mathbf{F}_v$, comes

$$
\mathbf{G} = \frac{1}{2\eta_0} \left( \varepsilon + \frac{1}{\mu} \right) \mathbf{F} - \frac{1}{2\eta_0} \left( \varepsilon - \frac{1}{\mu} \right) \mathbf{F}_v
$$

(48)

Then

$$
\mathbf{G} = \frac{1}{\eta} \exp(-\gamma r_v) \mathbf{F}
$$

(49)

With

$$
\mathbf{r}_v(\mathbf{F}) = \mathbf{F}_v = \mathbf{u}^\dagger \mathbf{F} \mathbf{u} = \frac{1}{c^2} \mathbf{u} \mathbf{F} \mathbf{u}
$$

(50)

The local form of the constitutive relation represents a projection of the constitutive relation on a particular observer, who is at rest regarding the media. In other words, it could be the media’s own observer. It is possible to split the constitutive relation $\mathbf{G} = \mathbf{G}(\mathbf{F})$ into two constitutive relations $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$ and $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$. This means that the media at hand is bianisotropic, and both $\mathbf{D}$ and $\mathbf{H}$ depend, not only on the electric field intensity $\mathbf{E}$, but also on magnetic field intensity $\mathbf{B}$. According to these results it can be concluded that an isotropic media on its own frame is considered as a bianisotropic medium on the rest frame.

From equation (49), one may obtain the following relation

$$
\mathbf{G} = \frac{1}{\eta} \mathbf{F}_v
$$

(51)

From equation It may be concluded that with the presented transformation, a constitutive relation is obtained, which has the same build as the constitutive relation concerning vacuum, therefore it is called VFR. With this transformation, the Minkowski spacetime structure has been changed, since it now corresponds to a fictitious spacetime, different from the original one.

Another field where STA excels is the one of electrodynamics and kinematics, where it is used to correct Einstein’s very well acknowledged mass-energy equivalence. But first one must describe the present characters on this application. Defining the proper linear momentum of a certain particle with mass $m$ will be given by

$$
\mathbf{p} = m\mathbf{u}
$$

(52)

Which is related to the particle’s total energy $E$. The mass-energy equivalence states the following

$$
\gamma m c^2 = \frac{p_0}{c}
$$

(53)

And

$$
\mathbf{p} = -\mathbf{e}_0 + \mathbf{p} = (\gamma mc)\mathbf{e}_0 + \gamma m\mathbf{u}
$$

(54)

Therefore

$$
0 = mc^2 \Rightarrow \begin{cases} \gamma = \frac{p_0}{\gamma m \mathbf{u}} \end{cases}
$$

(55)

This yields

$$
\gamma m \mathbf{u} = \sqrt{\frac{p_0^2}{\gamma} + (c\mathbf{p})^2}
$$

(56)

$\gamma$ is the particle’s proper energy, and one may attain the kinetic energy, represented by

$$
\mathbf{K} = \gamma m \mathbf{u}
$$

(57)

The following figure represents the application’s geometry.
reaches infinite, may be attained in an infinitesimal time fraction. Considering that one → ∞ and changes to γ.

This derivation was complete without This occurs because his velocity changes from $v$ to $-v$, since the acceleration suffered by the travelling twin is infinite. This problem is not feasible in real life, although one observer crosses two different frames while travelling, which results on the concluded time difference for both observers. This problem is not feasible in real life since the acceleration suffered by the travelling twin is infinite. This occurs because his velocity changes from $c$ to $-c$ in an infinitesimal time fraction. Considering that one observer travels at a speed given by $v$ and changes to $-v$, on the point where it loses reciprocity, the associated acceleration is given by $\lim_{\Delta t \to 0} \frac{v - (-v)}{\Delta t} = \frac{2v}{\Delta t} \to \infty$.

### 5. Conclusion

Regarding GA, some main features may be concluded from the performed investigation: the ease one has when applying rotations in spatial dimensions and boosts (or Lorentz transformations); the invertibility of all Clifford objects, which is a great advantage when applying to computational systems or simply performing calculations. Another important characteristic is the inherent geometric interpretation one has, which makes this algebra so intuitive to understand and use, hence attractive for beginners and motivating new users.

This work’s foundations rely on not only in GA, but on special relativity and Lorentz transformations as well, so several conclusions shall be presented. Simultaneity is a crucial concept when dealing with these matters: observers on different frames interpret events, each on a particular way. The correction factor allows a temporal or spatial transformation from one frame to another.

When performing a relativistic velocity addition for collinear vectors, one may conclude there is a velocity limit given by $c$. This derivation was complete without employing Einstein’s second postulate.

Regarding Doppler shift, one concluded its general expression, where it is possible to distinguish the longitudinal and transverse situations. $\partial F = \eta \gamma J$ is the most condensed way to write Maxwell equations in STA. It is possible to distinguish absolute $(F,G)$ and relative vectors $(E,B,D,H)$: the absolute vectors are don’t depend on the frame the observer is located; on the other hand relative vectors depend on the work frame.

As emphasized by David Hestenes, STA is the best framework to tackle electromagnetism in the context of Special Relativity, due to the inherent reduced complexity, as well as the coherent results obtained. In fact, the two Maxwell equations in STA are reduced to a single one, as the vacuum situation. This was able by applying VFR to a plane wave propagation in moving isotropic media. From the point of view of the laboratory, it may be concluded that it is actually a nonreciprocal bianisotropic media, which means its constitutive expressions $\vec{D}$ and $\vec{H}$ depend on $\vec{E}$ and $\vec{B}$.

Applying STA to a particle’s relativistic dynamics, allowed to correct some concepts, as the inertia of energy – Einstein’s expression $\phi = mc^2$ is incoherent, as well as an idea of a variable mass some physicists had.

One of the most famous paradoxes – Twins paradox has been disclosed, using the Doppler shift. It was concluded that the apparent inconsistency of results is acceptable, being explained by time dilation.