Summary

The aim of this text is to summarize the work expounded in 'The Welded Braids and The Crossed Module Invariant'. This will be done by presenting the principal concepts and results without the proofs and details.

1 Abstract

An automorphic crossed module can be defined as a triple: two groups, G and E, the second one abelian, and an action of G on E by automorphisms satisfying some conditions. This structure can be used to define an invariant of welded virtual knots.

The aim of the work was to compute, using *Mathematica*, the crossed module invariant of a large class of welded virtual knots arising from welded braids with three strands and up to ten crossings, using crossed modules with $G = GL(\mathbb{Z}_p, n)$ and $E = \mathbb{Z}_p^n$. We first define the welded braid group and, based on the work done by J. F. Martins and L. H. Kauffman, a definition of welded virtual graphs is also presented. We define the crossed module invariant for welded knots coming from welded braids with m strands using an action of the welded braid group on m strands on $(G \times E)^m$ and prove that it yields an invariant for welded virtual knots,by proving its invariance under the three types of moves that can be performed on a braid, which give a braid whose closure is the same as the original one.

We give an extension of the crossed module invariant for welded graphs, and we use it to calculate the invariant for a set of examples, obtained by closing a braid with no virtual crossings in two different ways that have the same knot group. In some cases we were able to distinguish the two closures using the crossed module invariant. We also computed another interesting example with trivial knot group. However, our calculations could not distinguish this example from the unknot. The issue of whether this should be expected on general grounds is not entirely settled, so this calculation contributes some evidence.

2 Welded Virtual Knots, Braids and Graphs

In this chapter we present some definitions, examples and results involving knots and braids that will be needed throughout this text.

A virtual knot diagram can be seen as a classical knot diagram in which an extra type of crossing is allowed, called the virtual crossings. These crossings can be represented as in figure 1.

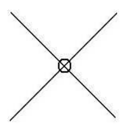

Figure 1: Representation of a virtual crossing

Thus we have the definitions:

Definition 2.1 *A **virtual knot diagram** is an immersion of $S^1 \bigsqcup ... \bigsqcup S^1 \to \mathbb{R}^2$ injective except at a finite number of double points that can be labeled either with a real or virtual crossing.*

Two virtual knot diagrams are said to be equivalent if one can be transformed into the other by planar isotopy and a finite sequence of real Reidemeister moves together with a version of the Reidemeister moves with virtual crossings and another move involving virtual and classical crossings (see figure 2).

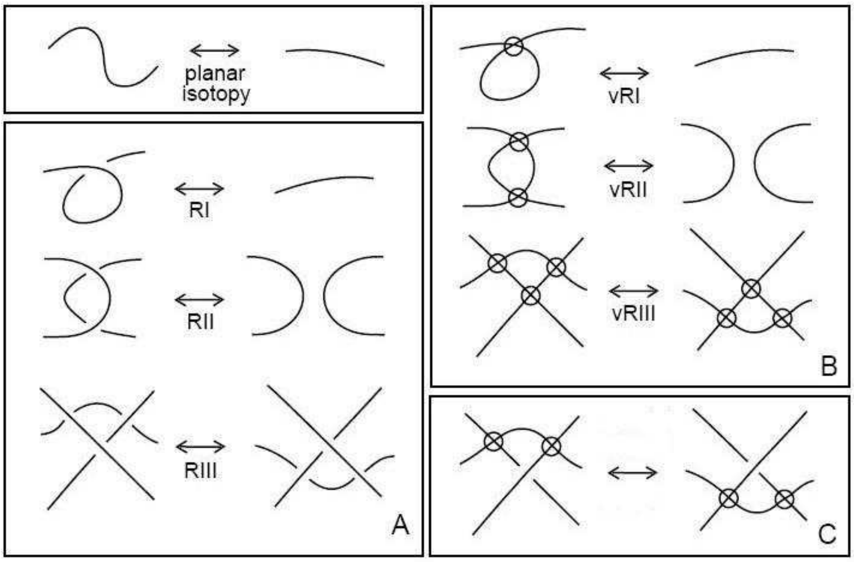

Figure 2: Reidemeister moves for virtual knots

Definition 2.2 *A **virtual knot** is an equivalence class of virtual knot diagrams, considering the equivalence relation to be the one described above.*

Notice there are two moves that do not figure in the list of movements shown above: one with an over arc and a virtual crossing and another one with an under arc and a virtual crossing. These two moves (see figure 3) are actually forbidden.

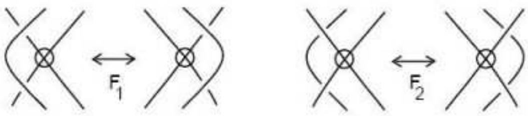

Figure 3: Forbidden moves

However, if we allow the move F_1 we obtain what are called ***welded knots***. The virtual knot and the welded virtual knot theories, despite being similar, are distinct.

A very useful presentation of (welded) knots is as a braid. This presentation was used in this work to compute the crossed module invariant.

In order to describe a braid, we can encode the crossings as follows: σ_i will represent the crossing in which the ith strand passes under the $(i+1)th$; σ_i^{-1} is the inverse crossing, when the ith goes over the $(i+1)th$; and finally, the virtual crossing between the ith and the $(i+1)th$ strands will be encoded as τ_i. This encoding is useful to define a group, considering as operation between two braids placing one on the top of the other.

Definition 2.3 *The **Welded Braid Group**, $\mathcal{WB}_m$, on m strands, $m \geq 2$, is the group generated by $\sigma_i, \sigma_i^{-1}, \tau_i$, $i = 1, ..., m-1$ with the following relations:*

1. $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$
2. $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| > 1$
3. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
4. $\tau_i^2 = 1$
5. $\tau_i \tau_j = \tau_j \tau_i$, $|i-j| > 1$
6. $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$
7. $\sigma_i \tau_j = \tau_j \sigma_i$, $|i-j| > 1$
8. $\sigma_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \sigma_{i+1}$
9. $\tau_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \tau_{i+1}$

Definition 2.4 *A **welded virtual arc diagram** is an immersion of unions of intervals $[0, 1]$ and circles into the plane $\mathbb{R}^2$, where the 4-valent vertices of the immersion can represent either classical or virtual crossings.*

We define a **welded virtual arc** as an equivalence class of welded virtual arc diagrams, considering two diagrams equivalent if they are related by the moves in figure 2 along with the following:

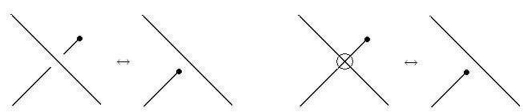

Figure 4: Additional moves for welded virtual arcs

Definition 2.5 *A **welded virtual graph** is an equivalence class of diagrams, considering the moves for the welded virtual knots and for welded virtual arcs along with the following (whenever a strand is represented with no orientation, the move is legal for any orientation):*

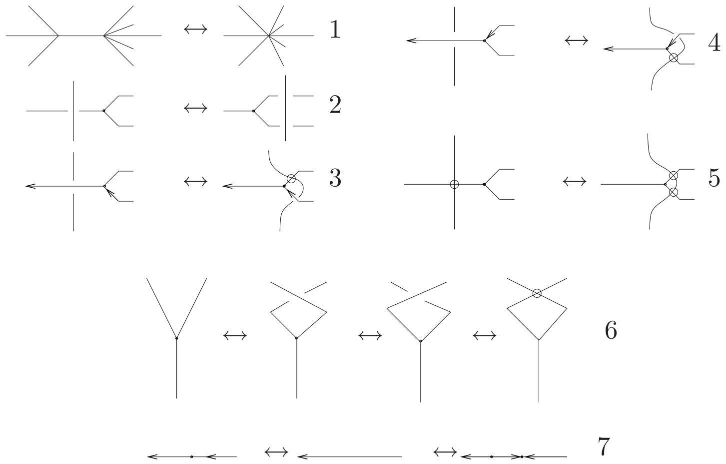

Figure 5: Additional moves for welded virtual graphs

Given a welded virtual graph K, one can obtain another graph by adding a trivial 1-handle, that consists in choosing a strand of K and doing the next move:

Figure 6: Addition of a trivial 1-handle to the graph K

Using the trivial 1-handle and the moves defining the welded virtual graphs, we can obtain a welded virtual knot from a welded virtual graph. For this, consider a graph K which is (topologically) the union of circles S^1 and intervals $I = [0,1]$. Now, if we add a trivial 1-handle to K and then apply a finite sequence of moves, as in figure 5, we can obtain another graph K' which is still a union of circles and intervals. However, if K has only one I-component, proceeding as before, we can obtain a graph only with S^1 components which is equivalent to K as a welded virtual graph.

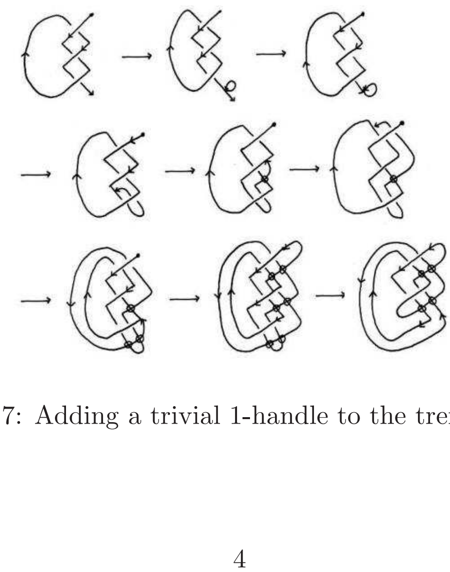

Figure 7: Adding a trivial 1-handle to the trefoil arc K

Definition 2.6 *The* ***knot group*** *of a (welded) virtual knot or a welded virtual arc is the group generated by all the arcs of a diagram of it considering the following relations, the so-called Wirtinger Relations:*

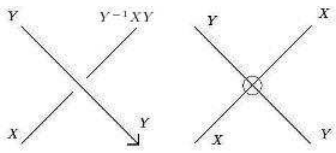

Figure 8: Wirtinger relations

We can also define the knot group for welded virtual graphs, adding the relation in the figure below for the vertices of the graph.

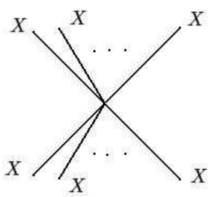

Figure 9: Relation of the knot group of a welded virtual graph at a vertex

It can be proved that for arcs with no virtual crossings and with only one component homeomorphic to $[0, 1]$,the closure of the arc and the arc itself have the same knot group. It can also be seen that the knot group is also invariant under addition of a trivial 1-handle.

An application to classical braids of these results leads to a set of very interesting examples that will be used in this work. If we regard an element of $\mathcal{B}_m$ as a welded virtual graph, we can close it in two different ways that will originate two different welded virtual knots with the same knot group. The interesting part is that the crossed module invariant, defined in the next chapter, will distinguish some of these pairs of knots.

Therefore, we will consider the usual closure (see figure 10), which will be called the **first trace**, and a second closure, which will be called the **second trace** defined in the following way: consider the usual closure of the braid, but only for the first $m-1$ strands, leaving the last strand untouched. We can close the braid completely by adding a trivial 1-handle to the last strand and then performing the moves in figure 5, as in the example of figure 7, thus producing a welded virtual knot.

Theorem 2.7 *Let $b \in \mathcal{B}_m$, be a classical braid. Let K and K' be the welded virtual knots obtained by the first and second traces of b, respectively. Then, we have that K and K' have the same knot group.*

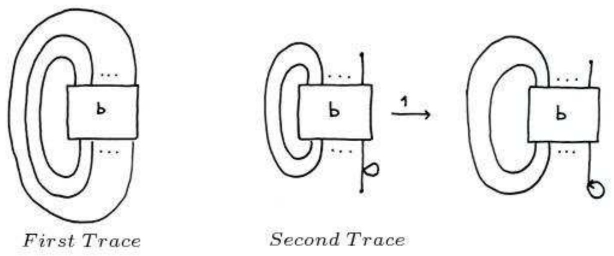

Figure 10: The two traces of the a braid b

3 The Crossed Module Invariant

In this chapter we present the definition of crossed module and crossed module invariant. We also summarize the two theorems proved in the original text.

Definition 3.1 *Let G and E be groups. A **Crossed Module**, $\mathcal{G} = (G, E, \partial, \triangleright)$, is given by a group morphism $\partial : E \to G$ and a left action $\triangleright$ of G on E by automorphisms. The conditions on ∂ and $\triangleright$ are:*

1. $\partial(X \triangleright e) = X\partial(e)X^{-1}$, $\forall X \in G$, $\forall e \in E$
2. $\partial(e) \triangleright f = efe^{-1}$, $\forall e, f \in E$

*A crossed module is called **automorphic** when $\partial(e) = 1, \forall e \in E$. In this case, the crossed module is given simply by two groups, G and E, where E is abelian, and a left action of G on E by automorphisms.*

Example 3.2 *Let G and E be $GL(\mathbb{Z}_p, n)$ and $\mathbb{Z}_p^n$, respectively, and the action $\triangleright$ simply the product between an element of G and an element of E. This automorphic crossed module will be denoted by $\mathcal{G}(n, p)$.*

Let $\mathcal{G} = (G, E, \triangleright)$ be an automorphic crossed module. We can define an action of the Welded Braid Group, $\mathcal{WB}_m$, on $(G \times E)^m$,$\psi : (G \times E)^m \times \mathcal{WB}_m \to (G \times E)^m$, such that each generator represented in the figure sends a pair in $(G \times E)^2$ to the pair shown below, leaving the other pairs unchanged:

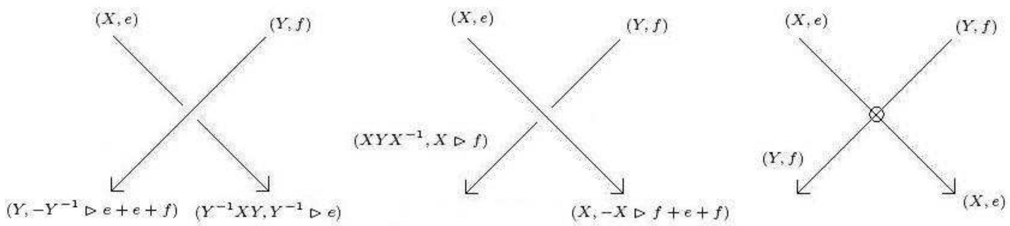

Figure 11: Relations at crossings

Theorem 3.3 *ψ is a well defined action for the $\mathcal{WB}_m$.*

Theorem 3.4 *Let $b \in \mathcal{WB}_m$. The number of fixed points of $\phi_b : (G \times E)^m \to (G \times E)^m$, which associates the initial assignment of elements of $G \times E$ to the m strands of b to the final one, according to the action defined above, is a welded virtual knot invariant for the knot that is the first closure of b.*

The crossed module invariant can be extended to welded virtual graphs. Since the aim of the work was to distinguish the two traces of the same classical braid, we present here the simplified expression for the invariant of the second trace:

$$\mathcal{H}_{\mathcal{G}}(K) = \#\{fixed\, points\ such\ that\ e_m = 0\}\#E$$

4 Computation of the invariant

The Algorithm developed in *Mathematica* computes the invariant $\mathcal{H}_{\mathcal{G}}$ mentioned previously with $G = GL(\mathbb{Z}_p, n)$, $E = \mathbb{Z}_p^n$ and the obvious action of G on E, given a welded virtual braid. The main goal was to compute the invariant for pairs of welded virtual knots that arise from the same classical braid by closing it in two different ways, yielding pairs of (possibly distinct) welded knots with the same knot group.

Since the crossed module invariant is a welded virtual graph invariant, we can calculate it for these pairs of welded virtual knots without needing their diagrams. We just need the representation of the braid b with m strands and no virtual crossings, n and p and then use the two different expressions for the invariant:

$$\mathcal{H}_{\mathcal{G}}(1st\ trace\ of\ b) = \#\{fixed\, points\ of\ \phi_b\}$$

$$\mathcal{H}_{\mathcal{G}}(2nd\ trace\ of\ b) = \#\{fixed\, points\ of\ \phi_b\ such\ that\ e_m = 0\}\#E$$

which makes the computation really simple.

The braids are represented as lists with two elements, the first being the number of strands and the second another list whose elements encode the crossings. Traveling along the braid from the top to the bottom, the crossings are represented in the list in the same order as they occur in the braid, the classic crossings as integers: the crossing σ_i corresponds to $-i$; the inverse crossing, σ_i^{-1}, corresponds to i; and the virtual crossings are encoded by τi.

Before computing the invariant we needed to generate all the invertible $n \times n$ matrices with entries in $\mathbb{Z}_p$ and all the elements of $\mathbb{Z}_p^n$. Afterwards, we split the calculation into two functions: one to calculate the invariant when the braids are closed according to the first trace and the other when the closure is done according to the second trace. The first function receives as input, as mentioned above, the representation of the braid, the dimension of the matrices and p. The calculation begins with the creation of all pairs (M, v) with $M \in GL(\mathbb{Z}_p, n)$ and $v \in \mathbb{Z}_p^n$. Then, all the tuples of m of these pairs (m being the number of strands that constitute the braid) are checked one by one: for each tuple (a tuple corresponds to a labeling of the strands of the braid) a function is applied that, after a crossing, relabels the strands according to the rules mentioned previously and returns the final labeling, which is compared with the original one. The output obtained is the number of these fixed points. The function used for the second trace is almost identical to this, except for the construction of the tuples - in this case the label for the final strand is a pair with the second component trivial - and in the end the number of fixed points needs to be multiplied by the cardinal of $\mathbb{Z}_p^n$.

5 Examples and Results

The first example is a non-trivial welded virtual knot with the same knot group as the trivial knot. This knot, let us call it K, is the closure of the following welded virtual braid that is equal to $\tau_1\sigma_2\sigma_1^{-1}\tau_1\sigma_1\sigma_2$ (thus we use as input $\{3, \{\tau 1, -2, 1, \tau 1, -1, -2\}\}$).

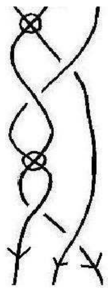

Figure 12: A braid presentation of K

The crossed module invariant does not distinguish K from the unknot when the crossed module is $\mathcal{G}(2,2)$, $\mathcal{G}(2,3)$ or $\mathcal{G}(2,4)$. Since K has the same knot group as the unknot and the map T preserves the knot group, it follows that $T(K)$ has the same fundamental group of the complement as the trivial knotted torus. The author of this example, R.Fenn, mentioned that there are some reservations about the proof that this implies that there exists an isotopy between the two corresponding knotted surfaces, but if this is the case then $\mathcal{H}_\mathcal{G}$ cannot distinguish K from the trivial knot since $\mathcal{H}_\mathcal{G}$ factors through the tube map. Nevertheless this calculation gives a coherence test to the invariant $\mathcal{H}_\mathcal{G}$.

As mentioned previously, the aim of the work was to compute the invariant for the two traces of classical braids, which has been done for all the braids with three strands and with up to ten crossings and with the crossed modules $\mathcal{G}(2,2)$, $\mathcal{G}(2,3)$ or $\mathcal{G}(2,4)$.

With the results obtained for the braids for which the closure (using the first trace) is a knot we concluded that the crossed module invariant with $\mathcal{G} = \mathcal{G}(2,2)$ does not distinguish any of the pairs obtained by closing the braids in the two different ways. However, when $\mathcal{G} = \mathcal{G}(2,3)$ it is already possible to see the non triviality of the invariant. In this case we were able to distinguish around 48% of the pairs. Unfortunately, the few pairs for which the crossed module invariant was calculated with $\mathcal{G} = \mathcal{G}(2,4)$ remained undistinguished by this invariant. On the other hand, for links all the cases were distinguished straight way by the simplest crossed module $\mathcal{G}(2,2)$. Some of the results are shown below:

Table 1: Knots with up to seven crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	$\mathcal{H}_{\mathcal{G}(2,3)}$	$\mathcal{H}_{\mathcal{G}(2,4)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	$\mathcal{H}_{\mathcal{G}(2,3)}$	$\mathcal{H}_{\mathcal{G}(2,4)}$
3_1	96	4320	24576	$c(3'_1)$	96	4752	27648
4_1	48	3024	15360	$c(4'_1)$	48	3024	15360
5_2	24	864	1536	$c(5'_2)$	24	864	1536
6_2	24	1296	1536	$c(6'_2)$	24	1296	1536
6_3	24	864	1536	$c(6'_3)$	24	864	1536
7_3	48	1728	–	$c(7'_3)$	48	2160	–
7_5	24	432	1536	$c(7'_5)$	24	432	1536

Table 2: Knots with eight crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	$\mathcal{H}_{\mathcal{G}(2,3)}$	$\mathcal{H}_{\mathcal{G}(2,4)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	$\mathcal{H}_{\mathcal{G}(2,3)}$	$\mathcal{H}_{\mathcal{G}(2,4)}$
8_2	24	864	1536	$c(8'_2)$	24	864	1536
8_5	144	5616	–	$c(8'_5)$	144	7776	–
8_7	24	432	1536	$c(8'_7)$	24	432	1536
8_9	24	432	1536	$c(8'_9)$	24	432	1536
8_{10}	144	7344	–	$c(8'_{10})$	144	9504	–
8_{16}	24	432	–	$c(8'_{16})$	24	432	–
8_{17}	24	864	–	$c(8'_{17})$	24	864	–
8_{18}	336	66096	–	$c(8'_{18})$	336	69120	–
8_{19}	144	9072	–	$c(8'_{19})$	144	9936	–
8_{20}	144	5616	–	$c(8'_{20})$	144	7776	–
8_{21}	144	6048	–	$c(8'_{21})$	144	8208	–

Table 3: Knots with nine crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	$\mathcal{H}_{\mathcal{G}(2,3)}$	$\mathcal{H}_{\mathcal{G}(2,4)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	$\mathcal{H}_{\mathcal{G}(2,3)}$	$\mathcal{H}_{\mathcal{G}(2,4)}$
9_3	24	864	–	$c(9'_3)$	24	864	–
9_6	96	4752	–	$c(9'_6)$	96	5184	–
9_9	24	432	–	$c(9'_9)$	24	432	–
9_{16}	144	7776	–	$c(9'_{16})$	144	8640	–

Table 4: Links with up to seven crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$
$L5a1$	336	$c(L5a1')$	756
$L6a4$	3168	$c(L6a4')$	9504
$L6n1$	1032	$c(L6n1')$	2700
$L7a1$	768	$c(L7a1')$	1728
$L7a3$	384	$c(L7a3')$	864
$L7a6$	144	$c(L7a6')$	432
$L7n1$	312	$c(L7n1')$	756
$L7n2$	336	$c(L7n2')$	756

Table 5: Links with eight crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$
$L8a16$	2688	$c(L8a16')$	8424	$L8a18$	1032	$c(L8a18')$	2700
$L8a19$	3000	$c(L8a19')$	7128	$L8n3$	1032	$c(L8n3')$	2700
$L8n4$	1272	$c(L8n4')$	3240	$L8n5$	3168	$c(L8n5')$	9504

Table 6: Links with nine crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$
$L9a2$	346	$c(L9a2')$	756	$L9a9$	1056	$c(L9a9')$	2376
$L9a14$	528	$c(L9a14')$	1188	$L9a20$	240	$c(L9a20')$	648
$L9a21$	120	$c(L9a21')$	378	$L9a22$	120	$c(L9a22')$	378
$L9a28$	288	$c(L9a28')$	1134	$L9a29$	264	$c(L9a29')$	702
$L9a31$	384	$c(L9a31')$	972	$L9a36$	288	$c(L9a36')$	702
$L9a38$	456	$c(L9a38')$	1026	$L9a39$	384	$c(L9a39')$	918
$L9a41$	1104	$c(L9a41')$	2538	$L9n4$	312	$c(L9n4')$	756
$L9n5$	384	$c(L9n5')$	864	$L9n6$	768	$c(L9n6')$	1728
$L9n13$	624	$c(L9n13')$	1512	$L9n14$	264	$c(L9n14')$	702
$L9n15$	288	$c(L9n15')$	702	$L9n16$	288	$c(L9n16')$	702
$L9n17$	264	$c(L9n17')$	702	$L9n18$	1080	$c(L9n18')$	2484
$L9n19$	432	$c(L9n19')$	1026				

Table 7: Links with ten crossings

knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$	knot	$\mathcal{H}_{\mathcal{G}(2,2)}$
$L10a138$	3360	$c(L10a138')$	10152	$L10a140$	5904	$c(L10a140')$	14364
$L10a145$	1032	$c(L10a145')$	2700	$L10a148$	1464	$c(L10a148')$	3672
$L10a156$	2112	$c(L10a156')$	5130	$L10a161$	1584	$c(L10a161')$	4914
$L10a162$	2424	$c(L10a162')$	8262	$L10a163$	2952	$c(L10a163')$	9234
$L10n77$	1032	$c(L10n77')$	2700	$L10n78$	1032	$c(L10n78')$	2700
$L10n79$	2688	$c(L10n79')$	8424	$L10n81$	3000	$c(L10n81')$	7128
$L10n92$	1584	$c(L10n92')$	4914	$L10n93$	1824	$c(L10n93')$	5400
$L10n94$	3336	$c(L10n94')$	8316				