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Galois Theory towards Dessins d'Enfants

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Abstract

This work is a journey through the main ideas and sucessive generalizations of Galois Theory, towards the origins of Grothendieck's theory of Dessins d'Enfants, firstly found in the late 70s and the 80s ([5]), as a tool to understant the absolute Galois group of the field of the rational numbers. This exposition follows a constructive approach, towards the definition of a *Galois category* and its *Fundamental Group*, as first introduced in *[4]- Exposé V.*

Resumo

Este trabalho consiste numa exposição das ideias e sucessivas generalizações da teoria de *Galois*, tendo em vista uma apresentação das origens da teoria dos "Desenhos de Criança", descoberta por Alexandre Grothendieck nos anos 70 e 80 ([5]), como possível via para a compreensão do grupo de Galois absoluto do corpo dos números racionais. A exposição aqui apresentada segue uma perspectiva construtiva que remete às origens dos conceitos, traçando um caminho em direcção à definição proposta por Grothendieck em [4]-Exposé V, de categoria *Galois* e do seu *Grupo Fundamental*.

Keywords

- Equivalence of Categories;
- Representable Functors;
- Schemes;
- Galois Theory of Fields;
- Galois Theory of Covering Spaces;
- Pro-Representable Functors
- Galois Categories;
- Fundamental Functor of a Galois Category;
- Fundamental Group of a Galois Category;
- Geometric Galois Action;
- Dessins d'Enfants

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0.1 Introduction

Galois Theory started with the study of roots of polynomial equations, long before Galois introduced his famous "Galois Group" as a tool to determine the properties of such roots. The modern formulation of Galois theory for number fields was designed by Emil Artin. However, as the other areas of mathematics progressed, other examples of "Galois Theories" were found. In topology, the classification of covering spaces of a given space and the role played by the fundamental group of that space, is an example of such a theory. In the late 50's, Grothendieck constructed a new categorical theory containing both cases, mostly inspired in the topological example, generalizing the role of the topological fundamental group. He introduced Galois Categories and Fundamental Functors on a highly general level and defined the fundamental group of a Galois Category, containing as particular cases both examples. Then, he applied his theory to schemes and obtained as a particular case the "old" Galois groups as fundamental groups of schemes and fundamental groups of curves defined over a certain number field. Later, he discovered his "Dessins d'Enfants" theory, as a tool to understand one of the most dificult objects in mathematics: The absolute Galois group of the field of the rational numbers. This work is intended as an introduction to his categorical formulation of Galois Theory and as a survey of the main results leading to "Dessins d'Enfants".

In the first chapter we introduce the language of category theory. In particular, the notion of an equivalence of two categories, and the fact that, by *Yoneda's Lemma*, any category $\mathbf{C}$ can be seen as a subcategory of the category of all functors

$$F : \mathbf{C} \longrightarrow \mathbf{Sets}$$

Moreover, we explain the notion of representability of such functors by an object X in $\mathbf{C}$. This property allows us to identify (for each object U in $\mathbf{C}$) every element of $F(U)$ with a unique morphism in $\mathbf{C}$. Also, it enables us to define limits and colimits of diagrams and functors, and as a particular case, products and sums of objects. We end by introducing group actions on objects and by defining the properties of a quotient object associated to such actions.

In the second chapter, we present the main definitions usually used in Algebraic Geometry, starting with some results of commutative algebra. Then we introduce (in a categorical approach) presheaves and sheaves over a topological space and consider pairs of, spaces equipped with preshaves, and their respective morphisms. We present the notion of a scheme as a generalization of a ring. At last, define geometric points of a scheme and a few considerations, concerning fibre products of two schemes over a third one, are made.

The third chapter concerns the classical Galois Theory of fields. Firstly we introduce algebraic extensions of a field and their natural morphisms. Then, Galois extensions and their Galois groups are introduced and a classification theorem of all subextensions of a given Galois extension is proved, first for finite extensions and then, through Krull's Theorem, for infinite extensions. We introduce the absolute Galois group of a field k, denote $Gal(k)$, and we prove that this group is a profinite group, the limit of a projective system of finite groups. The properties of profinite groups are explained in the Appendix Section. At last, we present a formulation of Galois Theory of fields, due to A. Grothendieck, stating that the category of finite separable extensions of a field k is equivalent to the category of finite sets with a continuous transitive action of $Gal(k)$.

In chapter 4 we introduce the topological fundamental group of a topological space X at a point $x \in X$, defined using paths on X (here denoted by $\pi_1^{top}(X, x)$.) After this introduction, we present the main results concerning the Galois theory of covering spaces. As in chatper 3, we introduce Galois covers and their Galois groups and again prove a theorem concerning the classification of all the intermediate covers of a Galois cover. This result establishes a bijection between subgroups of the Galois group and intermediate covers. Given a point $x \in X$, the correspondence assigning to each cover $p : Y \to X$ the set $p^{-1}(\{x\})$, is functorial. The algebraic fundamental group of a space X, at a point x, is defined as the automorphism group of this functor, here denoted has $\pi_1^{alg}(X,x)$. There is a natural action of $\pi_1^{top}(X,x)$ on the fiber sets, called the monodromy action. This action induces a group homomorphism

$$\pi_1^{top}(X,x) \to \pi_1^{alg}(X,x)$$

If X is a connected, path-connected, locally path-connected and semi-locally simply connected, then X has a universal cover. In this case, we prove that the fiber functor is representable by this cover and that the monodromy homomorphism above is an isomorphism. Moreover, in this case, we also prove that the category of covering spaces of X is equivalent to the category of sets with an action of $\pi_1^{alg}(X,x)$. As a corollary, the category of finite coverings is equivalent to the category of finite sets with a discrete topology, endowed with a continuous action $\pi_1^{alg}(X,x)$. In this case, this group is isomorphic to the profinite completion of $\pi_1^{top}(X,x)$. This result extends to every connected space X: the group $\pi_1^{alg}(X,x)$ is profinite and the category of finite coverings of X is equivalent to the category of finite sets with a discrete topology, endowed with a continuous action of $\pi_1^{alg}(X,x)$.

Chapter 5 is the main part of this work. The last result stated in chapter 4 is proved using categorical methods, by following the work done by A.Grothendieck in *[4]- Exposé V*. Given a category **C**, the category of pro-objects in **C**, denoted **Pro(C)**, is introduced. Its objects are projective systems with values in **C**. Generalizing the notion of representability of a functor

$$F : \mathbf{C} \longrightarrow \mathbf{Sets}$$

we define the notion of pro-representability, enabling us to identify, for each object U in **C**, every element of $F(U)$ with a unique morphism in **Pro(C)**. Following [2], we establish sufficient conditions for a given functor F, as above, to be pro-representable.

From here, we follow *[4]- Exposé V Section 4 - "Conditions axiomatiques d'une théorie de Galois"*, where six axiomatic conditions are considered over a pair Category **C** + functor F on **C** with values in the category of finite sets. Three of these conditions are related to **C** and the other three are related to F. Under this conditions we prove the following properties:

- F is pro-representable;
- The automorphism group of F, here denoted as π, is profinite;
- There is an equivalence between **C** and the category of finite sets, endowed with a discrete topology and a continuous action of π (here denoted as $\pi - \mathbf{FSets}$).

Then, we prove that $\pi - \mathbf{FSets}$, together with the natural forgetful functor to the category of finite sets, also verifies the six imposed conditions. Following A. Grothendieck, we present the definition of a Galois Category, as any category **C** equivalent to $\pi - \mathbf{FSets}$, where π is a profinite group. Indeed, this is equivalent to the following properties:

- **C** verifies all the three imposed conditions;

- The functor establishing the equivalence verifies the other three imposed conditions

π is called the *Fundamental Group* of the Galois category and F is called a *Fundamental Functor*. Any two fundamental functors are isomorphic and the category of fundamental functors on $\mathbf{C}$ is called the *Fundamental Groupoid* of the Galois category $\mathbf{C}$. Moreover, we present sufficient conditions for a given functor between two Galois categories to induce a group homomorphism between their respective fundamental groups. As examples:

- The category of finite covering spaces of a connected space X is a Galois category. The fiber functor over a point $x \in X$ is a fundamental functor. The algebraic fundamental group of X is precisely the fundamental group of this Galois category.
- Without exploring the details, the notion of an étale morphism of schemes is introduced. The category of such morphisms over a fixed connected and noetherian scheme X, is Galois. The fiber functor at a geometric point x is a fundamental functor. The algebraic fundamental group of X is, by definition, the fundamental group of this Galois category. Classical Galois theory concerning extensions of a field k is recovered by considering $X = Spec(k)$. The choice of a geometric point $s : Spec(K) \to spec(k)$ is indeed the choice of a separable closure of k. We have an isomorphism

$$Gal(k) \cong \pi_1(Spec(k), s)$$

Finally and briefly, in chapter 6, the main results leading to the Grothendieck's Theory of *Dessins d'enfants* are presented without proof. Given an extension of fields $L|k$, there is a pull-back functor from the category of curves over k to the category of curves over L, by the fact that all fibre products exist in the category of schemes. In particular, if X is a curve over k and $\bar{X}$ is the pull-back of X over $L = \bar{k}$, there is an exact sequence of profinite groups

$$1 \to \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x) \to Gal(k) \to 1$$

where x is a geometric point of X and $\bar{x}$ is its pull-back. This sequence allows us to define a natural action of $Gal(k)$ on $\pi_1(\bar{X}, \bar{x})$ (also called the geometric fundamental group of X), by outer automorphisms

$$Gal(k) \to Out(\pi_1(\bar{X}, \bar{x}))$$

At the same time, if $k = \overline{\mathbb{Q}}$ and $L = \mathbb{C}$, *Belyi's Theorem* says that a curve X over L is isomorphic to the pull-back of a curve over k if and only if there is a finite étale morphism of curves over L from X to $P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. Since the pull-back preserves finite étale morphisms, this results allows us to establish an equivalence between the Galois categories of étale morphisms over $P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ and those over $P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, and therefore, a natural isomorphism between the fundamental groups of these curves.

At the same time, to each curve X over $\mathbb{C}$, we can assign, in a functorial way, an analytic complex space, X^{an}. In [4], Grothendieck proved that the category of étale morphisms over X is equivalent to the category of finite topological covers over X^{an}, and therefore the algebraic fundamental group of X is isomorphic to the profinite completion of the topological fundamental group of X^{an}. If $X = P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$, we have $X^{an} = \mathbb{C}_\infty \setminus \{0, 1, \infty\}$, where $\mathbb{C}_\infty$ is the Riemann Sphere. Finally, we conclude that there is an isomorphism of fundamental groups

$$\pi_1^{alg}(P^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}) \cong \widehat{\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})} \quad (1)$$

and the action of $Gal(\overline{\mathbb{Q}})$ on $\pi_1^{alg}(P^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\})$, noticed in the first place, extends to an action on $\widehat{\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})}$:

$$Gal(\overline{\mathbb{Q}}) \to Out(\widehat{\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})})$$

After the results in chapter 4 and 5, this group classifies all the finite topological covers of $\mathbb{C}_\infty \setminus \{0,1,\infty\}$. *Dessins d'Enfants* are introduced as combinatorial structures assigned to each finite cover of $\mathbb{C}_\infty \setminus \{0,1,\infty\}$, enabling us to study this action.

Chapter 1

Categorical Preliminaries

This chapter was written as a general introduction to the main results and constructions concerning the language of category theory. As a general reference for standard facts, see [15].

1.1 Categories

1.1.1 Definitions and Properties

1. A Category $\mathbf{C}$ is a class of objects $Ob(\mathbf{C})$ and, for each pair of objects A and B, a set $Hom_{\mathbf{C}}(A,B)$, whose elements are called morphisms from the object A to the object B. We represent each element $f \in Hom_{\mathbf{C}}(A,B)$ as an arrow from A to B, $f : A \to B$. For every three objects A, B and C, we also require a binary operation (called *composition law*)

$$\begin{aligned} Hom_{\mathbf{C}}(A,B) \times Hom_{\mathbf{C}}(B,Z) \to& Hom_{\mathbf{C}}(A,Z) \\ (f,g) \mapsto& g \circ f \end{aligned}$$

   satisfying the following properties:

   - The composition is associative;
   - For each object A of $\mathbf{C}$ there is an *identity morphism* $I_A \in Hom_{\mathbf{C}}(A,A)$ i.e., for any other objects B and D and morphisms $f : B \to A$ and $g : A \to D$ we have $I_A \circ f = f$ and $g \circ I_A = g$.

   It follows that for each object there is only one identity morphism and therefore, we may identify the objects of a category with their identity morphisms.

2. Given a category, we consider also its dual, written $\mathbf{C}^{\circ}$, whose objects are the ones of $\mathbf{C}$ and the set of morphisms from A to B is now the set of morphism from B to A in $\mathbf{C}$.

$$Hom_{\mathbf{C}^{\circ}}(A,B) := Hom_{\mathbf{C}}(B,A)$$

3. A category is said to be small if the class of objects forms a set.

4. A subcategory of **C** is another category whose objects are some of the objects of **C** and the set of morphisms is a subset of those in C, closed with respect to the composition operation in **C** and incluing the identity morphism of every object in the subcategory.

5. Given a category **C**, a morphism $f : A \to B$ is said to be an isomorphism if there is another morphism $g : B \to A$ such that $f \circ g = I_B$ and $g \circ f = I_A$. Two objects are called isomorphic if there is an isomorphism between them.

6. A *Groupoid* is a category for which every morphism is an isomorphism.

7. Let A be an object of **C**. The elements of $End_{\mathbf{C}}(A) := Hom_{\mathbf{C}}(A, A)$ are called endomorphisms of A. The subset of endomorphisms of A which are isomorphisms are called automorphisms of A, and we write $Aut_{\mathbf{C}}(A) \subset End_{\mathbf{C}}(A)$.

   Proposition 1.1.1.1. *$Aut_{\mathbf{C}}(A)$ with the composition operation is a group.*

   Proof. By definition every element $f : A \to A$ of $Aut_{\mathbf{C}}(A)$ is an isomorphism, so there is a morphism $g : A \to A$ such that $f \circ g = g \circ f = I_A$. The composition operation is associative and I_A is a neutral element. Therefore $Aut_{\mathbf{C}}(A)$ is a group. □

   Proposition 1.1.1.2. *If two objects are isomorphic, their automorphism groups are isomorphic (in the category of Groups)*

   In fact, we have a more general result

   Proposition 1.1.1.3. *Let **C** be a category and $f : A \to A'$, $g : B \to B'$ two isomorphisms. Then*

$$Hom_{\mathbf{C}}(A, B) \cong_{Sets} Hom_{\mathbf{C}}(A', B') \tag{1.1}$$

1.1.2 Characterization of Morphisms

a) Monomorphisms

In set theory there are notions of injectivity and surjectivity. These may be translated to a pure categorical language:

Let **Sets** be the category of sets, whose objects are sets and whose morphisms are maps between sets.

Given two sets A and B, we say that a map $f : A \to B$ is injective if any two different elements of the first set are always mapped to different elements on the target set. We show the following equivalence:

Proposition 1.1.2.1. *A set-map $f : A \to B$ is injective if and only if the set-map*

$$Hom_{\mathbf{Sets}}(Z, A) \to Hom_{\mathbf{Sets}}(Z, B) \tag{1.2}$$

mapping $g \mapsto f \circ g$ is injective for all sets Z.

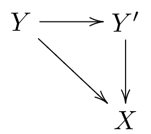
Proof. The map 1.2 being injective is equivalent to $f \circ g = f \circ h$ implying $g = h$, for any $g, h \in Hom_{\mathbf{Sets}}(Z, A)$. Suppose that there exists a set Z such that the 1.2 is not injective, which means that there are two different morphism g and h in $Hom_{\mathbf{Sets}}(Z, A)$ that verify $f \circ g = f \circ h$, or using elements of Z, $f \circ g(z) = f \circ h(z)$

for all $z \in Z$. Since g and h are different maps, there is at least one element of Z such that $g(z)$ is different from $h(z)$ (otherwise the functions would be equal). However, for this element, we have $f(g(z)) = f(h(z))$ and so the function f is not injective. Suppose now that the 1.2 is injective for all sets Z. In particular, it is injective for $Z = A$ and $g = I_A$, the identity map on A. By the injectivity of the above map, for any other map $h : A \to A$ such that $f \circ I_A = f = f \circ h$, $h = I_A$. We conclude that f has to be injective. $\square$

This allows us to generalize the notion of injectivity

Definition 1.1.2.2. *Let* $\boldsymbol{C}$ *be a category. A morphism* $f : A \to B$ *is called a* monomorphism *if, for any other object* Z *of* $\boldsymbol{C}$*, the map* $Hom_{\boldsymbol{C}}(Z, A) \to Hom_{\boldsymbol{C}}(Z, B)$ *defined by* $g \mapsto f \circ g$*, is injective. In this case* $Hom_{\boldsymbol{C}}(Z, A)$ *can be regarded as a subset of* $Hom_{\boldsymbol{C}}(Z, B)$*, for any Z.*

Here, we define a subobject of an object X in **C** as a monomorphism $Y \to X$, with Y some other object in **C**. We say two objects $Y \to X$ and $Y' \to X$ are the same, if they are isomorphic as morphisms over X. In other words, if there is an isomorphism $Y \to Y'$ making the following diagram commute



Notice that the composition of monomorphisms is still a monomorphism.

b) Epimorphisms

We will first formulate the notion of surjectivity on categorial terms within the category of sets and then generalize it for any category. Given two sets A and B, we say that a map $f : A \to B$ is surjective if any element of B is in the image of A by f, written $f(A)$, or, for any element b in B, there is at least one element a of A such that $b = f(a)$. We have:

Proposition 1.1.2.3. *A set-map* $f : A \to B$ *is surjective if and only if the set-map* $Hom_{\boldsymbol{Sets}}(B, Z) \to Hom_{\boldsymbol{Sets}}(A, Z)$ *defined by* $g \mapsto g \circ f$ *is injective for all sets Z.*

The proof is similar to the one of Prop. 1.1.2.1. The notion of surjectivity can then be generalized

Definition 1.1.2.4. *Let* $\boldsymbol{C}$ *be a category. A morphism* $f : A \to B$ *is called an* epimorphism *if, for any other object* Z *of* $\boldsymbol{C}$*, the map* $Hom_{\boldsymbol{C}}(B, Z) \to Hom_{\boldsymbol{C}}(A, Z)$ *defined by* $g \mapsto g \circ f$ *is injective. In this case we can see* $Hom_{\boldsymbol{C}}(B, Z)$ *as a subset of* $Hom_{\boldsymbol{C}}(A, Z)$*, for any Z.*

c) Isomorphisms

Previously, we have introduced the notion of an isomorphism as a morphism $f : A \to B$ for which there is a morphism $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ We shall prove the following property:

Proposition 1.1.2.5. *Let* $f : A \to B$ *be a morphism in* $\boldsymbol{C}$ *.The following conditions are equivalent*

- *(i)* f *is an isomorphism;*

- *(ii) For every object Z in $\boldsymbol{C}$, the map $(f \circ -) : Hom_{\boldsymbol{C}}(Z, A) \to Hom_{\boldsymbol{C}}(Z, B)$ is an isomorphism of sets;*

- *(iii) For every object Z in $\boldsymbol{C}$, the map $(- \circ f) : Hom_{\boldsymbol{C}}(B, Z) \to Hom_{\boldsymbol{C}}(A, Z)$ is an isomorphism of sets;*

Proof. $(i) \Rightarrow (ii)$: If f is an isomorphism (in the first sense) with an inverse $g : B \to A$, the map $(g \circ -) : Hom_{\mathbf{C}}(Z, B) \to Hom_{\mathbf{C}}(Z, A)$ is an inverse map of $(f \circ -)$; $(ii) \Rightarrow (i)$: If (ii) holds for every object Z, we choose $Z = B$. Then we have a bijection $(f \circ -) : Hom_{\mathbf{C}}(B, A) \to Hom_{\mathbf{C}}(B, B)$ and there is a unique $g : B \to A$ such that $f \circ g = I_B$. With this g in mind, fix $Z = A$ and the correspondent bijection $(f \circ -) : Hom_{\mathbf{C}}(A, A) \to Hom_{\mathbf{C}}(A, B)$. We now known $g \circ f$ to be mapped to f, but also I_A is mapped to f. By the injectivity of $(f \circ -)$, we conclude $g \circ f = I_A$. The other equivalence follows a similar proof. ☐

1.2 Functors

1.2.1 Definition and Properties

There is a notion of morphism between categories:

Definition 1.2.1.1. *A (*covariant*)* functor *F between two categories $\boldsymbol{C}$ and $\boldsymbol{D}$, written $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$, is a correspondence:*

- *for each object A of $\boldsymbol{C}$, an object $F(A)$ in $\boldsymbol{D}$;*

- *for each morphism $f : A \to B$, a morphism $F(f) : F(A) \to F(B)$ (in other words, a mapping $Hom_{\boldsymbol{C}}(A, B) \to Hom_{\boldsymbol{D}}(F(A), F(B))$) such that the composition law is preserved and $F(I_A) = I_{F(A)}$.*

Definition 1.2.1.2.

- *A* contravariant *functor F between $\boldsymbol{C}$ and $\boldsymbol{D}$ is a functor $F : \boldsymbol{C}^{\circ} \longrightarrow \boldsymbol{D}$, reversing the composition law: if $f : A \to B$ is a morphism in $\boldsymbol{C}$ then $F(f) : F(B) \to F(A)$.*

- *Given a functor $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$, the* opposite *functor of F, denoted F°, is the functor*

$$
\begin{aligned}
F^{\circ} : \boldsymbol{C}^{\circ} &\longrightarrow \boldsymbol{D}^{\circ} \\
X &\mapsto F^{\circ}(X) := F(X) \\
Hom_{\boldsymbol{C}^{\circ}}(X, Y) = Hom_{\boldsymbol{C}}(Y, X) &\longrightarrow Hom_{\boldsymbol{D}^{\circ}}(F^{\circ}(X), F^{\circ}(Y)) = Hom_{\boldsymbol{D}}(F(Y), F(X)) \\
f &\mapsto F(f)
\end{aligned}
$$

For each object X in $\mathbf{C}$, a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ induces a map $Hom_{\mathbf{C}}(X, X) \to Hom_{\mathbf{D}}(F(X), F(X))$, that preserves composition.

Proposition 1.2.1.3. *Let $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ be a functor. If $u : X \to Y$ is an isomorphism in $\boldsymbol{C}$ then its image by F, $F(u) : F(X) \to F(Y)$, is an isomorphism in $\boldsymbol{D}$.*

Proof. Suppose $u : X \to Y$ is an isomorphism and let $u^{-1} : Y \to X$ be its inverse. Since F is a functor, $F(u) \circ F(u^{-1}) = F(u \circ u^{-1}) = F(I_Y) = I_{F(Y)}$ and also $F(u^{-1}) \circ F(u) = I_{F(X)}$. So $F(u) : F(X) \to F(Y)$ is an isomorphism in $\mathbf{D}$.[1] □

We conclude that F induces a group homomorphism $Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{D}}(F(X))$, that is well defined, since by the last proposition, the image of an automorphism is an automorphism.

The notion of a functor, enables us to consider the category of all categories, here denoted as **Cat**.

1.2.2 The Image of a Functor

Given a functor, the objects of the form $F(A)$ in $\mathbf{D}$, for any object A in $\mathbf{C}$, and the morphisms of the form $F(f) : F(A) \to F(B)$, for any $f : A \to B$ morphism in $\mathbf{C}$, form a subcategory of $\mathbf{D}$, called the image of $\mathbf{C}$ by F.

1.2.3 Composition of Functors

There is a natural way to compose functors, by applying each one of them sequencially. Similarly, there is also a natural Identity Functor for each category which takes each object to itself and each morphism, again, to itself. The definition of a functor satisfies the definition of a morphism within a category. So, we are allowed to talk about the category of categories, whose objects are categories and morphisms are functors.

1.2.4 Categories of Functors

Given two categories $\mathbf{C}$ and $\mathbf{D}$ we can produce a notion of a morphism between two functors, $F : \mathbf{C} \longrightarrow \mathbf{D}$ and $G : \mathbf{C} \longrightarrow \mathbf{D}$:

Definition 1.2.4.1. *A* natural transformation *or* morphism of functors *between* $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ *and* $G : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ *is a collection of morphisms in* $\boldsymbol{D}$*,* $\{T_A : F(A) \to G(A)\}$*, one for each object* A *of* $\boldsymbol{C}$*, such that for each morphism* $f : A \to B$ *in* $\boldsymbol{C}$*, the following diagram commutes*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle T_A} & & \downarrow{\scriptstyle T_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

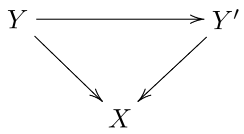
With this notion we have a *category of covariant functors between* $\boldsymbol{C}$ *and* $\boldsymbol{D}$, written **Fun(C,D)** where objects are Functors and morphism are natural transformations, presented as $T : F \to G$, for F and G functors. Denote by $Hom_{\mathbf{Fun(C,D)}}(F, G)$ the set of natural transformations from F to G. Moreover, we have isomorphisms of functors and we are able to define the automorphism group $Aut_{Fun(\mathbf{C},\mathbf{D})}(F)$ of a given functor F, called the group of self natural equivalences of F, and defined in exactly the same way as for a general category $\mathbf{C}$.

We denote by $\mathbf{Fun}(\mathbf{C}^{\circ}, \mathbf{D})$ the category of contravariant functors.

[1]The contravariant case can be proved with the same argument, reversing the order of composition

1.3 Categories of Morphisms over an object

1. Let **C** be a category and X one object in **C**. We define the category $\mathbf{C}/X$: Objects are morphisms $u : Y \to X$ in **C**, with Y some other object. A morphism of objects over X is a commutative diagram



   Sometimes we write an object $u : Y \to X$ over X, simply as a pair (Y, u).

2. There is a canonical source functor $s : \mathbf{C}/X \longrightarrow \mathbf{C}$ sending each morphism $Y \to X$ over X to its initial object Y;

3. If $f : X \to X'$ is a morphism in **C**, there is an induced functor $f_* : \mathbf{C}/X \longrightarrow \mathbf{C}/X'$ sending each morphism $Y \to X$ to the composition $Y \to X \to X'$.

1.4 Equivalence of Categories

1.4.1 Definition and Properties

The notion of an isomorphism between two categories **C** and **D** is naturally defined in the sense of the category of all categories, **Cat**. However, the isomorphism condition is too restrictive. In this sense we introduce an usefull weaker criterium

Definition 1.4.1.1. *Two categories* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *are said to be* equivalent categories *if there exist functors* $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ *and* $G : \boldsymbol{D} \longrightarrow \boldsymbol{C}$ *such that* $F \circ G$ *is a functor isomorphic (in* ***Fun(D,D)****) to the identify functor on* $\boldsymbol{D}$*,* $I_{\boldsymbol{D}}$*, and* $G \circ F$ *is isomorphic to the identify functor on* $\boldsymbol{C}$*,* $I_{\boldsymbol{C}}$*. We denote the equivalence by the pair* (F, G) *or simply by* $\boldsymbol{C} \cong \boldsymbol{D}$*.* F *and* G *are said to be* quasi-inverses.

Definition 1.4.1.2. *We say that an equivalence of categories* (F, G) *is an anti-equivalence if the functors in the pair are contravariant.*

We will only consider equivalences. The anti-equivalences can be treated analogously using dual categories. The following result can be easily proved,

Proposition 1.4.1.3. *Let* $\boldsymbol{C}$*,* $\boldsymbol{D}$ *and* $\boldsymbol{E}$ *be categories. If* $\boldsymbol{C} \cong \boldsymbol{D}$ *and* $\boldsymbol{D} \cong \boldsymbol{E}$ *then* $\boldsymbol{C} \cong \boldsymbol{E}$*.*

We describe some properties that are preserved:

Proposition 1.4.1.4. *Let* (F, G) *be an equivalence of categories as above (between* $\boldsymbol{C}$ *and* $\boldsymbol{D}$*). Then,* $u : X \to Y$ *is an isomorphism in* $\boldsymbol{C}$ *if and only if* $F(u) : F(X) \to F(Y)$ *is an isomorphism in* $\boldsymbol{D}$

Proof. If $u : X \to Y$ is an isomorphism, it follows directly from the fact that F is a functor, that $F(u) : F(X) \to F(Y)$ is also an isomorphism. If we start with $F(u) : F(X) \to F(Y)$ an isomorphism, since G is a functor, $G(F(u)) : G(F(X) \to G(F(Y))$ is also an isomorphism in **C**. But the pair (F, G) is an equivalence. So, $G \circ F$ is isomorphic as a functor to the identify on **C**, following that $G(F(X)) \cong X$ and $G(F(Y)) \cong Y$. Composing the isomorphisms, we conclude that $u : X \to Y$ is an isomorphism. □

In particular,

Corollary 1.4.1.5. *For any object X in $\boldsymbol{C}$, the map induced by an equivalence F, $Aut_{\boldsymbol{C}}(X) \to Aut_{\boldsymbol{D}}(F(X))$ sending $\phi : X \to X$ to $F(\phi) : F(X) \to F(X)$ is an isomorphism of groups.*

1.4.2 Conditions for Equivalence

Let us now establish an usefull criterium of equivalence of categories:

Definition 1.4.2.1. *A functor $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is said to be* essentially surjective *if every object in $\boldsymbol{D}$ is isomorphic to some object in the image of F.*

Definition 1.4.2.2. *A functor $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is* fully faithful *if the map $Hom_{\boldsymbol{C}}(A, B) \to Hom_{\boldsymbol{D}}(F(A), F(B))$, induced by the functor, is an isomorphism of sets, for every pair of objects A and B in $\boldsymbol{C}$.*[2]

We then have

Lemma 1.4.2.3. *A functor $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ establishes an equivalence of categories if and only if it is essentially surjective and fully faithful.*

Proof. Suppose first that (F, G) is an equivalence of categories:

- given an object O_D on $\mathbf{D}$, we know that $F(G(O_D)) \cong I_D(O_D) = O_D$, so O_D is isomorphic to the image of $G(O_D)$ under F and F is essentially surjective;
- if A and B are objects in $\mathbf{C}$ we can have a sequence of set maps induced by the functors F and G,

$$Hom_{\mathbf{C}}(A, B) \to^F Hom_{\mathbf{D}}(F(A), F(B)) \to^G Hom_{\mathbf{C}}(G \circ F(A), G \circ F(B))$$

  Since $G \circ F \cong I_{\mathbf{C}}$, the composite above is an isomorphism so each of the maps is an isomorphism. We conclude that $Hom_{\mathbf{C}}(A, B) \cong Hom_{\mathbf{D}}(F(A), F(B))$ and F is fully faithful.

Let's now suppose we are given a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ that is essentially surjective and fully faithful. To prove that F is an equivalence of categories we need to construct another functor $G : \mathbf{D} \longrightarrow \mathbf{C}$ such that the pair (F, G) is an equivalence (in the sense first defined). Consider the following construction: since F is essentially surjective, each object O_D of $\mathbf{D}$ is isomorphic to some object $F(A)$ on the image of $\mathbf{C}$, denote $\phi_{O_D} : F(A) \to O_D$ a choice of one such isomorphism, with one of those objects A. Define G by $O_D \mapsto G(O'_D) = A$. Moreover, given a morphism $u : O_D \to O'_D$, since $O_D \cong F(A)$ and $O'_D \cong F(A')$ for some objects A and A' in $\mathbf{C}$, we have by Prop 1.1.1.3 an isomorphism $Hom_{\mathbf{D}}(O_D, O'_D) \cong Hom_{\mathbf{D}}(F(A), F(A'))$. Since F is fully faithful (that is, $Hom_{\mathbf{D}}(F(A), F(A')) \cong Hom_{\mathbf{C}}(A, A')$), we define $G(u)$ to be the image of u under the composition of this isomorphisms. The composition of morphisms is then preserved and G is indeed a functor. Finally, we check that the pair (F, G) is indeed an equivalence. For one side, the maps $\phi_{O_D} : F(A) = F(G(O_D)) \to O_D$ define an isomorphism $\phi : F \circ G \to I_{\mathbf{D}}$. We construct now a natural transformation $T : G \circ F \to I_{\mathbf{C}}$ that is a natural isomorphism. We want to find an isomorphism $\psi_A : G(F(A)) \to A$, for each object A in $\mathbf{C}$. By the fact that F is fully faithful, $Hom_{\mathbf{D}}(F(G(F(A))), F(A)) \cong Hom_{\mathbf{C}}(G(F(A)), A)$, and since we have already seen that $F \circ G \cong I_D$, we define, for each A, ψ_A as the image of the isomorphism $F \circ G(F(A)) \to F(A)$ under the above isomorphism of sets.

[2] If F is contravariant, we ask for an isomorphism $Hom_{\mathbf{C}}(A, B) \to Hom_{\mathbf{D}}(F(B), F(A))$, reversing the arrows.

Observe that different choices of the isomorphisms $\phi_{O_D} : F(A) \to O_D$ give different functors G, but this construction, for any choice, produces a pair (F, G) defining an equivalence.

□

1.5 Adjoint Functors

Definition 1.5.0.4. *Let $\boldsymbol{C}$ and $\boldsymbol{C'}$ be two categories. A pair of functors* $F : \boldsymbol{C} \longrightarrow \boldsymbol{C'}$ *and* $G : \boldsymbol{C'} \longrightarrow \boldsymbol{C}$ *is called adjoint if there is an isomorphism of functors*

$$Hom_{\boldsymbol{C}}(G(.), .) \cong Hom_{\boldsymbol{C'}}(., F(.)) : \boldsymbol{C'} \times \boldsymbol{C} \longrightarrow \boldsymbol{Sets}$$

If $\mathbf{C}$ and $\mathbf{C}'$ are equivalent categories, the pair of functors F and G establishing the equivalence is easily seen to be an adjoint pair: Indeed, given two objects X in $\mathbf{C}$ and A in $\mathbf{C}'$, since $A = I_{\mathbf{C}'}(A) \cong F \circ G(A)$, we have an isomorphism

$$Hom_{\mathbf{C}'}(A, F(X)) \cong Hom_{\mathbf{C}'}(F(G(A)), F(X)) \tag{1.3}$$

and since F is fully faithfull (1.4.2.3), the right set in 1.3 is isomorphic to $Hom_{\mathbf{C}'}(G(A), X)$. All these isomorphisms have functorial properties.

1.6 Set-valued Functors

1.6.1 Functors h_X and h_X°

Let $\mathbf{C}$ be a category. Given any object X, the operations $h_X(-) := Hom_{\mathbf{C}}(X, -)$ and $h_X^\circ(-) := Hom_{\mathbf{C}}(-, X)$ define functors from $\mathbf{C}$ to the category of sets. h_X is a covariant functor: For each object Y in $\mathbf{C}$, $h_X(Y) = Hom_{\mathbf{C}}(X, Y)$ is a set and for each morphism $g : Y \to Z$ there is an induced set morphism (i.e. a set-map) $g \circ - : Hom_{\mathbf{C}}(X, Y) \to Hom_{\mathbf{C}}(X, Z)$ mapping $f : X \to Y$ to $g \circ f : X \to Z$. On the other hand $h_X^\circ = Hom_{\mathbf{C}}(-, X)$, is a contravariant functor: If Y is an object in $\mathbf{C}$ then $Hom_{\mathbf{C}}(Y, X)$ is a set. Also each $g : Y \to Z$ induces a set-map $- \circ g : Hom_{\mathbf{C}}(X, Y) \to Hom_{\mathbf{C}}(X, Z)$ defined by $f : Z \to X \mapsto f \circ g : Y \to X$. The composition law holds on both cases and identities map to identities.

1.6.2 Yoneda's Lemma

Let us concentrate our attention on h_X (the case of h_X° could be treated with dual arguments). Our purpose now is to identify some properties of the functor h_X and relate them with the object X. We start by proving a very general and important result, known as Yoneda's Lemma.

Lemma 1.6.2.1. *(Yoneda's Lemma) Let $\boldsymbol{C}$ be a category and F a covariant set-valued functor. Then:*

There is a functorial bijection between $F(X)$ and the set of natural transformations $h_X \to F$, given by the following correspondence:

$$Hom_{\textbf{\textit{Fun(C,Sets)}}}(h_X, F) \longrightarrow F(X)$$

$$\xi : h_X \to F \longmapsto \xi_X(I_X)$$

Proof. Given a natural transformation $\xi : h_X \to F$, we consider the particular morphism $\xi_X : h_X(X) \to F(X)$ and the image of the identity I_X, $\xi_X(I_X) \in F(X)$. This correspondence mapping $\xi \mapsto \xi_X(I_X)$ is a bijection. Let us construct an inverse:

Starting with an element $\zeta \in F(X)$ we construct a natural transformation ξ. We define $\xi_U : h_X(U) \to F(U)$ by $(u : X \to U) \mapsto F(u)(\zeta)$ for each object U in **C**, so that $\xi_X(I_X) = \zeta$. It follows that for any morphism $f : U \to V$ in **C**, the following diagram commutes

$$\begin{array}{ccc} h_X(U) & \xrightarrow{f\circ -} & h_X(V) \\ \downarrow{\scriptstyle \xi_U} & & \downarrow{\scriptstyle \xi_V} \\ F(U) & \xrightarrow{F(f)} & F(V) \end{array}$$

and the collection ξ_U, one for each object U, defines a natural transformation. This correspondence is an inverse of the one first presented.

□

From this lemma we can extract very important properties of functors h_X, such as:

Corollary 1.6.2.2. *Every natural transformation $h_X \to h_Y$ is induced by a unique morphism $Y \to X$. Moreover, the sets $Hom_{\textbf{\textit{Fun(C,Sets)}}}(h_X, h_Y)$ and $h_Y(X)$ are isomorphic.*

Proof. Consider $F = h_Y$ in the previous lemma. □

1.6.3 Embedding C in Fun(C,Sets)

The correspondence h, mapping each object X in **C** to the functor h_X and each morphism $f : X \to Y$ to the induced morphism $(- \circ f) : h_Y \to h_X$, defines a contravariant functor, embedding $\mathbf{C}^\circ$ in **Fun(C,Sets)**. Similary, the dual correspondence h° defines a covariant functor that embedding **C** in $\mathbf{Fun}(\mathbf{C}^\circ, \mathbf{Sets})$.

In fact,

Proposition 1.6.3.1. *The functors h° (resp. h) establish an equivalence of categories between **C**° and the subcategory of **Fun(C,Sets)** (resp., **C** in **Fun**(**C**°, **Sets**)) given by the image of the respective functor.*

Proof. A functor restricted to its image is obviously essentially surjective. Given X and Y in **C**, it follows from Yoneda's Lemma that $h_Y(X) = Hom_{\mathbf{C}}(Y, X) \xrightarrow{\sim} Hom_{\mathbf{Fun(C,Sets)}}(h_X, h_Y)$ is bijective and therefore h is fully faithful. □

As an immediate corollary we can conclude that any category **C** can be seen as a subcategory of **Fun**($\mathbf{C}^{\circ}$, **Sets**) through h° (resp. $\mathbf{C}^{\circ}$ of **Fun(C,Sets)** through h). We also conclude that a natural transformation $h_Y \to h_X$ is an isomorphism if and only if the morphism $X \to Y$ inducing it also is. Moreover, the correspondence $Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{Fun(C,Sets)}}(h_X)$ mapping each automorphism ϕ of X to the map $(- \circ \phi) : h_X \to h_X$ is an isomorphism of groups. Every automorphism of h_X is of this form.

1.6.4 Representable Functors

Through this section we will study the representability of a covariant functor (the dual case can be considered with similar arguments).

Definition 1.6.4.1. *Let **C** be a category, X an object and F a covariant functor to the category of sets. We say that the functor is* representable *by an object X if there exists an isomorphism $\xi : h_X \to F$. In this case the pair (X, ξ) is called a representation of F.*[3]

Remark 1.6.4.2. *If F is represented by an object X in **C***

$$\xi : h_X \to F$$

by Yoneda's Lemma*, ξ is uniquely determined by an element $\zeta \in F(X)$, given by*

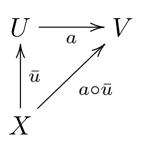
$$\zeta = \xi_X(I_X)$$

*In addition, since ξ is an isomorphism, for each object U in **C**, every element $u \in F(U)$ is identified with a unique morphism $\bar{u} : X \to U$ through the formula*

$$u = F(\bar{u})(\zeta)$$

where ζ is fixed.

As a result, for every morphism $a : U \to V$, the induced morphism $F(a) : F(U) \to F(V)$ mapping each $u \in F(U)$ to $F(a)(u) \in F(V)$ is naturally identified with the composition of morphisms $\bar{u} \to a \circ \bar{u}$



The fact that every element $u \in F(U)$ is obtained through $u = F(\bar{u})(\zeta)$ is equivalent to the fact that every morphism $\bar{u} : X \to U$ factors as $\bar{u} \circ I_X : X \to X \to U$

We will also write (X, ζ) (with $\zeta \in F(X)$) to denote the pair representing F.

Proposition 1.6.4.3. *If F is representable, any two objects that represent F are isomorphic.*

[3]It is also said that X is the solution to the universal problem posed by F, or equivalently, that X has the universal property posed by F.

Proof. F being representable by X and Y, objects in **C**, means that $F \cong h_X \cong h_Y$. This latter isomorphism is induced by a morphism $Y \to X$, which by corollary 1.6.2.2 is an isomorphism between the objects X and Y. □

Proposition 1.6.4.4. *If F is a functor represented by a pair (X, ξ) then the composition*

$$Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{Fun(C,Sets)}}(h_X) \to Aut_{\mathbf{Fun(C,Sets)}}(F)$$

defined by $\phi \mapsto (- \circ \phi) \mapsto \xi \circ (- \circ \phi) \circ \xi^{-1}$, is an anti-isomorphis of groups.

Proof. As already seen in the last section, the correspondence $X \to h_X$ is an anti-equivalence of categories and the induced map $Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{Fun(C,Sets)}}(h_X)$ is an isomorphism. For a fixed isomorphism $\xi : h_X \to F$ representing F, every automorphism of F induces an automorphism of h_X

$$\begin{array}{ccc} F & \longrightarrow & F \\ {\scriptstyle \xi^{-1}}\downarrow & & {\scriptstyle \xi^{-1}}\downarrow \\ h_X & \longrightarrow & h_X \end{array}$$

which is of the form $(- \circ \phi)$ for some automorphism ϕ of X. This concludes the proof. □

Notice that the equivalence stated in Section 1.6.3 between **C** and the image of h in **Fun(C,Sets)** extends to the full subcategory of representable functors, since the essential surjective property is ensured by the representability.

1.7 General Constructions on Categories

The notion of representable functors (covariant and contravariant) allows us to formalize many concepts, such as products of objects, sums of objects, limits and colimits.

1.7.1 Representable Functors with Values in Sets of Morphisms - Universal Properties

In this section we will study the results of remark 1.6.4.2 in the particular case of a functor F such that $F(U)$ takes values on a set of (possible families) morphisms either ending or starting on U (for covariant or contravariant functors, respectively), satisfying some property. Suppose F is covariant. For a morphism $f : U \to V$, $F(f) : F(U) \to F(V)$ is the composition of all the morphisms of a family on $F(U)$ with f, thus getting a family of morphisms on $F(V)$. If F is representable by a pair (X, ζ), then by remark 1.6.4.2:

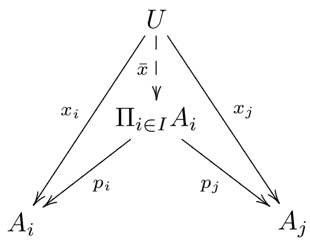
$$\forall x \in F(X) \quad \exists! \ (\bar{x} : X \to U) \ : \ x = F(\bar{x})(\zeta)$$

(meaning that x, as a family of morphisms, can be uniquely obtained by composing the family ζ with the uniquely determined morphism $\bar{x}$). This property is usually called the "Universal Property" imposed by the functor F. In the next sections we present some examples.

1. **Example I - Products and Sums**

To define a general product structure on an arbitrary category we use the cartesian product of sets.

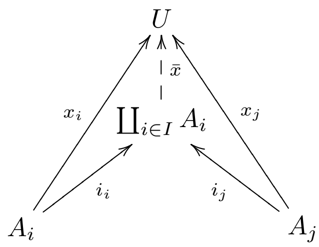
Let $(A_i)_{i\in I}$ be a family of objects of a category $\mathbf{C}$. We may consider, for each other object U, the family of sets $\{Hom_{\mathbf{C}}(U, A_i)\}_{i\in I}$ and the direct product of this sets, $\Pi_{i\in I} Hom_{\mathbf{C}}(U, A_i)$, which is also a set. The correspondence $F : U \mapsto \Pi_{i\in I} Hom_{\mathbf{C}}(U, A_i)$ is functorial and contravariant and it satisfies the conditions discussed in Section 1.7.1. We define the product of the family $(A_i)_{i\in I}$, if it exists, as a pair (X, ζ) representing this functor, unique up to isomorphism. We write the symbol $\Pi_{i\in I} A_i$ to denote X. In this case, ζ is a particular family of morphisms $\{\mathbf{p}_j : \Pi_{i\in I} A_i \to A_j\} \in \Pi_{j\in I} Hom_{\mathbf{C}}(\Pi_{i\in I} A_i, A_j)$ called "canonical projections". As noticed in 1.7.1, every element $x = (x_i : U \to A_i) \in F(U)$ is uniquely determined by a unique morphism $\bar{x} : \Pi_{i\in I} A_i \to U$ and the canonical projections:



We say that a category admits finite products if for any finite family this functor is representable and we say that it admits products, in general, if the representability of any family is possible.

As a product of a family of sets we recover the usual cartesian product.

With dual considerations, we can define the sum of a family of objects $(A_i)_{i\in I}$ as a pair $(\coprod_{i\in I} A_i, \zeta)$ representing the covariant functor $X \mapsto \Pi_{i\in I} Hom_{\mathbf{C}}(A_i, X)$. In this case, ζ is a family of morphisms $\{i_j : A_j \to \coprod_{i\in I} A_i\}$ called "canonical inclusions". The "universal property" of the "sum" structure can be encoded on the following diagram



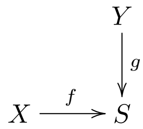
As a sum of two sets we recover the disjoint union of sets.

Given a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ between two categories both admiting products (or sums), we say that F preserves (or commutes) with products (or sums) if $F(\Pi_i A_i) = \Pi_i F(A_i)$ for every product of a familiy (A_i).

2. **Example II - Fiber Products and Pushouts**

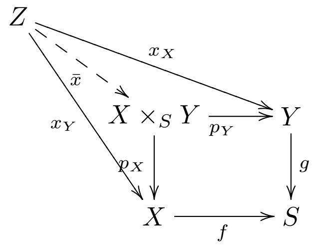
Let A, B and C be sets and let $u : A \to C$ and $v : B \to C$ be maps of sets. The product $A \times_C B$ is defined as the subset of $A \times B$ of pairs (a, b) such that $u(a) = v(b)$ in C.

Given X, Y and S objects in a category $\mathbf{C}$ and $f : X \to S$, $g : Y \to S$ morphisms, we may consider the diagram



(1.4)

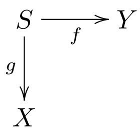
The correspondence $F : Z \mapsto F(Z) = Hom_{\mathbf{C}}(Z, X) \times_{Hom_{\mathbf{C}}(Z,S)} Hom_{\mathbf{C}}(Z, Y)$[4] defines a functor from $\mathbf{C}$ to **Sets**. The *fiber-product*[5] of the diagram is by definition, if it exists, a representative of this functor, $(X \times_S Y, \zeta)$, where ζ is an element of $Hom_{\mathbf{C}}(X \times_S Y, X) \times_{Hom_{\mathbf{C}}(X \times_S Y, S)} Hom_{\mathbf{C}}(X \times_S Y, Y)$, that is, a pair $(p_X : X \times_S Y \to X, p_Y : X \times_S Y \to Y)$ such that $f \circ p_X = g \circ p_Y$. Equivalently, and by 1.3, we could introduce fibre-products as the product of two objects in $\mathbf{C}/S$. After 1.7.1 we can write the universal property of fibre-products with the help of a diagram



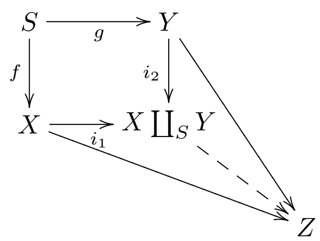
where $x = (x_X, x_Y)$ is an element of $F(Z)$.

Remark 1.7.1.1. *Notice that the fiber-product of a diagram of type 1.4 is in fact equivalent to the product (in the sense defined in the previous example) of two objects in $\mathbf{C}/S$. In this sense, we can define the fiber-product of an arbitrary family of morphisms over a given object S in $\mathbf{C}$, as the common fiber-product of every pair of morphisms over S in the family.*

Dually, given a pair of morphisms $f : S \to X$ and $g : S \to Y$,



for each object Z in $\mathbf{C}$ we assign the subset of $Hom_{\mathbf{C}}(X, Z) \times Hom_{\mathbf{C}}(Y, Z)$ of all pairs (u, v) with $u \circ f = v \circ g$. Again, this correspondence is functorial. We define the pushout of f and g as a pair representing this functor, given by an object $X \coprod_S Y$ and a pair of morphisms $i_1 : Y \to X \coprod_S Y$ and $i_2 : Y \to X \coprod_S Y$ with $i_1 \circ f = i_2 \circ g$:



[4] By defining $A = Hom_{\mathbf{C}}(Z, X)$, $B = Hom_{\mathbf{C}}(Z, Y)$ and $C = Hom_{\mathbf{C}}(Z, S)$ and maps $u := f \circ -$ and $v := g \circ -$, we are choosing only pairs of morphisms $Z \to X$ and $Z \to Y$ such that the square diagram commutes

[5] Fibre-Products are also called Pull-backs

Pull-Backs

Let **C** be a category for which all fiber-products of two objects over a third one exist. Let S be an object in **C** and consider the category $\mathbf{C}/S$. Let X be an object in $\mathbf{C}/S$, $X \to S$. Given another object S' and a morphism $\phi : S' \to S$ in **C**, we can see S' as an object over S and consider the fiber-product of $X \to S$ with $\phi : S' \to S$. In this case, the canonical morphism $X \times_S S' \to S'$ is an object in $\mathbf{C}/S'$. Moreover, from the properties of the fiber-product it follows that, given a morphism $Z \to X$ of objects over S, there is an unique induced morphism $Z \times_S S' \to X \times_S S'$ over S'. Therefore, the fiber-product in **C** induces a functor

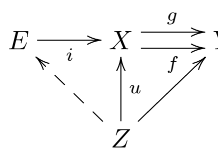
$$\phi^* : \mathbf{C}/S \longrightarrow \mathbf{C}/S' \tag{1.5}$$

for every morphism $\phi : S' \to S$. We call it the *pull-back* functor. Given S'' another object in **C** and a sequence $\phi \circ \phi' : S'' \to S \to S$ of morphisms in **C** , there is a canonical isomorphism between $(X \times_S S') \times_{S'} S''$ and $X \times_S S''$, obtained using the properties of fiber-products.

3. **Example IV - Equalizers and Coequalizers**

   •

   Example 1.7.1.2. *Given two set-maps* $X \underset{f}{\overset{g}{\rightrightarrows}} Y$ *we may consider the subset* $E \subseteq X$ *of all elements* $x \in X$ *such that* $f(x) = g(x)$*. This subset has the following property: If* Z *is another set with a morphism* $u : Z \to X$ *such that* $f \circ u = g \circ u$*, then it factors uniquely through* E *as:* $Z \dashrightarrow E \xrightarrow[i]{} X \underset{f}{\overset{g}{\rightrightarrows}} Y$ *, where* $i : E \to X$ *is the inclusion monomorphism, obviously with* $f \circ i = g \circ i$*. This property can be explained through a diagram as*

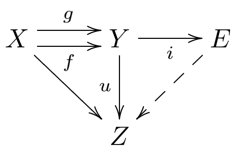


We generalize this construction to a general category: Given a pair of morphisms $X \underset{f}{\overset{g}{\rightrightarrows}} Y$ in **C** we want to find a subobject $i : E \to X$ such that every morphism $u : Z \to X$ with $f \circ u = g \circ u$, factors on a unique way through i. For every object Z we may consider the subset of $Hom_{\mathbf{C}}(Z, X)$ of all morphisms u with $f \circ u = g \circ u$. We denote this set by $Hom_{\mathbf{C}}(Z, X)_{f,g}$. The assignment of this set to each object Z establishes a functorial correspondence from **C** to **Sets**. We define the *equalizer* (also called *kernel*) of $X \underset{f}{\overset{g}{\rightrightarrows}} Y$ as a pair $(E, i \in Hom_{\mathbf{C}}(E, X)_{f,g})$ representing this functor.

Notice that $i : E \to X$ is a monomorphism: For every object Z, if two morphisms $u, v : Z \to E$ verify $i \circ u = i \circ v$ then we have $f \circ i \circ u = f \circ i \circ v$ and, since the factorization through E is unique, we must have $u = v$. In fact, the constructions yields the same universal property presented in the diagram above.

Example 1.7.1.3. *In the category of modules over a ring R (see 2.1.2), the kernel E of a morphism of R-modules $f : X \to Y$ is recovered as the equalizer of the pair $X \underset{f}{\overset{0}{\rightrightarrows}} Y$, where $0 : X \to Y$ is the null-morphism. In this case, E is an R-submodule of X and the canonical morphism $E \to X$ is the inclusion.*

- We dualize this construction and define the *coequalizer* (also called *cokernel*) of a pair $X \underset{f}{\overset{g}{\rightrightarrows}} Y$ in $\mathbf{C}$ as a pair $(E, i \in Hom_{\mathbf{C}}(Y,E)^{f,g})$ representing the functorial correspondence $Z \mapsto Hom_{\mathbf{C}}(Y,Z)^{f,g}$, where this last set denotes all morphisms $u : Y \to Z$ with $u \circ f = u \circ g$. The universal property posed by this functor can be visualized as



The canonical morphism $i : Y \to E$ is an epimorphism: Since $i \circ f = i \circ g$, given two morphisms $a, b : E \to Z$ with $a \circ i = b \circ i$ we have $a \circ i \circ f = b \circ i \circ g$ and since the factorization through i is unique, we must have $a = b$.

Example 1.7.1.4. *In the category of modules over a ring R, the cokernel of a morphism between R-modules, $f : X \to Y$ is recovered as the coequalizer of the pair $X \underset{f}{\overset{0}{\rightrightarrows}} Y$. In this case, this cokernel is isomorphic to the quotient module $Y/f(X)$.*

1.7.2 Diagrams in C

A directed graph is a set of *vertices* I, a set of *edges* E, a terminal map $t : E \to V$ assigning each edge to its ending vertex and an initial map $s : E \to V$ mapping each edge to its starting vertex. For a directed graph $D = (I, E, t, s)$, a diagram of type D or $D - Diagram$ on a category $\mathbf{C}$ is a map that to each vertex $i \in I$ assigns an object A_i in $\mathbf{C}$ and to each edge e starting on a vertex i and ending on j assigns a morphism $\rho_{ij} : A_i \to A_j$ in $\mathbf{C}$.

Sometimes we omit the reference to the graph D and simply see a diagram D as a collection of objects $(A_i)_{i \in I}$ and a collection of morphisms between those objects, $(\rho_{ij} : A_i \to A_j)_{(i,j) \in I \times I}$.

1.7.3 Limits and Colimits of Diagrams

Given a D-diagram and an object U of $\mathbf{C}$, we define a *cone* over D with vertex U as a collection of morphisms, from U to each one of the objects in the diagram $(u_i : U \to A_i)_{i \in I}$, commuting with all the morphisms $\rho_{ij} : A_i \to A_j$ defining the diagram (that is, $u_j = \rho_{ij} \circ u_i$, $\forall i, j \in I$). We write $Cone(U, D)$ to denote the subset of $\Pi_{i \in I} Hom_{\mathbf{C}}(U, A_i)$ of such families of morphisms that are cones.

Definition 1.7.3.1. *Given a D-diagram we may consider the map sending every object U of $\mathbf{C}$ to the set $Cone(U, D)$. This correspondence is functorial and contravariant. The limit of the diagram (also called left-limit), if it exists, is a pair $(limD, (x_i : limD \to A_i)_{i \in I})$ representing this functor.*[6]

$$Cone(U, D) \cong Hom_{\mathbf{C}}(U, limD)$$

[6] Sometimes we also use the notation $\underleftarrow{lim}D$ to denote this limit.

Considering the dual situation, given a D-diagram on $\mathbf{C}$, we define a co-cone of D with vertice on an object U, as a family of morphisms from each one of the objects in the diagram to U, $(u'_i : A_i \to U)_{i\in I}$, commuting with every morphisms ρ_{ij}. Denote by $CoCone(D,U) \subseteq \Pi_{i\in I} Hom_{\mathbf{C}}(A_i,U)$ the set of all this families. Like before, the correspondence $U \mapsto CoCone(D,U)$ is functorial, this time covariant, and we define the colimit of D , $colimD$[7] as a pair representing this functor.

Remark 1.7.3.2. *Since limits and colimits are defined using the representability of a certain functor, they are unique up to isomorphism.*

Remark 1.7.3.3. *The Product (resp. Sum) of an arbitrary family could be defined as a limit (resp. colimit) of a diagram whose vertices are represented by the objects of the family and whose edges correspond to the identity morphisms of those objects. Fibre products (resp. Pushouts) could defined as limits (resp. colimits) of diagrams of the type* $\bullet \longrightarrow \bullet \longleftarrow \bullet$ *(resp.* $\bullet \longleftarrow \bullet \longrightarrow \bullet$ *), on* $\mathbf{C}$*. Also, equalizares (resp. coequalizers) have a similar definition, as limits (resp. colimits) of diagrams of type* $\bullet \rightrightarrows \bullet$.

We use the following terminology concerning $\mathbf{C}$:

- has finite products: if all limits of diagrams with a finite set of vertices and an empty set of edges, exist in $\mathbf{C}$.
- admits finite limits: if all limits of diagrams with a finite set of vertices and a finite set of edges exist in $\mathbf{C}$
- has all limits: if all diagrams have a limit in $\mathbf{C}$.

We apply the same terminology for sums and colimits.

1.7.4 Terminal and Initial Objects

An *initial* object $\emptyset_{\mathbf{C}}$ in $\mathbf{C}$ is an object such that for every other object X of $\mathbf{C}$, the set $Hom_{\mathbf{C}}(\emptyset_{\mathbf{C}}, X)$ has precisely one element. Similarly, we define a *terminal* (or final) object $1_{\mathbf{C}}$ in $\mathbf{C}$ as an object such that for every other object X, the set $Hom_{\mathbf{C}}(X, 1_{\mathbf{C}})$ is a one-element set. Both initial and terminal objects, when they exist, are unique up to isomorphism. There is an alternative description of both terminal and initial objects as limits, resp. colimits of empty diagrams. For a detailed exposition see [15].

We focus now on the properties of these objects with respect to sums and products.
If we add a terminal object $1_{\mathbf{C}}$ to some family $(A_i)_{i\in I}$ of objects, to obtain the product of this new family we need to find a representation of the functorial correspondence given by $U \mapsto \Pi_{i\in I} Hom_{\mathbf{C}}(U, A_i) \times Hom_{\mathbf{C}}(U, 1_{\mathbf{C}})$. The last set on this product is a one-element set, and we immediately conclude that this functor is isomorphic to the one constructed considering the original family. Since the functors are isomorphic, their representing objects are also isomorphic and

$$(\Pi_{i\in I} A_i) \times 1_{\mathbf{C}} \cong \Pi_{i\in I} A_i$$

We conclude that $1_{\mathbf{C}}$ plays the role of a neutral element with respect to products of objects. Using similar arguments we conclude that an initial object $\emptyset_{\mathbf{C}}$ is a neutral element with respect to sums.[8]

[7] (Sometimes we write $limD = \varprojlim D$ and $colimD = \varinjlim D$)

[8] In some literature it is common to find the term "left-unity" to denote terminal objects and "right-unity" to denote initial objects.

Example 1.7.4.1. *In* ***Sets****, the empty set is an initial object and a set with one element is a terminal object.*

We introduce a useful definition.

Definition 1.7.4.2. *We say that an object X in a category $\boldsymbol{C}$ is connected if everytime we have $X \cong A \coprod B$ then either $X \cong B$ and $A \cong \emptyset_{\boldsymbol{C}}$ or $B \cong \emptyset_{\boldsymbol{C}}$ and $X \cong A$.*

Example 1.7.4.3.

- *In the category of sets the sum operation corresponds to the disjoint union of sets. The only connected objects are the one-element sets since every set S can be written as a disjoint union $S = \coprod_{s \in S}\{s\}$;*
- *In the category of topological spaces with continuous maps, the sum operation is also given by the disjoint union of topological spaces. However, in this case we need to preserve the topological structure. It can be proved that a topological space is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself (See [7]).*
- *In the category of vector spaces over $\mathbb{R}$, with linear transformations, the sum operation is given by the directed sum of copies of $\mathbb{R}$. In this case, the only connected object is precisely $\mathbb{R}$.*

1.7.5 Equivalence of Constructions

Up to now, we have introduced many different constructions on a category $\mathbf{C}$, all of them using representations of functors with values on sets of morphisms. Some of these constructions can be shown to be equivalent:

- (i) - The existence of all finite products and all equalizers implies the existence of all fibre-products and of a terminal object: As mentioned above, a terminal object can be obtained as a product of an empty family of objects; To obtain the fiber-product of two morphisms over an object B, $f : A \to B$ and $g : C \to B$, consider the product $A \times C$ and the projections $p_1 : A \times C \to A$ and $p_2 : A \times C \to C$. Since we are supposing that all equalizers exist, we recover the fiber product as the equalizer of $A \times C \underset{f \circ p_1}{\overset{g \circ p_2}{\rightrightarrows}} B$.
- (ii) - The existence of all fibre-products and of a terminal object implies the existence of all finite products and all equalizers: The product of two objects A and B is recovered as the fiber-product of the unique morphisms from A and B to the terminal object; Since we now know that finite products exist, we use them to prove the existence of all equalizers. Given a pair of morphisms $A \underset{f}{\overset{g}{\rightrightarrows}} B$ we consider the fiber-product of f and g, $A \times_B A$ along with $p_1, p_2 : A \times_B A \to A$ with $p_1 \circ f = p_2 \circ g$, the canonical morphisms. Moreover, since the product $A \times A$ exists, the pairs p_1 and p_2 factors through a unique morphism $A \times_B A \to A \times A \to A$, where the last arrow is the unique canonial projection given by the product. Similarly, the pair (I_A, I_A), where $I_A : A \to A$ is the identity morphism, also factors uniquely through $A \to A \times A \to A$. With this, we have a pair of morphism over $A \times A$ and the fiber-product of this pair, $A \times_{A \times A} (A \times_B A)$, is the desired equalizer of the diagram $A \underset{f}{\overset{g}{\rightrightarrows}} B$.

We summarize the above results in the following proposition

Proposition 1.7.5.1. *A category $\boldsymbol{C}$ admits finite products and equalizers if and only if admits all fibre-products and has a terminal object.*

In fact, the existence of all finite products and all equalizers implies all limits of finite diagrams to exist:

Proposition 1.7.5.2. *The limit of any finite diagram in **C** is constructible from the existence of all finite products and all equalizers.*

Proof. If all limits of finite diagrams exist in **C**, by the precedding remarks, all finite products and all equalizers also exist. Conversely suppose that all finite products and all equalizers exist in **C**. Let $(A_i)_{i\in I}$ and $(\rho_{i,j} : A_i \to A_j)_{(i,j)\in I\times I}$ be a D finite type diagram in **C**. Given an object U and a cone $u = (u_i : U \to A_i)$ over the diagram, since we are assuming that all finite products exist, the product $\Pi_{i\in I}A_i$ exists together with the canonical projections $(p_j : \Pi_{i\in I}A_i \to A_j)_{j\in I}$. By the universal property of the product, u factors in a unique way through a morphism $\bar{u} : U \to \Pi_{i\in I}A_i$, and we have $u_i = p_i \circ \bar{u}$. However, we notice that the family (p_i) is not necessarily a cone (Indeed, if it was a cone, then the product $\Pi_{i\in I}A_i$ would already be the limit of the diagram).
For each morphism $\rho_{i,j} : A_i \to A_j$ in the diagram, we may consider the pair of morphisms $\rho_{i,j}\circ p_i : \Pi_{i\in I}A_i \to A_i \to A_j$ and $p_j : \Pi_{i\in I}A_i \to A_j$. This pair induces another pair

$$\Pi_{i\in I}A_i \rightrightarrows \Pi_{(\rho_{i,j})}A_j$$

Let $(X, x : X \to \Pi_{i\in I}A_i)$ be the equalizer of this last pair. We notice that X is precisely the limit of the diagram: Indeed, if u is a cone as introduced above, u factors in a unique way through the canonical projections and since u is a cone it has to factor in a unique way through x. □

Using similar arguments we could easily prove that if all finite sums and all coequalizers exist in **C**, then all colimits also exist. Before ending this section, let us make some important observations:

Remark 1.7.5.3. *The finite property only concerns products and the type of the diagram. Suppose **C** has all equalizers. Thus, if products of a certain cardinality exist then limits for diagrams with the same cardinality, also exist. In particular, if all products exist then all diagrams have a limit, constructible as above.*

Remark 1.7.5.4. *All the previous statements concerning limits, products, fibre products, equalizers and terminal objects, could be dualized, refering respectively to colimits, sums, pushouts, coequalizers and initial objects.*

Remark 1.7.5.5.
1. *In **Sets** all limits (resp. colimits) exist. This follows from the fact that all products* [9] *and all equalizers (resp. sums and coequalizers) exist in **Sets**;*
2. *All limits exist in **Fun(C,Sets)**: Given a family of set-valued functors $(F_i)_{i\in I}$, the assignment defined by*

$$(\Pi_{i\in I}F_i) : X \mapsto \Pi_{i\in I}(F_i(X))$$

is a functorial correspondence. For each object X, the product $\Pi_{i\in I}(F_i(X))$ comes equipped with a family of canonical projections $p_i(X) : \Pi_{i\in I}(F_i(X)) \to F_i(X)$. We easily conclude that the assignment $X \mapsto p_i(X)$ defines a morphism of functors $p_i : (\Pi_{i\in I}F_i) \to F_i$. Given a third set $Z(X)$ and a family of set morphisms $(u_i(X) : Z \to F_i(X))_{i\in I}$, by the product's universal property, there is a unique factorization $u_i(X) : Z \xrightarrow{\Psi_X} \Pi_{i\in I}F_i(X) \xrightarrow{p_i(X)} F_i(X)$. If Z is another set-valued functor and $(u_i : Z \to F_i)_{i\in I}$ is a family of natural transformations, we easily see that the assignment $X \to \Psi_X$ defines a (uniquely determined) natural transformation $Z \to \Pi_{i\in I}F_i$ such that

[9] By the axiom of choice

$$u_i : Z \overset{}{\underset{\Psi}{-\,-\!\!\!>}} \Pi_{i\in I}F_i \xrightarrow[p_i]{} F_i$$

and $\Pi_{i\in I}F_i$ *is the product of* $(F_i)_{i\in I}$.

Given a diagram $F_1 \underset{f}{\overset{g}{\rightrightarrows}} F_2$ *in* ***Fun(C,Sets)****, we consider the assignement*

$$X \longmapsto Equalizer(F_1(X) \underset{f_X}{\overset{g_X}{\rightrightarrows}} F_2(X))$$

where the equalizer is taken in the category of sets. Using similar arguments as above, we can easily see that this correspondence is functorial and is the equalizer of the diagram above.

1.7.6 Functors Preserving Limits

Let $F : \mathbf{C} \longrightarrow \mathbf{C}'$ be a functor. Considering a diagram of type D in $\mathbf{C}$ given by a collection $(X_i)_{i\in I}$ of objects and morphisms $(\rho_{ij})_{i,j\in I}$, the image under F of this collection forms another D-type diagram, this time in $\mathbf{C}'$. We denote it by F_D.

Definition 1.7.6.1. *We say that F preserves limits if everytime a D-diagram has a limit in* ***C****, given by a pair* $(limD, \phi \in Cone(limD, D))$*, the image of this pair,* $(F(limD), F(\phi) \in Cone(F(limD), F_D))$ *is a limit of* F_D *in* ***C'****.*

The previous section gives us a tool to determine wether a given functor preservers limits. We saw that a limit of a diagram of type D in $\mathbf{C}$ can always be constructed using products and equalizers or equivalently, fibre-products and a terminal object. This implies:

Proposition 1.7.6.2. *F preserves finite limits if and only if F maps terminal objects to terminal objects and preserves fibre-products (or equivalently, preserves products and equalizers).*

Remark 1.7.6.3. *We notice that if two functors are isomorphic and one of them preserves limits, then the other also preserves limits. This follows immediatly from the last proposition, because both must map terminal objects to terminal objects and preserve fibre-products.*

Example 1.7.6.4. *From the previous remark, we conclude that to address the case of representable functors, it suffices to study functors of type* $h_X : \mathbf{C} \longrightarrow \mathbf{Sets}$ *:*

- h_X *naturally preserves products: In fact, by the definition of a product we have* $Hom_{\mathbf{C}}(X, \Pi_{i\in I}A_i) \cong \Pi_{i\in I}Hom_{\mathbf{C}}(X, A_i)$*;*

- h_X *preserves equalizers: Given a diagram* $E \xrightarrow[i]{} A \underset{f}{\overset{g}{\rightrightarrows}} B$ *with* $i : E \to A$ *an equalizer, we have*

  $Hom_{\mathbf{C}}(X, E) \xrightarrow[i_*]{} Hom_{\mathbf{C}}(X, A) \underset{f*}{\overset{g*}{\rightrightarrows}} Hom_{\mathbf{C}}(X, B)$ *also an equalizer in sets, by definition.*

Thus, all covariant representable functors preserve all limits existing in ***C****. Dually, contravariant representable functors map colimits to limits.*

Seeing X as a variable in the formulas above, we conclude that the functor $h^\circ : \boldsymbol{C} \longrightarrow$ ***Fun(C,Sets)*** *mapping $Y \to h^\circ_Y$, preserves products, $h^\circ_{A\times B} \cong h^\circ_A \times h^\circ_B$, and equalizers and so, all limits existing in* ***C****.*

Dually, we conclude that the contravariant functor $h : \boldsymbol{C} \longrightarrow$ ***Fun(C,Sets)*** *sends sums to products and coequalizers to equalizers, thus mapping all colimits to limits.*

Example 1.7.6.5. *Each object X in* ***C*** *naturally defines a functor* ***Fun(C,Sets)*** $\longrightarrow$ ***Sets*** *mapping each F to $F(X)$. We conclude that this functor maps terminal objects to terminal objects and preserves fibre products, and therefore preserves all finite limits. The same situation happens with initial objects and pushouts. In this sense, we conclude that all limits/colimits in* ***Fun(C,Sets)*** *are fully determined "object by object".*

We use the following terminology concerning a functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$:

- is right exact if it preserves colimits;
- is left exact if it preserves limits;
- is exact, it it is both right and left exact.

1.7.7 Inductive and Projective Systems

Let **I** be a small category. We can think of **I** as a graph whose vertices and edges are, respectively, the objects and morphisms of **I**. With this in mind, we can describe a diagram in **C** of the same type of the graph encoding **I**, simply as a functor from **I** with values in **C**.

Definition 1.7.7.1. *Let* ***I*** *be a small category.*

- *An* inductive system *in* ***C*** *of type* ***I*** *is a covariant functor*

$$\alpha : \boldsymbol{I} \longrightarrow \boldsymbol{C}$$

- *A* projective system *in* ***C*** *of type* ***I*** *is a contravariant functor:*

$$\beta : \boldsymbol{I}^\circ \longrightarrow \boldsymbol{C}$$

Definition 1.7.7.2.

- *The* inductive limit *of an inductive system* $\alpha : \boldsymbol{I} \longrightarrow \boldsymbol{C}$ *is the colimit of the diagram in* ***C*** *spanned by the image of α. We use the notation $\varinjlim_I \alpha$ to denote this colimit.*
- *The* projective limit *of a projective system* $\beta : \boldsymbol{I}^\circ \longrightarrow \boldsymbol{C}$ *is the limit of the diagram in* ***C*** *spanned by the image of β; We use the notation $\varprojlim_I \alpha$ to denote this limit.*

We will frequently omit the functor F and write a projective system simply by indicating the collection of objects $(F(i))_{i\in I}$ and transition morphisms $(\rho_s = F(i,j) : F(i) \to F(j))_{s:j\to i}$.

Example 1.7.7.3. *In* ***Sets****, inductive and projective limits always exists.*

- *Let* $\beta : \boldsymbol{I}^\circ \longrightarrow$ ***Sets*** *be a projective system. We notice that the set $S = \{(x_i) \in \Pi_{i\in I}\beta(i) : \beta(s)(x_i) = x_j,$ for all $s \in Hom_{\boldsymbol{I}}(j,i)\} \subseteq \Pi_{i\in I}\beta(i)$, together with the canonical projections inherited from the product, $S \to \beta(j)$, satisfies all the necessary properties of a limit:*

- *By construction, each canonical projection commutes with every composable* ρs*;*
- *If* T *is another set with a family of maps* $u_i : T \to \beta(i)$ *commuting with the transition morphism* $\beta(s)$ *then there is a uniquely determined factorization* $T \to S \to \beta(j)$ *defined by mapping each* $t \in T \mapsto (u_i(t))_{i\in I}$*;*

In fact, there is another equivalent description of this limit set: Each point $x = (x_i)_{i\in I} \in S$ *defines a natural transformation, between the projective system* $* : \mathbf{I}^\circ \longrightarrow \mathbf{Sets}$ *mapping every* i *to a one-element set* $\{*\}$*, and* β*. The converse is also true and we have a bijection*

$$S \cong Hom_{\mathbf{Fun(I,Sets)}^\circ}(*, \beta)$$

- *If we have an inductive system instead, with* $(\rho_s : S_i \to S_j)_{s:i\to j}$*, we can produce a colimit set by considering an equivalence relation on* $\coprod_{i\in I} S_i$ *by defining* $x \in S_i$ *equivalent to* $y \in S_j$ *if and only if there exists* $s : i \to k$ *and* $s' : j \to k$ *such that* $\rho_s(x) = \rho_{s'}(y)$*. The quotient space of* $\coprod_{i\in I} S_i$ *under this relation together with the inclusion maps* $S_i \to \coprod_{i\in I} S_i$ *satisfy all the conditions of a colimit.*

Note that limits of inductive and projective systems on an arbitrary category $\mathbf{C}$ can be obtained using projective limits of sets. To understand this, consider a projective system

$$\beta : \mathbf{I}^\circ \longrightarrow \mathbf{C}$$

For each object U in $\mathbf{C}$, the family of sets $(Hom_{\mathbf{C}}(U, \beta(i)))_{i\in I}$ together with the composition maps $(\rho_s \circ -) : Hom_{\mathbf{C}}(U, \beta(i)) \to Hom_{\mathbf{C}}(U, \beta(j))_{s:j\to i}$ defines a projective system of sets:

$$\phi^U_\beta : \mathbf{I}^\circ \longrightarrow \mathbf{Sets} \qquad \text{defined by } i \longmapsto Hom_{\mathbf{C}}(U, \beta(i))$$

As seen in the last example, the limit of a projective system of sets always exists and, we have another functorial correspondence

$$\mathbf{C}^\circ \longrightarrow \mathbf{Sets} \qquad \text{sending } U \longmapsto \varprojlim_I \phi^U_\beta = \varprojlim_I Hom_{\mathbf{C}}(U, X_i)$$

Definition 1.7.7.4. *This functor is called the limit functor of* β *and denoted simply as* $\varprojlim_I \beta$*.*

Notice that the set $\varprojlim_I \phi^U_\beta$ is easily identified with what we defined as $Cone(U, D)$, where D is the diagram in $\mathbf{C}$ spanned by the image of β. Therefore, according to definition 1.7.3.1 the limit of β exists in $\mathbf{C}$ if and only if this limit functor is representable.

Remark 1.7.7.5. *When* $\mathbf{C} = \mathbf{Sets}$ *and* $\beta : \mathbf{I}^\circ \longrightarrow \mathbf{Sets}$ *is a projective system, we may still consider the limit functor*

$$\varprojlim_I \beta : \mathbf{I}^\circ \longrightarrow \mathbf{Sets}$$

In this case, this functor is representable by the limit set found in 1.7.7.3.

Similary, given an inductive system $\alpha : \mathbf{I} \longrightarrow \mathbf{C}$, for each object U in $\mathbf{C}$ the collection of sets $(Hom_{\mathbf{C}}(X_i, U))_{i\in I}$ together with the appropriate composition maps defines a projective system of sets

$$i \mapsto Hom_{\mathbf{C}}(\alpha(i), U)$$

For each U, the limit of this projective system coincides with what we have defined as $CoCone(D, U)$, where D denotes the diagram spanned by the image of α in $\mathbf{C}$.

Example 1.7.7.6. *In particular, if α is an inductive system of sets, the limit-set $(\coprod_I \alpha(i)/\sim)$ constructed in 1.7.7.3 represents the limit-functor of α.*

Remark 1.7.7.7. *Given a functor f between two categories $\boldsymbol{C}$ and $\boldsymbol{D}$, any inductive system $\alpha : \boldsymbol{I} \longrightarrow \boldsymbol{C}$ induces another inductive system on $\boldsymbol{D}$ through the composition of functors, $f \circ \alpha$. If f is contravariant, it turns projective systems to inductive systems and vice-versa.*

1.7.8 Cofinal Functors

Definition 1.7.8.1. *Let $\boldsymbol{I}$ and $\boldsymbol{J}$ be small categories. A functor*

$$\varphi : \boldsymbol{J} \longrightarrow \boldsymbol{I}$$

is called cofinal *if for every functor $\beta : \boldsymbol{I}^\circ \longrightarrow \boldsymbol{Sets}$ the natural map between the limit sets (1.7.7.3)*

$$\varprojlim \beta \to \varprojlim (\beta \circ \varphi^\circ)$$

is a bijection [10]*. Whenever φ is an inclusion functor of a subcategory $\boldsymbol{J}$ of $\boldsymbol{I}$, we say that $\boldsymbol{J}$ is a cofinal subcategory if ϕ is cofinal.*

Proposition 1.7.8.2. *Let $\varphi : \boldsymbol{J} \longrightarrow \boldsymbol{I}$ be a functor. The following properties are equivalent:*

1. *φ is cofinal;*
2. *For every category $\boldsymbol{C}$ and every functor $\beta : \boldsymbol{I}^\circ \longrightarrow \boldsymbol{C}$ the canonical natural transformation between the limit functors (1.7.7.4)*

   $$\varprojlim \beta \to \varprojlim (\beta \circ \varphi^\circ)$$

   is an isomorphism in $\boldsymbol{Fun}(\boldsymbol{C}^\circ, \boldsymbol{Sets})$.
3. *for every functor $\alpha : \boldsymbol{I} \longrightarrow \boldsymbol{Sets}$ the natural map between the limit sets (1.7.7.3)*

   $$\varinjlim (\alpha \circ \varphi) \to \varinjlim (\alpha)$$

   is a bijection.
4. *For every category $\boldsymbol{C}$ and every functor $\alpha : \boldsymbol{I} \longrightarrow \boldsymbol{C}$ the canonical natural transformation between the limit functors (1.7.7.4)*

   $$\varinjlim (\alpha \circ \varphi) \to \varinjlim (\alpha)$$

   *is an isomorphism in **Fun(C, Sets)**.*

[10] φ° denotes the opposite functor of φ

Proof.

- (1) ⇒ (2): Given a functor $f : \mathbf{C}^\circ \longrightarrow \mathbf{Sets}$, the limit functor of f is given by the correspondence $X \mapsto \varprojlim_I \phi_f^X$, where $\phi_f^X : \mathbf{I}^\circ \longrightarrow \mathbf{Sets}$ is the projective system introduced in 1.7.7. Since φ is cofinal, both sets $\varprojlim \phi_f^X$ and $\varprojlim(\phi_f^X \circ \varphi^\circ)$ are isomorphic. This isomorphism is functorial and so, the limit functors $\varprojlim f$ and $\varprojlim(f \circ \varphi^\circ)$ are isomorphic in $\mathbf{Fun}(\mathbf{C}^\circ, \mathbf{Sets})$.

- (2) ⇒ (1): To prove this direction, consider the case when $\mathbf{C} = \mathbf{Sets}$. By (2), for any projective system $\beta : \mathbf{I}^\circ \longrightarrow \mathbf{Sets}$, there is an isomorphism between the limit functors $\varprojlim \beta$ and $\varprojlim(\beta \circ \varphi^\circ)$, in $\mathbf{Fun}(\mathbf{C}^\circ, \mathbf{Sets})$. By 1.7.7.3, we know that all projective limits exist in **Sets**, given by a set representing the limit functors. Therefore, since the limit functors are isomorphic, the sets representing them are also isomorphic and φ is cofinal.

- The results (1) ⇒ (3) and (3) ⇒ (1) can be proved using the fact that inductive limits can be obtained using projective limits of sets, as explained in 1.7.7.

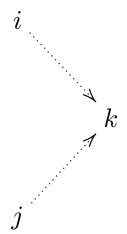
□

1.7.9 Filtrant Categories

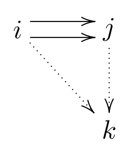
Definition 1.7.9.1. *A category* $\boldsymbol{I}$ *is called* filtrant *if it satisfies the following conditions:*

1. $\boldsymbol{I}$ *is non-empty;*

2. *for every objects* i *and* j *in* $\boldsymbol{I}$ *there exists another object* k *in* $\boldsymbol{I}$*, such that the sets* $Hom_{\boldsymbol{I}}(i,k)$ *and* $Hom_{\boldsymbol{I}}(j,k)$ *are non-empty. Visualized on a diagram:*



3. *for any pair of morphisms* $f, g : i \rightrightarrows j$ *, there exists another morphism* $h : j \to k$ *such that* $h \circ f = h \circ g$*:*



Example 1.7.9.2. *A partial order* $\leqslant$ *on a set* I *is said to be directed if for any* $i, j \in I$ *exists* $k \in I$ *such that* $i, j \leqslant k$*. In this case, we call the pair* $(I, \leqslant)$ *a directed set. Any partially ordered directed set* $(I, \leqslant)$ *defines a filtrant category* $\boldsymbol{I}$*, whose objects are the elements of* I *and for* $i, j \in I$ *the set* $Hom_{\boldsymbol{I}}(i,j)$ *has exactly one element whenever* $i \leqslant j$ *and is otherwise empty. The third condition in the definition above is trivially satisfied. If* $i \leqslant j$ *and* $j \leqslant k$*, the composition rule satisfy* $(j,k) \circ (i,j) = (i,k)$*, inheriting the order relation. We also have* $(i,i) = I_i$*.* [11]

[11] This procedure is used, for example, when defining presheaves over a topological space X, where we take for I the open sets of X together with the inclusion relation.

We have the following important property concerning filtrant categories:

Lemma 1.7.9.3. *Given a functor* $\varphi : \boldsymbol{J} \longrightarrow \boldsymbol{I}$ *between small categories, with* $\boldsymbol{I}$ *filtrant, suppose the following conditions:*

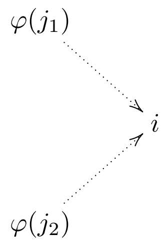
- φ *is fully faithful;*
- *for every object* i *in* $\boldsymbol{I}$ *there is an object* j *in* $\boldsymbol{J}$ *and a morphism* $i \to \varphi(j)$*.*

In this case

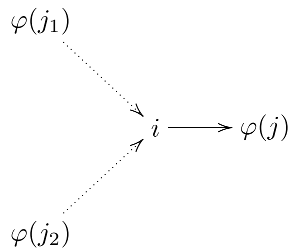
1. $\boldsymbol{J}$ *is also filtrant;*
2. φ *is cofinal.*

Proof.

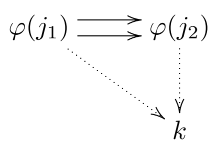
1. Since $\mathbf{I}$ is filtrant, $\mathbf{I}$ is non-empty and by the second condition, $\mathbf{J}$ is also non-empty; Again, since $\mathbf{I}$ is filtrant, given two objects j_1 and j_2 in $\mathbf{J}$, there is another object i in $\mathbf{I}$ with



The second condition implies the existence of a third object j in $\mathbf{J}$ with



Since φ is fully faithful, the fact that the sets $Hom_{\mathbf{I}}(\varphi(j_1), \varphi(j))$ and $Hom_{\mathbf{I}}(\varphi(j_2), \varphi(j))$ are non-empty, implies that $Hom_{\mathbf{J}}(j_1, j)$ and $Hom_{\mathbf{J}}(j_2, j)$ are also non-empty sets; At last, given any diagram $j_1 \rightrightarrows j_2$ in $\mathbf{J}$, there is a third object k in $\mathbf{I}$ with



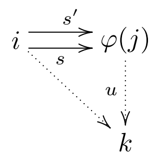
Again, the second condition is enough to ensure the existence of a third object j in $\mathbf{J}$ and a morphism $k \to \varphi(j)$. The fact that φ is fully faithful allows us to conclude the proof.

2. According to the definition 1.7.8.1, to prove that φ is cofinal it suffices to prove that for every projective system $\beta : \mathbf{I}^\circ \longrightarrow \mathbf{Sets}$ the natural set-map (see 1.7.7.3)

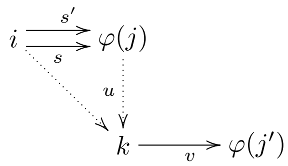
$$\underleftarrow{lim}\beta \longrightarrow \underleftarrow{lim}(\beta \circ \varphi) \qquad \text{sending} \qquad (x_i)_{i\in I} \longmapsto (x_{\varphi(j)})_{j\in J}$$

The first family satisfies $\beta(s)(x_{i'}) = x_i$ for all morphisms $s : i \to i'$ in $\mathbf{I}$ and its restriction $(x_{\varphi(j)})_{j\in J}$ holds $\beta(s)(x_{\varphi(j')}) = (x_{\varphi(j)})$, for all morphisms $s : \varphi(j) \to \varphi(j')$ in $\mathbf{I}$ and since φ is fully faithful, in fact, this happens for all morphisms $j \to j'$ in $\mathbf{J}$. The second imposed condition implies that this map is a bijection since we can recover the whole family, starting with the values $(x_{\varphi(j)})_{j\in J}$. For every $i \in \mathbf{I}$, by the second condition, there exists a morphism $s : i \to \varphi(j)$, for some $j \in \mathbf{J}$. We define for each $i \in I$, $x_i = \beta(s)(x_{\varphi(j)})$. This is well-defined:

- Given two morphisms $i \underset{s}{\overset{s'}{\rightrightarrows}} \varphi(j)$, since $\mathbf{I}$ is filtrant, by 1.7.9.1-3 there is a $k \in I$ such that

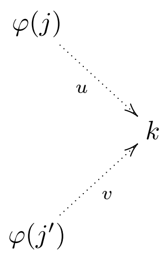


By the second hypothesis, there is another $j \in J$ and a morphism $v : k \to \varphi(j')$ and we have

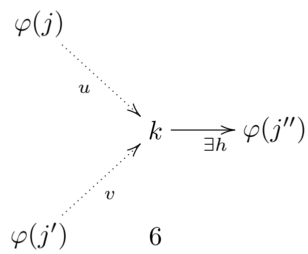


we have $\beta(v \circ u \circ s') = \beta(v \circ u \circ s) \Leftrightarrow \beta(s') \circ \beta(v \circ u) = \beta(s) \circ \beta(v \circ u)$ and therefore $\beta(s)(x_{\varphi(j)}) = \beta(s')(x_{\varphi(j)})$.

- If we have $s : i \to \varphi(j)$ and $s' : i \to \varphi(j')$, by 1.7.9.1-2, there is a $k \in \mathbf{I}$ with



and by the second hypothesis



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This way, we find a pair of morphisms $i \underset{houos}{\overset{houos'}{\rightrightarrows}} \varphi(j'')$ and by applying the previous step we conclude the proof.

□

After this lemma, whenever we have an inclusion functor $\varphi : \mathbf{J} \longrightarrow \mathbf{I}$ of $\mathbf{J}$ a subcategory of $\mathbf{I}$ filtrant, to prove that $\mathbf{J}$ is cofinal, we simply need to check the second condition in the lemma.

1.8 Strict Morphisms

Let **C** be a category and suppose that all equalizers and coequalizers exist in **C**.

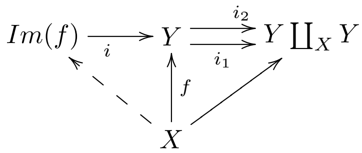
Definition 1.8.0.4. *Let* ***C*** *be a category and $f : X \to Y$ a morphism in* ***C***. *The image of f, denoted $Im(f)$, is by definition, constructed as*

$$Im(f) := Equalizer(Pushout(X \underset{f}{\overset{f}{\rightrightarrows}} Y))$$

The pushout of $X \underset{f}{\overset{f}{\rightrightarrows}} Y$ is given by an object $Y \coprod_X Y$ and a pair of morphisms $i_1, i_2 : Y \to Y \coprod_X Y$ and,

$$Im(f) = Equalizer(\ Y \underset{i_1}{\overset{i_2}{\rightrightarrows}} Y \coprod_X Y\ )$$

Moreover, we have $i_1 \circ f = i_2 \circ f$ and by the universal property of an equalizer (see 1.7.1-3) there is a canonical factorization of f



and, as discussed in 1.7.1-3, the canonical morphism $i : Im(f) \to Y$ is a monomorphism.

Example 1.8.0.5.

- ***Sets****: Consider a set-map $f : X \to Y$. The image of f is the set of all elements $y \in Y$ such that, there is some $x \in X$ with $y = f(x)$, denoted $f(X)$. This set can be recovered using the above definition. $Y \coprod_X Y$ is the disjoint union of two copies of Y under the following equivalence relation: if a point $y \in Y$ is equal to $f(x)$ for some $x \in X$, we identify the y in the first copy with the y point in the second copy. The canonical morphisms $i_1, i_2 : Y \to Y \coprod_X Y$, respectively, maps Y to the first, (resp. second) copy of Y in $Y \coprod_X Y$. Given another set Z together with a map $\phi : Z \to Y$ with $i_1 \circ \phi = i_2 \circ \phi$, we have for each point $z \in Z$, $i_1(\phi(z)) = i_2(\phi(z)$. This is possible only if $\phi(z)$ is in both copies of Y, or, in other words, if $\phi(z)$ equals $f(x)$ for some $x \in X$. In this case, there is a canonical factorization of ϕ through $Z \to f(X) \to Y$, where the first arrow sends $z \mapsto f(x)$ and the second arrow is the canonical inclusion of $f(X)$ in Y. We conclude that $Im(f)$ is the set $f(X)$.*

- *Topological Spaces: Using a procedure similar to the one presented in the previous example we can prove that the image of a morphism $f : X \to Y$ between topological spaces is isomorphic to the set $f(X)$ equipped with the subspace topology, together with the canonical inclusion $f(X) \to Y$.*

- *Modules over a ring R: Given a morphism of R-modules (see 2.1.2), $f : X \to Y$, the pushout of $X \underset{f}{\overset{f}{\rightrightarrows}} Y$ is isomorphic to $Y \oplus Y$ under the identification of all pairs $(f(x), -f(x))$ with $(0, 0)$.*

Given a third R-module Z together with a morphism $\phi : Z \to Y$ with $i_1 \circ \phi = i_2 \circ \phi$, for each element $z \in Z$ we have $(\phi(z), 0) = (0, \phi(z)) \Leftrightarrow (\phi(z), -\phi(z)) = (0, 0)$. With this, we easily conclude that the equalizer of the diagram $Y \underset{i_1}{\overset{i_2}{\rightrightarrows}} Y \coprod_X Y$ (where i_1 is the inclusion $y \mapsto (y, 0)$ and $i_2 : y \to (0, y)$) is precisely the image submodule of f in Y.

We introduce the coimage of a morphism:

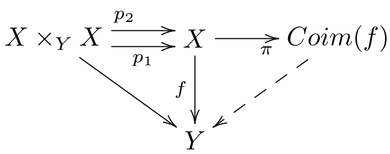
Definition 1.8.0.6. *Let $\boldsymbol{C}$ be a category and $f : X \to Y$ a morphism in $\boldsymbol{C}$. The coimage of f, denoted $CoIm(f)$, is by definition, constructed as*

$$Coim(f) := Coequalizer(FiberProduct(X \underset{f}{\overset{f}{\rightrightarrows}} Y))$$

Again, the fiber-product of $X \underset{f}{\overset{f}{\rightrightarrows}} Y$ is given by an object $X \times_Y X$ and a pair of morphisms $p_1, p_2 : X \times_Y X \to X$. We have

$$Coim(f) = Coequalizer(X \times_Y X \underset{p_1}{\overset{p_2}{\rightrightarrows}} X))$$

and since $f \circ p_1 = f \circ p_2$, by the universal property of coequalizers, (see 1.7.1-3) there is a canonical factorization of f



where, as discussed in 1.7.1-3, the canonical morphism $\pi : X \to Coim(f)$ is an epimorphism.

Example 1.8.0.7.

- ***Sets**: Consider a set-map $f : X \to Y$. The fiber-product $X \times_Y X$ is the set $\{(x_1, x_2) : f(x_1) = f(x_2)\} \subseteq X \times X$, together with the natural coordinate projections $p_1, p_2 : X \times X \to X$. Given a third set Z and a map $\phi : X \to Z$ with $\phi \circ p_1 = \phi \circ p_2$. for each pair (x_1, x_2) with $f(x_1) = f(x_2)$ we have $\phi(x_1) = \phi(x_2)$. We consider an equivalence relation on X: x is equivalent to x' if and only if $f(x) = f(x')$. There is natural quotient map $\pi : X \to X/\sim$. Notice that every ϕ as above, factors through*

$$X \to X/\sim \to Z$$

where the last arrow maps $[x] \mapsto \phi(x)$. Therefore, $Coim(f)$ is given by $X/\sim$. Notice also that in this case, $Coim(f)$ is isomorphic to $f(X)$.

- *Topological Spaces: Following the previous item, if $f : X \to Y$ is a continuous map between topological spaces, we find that $Coim(f)$ is given by $X/\sim$ equipped with the quotient topology with respect to $\pi : X \to X/\sim$. Notice that, in general, $Coim(f)$ is not isomorphic to $Im(f)$.*

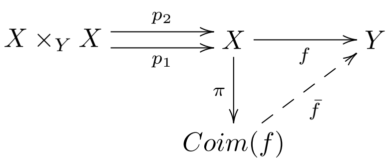
- *Modules over a ring R: Given a morphism of R-modules (see 2.1.2), $f : X \to Y$, the fiber-product of* $X \underset{f}{\overset{f}{\rightrightarrows}} Y$ *is the R-submodule of $X \times X$ of all pairs (x_1, x_2) with $(x_1 - x_2) \in ker(f)$. Following this, it can easily be proved that $Coim(f)$ is isomorphic to $X/kern(f)$.*

Proposition 1.8.0.8. *There is a canonical unique factorization of $f : X \to Y$ as*

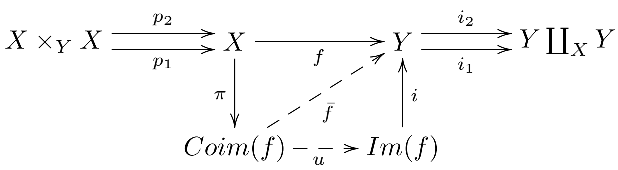
$$X \xrightarrow{\pi} Coim(f) \dashrightarrow Im(f) \xrightarrow{i} Y$$

where i is a monomorphism and π is an epimorphism.

Proof. We use the universal properties of $Im(f)$ and $Coim(f)$: Since we have $f \circ p_1 = f \circ p_2$, there is a factorization of f through



and we know that the vertical arrow $\pi : X \to Coim(f)$ is an epimorphism. This implies that the factorization $\bar{f}$ is unique. At the same time, we have $i_1 \circ f = i_2 \circ f$ and $f = \bar{f} \circ \pi$. Since π is an epimorphism we conclude that $i_1 \circ \bar{f} = i_2 \circ \bar{f}$ and therefore $\bar{f}$ has a canonical factorization



Now, we know by a previous discussion that $i : Im(f) \to Y$ is a monomorphism. This implies that this second factorization, u, is also unique.

□

Definition 1.8.0.9. *Let $\boldsymbol{C}$ be a category where all finite limits and colimits exist. A morphism $f : X \to Y$ in $\boldsymbol{C}$ is called strict if the canonical morphism*

$$Coim(f) \dashrightarrow Im(f)$$

in 1.8.0.8, is an isomorphism.

Example 1.8.0.10. *In the category of modules over a ring R (see 2.1.2), every morphism is strict. This fact is usually called the* 1st Isomorphism Theorem*: Every morphism of R-modules $f : X \to Y$ factors in a unique way as $X \to X/ker(f) \to Im(f) \to Y$, where the first and the last arrows are respectively, an epimorphism and a monomorphism, and the middle one is an isomorphism.*

Proposition 1.8.0.11. *Let $\boldsymbol{C}$ be a category where all finite limits and colimits exist. Let $f : X \to Y$ be a morphism in $\boldsymbol{C}$. Then:*

1. *f is an epimorphism if and only if $i : Im(f) \to Y$ is an isomorphism;*

2. *f is a monomorphism if and only if $\pi : X \to Coim(f)$ is an isomorphism;*

3. *f is a strict epimorphism if and only if $Coim(f) \to Y$ is an isomorphism;*

4. *f is a strict monomorphism if and only if $X \to Im(f)$ is an isomorphism;*

5. *If f is simultaneously a monomorphism, epimorphism and strict morphism, then f is an isomorphism.*

Proof. 1. By construction, we have $Im(f) = Eq(Y \overset{i_2}{\underset{i_1}{\rightrightarrows}} Y \coprod_X Y)$, with $i_1 \circ f = i_2 \circ f$. If f is an epimorphism, we must have $i_1 = i_2$ and therefore Y is also an equalizer of the mentioned diagram and $i : Im(f) \to Y$ is an isomorphism. Conversely, if i is an isomorphism, Y is also an equalizer of $Y \overset{i_2}{\underset{i_1}{\rightrightarrows}} Y \coprod_X Y$ and we must have $i_1 = i_2 = h$. Given any other object Z in $\mathbf{C}$ and any pair of morphisms $a, b : Y \to Z$ with $a \circ f = b \circ f$, there is a unique factorization $Y \xrightarrow[i_1=i_2=h]{} Y \coprod_X Y \overset{}{\underset{u}{\dashrightarrow}} Z$, and we have $a = u \circ i_1 = u \circ h = u \circ i_2 = i_b$.

2. Follow the previous proof using dual arguments.

3. This is a corollary of 1.

4. Corollary to 2;

5. Follows from 1. and 2. and 1.8.0.9

□

1.9 Group Actions

1.9.1 Group actions on objects and morphisms

Given a category $\mathbf{C}$, an action of a group G on some object X of $\mathbf{C}$ by automorphisms, is a group homomorphism $\phi : G \to Aut_{\mathbf{C}}(X)$. The kernel of a G action $\{g \in G : \phi(g) = I_X\} \subseteq G$ is a normal subgroup of G.

Let G act on X. For every object Z, the action of G can be extended to the sets $Hom_{\mathbf{C}}(X, Z)$ and $Hom_{\mathbf{C}}(Z, X)$, defined respectively by mapping $(u : X \to Z) \mapsto (u \circ g : X \to Z)$ and $(v : Z \to X) \mapsto (g \circ v : Z \to X)$, $\forall g \in G$. We denote by $Hom_{\mathbf{C}}(X, Z)^G$ and $Hom_{\mathbf{C}}(Z, X)^G$ the set of morphisms invariant under this action.

1.9.2 Compatible Morphisms

Let $\mathbf{C}$ be a category and consider an action of G on two objects X and Y. We say a morphism $f : X \to Y$ is compatible with the actions of G if for all $g \in G$ we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow[f]{} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

In particular, we say that the actions of G on X and Y are equivalent if there is an isomorphism $f : X \to Y$ compatible with the actions.

Example 1.9.2.1. *Let F be a set-valued functor on a category $\boldsymbol{C}$, represented by an object X. From 1.6.4.4 we know that the group of automorphisms of F and X are isomorphic. It follows directly from this construction that the action of $Aut_{\boldsymbol{Fun(C,Sets)}}(F)$ on the sets $F(U)$ is equivalent to the natural action of $Aut_{\boldsymbol{C}}(X)$ on $Hom_{\boldsymbol{C}}(X, U)$.*

1.9.3 Quotients by Group Actions

Let X be an object with an action of a group G by automorphisms. The correspondence $Z \mapsto Hom_{\mathbf{C}}(X, Z)^G$ is functorial and it is natural to ask wether it is representable by a pair (Y, p), where Y is an object in $\mathbf{C}$ and $p \in Hom_{\mathbf{C}}(X, Y)^G$, such that the map $Hom_{\mathbf{C}}(Y, Z) \to Hom_{\mathbf{C}}(X, Z)^G$ mapping $g \to g \circ p$ is an isomorphism. Such object Y will be unique up to isomorphism. This is equivalent to say that any morphism $f : X \to Z$ invariant under the action of G has a unique factorization $\bar{f}$, through $p : X \to X/G$ (also G-invariant). We denote it by X/G and call it the quotient of X by the action of G.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle p}\downarrow & \nearrow_{\bar{f}} & \\ X/G & & \end{array} \tag{1.6}$$

We say that $\mathbf{C}$ admits quotients if this construction is always possible. As an example, quotients by group actions do exist in categories like **Sets** and **Groups**. An equivalent particular description of a quotient of X by a group G action can be recovered as the colimit of a "flower" shapped diagram in $\mathbf{C}$ with one vertex X and arrows $g \in Aut_{\mathbf{C}}(X)$.

As we have seen in section 1.2, a functor F between two categories $\mathbf{C}$ and $\mathbf{D}$ induces a group homomorphism $Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{D}}(F(X))$, sending each automorphism ϕ to $F(\phi)$. In particular, a functor induces an action of the automorphisms of X on $F(X)$. Also, each action of a group G on X can be extended to an action on $F(X)$, composing $G \to Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{D}}(F(X))$.

We say that F preserves quotients if $F(X)/G$ exists and is isomorphic to $F(X/G)$, for all X in $\mathbf{C}$, whenever X/G exists.

1.9.4 G-Categories

We say a group G acts on a category $\mathbf{C}$ (or that $\mathbf{C}$ has a G-action) if it acts on each object X in $\mathbf{C}$. Equivalently, this action on $\mathbf{C}$ is given by a collection $\{G \to Aut_{\mathbf{C}}(X)\}$. We introduce the following definition:

Definition 1.9.4.1. *A G-category is a category $\boldsymbol{C}$ with a G-action, such that all the morphisms are compatible with the action. We use the notation $\boldsymbol{C}(G)$.*

Given a category with a group action we may consider the subcategory of $\mathbf{C}$, whose objects are still the objects of $\mathbf{C}$ but for morphisms we choose only the ones compatible with the action.

Chapter 2

Fundamental Notions from Algebraic Geometry

In this chapter we review the concepts and results of Algebraic Geometry, which will be needed in the next chapters. Regarding commutative algebra, the reader is recommeded to follow [12] and [6] and for a detailed exposition on sheaves and schemes, the fundamental work done by A.Grothendieck on [3]

2.1 Rings and Modules

2.1.1 Ideals and Quotients

Let R be a ring (in this work we will always assume rings to be commutative). A subset I of R is called an ideal if $(I, +)$ is an abelian group and for all $r \in R$, $a \in I$, ar is also in I. The kernel of a morphism of rings $f : R \to S$ is the subset $\{r \in R : f(r) = 0\} \subset R$. The image of f is the set $\{s \in S : \exists r \in R$ such that $s = f(r)\}$ We just gather now a collection of facts which we will use later

Proposition 2.1.1.1. *Let R and S be rings and $f : R \to S$ a ring morphism.Then*

- *If J is an ideal of S then $f^{-1}(J)$ is and ideal of R.*
- *In general, if I is an ideal of R then $f(I)$ may not be an ideal in S.*
- *The kernel of a morphism of rings $f : R \to S$ is an ideal of R.*
- *f is injective if and only if the kernel of f is $\{0\}$.*

Proof. See [6]. □

Let R be a ring. Given an additive subgroup I we may form the quotient R/I whose elements are equivalence classes $[r]$ of elements of $r \in R$, such that $[r] = [r']$ iff $(r' - r) \in I$. We denote $[r]$ as $r + I$. If I is an ideal then R/I is a ring whose operations are inherited from those in R and are such that the surjective map $\pi : R \to R/I$ mapping $r \mapsto r + I$, is a ring morphism.

Proposition 2.1.1.2. *There is a one-to-one correspondence between the ideals of R/I and the ideals in R that contain I.*

Proof. See [6]. □

2.1.2 Modules over a Ring

Given an abelian group $(M, +)$, the set of its endomorphism $End(M)$ inherits a group operation, given by adding endomorphisms. This operation, together with the operation of composition give $End(M)$ the structure of a ring.

Definition 2.1.2.1. *Given a ring R, a R-module, also called,a module over R, is an abelian group M together with a ring homomorphism $R \to End(M)$.*

2.1.3 Primes and Maximal ideals

Let R be a ring. We introduce some standard definitions: An ideal I is said to be *prime* if for every pair $a, b \in R$ such that $(a.b) \in I$, then one of them has to be already in I. An ideal I is said to be *maximal* if for every other ideal $J \neq R$ such that $I \subset J$ we have $I = J$. We enumerate some properties:

Proposition 2.1.3.1. *Let R be a ring*

1. *Every maximal ideal of R is prime.*
2. *R is a field if and only if (0) is the only prime ideal.*

Some more terminology: the radical of an ideal I, wich we denote by $\sqrt{I}$, is by definition the set of elements $x \in R$ such that $x^n \in I$, for some $n \in \mathcal{N}$. The radical of the ideal (0) is called the nilradical of R.

We also refer to some functorial properties of ideals. If $f : R \to R'$ is a morphism of rings, from Prop. 2.1.1.1, it follows that, the preimage of an ideal in R' is an ideal in R. In particular, the preimage of a prime ideal in R' is a prime ideal in R.

2.1.4 Local Rings

A local ring is by definition a ring with only one maximal ideal. If R is local with maximal ideal m, we call the field R/m the residue field of R. A morphism of local rings $u : R' \to R$ is said to be local if the preimage of the unique maximal ideal m' in R is the maximal ideal m of R'. In such case, u induces a morphism between the residue fields $R'/m' \to R/m$ defined by mapping each class $r' + m'$ to $u(r) + m$.

2.1.5 Localization

1. Let R be a ring. A multiplicatively closed subset W of R is a subset containing the identity of R, not containing 0, and closed under multiplication. The localization of R with respect to W is a quotient of the product $R \times W$, under the relation $(r, w) \cong (r', w')$ iff $\exists s \in W$ such that $s.(r.w' - w'.r) = 0$. The set of such equivalence classes, denoted by R_W, has a well-defined ring structure inherited from R and

following from the fact that W is closed under multiplication. We write $\frac{r}{w}$ to denote each equivalence class $[(r,w)]$. This procedure turns all the elements of W to invertible elements in R_W. We have a canonical homomorphism $i_W^R : R \to R_W$ mapping each r to $\frac{r}{1}$, called the locatization homomorphism.

Some relevant localizations:

Proposition 2.1.5.1. *(a) Let P be an ideal of R. Then P is prime if and only if $W = R - P$ is a multiplicatively closed subset. We write R_P for the localization with respect to $R - P$ and call R_P the localization of R at P. In particular R_P is a local ring with a unique maximal ideal, P_P.*

(b) For $f \in R$, $f \neq 0$, the set of its powers $W = \{f^n\}_{n \in N}$ is a multiplicatively closed subset. We write R_f when localizing with respect to such a set.

Given M an R-module, we introduce a process of localizing M on a multiplicatively closed subset $W \subset R$, by defining an equivalence relation on the product $M \times W$: we identify (m,w) with (m',w') if and only if there is an element $s \in W$ such that $s.w'.m = s.w.m'$. This is an equivalence relation. We denote by $\frac{m}{w}$ the equivalence class of (m,w) and write M_W for this quotient set. By introducing the operations $\frac{m}{w} + \frac{m'}{w'} := \frac{w'.m + w.m'}{w.w'}$ and $\frac{r}{w}.\frac{m}{w'} := \frac{rm}{w.w'}$, M_W acquires the structure of a R and R_W modules.

We refer to some functorial properties of this process: If $f : M \to N$ is a morphism of R-modules and $W \subset R$ is a multiplicatively closed subset, there is a natural morphism of R_W-modules, $f_W : M_W \to N_W$, defined by $\frac{r}{w} \mapsto \frac{f(r)}{w}$. Using the notation $L_W^R(M) = M_W$ and $L_W^R(f) = f_W$. We conclude that L_W^R, the process of localizing with respect to W, defines a functor.

$$L_W^R : R - \mathbf{Mod} \longrightarrow R_W - \mathbf{Mod}$$

2. Let R be a ring and W a multiplicatively closed subset. Consider the correspondence

$$R' \longmapsto \{u \in Hom_{\mathbf{Rings}}(R, R') : \text{all elements in } u(W) \text{ are invertible in } R'\}$$

This correspondence is functorial. We notice that this functor is represented precisely by R_W, the localization, together with the canonial morphism i_W^R introduced above. Therefore every morphism of rings $u : R \to R'$ with $u(W)$ invertible in R' factors uniquely as $R \xrightarrow[i_W^R]{} R_W \dashrightarrow R'$.

3. (Dependence on the multiplicative subset) For a fixed ring R, consider W and T two multiplicatively closed subsets of R with $W \subseteq T$. In this case, there is a canonical morphism of rings $\rho_{T,W}^R : R_W \to R_T$ defined by mapping each element a/w in R_W to the same element a/w in R_T. This morphism satisfies $i_T^R = \rho_{T,W}^R \circ i_W^R$.

For M an R-module, there is a similar subset change morphism $\rho_{T,W}^M : M_W \to M_T$ and for $f : M \to N$ a morphism of R-modules, the diagram

$$\begin{array}{ccc} M_W & \xrightarrow[f_W]{} & N_S \\ {\scriptstyle \rho_{T,W}^M}\downarrow & & \downarrow{\scriptstyle \rho_{T,W}^N} \\ M_T & \xrightarrow[f_T]{} & N_T \end{array}$$

commutes. Therefore, the collection $\rho_{T,W}^M$ gives a natural transformation $\rho_{T,W}^R : L_W^R \to L_T^R$ between the localization functors.

4. We say that a multiplicatively closed subset $W \subseteq R$ is saturated if all elements $r \in R$ dividing an element $w \in W$ are already in W. (we say a divides b if there is some element $u \neq 1$ such that $b = ua$).

   If W is multiplicatively closed, we denote by W' the set of all divisors of W in R. W' is also multiplicatively closed and $W \subseteq W'$, inducing a natural transformation $\rho_{W',W} : L_W \to L'_W$. In this case, this is a natural isomorphism: For M an R-module, we have $\rho^M_{W',W} : M_W \to M_{W'}$ defined as in 3).

   - $\rho^M_{W',W}$ is injective: if $m/w = 0$ in $M_{W'}$, there is some $w' \in W'$ with $w'.m = 0$, with $u.w' \in W$ for some $u \in R$. Therefore, $u.w'.m = 0$ and so m/w is already 0 in M_W;
   - $\rho^M_{W',W}$ is surjective: given $m/w' \in M_{W'}$, there is some $u \in R$ such that $u.w'$ is in W and so $m/w' = \rho^M_{W',W}(um/uw')$.

   This allows us to consider only saturated multiplicatively closed subsets.

5. If W, T and U are multiplicatively closed subsets of R with $W \subseteq T \subseteq U$, we have $\rho_{U,W} = \rho_{U,T} \circ \rho_{T,W}$.

6. (Change of ring) Let R' and R be two rings and $u : R' \to R$ a morphism of rings. Let W' and W be multiplicatively closed subsets, in R', resp. R, with $u(W') \subseteq W$. Considering the composition $i^R_W \circ u : R' \to R \to R_W$, the image of W' is invertible in R_W and by 2) we conclude that this composition to factors uniquely as $R' \to R'_{W'} \to R_W$, where the last arrow maps $u_{W'} : a'/w' \to u(a')/u(w')$.

7. Following 3) and 6), having $W' \subseteq T' \subseteq R'$ and $W \subseteq T \subseteq RA$, respectively, multiplicatively closed subsets with $u(T') \subseteq T$ and $u(W') \subseteq W$, we naturally have a commutative diagram

$$\begin{array}{ccc} R'_{W'} & \longrightarrow & R_W \\ \downarrow & & \downarrow \\ R'_{T'} & \longrightarrow & R_T \end{array}$$

2.1.6 Extensions of Rings and Fields

1. Let R be a ring. We denote by $R[X]$ the ring of all polynomials with coefficients in R.

   Proposition 2.1.6.1. *If R is a field, $R[X]$ is a principal ideal domain.*

2. An *extension* of a ring R, is another ring S, such that R is a subring of S: R is contained in S, is closed with respect to all operations and the neutral elements of R are the neutral elements of S. Indeed, this is equivalent to have a monomorphism of rings $R \to S$. An *extension* of a field k is another field L, such that k is a subfield of L. In this case, since any non-null morphism of fields $k \to L$ is necessarily a monomorphism (because its kernel is an ideal of k, and since k is a field, its only proper ideal is (0)), an extension of fields is simply a morphism $k \to L$.

3. Notice that if $\phi : R \to S$ is an extension of rings, S has naturally the structure of an $R - Module$, given by $R \times S \xrightarrow{(\phi \times I_S)} S \times S \longrightarrow S$, where the last arrow is the product in S. Therefore, if R is a field S becomes a R-vector space. We write $[S : R]$ to denote the dimension of S as a R vector space and we say that the extension is finite if this dimension is finite.

4. Let $\phi : R \to S$ be a morphism of rings. ϕ induces a morphism $\bar{\phi}$ between the polynomial rings $R[X]$ and $S[X]$, defined by sending each polynomial $p(X) = r_n X^n + ... + r_0 \in R[X]$ to the polynomial $(\bar{\phi}(p))(X) = \phi(r_n)X^n + ... + \phi(r_0)$, which we denote as $\bar{p}(X)$. Using the properties of ϕ as a morphism

of rings we conclude that this correspondence is also a morphism of rings. If ϕ is an extension of rings, we immediately conclude that $\bar{\phi}$ defines an extension of rings $R[X] \subseteq S[X]$.

5. Let R be a ring and consider $R[X]$ the polynomial ring in one-variable. A root of a polynomial $f(X) \in R[X]$ in R is an element $u \in R$, such that $f(u) = 0$.

   Let $\phi : R \to S$ be a morphism of rings, $p(X) = r_n X^n + ... + r_0$ a polynomial in $R[X]$ and u a root of $p(X)$ in R. In this case, we prove that the element $\phi(u)$ in S is a root of the polynomial $\bar{p}(X)$ in $S[X]$: Since ϕ is a morphism of rings, we have $\phi(r_n)\phi(u)^n + ... + \phi(r_0) = \phi(r_n u^n + ... + r_0) = \phi(0) = 0$; If ϕ is an extension of rings, we easily conclude that $u \in R$ is a root of $p(X)$ in R if and only if $\phi(u)$ is a root of $\bar{p}(X)$ in S.

   It is possible for a polynomial $p(X)$ in $R[X]$ to have no roots in R.

   Definition 2.1.6.2. *We say that a field R is algebraically closed if every polynomial in $R[X]$ as one root in R.*

   Given an extension of rings $R \to S$, we have $R[X] \subseteq S[X]$ and so, we may search for roots of $p(X)$ in S, in other words, elements $u \in S$ such that $p(u) = 0$ for $p(X) \in R[X]$.

   Definition 2.1.6.3. *An extension of rings $R \to S$ is said to be integral if every element $s \in S$ is the root of a monic polynomial in $R[X]$.*

6. Each element $a \in R$ naturally defines an evaluation morphism $Ev_a : R[X] \to R$ defined by mapping each polynomial f to the value $f(a)$ in R. If we have an extension of rings $R \hookrightarrow S$, Ev_a is defined not only for elements $a \in R$ but for all elements in $a \in S$, since $R[X] \subseteq S[X]$. We write $Ev_a : R[X] \to S$ to denote the evaluation map induced by an extension of rings $R \hookrightarrow S$. Moreover, since it is a ring homomorphism, its image is always a subring of S. We write $R[a]$ to denote the image of R[X] under Ev_a, in S.

2.1.7 Noetherian Rings

Definition 2.1.7.1. *We say a ring R is* Noetherian *if every ascending chain of prime ideals*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq I_3 ....$$

stops. In other words, if there is some natural number n such that $I_k = I_n$ for all $k \geqslant n$

As an example, we find all fields to be noethering rings, since the only prime ideal is (0). There is another possible characterization of Noetherian rings:

Proposition 2.1.7.2. *Let R be a ring. The following conditions are equivalent:*

- *R is Noetherian;*
- *Every ideal I of R is finitely generated;*
- *Every non-empty set of ideals in R, partially ordered by inclusion, has a maximal element.*

Proof. See [6]. □

And the following important result, known as *Hilbert Basis Theorem*:

Proposition 2.1.7.3. *If R is a Noetherian ring then $R[X]$ is also a Noetherian ring.*

Proof. See [6]. □

By induction, we find that if R is Noetherian then all polynomial rings $R[X_1, ..., X_n]$ are also Noetherian. As a corollary we find that for a field k, every prime ideal of $k[X_1, ..., X_n]$ is finitely generated

2.2 Sheaves

Again, this section is a brief presentation of the main ideias and notions regarding sheaf theory and the theory of schemmes. Here we recommend the reader to follow all the exposition in [3]- chapter 0 section 3.

2.2.1 Presheaves

1. For X a topological space, we define the category **Open(X)**: Each open set U is seen as an object and we define $Hom_{\mathbf{Open(X)}}(U, V)$ as a one-element set, the continuous inclusion $U \subseteq V$ whenever U is a subset of V. Otherwise we define this set to be empty. We introduce presheaves on X:

   Definition 2.2.1.1. *A presheaf on X with values in a category **C** is a functor*

   $$\mathcal{F} : \boldsymbol{Open(X)}^{\circ} \longrightarrow \boldsymbol{C}$$

   In other words, a presheaf is simply a collection $F(U)$ of objects in **C** indexed by the open sets of X and, whenever $U \subseteq V$ a "restriction" morphism $\rho_U^V := F(U \subseteq V) : F(V) \to F(U)$ such that $\rho_U^U = I_{F(U)}$ and $\rho_U^W \circ \rho_W^V = \rho_U^V$ whenever $U \subseteq W \subseteq V$.

2. We introduce a morphism of presheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ over X with values in **C** as a natural transformation $\mathcal{F}_1 \to \mathcal{F}_2$. With such morphisms we have a category of presheaves over X, which we denote by **Presheaves(X,C)**.

2.2.2 Sheaves

Let $\mathcal{F}$ be a presheaf over X with values in **C**. Let U_i and U_j be two open sets in X with a non-empty intersection, $U_i \cap U_j \neq \emptyset$. Each set U_i and U_j indexes some object, respectively, $F(U_i)$ and $F(U_j)$ in **C**. Since the union and finite intersection of open sets is open, $U_i \cap U_j$ and $U_i \cup U_j$ both index some object $F(U_i \cap U_j)$ and $F(U_i \cup U_j)$. Moreover, we have two chains of inclusions $U_i \cap U_i \subseteq U_i \subseteq U_i \cup U_j$ and $U_i \cap U_j \subseteq U_i \subseteq U_i \cup U_j$ inducing a composition of morphisms in **C**, $\rho^{U_i}_{U_i \cap U_j} \circ \rho^{U_i \cup U_j}_{U_i}$ and $\rho^{U_j}_{U_i \cap U_j} \circ \rho^{U_i \cup U_j}_{U_j}$. We say that $\mathcal{F}$ has a sheaf property with respect to (U_i, U_j) if $F(U_i \cup U_j)$ together with the restriction morphisms $\rho^{U_i \cup U_j}_{U_i} : F(U_i \cup U_j) \to F(U_i)$ and $\rho^{U_j \cup U_j}_{U_i} : F(U_i \cup U_j) \to F(U_j)$ is a fibre product in **C** of the diagram

$$\begin{array}{ccc} & & F(U_j) \\ & & \downarrow \rho^{U_j}_{U_i \cap U_j} \\ F(U_i) & \xrightarrow{\rho^{U_i}_{U_i \cap U_j}} & F(U_i \cap U_j) \end{array}$$

We introduce sheaves:

Definition 2.2.2.1. *Let $\mathbf{C}$ be a category, X a topological space and $\mathcal{F}$ a $\mathbf{C}$-valued presheaf over X. We say $\mathcal{F}$ is a sheaf if for every open cover $\{U_i\}_{i\in I}$ of an open set $U \subseteq X$, the object $F(U)$ together with the family of morphisms $\rho^U_{U_i} : F(U) \to F(U_i)$, is the limit all diagrams of the form above.*

A morphism of sheaves is a morphism of their respective presheaves. We denote by **Sheaves(X,C)** the subcategory of presheaves over X with values in a category $\mathbf{C}$, that are sheaves.

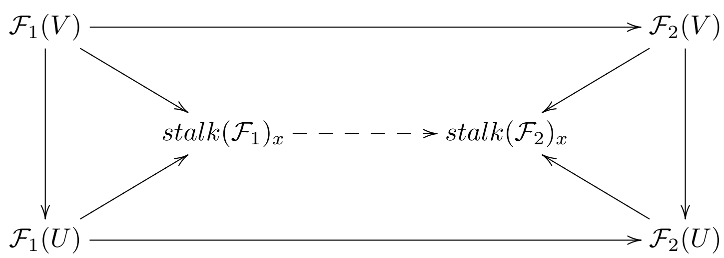
2.2.3 The stalk of a sheaf

Let $\mathcal{F}$ be a $\mathbf{C}$-valued presheaf over X. Given a point $x \in X$, the open sets $U \subset X$ containing x, form a subcategory of **Open(X)**. Consider $\mathcal{F}|_x$, the restriction of $\mathcal{F}$ to this subcategory.

Definition 2.2.3.1. *We define the stalk of $\mathcal{F}$ at point $x \in X$ as the colimit of $\mathcal{F}|_x$.*

This colimit is given by an object in $\mathbf{C}$, which we denote as $stalk(\mathcal{F})_x$, and a family of morphisms $\rho_U : \mathcal{F}(U) \to stalk(\mathcal{F})_x$, one for each open set $U \subseteq X$ containing x and commuting with every restriction morphism $\rho^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ whenever $U \subseteq V$ and $x \in U$.

If we consider two presheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ over X with values in $\mathbf{C}$, for any morphism $u : \mathcal{F}_1 \to \mathcal{F}_2$ we have a commutative diagram for every pair (U, V) with $U \subseteq V$ and $x \in U$. This induces a unique canonical morphism u_x between the two stalks at x:



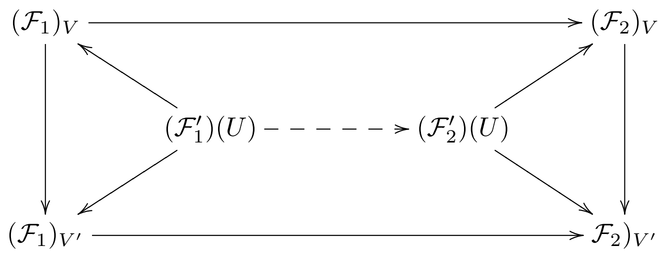
2.2.4 Construction of Presheaves

1. Let X be a topological space and suppose $\mathcal{B}$ is a basis for the topology on X. Let $\mathbf{C}$ be a category where all limits exist. A *presheaf defined over* $\mathcal{B}$ is a collection of objects $\mathcal{F}_V$ in $\mathbf{C}$ indexed by the open sets $V \in \mathcal{B}$ and a family of morphisms in $\mathbf{C}$, $\rho^V_U : \mathcal{F}_V \to \mathcal{F}_U$ everytime $U \subseteq V$, for $U, V \in \mathcal{B}$, satisfying $\rho^V_V = id$ and $\rho^W_U \circ \rho^V_W = \rho^V_U$, everytime $U \subseteq V \subseteq W$ in $\mathcal{B}$.

   From this collection we construct a presheaf $\mathcal{F}'$ over X with values in $\mathbf{C}$, defining for each open set U in X, $\mathcal{F}'(U)$, the limit of the diagram in $\mathbf{C}$ formed by all $\mathcal{F}_V$ with $V \in \mathcal{B}$ and $V \subseteq U$ and their respective transition morphism $\rho^V_{V'}$. If $U \subseteq U'$ in X, there is a canonical morphism $\mathcal{F}'(U') \to \mathcal{F}'(U)$ since every morphism $\mathcal{F}'(U') \to \mathcal{F}_V$ with $V \in \mathcal{B}$ and $V \subseteq U$, factors uniquely through $\mathcal{F}'(U)$.

2. If $\mathbf{C}$ has all colimits, $\mathcal{F}$ is a presheaf over $\mathcal{B}$ and $\mathcal{F}'$ is the associated presheaf over X as above, the stalk of $\mathcal{F}'$ at a point $x \in X$ is uniquely determined by $\mathcal{F}$ as the colimit of the diagram in $\mathbf{C}$ formed by all $\mathcal{F}_V$ with $V \in \mathcal{B}$ containing x, together with their transition morphisms.

3. We introduce a morphism of presheaves $\mathcal{F}_1 \to \mathcal{F}_2$ over a base $\mathcal{B}$, as a collection of morphisms in $\mathbf{C}$, $(u_V : (\mathcal{F}_1)_V \to (\mathcal{F}_2)_V)_{V\in\mathcal{B}}$, commuting with all transition morphisms defining $\mathcal{F}_1$ and $\mathcal{F}_2$. Such collection induces a unique morphism of the associated presheaves $\mathcal{F}_1' \to \mathcal{F}_2'$ over X, obtained using the properties of $(\mathcal{F}_1')(U)$ and $(\mathcal{F}_2')(U)$ as limits of diagrams:



everytime $V' \subseteq V$, for V and V' in $\mathcal{B}$.

2.2.5 Direct image of a presheaf

1. Let X and Y be topological spaces and $\psi : X \to Y$ a continuous map. Let $\mathcal{F}$ be a presheaf over X with values on a category $\mathbf{C}$. We introduce a presheaf $\psi_*\mathcal{F}$ on Y as follows: For every open set $U \subseteq Y$, define $(\psi_*\mathcal{F})(U) := \mathcal{F}(\psi^{-1}(U))$. Since ψ is continuous and U is open in Y, $\psi^{-1}(U)$ is also open in X. If $U \subseteq V$ in Y, we have $\psi^{-1}(U) \subseteq \psi^{-1}(V)$ in X and so, since $\mathcal{F}$ is a presheaf, there is a restriction morphism $(\psi_*\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V) \to (\psi_*\mathcal{F})(U) := \mathcal{F}(\psi^{-1}(U)$. We immediately conclude that $\psi_*\mathcal{F}$ is a presheaf over Y, with values in $\mathbf{C}$. We call it the direct image (or push-forward) of $\mathcal{F}$ by ψ.

2. Given $\mathcal{F}_1$ and $\mathcal{F}_2$ two presheaves over X with values in $\mathbf{C}$, every morphism $u : \mathcal{F}_1 \to \mathcal{F}_2$ induces a morphism $\psi_*(u) : \psi_*\mathcal{F}_1 \to \psi_*\mathcal{F}_2$ over Y, defined by $\psi_*(u)_U = u_{\psi^{-1}(U)}$, for every open set $U \subseteq Y$. If $v : \mathcal{F}_2 \to \mathcal{F}_3$ is another morphism of presheaves over X composable with u, we easily find that $\psi_*(v \circ u) = \psi_*(v) \circ \psi_*(u)$.

3. We conclude that ψ_* acts as a covariant functor from the category of presheaves over X with values in $\mathbf{C}$ to the category of presheaves over Y with values in $\mathbf{C}$.

4. The stalk of $\psi_*\mathcal{F}$ at a point $y = \psi(x)$ is defined by taking the colimit of $\psi_*\mathcal{F}(V)$ over all open sets $V \subseteq Y$, containing y.

$$stalk(\psi_*\mathcal{F})_{\psi(x)} = \varinjlim_{\psi(x)\in V\subseteq Y}(\psi_*\mathcal{F})(V) = \varinjlim_{x\in\psi^{-1}(V)\subseteq X}\mathcal{F}(\psi^{-1}(V))$$

and we immediately conclude the existence of a canonical morphism

$$\psi_x : stalk(\psi_*\mathcal{F})_{\psi(x)} = \varinjlim_{x\in\psi^{-1}(V)}\mathcal{F}(\psi^{-1}(V)) \to \varinjlim_{x\in U}(U) = stalk(\mathcal{F})_x$$

2.2.6 Presheaved Spaces

1. Under the conditions considered in 2.2.5, if $\mathcal{F}$ is a presheaf over X with values in $\mathbf{C}$, we call a morphism $u : \mathcal{G} \to \psi_*(\mathcal{F})$ of presheaves over Y a ψ-morphism from $\mathcal{G}$ to $\mathcal{F}$ and denote it simply by $\mathcal{G} \to \mathcal{F}$.

   Considering pairs $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a presheaf over X (with values in $\mathbf{C}$), we introduce a morphism between such objects, $\Psi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ as a pair $\Psi = (\psi, \theta)$ where

$\psi : X \to Y$ is a continuous map and $\theta : \mathcal{G} \to \psi_*(\mathcal{F})$ is a morphism of presheaves over Y, a ψ-morphism. If $\Psi' = (\psi', \theta') : (Y, \mathcal{G}) \to (Z, \mathcal{H})$ is another morphism, the composition $\Psi' \circ \Psi$ is naturally defined by considering the pair $(\psi' \circ \psi, \psi'_*(\theta) \circ \theta')$. Thus, we can speak about the category of spaces with a **C**-valued presheaf, denoted by **PresheavedSpaces(C)**.

2. If $(X, \mathcal{F})$ is a space with a **C**-valued presheaf and $U \subseteq X$ is an open set, there is an induced presheaved space obtained by restricting $\mathcal{F}$ to $\mathcal{F}|U$: the pair $(U, \mathcal{F}|U)$ is again an object in **PresheavedSpaces(C)**. The canonical inclusion $U \hookrightarrow X$ induces a morphism $(U, \mathcal{F}|U) \to (X, \mathcal{F})$

3. For a morphism $(\psi, \theta) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ between presheaved spaces, we denote by $\theta^\sharp_x$ the composition

$$stalk(G)_{\psi(x)} \to stalk(\psi_*\mathcal{F})_{\psi(x)} \to stalk(\mathcal{F})_x$$

where the last arrow comes from 2.2.5-5.

2.3 Schemes

Here we present the definition of a scheme. Again, the reader is recommended to follow [3].

2.3.1 Ringed Spaces

Following 2.2.6, we introduce ringed spaces as objects in **PresheavedSpaces(Rings)**. For a ringed space $(X, \mathcal{O}_X)$ we call X the base space and $\mathcal{O}_X$ the structure sheaf. Also, for a point $x \in X$ we denote by $(\mathcal{O}_X)_x$ the stalk of the structure sheaf at x. Again following 2.2.6, a morphism of ringed spaces is a pair $\Psi = (\psi, \theta) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ where $\psi : X \to Y$ is a continuous map and $\theta : \mathcal{G} \to \psi_*(\mathcal{F})$ is a ψ-morphism between presheaves of rings.

We say that a ringed space $(X, \mathcal{O}_X)$ is locally ringed if the stalk of $\mathcal{O}_X$ at each point $x \in X$ is a local ring.

2.3.2 Affine Schemes

1. (Prime Spectrum) Let R be a ring. We introduce the prime spectrum of R, denoted $Spec(R)$, as the set of all prime ideals of R. For a point $x \in Spec(R)$ we use the notation i_x to describe x as a subset of R.

   We introduce some terminology: R_x denotes the localization of R at $R - i_x$; $m_x = \frac{i_x}{1}$ is the only maximal ideal of R_x; $k(x)$ is the field R_x/m_x; for each $r \in R$ we use the notation $r(x)$ to describe the equivalence class of r in the quotient ring R/i_x. Also, for $Y \subseteq Spec(R)$ we introduce the set $i(Y)$ in R, the set of all $r \in R$ such that $r(y) = 0$, for all $y \in Y$). In particular if $Y = \{x\}$, we have $i_x = i(\{x\})$.

   For each subset $E \subseteq R$ we introduce the sets $V(E) =$ set of prime ideals of R containing E .

   Proposition 2.3.2.1. *These sets have the following properties:*

   (a) $V(\{0\}) = Spec(R)$ *and* $V(R) = \emptyset$*;*

   (b) if $E \subseteq E'$ *then* $V(E') \subseteq V(E)$*;*

(c) *for all families* $\{E_\lambda\}_{\lambda\in I}$ *we have* $V(\cup_\lambda E_\lambda) = V(\sum_\lambda E_\lambda) = \cap_\lambda V(E_\lambda)$;

(d) $V(E.E') = V(E) \cup V(E')$;

(e) $V(E) = V(\sqrt{E})$, *where we define* $\sqrt{E} = \cap_{I\in V(E)} I$, *the radical of* E.

Proof. For a detailed proof see [3]. □

This implies that the collection of sets $V(E)$ satisfies all the properties of closed sets on a topological space. We introduce the Zariski spectral topology on the *Spec*(R) as the topology generated by these closed subsets. From now on, whenever we speak about the spectrum of a ring we assume this topology. Let us present some properties:

Proposition 2.3.2.2. (a) *For* $V(I)$ *some subset of Spec*(R) *with* I *an ideal in* R, *we have* $i(V(I)) = \sqrt{I}$;

(b) *There is a bijection between the set of closed subsets of Spec*(R) *and the set of radical ideals in* R *(ideals* I *with* $\sqrt{I} = I$*);*

(c) *The open sets of the form* $U_f := Spec(R) - V(\{f\})$ *with* $f \in R$, *are identified with the spectrum of* R_f

(d) *The open sets* U_f *form a basis for the Zariski topology;*

(e) *Spec*(R) *is compact*[1], *in the sense that any open cover admits a finite subcovering.*

Proof. See [3]-I □

2. Given a morphism of rings $\phi : R' \to R$, there is a natural map $\tilde{\phi} : Spec(R) \to Spec(R')$ defined by sending each prime ideal I in R to its preimage under ϕ, $\phi^{-1}(I)$, which, as we mentioned before, is a prime ideal in R'.

Proposition 2.3.2.3. $\tilde{\phi}$ *is continuous, with respect to the Zariski topology.*

Proof. This follows immediately from the fact that $\tilde{\phi}^{-1}(V(E')) = V(\phi(E))$, for all $E' \subseteq R'$. In particular this implies $\tilde{\phi}^{-1}(U_{f'}) = U_f$ and obviously $\tilde{\phi}$ is continuous. □

3. (Presheaf over Spec(R) associated to an R-Module)
Let R be a ring and M an R-module. We construct a presheaf over *Spec*(R) that encodes and recovers M: From 2.3.2.2 we known that the open sets U_f, with $f \in R$, form a basis for the Zariski topology in *Spec*(R). Denoting by $\mathcal{B}$ this basis and following 2.2.4, we introduce a presheaf of modules over $\mathcal{B}$ by assigning to each open set U_f the R_f-Module M_f, the localization of M with repect to the multiplicatively closed subset W_f of all natural powers of f. The following properties ensure sufficient conditions for this to be a well-defined presheaf over $\mathcal{B}$:

Lemma 2.3.2.4. (a) *if* $U_f = U_g$ *then* $M_f = M_g$;

(b) *if* $U_f \subseteq U_g$ *then* $W_g \subseteq W_f$ *and from 2.1.5 3), there is a natural morphism* $\rho^g_f : M_g \to M_f$. *Together with 2.1.5 4), we conclude* $\rho^g_g = Id$ *and* $\rho^h_f = \rho^g_f \circ \rho^h_g$, *everytime* $U_f \subseteq U_g \subseteq U_h$.

Proof. See [3]-I □

We have

[1]Meaning that every open cover has a finite subcover

Proposition 2.3.2.5. *(a) $\tilde{M}$ is a sheaf;*

(b) the stalk $\tilde{M}_x$ at a point $x \in Spec(R)$ is the localization of M at the prime ideal x, M_x.

(c) $\tilde{M}(Spec(R))$ is precisely the module M.

Proof. See [3]-I ☐

4. Applying the construction in the previous item to the ring R itself, seen as an R-Module, we obtain a sheaf of rings $\tilde{R}$ over $Spec(R)$, with $\tilde{R}(U_f) = R_f$ and stalk R_x, for any $x \in X$. Also, we recover the ring R as $\tilde{R}(Spec(R))$. We call $\tilde{R}$ the structure sheaf of R.

   From this, every ring R induces a ringed space $(Spec(R), \tilde{R})$. We introduce affine schemes:

   Definition 2.3.2.6. *An affine scheme is a ringed space $(X, \mathcal{O}_X)$ isomorphic (as a ringed space) to one of the form $(Spec(R), \tilde{R})$ constructed as above for some ring R.*

5. Let $u : R' \to R$ be a morphism of rings and consider $(X', \mathcal{O}_{X'})$ and $(X, \mathcal{O}_X)$ their respective affine schemes with $X' = Spec(R')$ and $X = Spec(R)$. As seen above in 2., u induces a continuous morphism $\tilde{u} : X \to X'$. In fact, this extends to a morphism of ringed spaces $(\tilde{u}, \theta) : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$, where θ is a $\tilde{u}$-morphism of presheaves over X':

   Writing $f = u(f')$, we start by defining θ, as in 2.2.4 3), over the basis of open sets $U_{f'}$ for the topology in X'. First we notice the identification of $\mathcal{O}_{X'}(U_{f'})$ with $R'_{f'}$ and $\tilde{u}_*(\mathcal{O}_X)(U_{f'}) = \mathcal{O}_X(\tilde{u}^{-1}(U_{f'})) = \mathcal{O}_X(U_f)$ with R_f. Applying 2.1.5.6 to $W' = \{(f')^n\}_{n \geqslant 0} \subseteq R'$ and $W = \{f^n\} \subseteq R$, we find a canonical morphism of rings

$$\theta_{U_{f'}} := \tilde{u}_{W'} : \mathcal{O}_{X'}(U_{f'}) \to \mathcal{O}_X(\tilde{u}^{-1}(U_f))$$

   From 2.1.5.7 we conclude that this collection of maps defines a morphism of presheaves over the base $\mathcal{B}$ and following 2.2.4 3) we find the desired morphism u' of presheaves over X'.

   Proposition 2.3.2.7. *([3] - 1.6.1) With the conditions above, the map $\theta_x^\sharp : stalk(\mathcal{O}_{X'})_{\psi(x)} = R'_{\psi(x)} \to stalk(\mathcal{F})_x = R_x$ in 2.2.6-3 is precisely the localization map $u_{\psi(x)}$ obtained in 2.1.5-6, with $W' = R' - i_{\psi(x)}$ and $W = R - i_x$.*

6. From 4), 5) and 6) we conclude that the correspondence

$$R \longmapsto (Spec(R), \tilde{R})$$

   acts as a functor, from the category of rings to the category of ringed spaces. We call it the *Spec* functor.

7. Let $\Psi = (\psi, \tilde{\psi}) : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a morphism of ringed spaces between affine schemes, with $X = Spec(R)$ and $X' = Spec(R')$.

   Lemma 2.3.2.8. *A morphism $\Psi = (\psi, \tilde{\psi})$ as above is induced by a unique ring homomorphism $u : R' \to R$, constructed as in 5), if and only if every stalk map $\tilde{\psi}_x^\sharp : stalk(\mathcal{O}_{X'})_{\psi(x)} = R'_{\psi(x)} \to stalk(\psi_*(\mathcal{O}_X))_{\psi(x)} = R_x$ is a local homomorphism of rings.*

   Proof. See [3] Prop. 1.7.3 ☐

We proceed now by defining a morphism of affine schemes $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ as a morphism of ringed spaces satisfying the condition in 2.3.2.8. The composition of two such morphism is again one of the same type, since the composition of local homomorphisms is a local homomorphism. Affine schemes, with such morphisms, form a category here denoted as **AScheme**.

In this case, it follows immediately as a corollary

Corollary 2.3.2.9. *The Spec functor induces an equivalence between the category of rings and the category of affine schemes.*

Proof. The definition above for a morphism of affine schemes, following lemma 2.3.2.8, implies Spec to be fully faithfull. Essential surjectiveness is immediate. □

2.3.3 Schemes and Morphisms of Schemes

1. A Scheme is a generalization of an affine scheme. Given a ringed space $(X, \mathcal{O}_X)$, we say that an open set $U \subseteq X$ is affine if the restricted ringed space $(U, \mathcal{O}_X|U)$ is an affine scheme.

   Definition 2.3.3.1. *A ringed space $(X, \mathcal{O}_X)$ is said to be a Scheme if every point $x \in X$ has an affine neighbourhood.*

   We identify some properties

   Proposition 2.3.3.2. *Let $(X, \mathcal{O}_X)$ be a scheme. Then:*

   (a) ([3]- 2.1.3) -The affine open sets form a basis for the topology of X;

   (b) ([3]- 2.1.7) -For every open set $U \subseteq X$, $(U, \mathcal{O}_X|U)$ is also a scheme.

2. A morphism between two schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces,i.e. a pair $\Psi = (\psi, \tilde{\psi})$ with $\psi : X \to Y$ a continuous map and $\tilde{\psi} : \mathcal{O}_Y \to \psi_*(\mathcal{O}_X)$ a ψ-morphism of presheaves over Y, with one extra condition: for all $x \in X$, the map $\tilde{\psi}^\sharp_x : stalk(\mathcal{O}_Y)_{\psi(x)} \to stalk(\mathcal{O}_X)_x$ (2.2.6-3) has to be a local homomorphism, in order for Ψ to be, locally, a morphism of affine schemes as defined 2.3.2 7).

3. Since a scheme $(X, \mathcal{O}_X)$ is locally isomorphic to an affine scheme $(Spec(A), \tilde{A})$, the stalk of $\mathcal{O}_X$ at a point $x \in Spec(A) \subseteq X$ is locally isomorphic to the localization of A at the prime ideal corresponding to x under the mentioned isomorphism. We denote its maximal ideal by m_x and use the notation $k(x)$ for the residue field $stalk(\mathcal{O}_X)_x/m_x$.

4. As defined in 2. and following 2.1.4 a morphism of schemes $\Psi = (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces a morphism between the residue fields $k(\psi(x)) \to k(x)$, for every $x \in X$. Such morphism is always a monomorphism, allowing us to understand $k(x)$ as a field extension of $k(\psi(x))$.

5. From 2.2.6 and the fact that the composition of local homomorphisms is again a local homomorphism, we obtain that the composition of two morphisms of schemes is again a morphism of schemes. Schemes with these morphisms, form a category, denoted **Schemes**.

6. We introduce some terminology: Let X be a scheme:

   (a) X is called locally noetherian if there is an open cover of X by affine schemes such that each structure ring is noetherien;

(b) We say that X is connected if the underlying space is connected;

(c) X is called integral if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain;

(d) X is said to be reduced if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotente elements;

(e) A topological space Y is said to be irreducible if it is non-empty and it can not be written as an union of two nonempty closed subspaces; In particular, if $Y = \overline{\{x\}}$, then Y is irreducible. In this case, x is called a *generic point* of Y. Considering the Zariski topology, every affine scheme whose struture ring is a domian has a generic point corresponding to the prime ideal (0) and therefore is irreducible.

Theorem 2.3.3.3. *A scheme X is integral if and only if it is reduced and its underlying topological space is irreducible.*

Proof. See [3]- I.2.1.7 □

2.3.4 Points of a Scheme with values on a Scheme

We address the following problem: given a category $\mathbf{C}$, what is a "point" of an object X in $\mathbf{C}$? To find a usefull definition for a point of an abstract object X, we may look at the example given by the category of topological spaces: If X is a topological space, the points of X are easily identified with morphisms

$$\{*\} \to X$$

where $\{*\}$ is a one-point topological space. Grothendieck proposed a generalization of this fact by defining geometric points:

Definition 2.3.4.1. *Let $\mathbf{C}$ be a category and $1_{\mathbf{C}}$ a terminal object. A geometric point of an object X in $\mathbf{C}$ is a morphism*

$$1_{\mathbf{C}} \to X$$

Definition 2.3.4.2. *Let X be a scheme. A* point of X with values on a scheme T *is a morphism of schemes $s : T \to X$. We write $X(T)$ to denote the set of points in X with values in T. If $T = (Spec(R), \tilde{R})$ for some ring R, we say that $s : T \to X$ is a point in X with values in the ring R. In this case we simply write $X(R)$ to denote the set of R-valued points in X.*

Let Ω be a field and consider the associated affine scheme $(Spec(\Omega), \tilde{\Omega})$. In this case, $Spec(\Omega)$ is a one-element set, containing the unique prime ideal $x = (0)$ of Ω and we have $\tilde{\Omega}_x = \Omega_{(0)} \cong \Omega$. Therefore, given a point s of X with values in $Spec(\Omega)$

$$s : (Spec(\Omega), \tilde{\Omega}) \to (X, \mathcal{O}_X)$$

we have a morphism of presheaves over X, $\mathcal{O}_X \to s_*\tilde{\Omega}$, extending to a local homomorphism $stalk(\mathcal{O}_X)_{s(0)} \to s_*(\tilde{\Omega})_{s(0)} \cong \Omega$, inducing an extension of the residual field $k(s(0)) \to \Omega$.

Example 2.3.4.3. *If k is a field, a point of $(Spec(k), \tilde{k})$ with values on a field Ω, $(Spec(\Omega), \tilde{\Omega}) \to (Spec(k), \tilde{k})$, following 2.3.2.9 is equivalent to an extension of fields $k \to \Omega$.*

We introduce geometric points of a scheme:

Definition 2.3.4.4. *A geometric point of a scheme X is a point of X with values on an algebraically closed field.*

It follows that to give a geometric point of $(Spec(k), \tilde{k})$ it is equivalent to give an algebraically closed extension of k.

2.3.5 Fibre Product of Schemes

1. Let S be a scheme. Following 1.3 we introduce the category **Schemes**$/S$ of schemes over S, also called S-schemes, of all morphisms $X \to S$, for X another scheme. We say $X \to S$ is a scheme over S. When S is an affine scheme with structure ring R we say $X \to S$ is an R-scheme, or X is defined over R.

   As in 2, we refer to the product of two S-schemes $X \to S$ and $Y \to S$ as their fiber-product in **Schemes**, given by a scheme $X \times_S Y$ and two canonical morphisms $p_X : X \times_S Y \to X$ and $p_y : X \times_S Y \to Y$.

   Proposition 2.3.5.1. *([3]-3.2.2) Let $X \to S$ and $Y \to S$ be two S-schemes with X, Y and S all affine schemes with strucure rings, respectively, B, C and A. The affine scheme $(Spec(B \otimes_A C), \widetilde{B \otimes_A C})$ together with $p_Y : (Spec(B \otimes_A C), \widetilde{B \otimes_A C}) \to Y$ and $p_X : (Spec(B \otimes_A C), \widetilde{B \otimes_A C}) \to X$ respectively induced (see* 2.3.2.9*) by the canonical morphisms $B \to B \otimes_A C$ and $C \to B \otimes_A C$, is a fiber product of $X \to S$ and $Y \to S$.*

   we then have

   Theorem 2.3.5.2. *([3] - Chap. I 3.2) The fiber-product of any two schemes $Y \to S$ and $Z \to S$ over a scheme S, exists.*

2. Let $X \to S$ be a scheme over S and $p \in S$ a point on the underlying topological space of S. There is an inclusion morphism $Spec(k(p)) \to X$, where $k(p)$ is the residual field of the stalk at p: If $U = Spec(A)$ is an affine open subset of X containing p, then p is identified with a prime ideal of A and we have a morphism $A \to A_p$ that we may compose with the natural projection $A_p \to A_p/p_p \to k(p)$, corresponding to a morphism $Spec(k(p)) \to U \to X$.

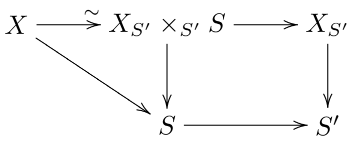
   Definition 2.3.5.3. *Given a morphism $\phi : Y \to X$ and a point p of X, the fibre of ϕ at p is the scheme $Y_p = Y \times_X Spec(k(p))$*

3. Let $\phi : S \to S'$ be a morphism of schemes. Following 2, this morphism induces a pullback functor

   $$\phi^* : \mathbf{Schemes}/S' \longrightarrow \mathbf{Schemes}/S$$

   assigning to each scheme X over S', the fibre product $X \times_{S'} S \to S$ over S, where this arrow is the canonical morphism given by the fiber product.

   Definition 2.3.5.4. *We say that a scheme X over S is defined over S' (or that X has a model over S') if X is isomorphic (as a scheme over S) to some scheme over S in the image of ϕ^*, $X_{S'} \times_{S'} S$, where $X_{S'}$ is some scheme over S'.*



Also, we say that a morphism of schemes $f : X \to Y$ over S is defined over S' if both X and Y are defined over S', respectively, $X \cong X_{S'} \times_{S'} S$ and $Y \cong Y_{S'} \times_{S'} S$, where both schemes $X_{S'}$ and $Y_{S'}$ are defined over S' and there is a morphism $f_{S'} : X_{S'} \to Y_{S'}$ over S' such that the following diagram commutes

$$\begin{array}{ccc} X_{S'} \times_{S'} S & \xrightarrow[\phi^*(f_{S'})]{} & Y_{S'} \times_{S'} S \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Chapter 3

Galois Theory of Fields

The first part of this chapter is a brief presentation of the main classical results of the Galois Theory of Fields for finite and non-finite extensions. The second part presents a reformulation of this, due to Grothendieck.

3.1 Category of Algebraic Extensions

3.1.1 Algebraic and Transcendental Field Extensions

Following 2.1.6-2, an extension of a field k is given by another field L, such that k is a subfield of L, or equivalentely, by a morphism of fields $k \to L$. If M is extension of k and L extends M, then L is also an extension of k and we write $L|M|k$. In this case we say that $M|k$ is a subextension of $L|k$.

An extension $L|k$ is said to be algebraic if all the elements of L are algebraic over k, i.e. every element of L is a root of some polynomial with coefficients in k, ($\in k[X]$). If there exists an element in L not algebraic over k, we use the expression "transcendental" to denote such an element and also, the whole extension. For an algebraic element $u \in L$ we may define the evaluation map $k[X] \to L$ sending each polynomial $f(X)$ to $f(u)$. Since k is a field, $k[X]$ is a principal ideal domain and the kernel of this map is of the form $(g(X))$ for a unique polynomial $g(X)$. We call it the minimal polynomial of $u \in L$ over k. Two elements are said to be conjugated if they have the same minimal polynomial.

We introduce another possible characterization: As we saw in 2.1.6-2, an extension $k \to L$ endows L with the structure of a k-vector space. We say the extension is finite if the dimension $[L : k]$ is finite.

Proposition 3.1.1.1. *Let $M|k$ and $L|M$ be field extensions.*

- *$L|k$ is algebraic if and only if $M|k$ and $L|M$ are both algebraic;*
- *$L|k$ is finite if and only if $M|k$ and $L|M$ are both finite. In this case, $[L : k] = [L : M][M : k]$.*

and also:

Proposition 3.1.1.2. *Let $L|k$ be an extension of fields. If it is finite then it is algebraic.*

3.1.2 Constructions and Examples of Fields Extensions

1. **I-Extensions generated by subsets**
   Given an extension $L|k$ and and a subset $S \subseteq L$, we denote by $k(S)$, the smallest subfield of L containing both S and k. The particular case when S is a set with a single element $u \in L$ we use the notation $k(u) = k(\{u\})$, and call it a simple extension. Some properties

   Proposition 3.1.2.1. *Let $k(u)|k$ be a simple extension. Then*

   - *The extension is algebraic if and only if $u \in L$ is algebraic and in this case $k(u) \cong k[X]/kernel(Ev_u)$, where $Ev_u : k[X] \to L$ is the evaluation map on u. Moreover its dimension is equal to the degree n of the minimal polynomial at u and the set $\{1, u, ...u^{n-1}\}$ is a basis as k-vector space.*[1]
   - *The extension is transcendental if and only if u is transcendental. In this case, $k[u] \cong k[X]$.*

2. **II-Splitting fields**
   Given a field k and a polynomial $f(X) \in k[X]$ of degree n, we may be interested on finding roots of f in k, which may not exist. However, we may search for solutions in fields L extending k. We introduce the notion of a *splitting field* for f as an extension $L|k$ such that $f(X)$, as a polynomial in $L[X]$, can be factored in a product of linear polynomials, $f(X) = \Pi_{i=1}^{n}(X - r_i)$ with r_i roots of $f(X)$ in L. Moreover, we ask L to be the smallest extension with this property. That is, $L = k(r_1, ..., r_n)$ is the smallest field containing k and all the roots of f. We gather some important facts:

   Proposition 3.1.2.2. *Let k be a field. Then*

   - *Every polynomial $f(X) \in k[X]$ admits a finite splitting field with finite dimension, therefore algebraic.*
   - *Every two splitting fields of the same polynomial are isomorphic.*

3. **III- Algebraic Closure of a field**
   A field L is said to be algebraically closed if it has no algebraic extensions other than itself. Indeed, this is equivalent to say that every polynomial in $L[X]$ has a root in L. The existence of an algebraically closed extension of a field k can only be proved by means of *Zorn's Lemma*

   Proposition 3.1.2.3. *Every field k admits an extension $L|k$ where L is algebraically closed.*

   We define the *Algebraic Closure* of a field k as an algebraically closed extension $L|k$ which is also algebraic over k. The existence of such an extension is a consequence of the following lemma

   Lemma 3.1.2.4. *Let $L|k$ be an extension of fields. The subset of all algebraic elements in L, over k, is a subfield of L, containing k. We denote this subfield by L^a.*

   In order to obtain an algebraic closure we consider the subfield of algebraic elements inside an algebraically closed extension of k. Also, it can be proved that any two algebraic closures of k are isomorphic, so that there is no ambiguity in denoting it by $\bar{k}$.

[1] Recall that if k is a field, the polynomial ring $k[X]$ is a principal ideal domain, and therefore $kernel(Ev_u)$ is an ideal of the form $(g(X))$, for some irreducible polynomial $g(X) \in k[X]$.

3.1.3 Morphisms of Algebraic Extensions

A morphism between extensions of a particular field k, $L_1|k \to L_2|k$ is a morphism of fields $\sigma : L_1 \to L_2$, such that $\sigma(x) = x, \forall x \in k$. Clearly, such a morphism has to be injective. Field extensions of k form a category, which we denote by $\mathbf{E}(k)$.

These morphisms preserve the roots of polynomials in $k[X]$: If $u \in L$ is a root of a polynomial $f(X) = (a_nX^n + ...a_0) \in k[X]$, since all the coefficients a_i are in k, they are all fixed by any morphism of extensions σ. Therefore , $\sigma(u)$ is also a root of $f(X)$, since $a_n\sigma(a)^n + ... + a_0 = \sigma(a_nu^n + ... + a_0) = \sigma(0) = 0$.

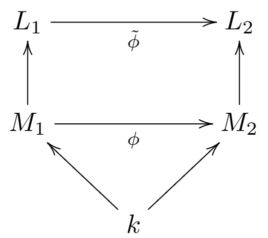
Proposition 3.1.3.1. *Every endomorphism of an algebraic extension is an automorphism.*

Proof. Let $\sigma : L|k \to L|k$ be an endomorphism. Cleary, σ is injective. Let u be an element in L, consider $g(X)$ its minimal polynomial over k and denote by S_g the set of its roots in L. Since the image of a root is still a root, the restriction $\sigma_{S_g} : S_g \to S_g$ is a well-defined injective map. Also, since the set S_g is finite, the restriction $\sigma|S_g$ has to be surjective and u equals $\sigma(a)$ for some $a \in L$, other root of $g(X)$. Therefore, σ is surjective. □

We use the notation $Aut_{\mathbf{E}(k)}(L|k)$ to denote the group of automorphisms of an extension $L|k$. This group coincides with the subgroup of $Aut_{\mathbf{Fields}}(L)$ of field automorphisms $L \to L$ preserving k.

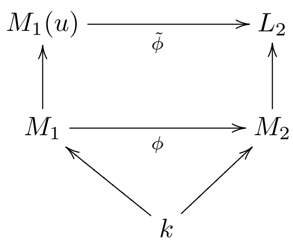
3.1.4 Lifting Morphisms of Field Extensions

Let $L_1|M_1|k$ and $L_2|M_2|k$ be two sequences of field extensions of k. Given an isomorphism of extensions $\phi : M_1|k \to M_2|k$ we look for conditions to extend ϕ to a morphism $\tilde{\phi} : L_1|k \to L_2|k$.



where the vertical arrows denote the respective inclusions.

Lemma 3.1.4.1. *(Lifting isomorphisms on the base fields to morphisms of extensions)*
Given the above conditions, suppose that $L_1|M_1$ and $L_2|M_2$ are field extensions and $\phi : M_1|k \to M_2|k$ is an isomorphism of extensions. Let $u \in L_1$ be an algebraic element over M_1 with minimal polynomial $f(X) \in M_1[X]$. Then ϕ can be extended to a morphism



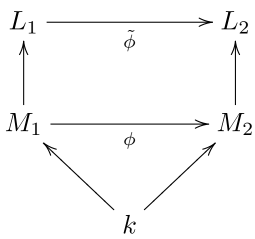
if and only if the polynomial $\overline{f}(X) \in M_2[X]$ *(see 2.1.6-4) has at least one root in* L_2*. Moreover, the number of such extensions is equal to the number of roots of* $\bar{f}(X)$ *in* L_2*.*

Proof. We notice that if ϕ extends to $\tilde{\phi}$, the element $\tilde{\phi}(u)$ is a root of $\bar{f}(X)$ in $M_2[X]$. Conversely, if $\bar{f}(X)$ already has a root α in L_2, then we define a morphism $M_1(u) \to L_2$ by sending u to α and every element of M_1 to its image through ϕ. We immediately conclude the number of extensions of ϕ to be precisely the number of roots of $\bar{f}(X)$ in L_2

□

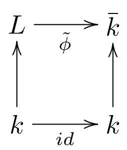
The previous lemma, together with Zorn's lemma, allows us to prove the following result:

Theorem 3.1.4.2. *Let* $L_1|M_1|k$ *and* $L_2|M_2|k$ *be extensions and* $\phi : M_1|k \to M_2|k$ *an isomorphism. Suppose that* $L_1|M_1$ *is algebraic and* $L_2|M_2$ *is algebraically closed. Then there exists an inclusion morphism of extensions*



Moreover, if L_1 *is algebraically closed and* L_2 *is also algebraic,* $\tilde{\phi}$ *is an isomorphism.*

It follows immediately as a corollary, that any two algebraic closures of a field are isomorphic, and also that for any algebraic extension L of a given field k, there is an inclusion morphism $L \to \bar{k}$ into the algebraic closure of k and we may assume that every algebraic extension $L|k$ is embedded as an intermediate extension $\bar{k}|L|k$:



Also,

Corollary 3.1.4.3. *Any automorphism* σ *of an algebraic extension of* k*, lifts to an automorphism of the algebraic closure of* k*.*

Proof. This follows immediately from the last theorem applied to the diagram

$$\begin{array}{ccc} \bar{k} & \xrightarrow[\tilde{\sigma}]{} & \bar{k} \\ \uparrow & & \uparrow \\ L & \xrightarrow[\sigma]{} & L \\ \uparrow & & \uparrow \\ k & \xrightarrow[id]{} & k \end{array}$$

□

3.1.5 Separable Extensions

Since splitting fields are algebraic, they can be embedded in an algebraic closure. In particular, a polynomial $f \in k[X]$ can be decomposed as a product of linear terms in $\bar{k}[X]$. We define the multiplicity of a root $u \in \bar{k}$ to be the greatest natural number m such that $(X-u)^m$ divides f. If $m = 1$ we say that u is a simple root. A polynomial is said to be separable if it has no multiple roots in $\bar{k}$. In other words, there are as many roots as the degree of the polynomial, all of them distinct. Given an extension $L|k$ we say that an element $u \in L$ is separable if its minimal polynomial is separable.

Definition 3.1.5.1. *An extension $L|k$ is said do be separable if it is algebraic and all its elements are separable. A field k is said to be perfect if all algebraic extensions of k are separable.*

Proposition 3.1.5.2. *For a sequence of extensions $L|M|k$, if $L|k$ is separable then $L|M$ is also separable.*

Finite separable extensions have a particular characterization:

Proposition 3.1.5.3. *Let $L|k$ be a finite extension of degree n. Then $L|k$ has at most n distinct morphisms to $\bar{k}|k$ with equatlity if and only if $L|k$ is separable.*

Proof. This follows immediately from 3.1.4.1, using induction on the number of generators of L over k. □

In particular, we find that any finite separable extension of k is of the form $k(a_1, ..., a_n)$ for $a_1, ..., a_n$ separable elements over k.

Let k_S denote subset of all separable elements over k, in $\bar{k}$. Obviously k is contained in this set, since all elements of k are separable over k. In fact, k_S is a subfield of $\bar{k}$: from the previous lemma, given two separable elements a and b, the extension $k(a, b)$ is separable and therefore, the elements $a.b$, $a - b$, $a + b$ and a/b are all separable elements.

Definition 3.1.5.4. *The field k_S is called the separable closure of k in $\bar{k}$.*

In fact, a field k is perfect if and only if the separable closure of k is equal to the algebraic closure.

3.1.6 Exact Extensions

We proceed now to relate the groups $Aut_{\mathbf{E}(M)}(L|M)$, $Aut_{\mathbf{E}(k)}(M|k)$ and $Aut_{\mathbf{E}(k)}(L|k)$ for a sequence of algebraic extensions $L|M|k$.

1. The group $Aut_{\mathbf{E}(M)}(L|M)$ is naturally included as a subgroup of $Aut_{\mathbf{E}(k)}(L|k)$: each automorphism σ of $L|M$ is given by an automorphism of L preserving all the elements in M, and since k is a subfield of M, σ also fixes k;

2. We can restrict every automorphism $\sigma \in Aut_{\mathbf{E}(k)}(L|k)$ to M, $\sigma|_M$. We easily conclude that if, for all $\sigma \in Aut_{\mathbf{E}(k)}(L|k)$, we have $\sigma(M) \subseteq M$, then the restriction $\sigma|_M$ is an endomorphism $M|k \to M|k$ and since every endomorphism of an algebraic extensions is an automorphism, we have natural morphism of groups

$$Aut_{\mathbf{E}(k)}(L|k) \to Aut_{\mathbf{E}(k)}(M|k)$$

mapping σ to its restriction $\sigma|_M$. The kernel of this map is the subgroup of all field automorphisms of L leaving M invariant, by definition, $Aut_{\mathbf{M}}(L|M)$. In this case, we have a left-exact sequence of groups

$$1 \to Aut_{\mathbf{M}}(L|M) \to Aut_{\mathbf{E}(k)}(L|k) \to Aut_{\mathbf{E}(k)}(M|k)$$

3. Surjectivity is equivalent to the possibility of extending any automorphism $\phi : M|k \to M|k$ to an automorphism $\Phi : L|k \to L|k$. In this case we have an exact sequence of groups

$$1 \to Aut_{\mathbf{E}(M)}(L|M) \to Aut_{\mathbf{E}(k)}(L|k) \to Aut_{\mathbf{E}(k)}(M|k) \to 1 \tag{3.1}$$

and therefore $Aut_{\mathbf{E}(k)}(M|k)$ is isomorphic to the quotient $Aut_{\mathbf{E}(k)}(L|k)/Aut_{\mathbf{E}(M)}(L|M)$. We introduce the following definition:

Definition 3.1.6.1. *We say that a sequence of extensions $L|M|k$ is exact if the restriction operation $Aut_{\mathbf{E}(k)}(L|k) \to Aut_{\mathbf{E}(k)}(M|k)$ is a well-defined surjective group homomorphism, inducing a short exact sequence of groups, as above.*

3.1.7 Normal Extensions

1.

Definition 3.1.7.1. *We say that an extension $L|k$ is normal if it is algebraic and if all the irreducible polynomials having one root in L also have all the other roots in L.*

In fact, this definition is equivalent to ask for L to contain the splitting field of the minimal polynomial of every element $l \in L$. Any algebraically closed extension $L|k$ is normal, since every polynomial has a root in L, therefore splitting in linear terms. Also, given an intermediate extension $L|M|k$, if $L|k$ is normal then clearly $L|M$ is also normal.

Proposition 3.1.7.2. *let $L|k$ be an algebraic extension. Considered as a subextension of $\bar{k}|k$, $L|k$ is normal if and only if every automorphism σ of $\bar{k}|k$ restricts to an automorphism of $L|k$, in other words, $\sigma(L) = L$.*

Proof. Suppose $L|k$ is normal. As seen before, $\sigma(l)$ is a root of the minimal polynomial of l. Since the extension is normal, by definition, $\sigma(l)$ has to be in L. The equality is true because every endomorphism of an algebraic extension is in fact an automorphism. Conversely, suppose that $\sigma(L) = L$, for every automorphism σ of $\bar{k}$. Let $l \in L$ be an element with minimal polynomial $g(X) \in k[X]$. For every automorphism σ, the values $\sigma(l)$ are roots of $g(X)$ and they are all in L. Suppose there is a root u of $g(X)$ not in L. In this case we consider the field homomorphism $k(l) \to k(u)$, defined by mapping l to u and fixing the elements of k. Both extensions are algebraic and this isomorphism extends to an automorphism σ of $\bar{k}$, by 3.1.4.2. Restricted to L, we conclude that $u = \sigma(l)$ is also in L. □

This proposition implies the following results:

2. For normal a extension $L|k$, the restriction map (3.1.6-2)

$$Aut_{\mathbf{E}(k)}(\bar{k}|k) \to Aut_{\mathbf{E}(k)}(L|k)$$

is well-defined; Moreover, from 3.1.4.3, every automorphism of an algebraic extension $L|k$ lifts to an automorphism of $\bar{k}|k$, and therefore the restriction map is surjective. We conclude that if $L|k$ is normal then the sequence of extensions $\bar{k}|L|k$ is exact.

3. Considering a sequence of extensions $L|M|k$ where $L|k$ and $M|k$ are both normal, from 3.1.4.3, we know that any automorphism σ of $L|k$ lifts to an automorphism $\tilde{\sigma}$ of $\bar{k}|k$. Since $M|k$ is also normal, from the previous proposition, the restriction $\tilde{\sigma}|_M = \sigma|_M$ is an automorphism of $M|k$ and the restriction map

$$Aut_{\mathbf{E}(k)}(L|k) \to Aut_{\mathbf{E}(k)}(M|k)$$

is well-defined.

Moreover, any automorphism of $M|k$ lifts to an automorphism of $\bar{k}|k$, which, since $L|k$ is normal, restricts again to an automorphism of $L|k$. Therefore, the restriction map is surjective. The kernel is precisely the group $Aut_{\mathbf{E}(M)}(L|M)$.

We conclude that any sequence of normal extensions $L|M|k$ is exact.

3.2 Galois Extensions and Classification of their subextensions

3.2.1 Group Actions and Galois Extensions

1. Let $L|k$ be an extension of fields. The action of a group G on an extension $L|k$ is given by a homomorphism of groups $G \to Aut_{\mathbf{E}(k)}(L|k)$.

2. Given an action of a group G on $L|k$ we consider the set $Fix(G)$ of all elements $l \in L$, invariant under the action of $G : g(l) = l$, $\forall g \in G$.

Proposition 3.2.1.1. *$Fix(G)$ is a subfield of L containing k.*

Therefore, to every group action we assign an intermediate subextension $L|Fix(G)|k$. In particular, if H is a subgroup of G we naturally have $Fix(H)$ extending $Fix(G)$. We introduce Galois Extensions:

Definition 3.2.1.2. *An algebraic extension $L|k$ is said to be Galois if the set of elements in L, invariant under the action of $Aut_{\mathbf{E}(k)}(L|k)$ coincides with k. In this case we say $Aut_{\mathbf{E}(k)}(L|k)$ is the Galois Group of the extension.*

We immediately conclude that for a sequence $L|M|k$, if $L|k$ is a Galois extension then $L|M$ is also Galois.

3. Let $L|k$ be an algebraic field extension with automorphism group G. To each element $l \in L$ we assign the subset of G of all automorphisms fixing l. This subset is a subgroup. It is called the stabilizer of l, and denoted by H_l. For each l, we write the polynomial $f(X) = \Pi_{[\sigma]\in G/H}(X - \sigma(x))$, where G/H_l is the set of all equivalence classes where two automorphisms σ and σ' are equivalent if and only if they differ by some automorphism fixing l, $\sigma = \sigma' \circ h$. Therefore, $f(X)$ is well-defined. We notice that all roots of $f(X)$ are conjugated elements of l. Furthermore, the action of G on the coefficients of $f(X)$, leaves $f(X)$ invariant.

   If the extension $L|k$ is Galois, since $f(X)$ is invariant under the action of G, $f(X)$ has to be in $k[X]$. Moreover, $f(X)$ divides the minimal polynomial of l in $k[X]$, which is, by definition, irreducible. Therefore, $f(X)$ coincides with the minimal polynomial. Moreover, $f(X)$ is separable, since all the roots are different by construction. We conclude

   Proposition 3.2.1.3. *Any Galois extension is normal and separable.*

4. After the previous item, we conclude that any Galois extension is a subextension of the separable closure k_S of k. In fact

   Proposition 3.2.1.4. *Any separable closure $k_S|k$ is a Galois extension.*

   Proof. We prove that any element $\alpha \in k_S$, not in k, is moved by the action of $G = Aut_{\mathbf{E}(k)}(k_S|k)$. By definition $k_S|k$ is algebraic and separable. Therefore, all the roots of the minimal polynomial of any element α, are different. Since we choosed $\alpha \in k_S \backslash k$, its minimal polynomial can not have degree 1. So, there is another root α', different from α, $\alpha' \in k_S \setminus k$. We consider the isomorphism $\phi : k(\alpha) \to k(\alpha')$, mapping α to α' and the identity on k. From theorem 3.1.4.2, ϕ lifts to an automorphism $\tilde{\phi}$ of $\bar{k}$. It is sufficient to prove that $\tilde{\phi}(k_S) \subseteq k_S$. This follows from that fact that $\tilde{\phi}(\beta)$ is still a root of the minimal polynomial of β. If β is separable, so is $\tilde{\phi}(\beta)$. □

   Definition 3.2.1.5. *The absolute Galois group of k is the group $Aut_{\mathbf{E}(k)}(k_S|k)$, denoted as $Gal(k)$.*

3.2.2 Classification of subextensions of a finite Galois Extension

1. We start by noticing some facts about finite extensions. After lemma 3.1.4.1 we know that

$$|Hom_{\mathbf{E}(k)}(L|k, \bar{k}|k)| \leqslant [L : k]$$

   with an equality if and only if the extension is separable. If the extension is normal, from Prop 3.1.7.2 every morphism $L|k \to \bar{k}|k$ is in fact an automorphism of $L|k$. Therefore,

   Proposition 3.2.2.1. *If an extension $L|k$ is finite and Galois (=> normal + separable) then its Galois group is finite, with*

$$|Aut_{\mathbf{E}(k)}(L|k)| = [L : k]$$

   There is also an important result concerning the action of a finite group on a field extension:

   Proposition 3.2.2.2. *(Artin's Lemma) Let G be a finite group acting on a field L by automorphism. Then the extension $L|Fix(G)$ is finite with $[L : Fix(G)] \leqslant |G|$.*

   Proof. See [14] - 7.7.5. □

2. The last sequence of results allows us to prove the following classification theorem

   Theorem 3.2.2.3. *Let $L|k$ be a finite Galois extension with Galois group G. There is a bijection between the set of all intermediate field extensions $L|M|k$ and the set of subgroups of G.*

   Proof. As we have already seen in 3.1.6, for a sequence of extensions $L|M|k$, the group $Aut_{\mathbf{E}(M)}(L|M)$ is a subgroup of G. Conversely, each subgroup $H \subseteq G$ as an associated intermediate field extension $L|Fix(H)|Fix(G)$, where $Fix(G) = k$ because the extension is Galois. We prove the assignments $L|M|k \mapsto Aut_{\mathbf{E}(M)}(L|M) \subseteq G$ and $H \subseteq G \mapsto L|Fix(H)|k$ are inverse maps:

   (a) If $L|M|k$ is an intermediate extension then $Fix(Aut_{E(M)}(L|M)) = M$: Since $L|k$ is finite and galois $L|M$ is also finite and galois and we immediately conclude $M = Fix(Aut_{E(M)}(L|M)$;

   (b) For a subgroup $H \subseteq G$ we have $Aut_{\mathbf{E}(Fix(H)}(L|Fix(H)) = H$: Following from Artin's lemma we have $[L : Fix(H)] \leqslant |H|$. Now, since $L|k$ is a galois extension we conclude $L|Fix(H)$ to be also galois, and $|Aut_{\mathbf{E}(Fix(H)}(L|Fix(H))| = [L : Fix(H)]$. Since $H \subseteq Aut_{\mathbf{E}(Fix(H)}(L|Fix(H))$ we conclude $|H| \leqslant Aut_{\mathbf{E}(Fix(H)}(L|Fix(H))$. Together with the first relation obtained, we conclude that both groups have the same order and since they are finite, they are equal.

   □

3. If $M|k$ is an intermediate Galois subextension of $L|k$ Galois, from 3.1.7-3, the sequence $L|M|k$ is exact, we find

$$Aut_{\mathbf{E}(k)}(M|k) \cong G/Aut_{\mathbf{E}(M)}(L|M)$$

   and $Aut_{\mathbf{E}(M)}(L|M)$ is a normal subgroup of G. Conversely, if H is a normal subgroup of G, the action of G on $Fix(H)|k$ factors through the quotient G/H and we have $Fix(H)|Fix(G/H)|k$. Since $L|k$ is Galois, we conclude that $Fix(G/H) = Fix(G) = k$ and so, $Fix(H)|k$ is Galois with Galois group G/H. Therefore

   Proposition 3.2.2.4. *The correspondence in 3.2.2.3 establishes a bijection between Galois intermediate extensions of a Galois extension $L|k$ and normal subgroups of G.*

3.2.3 Classification of subextensions of an infinite Galois Extension

We begin with the fact that any Galois extension $L|k$, not necessarily finite, is the union of all its intermediate finite Galois subextensions:

Proposition 3.2.3.1. *Let $L|k$ be a Galois extension. Then $L = \cup_{M|k} M$, where M runs over all the intermediate finite Galois subextensions of L.*

Proof. Indeed, if $L|k$ is Galois, we know that it is also normal and separable, containing all the splitting fields of the minimal polynomials of all elements in L. The inclusion $\cup_{M|k} M \subseteq L$ is obvious. The second inclusion follows from the fact that the splitting field of every element in L is contained in a finite Galois extension that is still contained in L. □

This fact implies that any automorphism of a Galois extension is completely determined by its restriction to each finite Galois subextension. Moreover, we conclude that any finite subextension $M|k$ of a Galois extension $L|k$ is contained on a finite Galois subextension $L|P|M|k$, obtained by considering the union of the splitting fields of the minimal polynomials of each generator of $M|k$.

Let $L|k$ be a Galois extension, not necessarily finite. Our goal is to prove that the Galois group $Aut_{\mathbf{E}(k)}(L|k)$ can be obtained as the limit of a projective system of finite groups: the Galois groups of intermediate finite Galois field extensions of $L|k$.

1. Given an intermediate finite Galois extension $M|k$ of $L|k$, the sequence $L|M|k$ is exact. Therefore, the restriction map

$$\psi_M : Aut_{\mathbf{E}(k)}(L|k) \rightarrow Aut_{\mathbf{E}(k)}(M|k)$$

is well-defined and each group $Aut_{\mathbf{E}(k)}(M|k)$ is a finite quotient of $Aut_{\mathbf{E}(k)}(L|k)$

2. Given a tower of intermediate extensions $L|M_1|M_2|k$ with $M_1|k$ and $M_2|k$ both finite and Galois, we know that the sequence $M_1|M_2|k$ of finite extensions is also exact. Therefore, the restriction maps

$$\phi_{M_2}^{M_1} : Aut_{\mathbf{E}(k)}(M_1|k) \rightarrow Aut_{\mathbf{E}(k)}(M_2|k)$$

are well-defined surjective group morphisms. If $M_3|k$ is a third finite Galois subextension with $L|M_1|M_2|M_3|k$ then we easily conclude that $\phi_{M_3}^{M_1} = \phi_{M_3}^{M_2} \circ \phi_{M_2}^{M_1}$.

The collection of finite groups $Aut_{\mathbf{E}(k)}(M|k)$ and maps $\phi_{M'}^{M}$ indexed by the set of all finite Galois intermediate extensions $M|k$ of $L|k$, is a projective system of groups. We denote it by $\mathcal{P}$.

3. Notice that the restriction map ψ_M commutes with every composable transition morphism $\phi_{M'}^{M}$:

$$\psi_{M'} = \phi_{M'}^{M} \circ \psi_M$$

From A.2.3.1 we know how to construct the limit of any projective system of groups $\varprojlim_I \mathcal{P}$ as a subgroup of the product $\Pi_{M|kGaloisFinite}\mathcal{P}(M)$, consisting on all families (g_M) with $\phi_{M'}^{M}(g_M) = g_{M'}$. We recover the Galois group $Aut_{\mathbf{E}(k)}(L|k)$:

Theorem 3.2.3.2. *The canonical map determined by the collection of morphisms ψ_M,*

$$Aut_{\mathbf{E}(k)}(L|k) \longrightarrow \varprojlim_{M|k}\mathcal{P} \qquad \textit{with } M|K \textit{finite Galois subextension of } L|k$$

is an isomorphism of groups.

Proof. Consider the map $\phi : Aut_{\mathbf{E}(k)}(L|k) \rightarrow \Pi_{M|kfiniteGalois}Aut_{\mathbf{E}(k)}(M|k)$ sending each automorphism σ to $(\sigma|_M)_{M|k-Galois}$. This map is injective: If σ restricts to the identity map on one of the galois subextensions $M|k$, then M is σ invariant. Since $L|k$ is Galois, we conclude that M has to be equal to L and so σ has to be the identity map. We prove that the image of ϕ is precisely $\varprojlim_{M|k}\mathcal{P}$ as constructed in A.2.3.1 where the limit is taken over all finite galois subextensions $M|k$ of $L|k$: Since ψ_M commutes with every composable transition map $\phi_{M'}^{M}$, we conclude that the image of ϕ is contained in $\varprojlim_{M|k}\mathcal{P}$. To prove that this is an equality of sets, given a family $(\sigma|_M)_{M|k}$ in $\varprojlim_{M|k}\mathcal{P}$ we define an automorphism of L by $\sigma(\alpha) := \sigma|_M(\alpha)$. The fact that σ is well-defined follows from the fact that any automorphism of a Galois extension is determined by its restrictions to each finite intermediate Galois subextension. ∎

From A.2 we see that the Galois Group of an infinite Galois extension is a profinite group, determined by all its finite quotients, and has a natural topology induced by the discrete topology on each finite Galois group $Aut_{\mathbf{E}(k)}(M|k)$. Moreover, the restriction map ϕ_M is continuous with respect to this topology

4. Our aim is to present a classification theorem for Galois intermediate extensions of an infinite Galois extension $L|k$. To begin with, notice that any finite subextension of $L|k$, being generated by a finite number of elements, can always be embedded in a finite Galois subextension, the splitting field of the generators. We prove the following

Proposition 3.2.3.3. *Let $L|M|k$ be an intermediate extension of an infinite Galois extension of $L|k$ with Galois group G. Then $Aut_{\mathbf{E}(M)}(L|M)$ is a closed subgroup of G.*

Proof. Starting with a finite subextension $L|M|k$, there is a finite Galois subextension $L|P|M|k$, with Galois group $G_P = Aut_{\mathbf{E}(k)}(P|k)$ a finite quotient of G by $Aut_{\mathbf{E}(P)}(L|P)$ endowed with the discrete topology. Moreover, we know that the group $Aut_{\mathbf{E}(M)}(P|M)$ is an open subgroup of G_P (because G_P has the discrete topology). Therefore its inverse image U_M through the continuous map $\psi_P : G \to G_P$, is open in G. We prove that $U_M = Aut_{\mathbf{E}(M)}(L|M)$. Indeed every element of U_M fixes M and the image of $Aut_{\mathbf{E}(M)}(L|M)$ through ψ_P is contained in $Aut_{\mathbf{E}(M)}(P|M)$. Following the results in the appendix, $Aut_{\mathbf{E}(M)}(L|M)$ is a closed and finite subgroup of G. For an arbitrary intermediate subextension $L|M|k$, since $M|k$ is the union of all its finite subextensions, $L_\alpha|k$, we know that each $Aut_{\mathbf{E}(k)}(L_\alpha|k)$ is closed. The intersection of all these groups, for all α, is precisely the subgroup $Aut_{\mathbf{E}(M)}(L|M)$ and so, it is also closed. ☐

At last, we present the classification theorem for subextensions of an infinite Galois extension

Theorem 3.2.3.4. *(Krull's Theorem) Let $L|k$ be an infinite Galois extension with Galois group G. Then, there is a bijection between the set of intermediate subextension $L|M|k$ and the set of closed subgroups of G.*

Proof. See [16]. ☐

The relation between Galois subextensions and normal subgroups is proved with the same arguments of 3.2.2.4.

3.2.4 Grothendieck's Formulation of the Galois Theory of Fields

Let k be a field and k_S a separable closure of k in some algebraic closure $\bar{k}$ and consider $Gal(k)$ the absolute Galois group. Given a finite separable extension $L|k$, the set $Hom_{\mathbf{E}(k)}(L|k \to k_S|k)$ is finite, of cardinality $[L : k]$. This set has a natural action of $Gal(k)$ defined by $g : \phi \mapsto g \circ \phi$, for $g \in Gal(k)$ and $\phi : L|k \to k_S|k$.

Proposition 3.2.4.1. *This action of $Gal(k)$ on $Hom_{\mathbf{E}(k)}(L|k \to k_S|k)$ is continuous and transitive.*

We then have the main result which makes possible the Grothendieck's formulation of Galois theory for fields

Theorem 3.2.4.2. *Let k be a field and k_S a separable closure. Using the notations of the first chapter, the contravariant functor*

$$h^{\circ}_{k_S} : \boldsymbol{E}(k) \longrightarrow \boldsymbol{Sets}$$

induces an equivalence between the category of finite separable extensions of k and the category of sets endowed with a transitive and continuous action of $Gal(k)$.

Proof. See [16]. □

Chapter 4

Galois Theory of Covering Spaces

Our aim is to introduce the Theory of Covering Spaces as an example of a "Galois Theory", and the definition of the algebraic fundamental group of a topological space X. In this chapter, we introduce two different fundamental groups on a topological space X: the familiar topological version, built on paths on X and a new definition, constructed through a pair category+functor, built on coverings of the space X, generalizing the role of the topological version.

4.1 Fundamental Group of a topological space

4.1.1 Topological Fundamental Group

Let X be a topological space. A path on X is a continuous map $\gamma : I = [0,1] \to X$. $\gamma(0)$ is called the starting point and $\gamma(1)$ the ending point. A loop is a path having the same initial and final points. The image of the path is the set $\gamma(I) \subseteq X$. An homotopy between two paths γ_1 and γ_2 with the same extreme points, is a continuous map $H : I \times I \to X$ such that $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma_2(t)$. Morever we ask for the extreme points to be fixed, that is, $H(s,1) = \gamma_1(1) = \gamma_2(1)$ and $H(s,0) = \gamma_1(0) = \gamma_2(0)$. More generally, given two topological spaces Y and Z and two maps $f, g : Y \to Z$, an homotopy between f and g is a continuous map $H : I \times Y \to Z$ such that $H(0,y) = f(y)$ and $H(1,y) = g(y)$, $\forall y \in Y$. The existence of an homotopy between two paths is an equivalence relation on the set of paths on X. We denote by $[\gamma]$ the class of paths homotopicaly equivalent to γ.

Given two paths γ_1 and γ_2 such that γ_2 starts where γ_1 ends ($\gamma_1(1) = \gamma_2(0)$) we obtain a third path, the composition $\gamma_2 \circ \gamma_1$, defined by $t \mapsto \gamma_1(2t)$ for $t \in [0, \frac{1}{2}]$ and $t \mapsto \gamma_2(2t-1)$ if $t \in [\frac{1}{2}, 1]$. This composition extends to equivalence classes: given two composable paths γ and α and $[\gamma]$ and $[\alpha]$ their respective equivalence classes, we define the composition $[\gamma] \circ [\alpha] := [\gamma \circ \alpha]$. This composition is well-defined: if γ' (resp. α') is another path representing the same class of γ (resp. α), there are homotopies between, respectively γ and γ' and α and α' we can use them to construct an homotopy between $\gamma \circ \alpha$ and $\gamma' \circ \alpha$.

We say that a topological space is path-connected if there is a path joining every two points of X. Assume X is path connected- The above operation induces a categorical structure on the set of points of a path-connected space X. Viewing points x and y as objects we define the set of morphism $Hom(x,y)$ to be the set of equivalence classes of paths starting at x and ending on y. We have $Hom(x,x)$ as the set of equivalence classes of homotopic loops starting and ending at x. We observe that there is an identity morphism I_x for each point $x \in X$: it is represented by the homotopy class of the constant path at x, that is, the path mapping $t \mapsto x$, $\forall t \in I$.

The above properties of path-composition and the existence of an identity morphism, ensure sufficient conditions for the composition of these morphisms to be well-defined and for this construction to be a category. We denote it by $\Pi_1(X)$.

In the above sense, an isomorphism $[\gamma] : x \to y$ is an equivalence class of paths from x to y such that there is another class $[\beta] : y \to x$, verifying $[\beta] \circ [\gamma] := [\beta \circ \gamma] = I_x$ and $[\gamma] \circ [\beta] := [\gamma \circ \beta] = I_y$. This is equivalent to ask for a path β from y to x such that both possible compositions are homotopicaly equivalent to constant paths on x and y, respectively. Such path always exists, defined by $t \mapsto \gamma(1-t)$. It has the same image as γ but it is travelled backwards. It can be shown that the composition of these two paths is homotopic to a constant path on the starting point of γ.

This property turns $\Pi_1(X)$ into a groupoid. We call it the fundamental groupoid of X. For each point $x \in X$ we have its automorphism group $Aut_{\Pi_1(X)}(x)$, which we denote by $\pi_1(X, x)$. We call it the *topological fundamental group of X at x*. Remember that such automorphisms are simply equivalence classes of loops starting and ending on x.

Since any two points are isomorphic in $\Pi_1(X)$, their automorphism groups are also isomorphic.

4.1.2 Algebraic Fundamental Group

This section is an overview of what is coming next. Having defined a topological fundamental group, we will introduce a new group generalizing the role of the fundamental group.We will introduce a category over X, considering as objects a certain kind of morphisms to X- which we call covering maps. We will denote it by **Cov(X)**. We will explore some fundamental features of this category, introduce Galois objects and a Galois theory concerning the classification of their subobjects.

This category is naturally equipped with a set-valued functor Fib_x - The Fiber Functor, depending on a point $x \in X$. After exploring some properties, we define the algebraic fundamental group of a space X as the group of automorphisms of this functor. We will denote it by $\pi_1^{alg}(X, x)$. Our aim will then be to compare both fundamental groups.

For a connected, path-connected, locally path connected and semi-locally simply connected topological space X, the fiber functor is representable and the group of automorphisms of the object representing it, is precisely $\pi_1(X, x)$. This result not only allows us to conclude that both fundamental groups are isomorphic but also, their actions on the fiber sets are equivalent. As a final result, we prove that the category of covering spaces is equivalent to the category of $\pi_1(X, x)$-sets.

This result depends in an essential way on the representability property, which in general does not hold. However, a particular version is still valid: For a connected space X, the category of finite covering spaces is equivalent to the category of $\pi_1^{alg}(X, x)$-sets, and in this case, $\pi_1^{alg}(X, x)$ is a profinite group.

4.2 Theory of Covering Spaces

4.2.1 Covering Spaces

Let X be a topological space. A space over X is a topological space Y together with a continuous map $p : Y \to X$. Sometimes we simply denote it by (Y, p). It is said to be connected if the space Y is connected.

We introduce a particular kind of spaces over X:

Definition 4.2.1.1. *A covering space (or cover) of X, is a space over X, $p : Y \to X$, such that each point x of X admits an open neighbourhood V_x for which $p^{-1}(V_x)$ is a collection of disjoint non-empty open sets $\{U_i\}$ and the restriction of p to U_i is a homeomorphism to V_x.*

As in section 1.3, a *morphism of spaces over* X, $(Y, p) \to (Z, p')$, is a continuous map $\phi : Y \to Z$ such that $p' \circ \phi = p$. Such maps preserve the fiber structure: given a point $y \in Y$ over x, its image $\phi(y)$ is a point in Z also over x, $p'(\phi(y)) = p(y) = x$.

The category of covering spaces of X, **Cov(X)** forms a subcategory of the category of (continuous) spaces over X.

Definition 4.2.1.2. *Let $p : Y \to X$ be a space over X. The fiber of a point $x \in X$ is the set of points on Y being mapped to x, $p^{-1}(\{x\})$. We use the expression "point over x" to denote a point $y \in Y$ in the fiber of x.*

For a covering, the fiber of a point is a discrete subset of Y. Also, the cardinality of $p^{-1}(\{x\})$ is a locally constant function and if X is connected, it is constant. It is called the degree of the cover. More generally, for a connected cover, the fiber sets are all homeomorphic to a single discrete topological space I. The cover is called finite if the fiber sets are finite. We restrict our study to covers of connected spaces X.

Let us proceed with some properties of morphisms between covering maps.

We say that a cover $q : Z \to X$ is an intermediate cover of $p : Y \to X$ if there is a covering morphism $f : (Y, p) \to (Z, q)$. Notice that

Proposition 4.2.1.3. *For $p : Y \to X$ any covering map and $Z \to X$ some connected cover, any morphism of covers given by a map $Y \to Z$ is itself a covering map of Z*

Proof. For a point $z \in Z$, we may consider a neighbourhood V of $q(x)$ with the covering property for both p and q. This way, $p^{-1}(V)$ is a collection of disjoint open sets U_i in Y, each one mapped homeomorphically to V. The same for $q^{-1}(V) = \cup V_i$. Each U_i is connected and because f is continuous, the sets $f(U_i) \subset Y$ are also connected. The commutation relation $q \circ f = p$ imposes each $f(U_i)$ to be equal to one of the sets V_j, therefore, open. For each point $y \in Y$ we could apply this argument and we conclude that $f(Y)$ is an open set on Z This proves that f verifies the covering condition. We still need to prove surjectiveness. SInce $f(Y)$ is open on Z and Z is connected, this resumes to prove that $f(Y)$ is also a closed set. Being open and closed on a connected set Z imposes $f(Y) = Z$. For this, we prove that $Z - f(Y)$ is open. Given a point $z \in Z - f(Y)$ we consider a neighbourhood V of $q(x)$ with the covering property for q, $q^{-1}(V) = \cup V_j$. One of this disjoint V_j containes z and we notice that this open set is disjoint from $f(Y)$. Again, the relation $q \circ f = p$ would force all the V_j to be contained on $f(Y)$. Finally, since every point $z \in Z - f(Y)$ admits an open neighbourhood V_j not intersecting $f(Y)$, $Z - f(Y)$ is indeed open.

□

We conclude this section, by noticing that the identity map $X \to X$ is a covering map. This covering is a final object in **Cov(X)**.

4.2.2 Group Actions, Quotient Spaces and Coverings

Let us briefly introduce a procedure to construct covering maps

Consider an action of a group G on a topological space Y. The quotient space Y/G is the set of equivalence classes $\{[y] : y \in Y\}$ and we have a natural projection $\pi : Y \to Y/G$ sending each y to its equivalence class $[y]$. We equip this space with the finest topology that makes this projection continuous (the quotient topology (See [7])). Notice that this construction is an example of the categorical notion of a quotient of an object by a group action, introduced in section 1.9.3. Therefore, every invariant G-morphism f from Y to another topological space Z, always factors uniquely, through π

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle\pi}\downarrow & \nearrow_{\bar{f}} & \\ Y/G & & \end{array} \tag{4.1}$$

where $\bar{f}$ is a morphism of topological spaces, in other words, continuous. We introduce a particular kind of actions:

Definition 4.2.2.1. *Let G be a group acting on a topological space Y. We say that the action of G on Y is* even *if every point $y \in Y$ admits an open neighbourhood U such that $g(U) \cap g'(U)$ is empty for all $g \neq g'$.*

Such actions produce covering maps:

Proposition 4.2.2.2. *If G acts evenly on Y, then the quotient map $Y \to Y/G$ is a covering.*

Proof. For each point $y \in Y$ pick an open neighbourhood U verifying the condition imposed by the even action of G. The collection of subsets $g(U)$, for different $g \in G$, is disjoint. The quotient map $\pi : Y \to Y/G$ identifies all these subsets with a single open subset $\pi(U)$ in Y/G. By the definition of the quotient topology on Y/G, π restricts to a homeomorphism from $g(U)$ to $\pi(U)$ and therefore is a covering map. □

4.2.3 Automorphisms of Covering Spaces

Automorphisms of a covering $p : Y \to X$ are homeomorphisms $Y \to Y$ commuting with the covering map p.

Lemma 4.2.3.1. *Let $p : Y \to X$ be a covering map. Let Z be a connected topological space and consider maps $f, g : Z \to Y$ with $p \circ f = p \circ g$. If f and g are equal at one point they have to be the same map.*

Proof. We prove the subset of points in Z where f and g are equal is both open and closed and since Z is connected, this set has to be the whole Z. It is open: Let $z \in Z$ be a point such $f(z) = g(z)$ on Y. Call this point y. The covering condition ensures the existence of a neighbourhood V of $p(y)$ on X such that $p^{-1}(V)$ is a disjoint union of open subsets U_i in Y, each one homeomorphic to V. Since $f(z)$ and $g(z)$ are equal they have to be on the same open set U_i. By continuity, there is an whole open neighbourhood W

of $z \in Z$ is mapped in U_i, and since $p(f(z))$ is equal to $p(g(z))$, we must have $f(z) = g(z)$ everywhere in W. It's closed: we prove the complementary set of points where $f(z) \neq g(z)$ is also open: Again, since p(f(z))=p(g(z)), $f(z)$ and $g(z)$ lie over the fiber of a same point $x \in X$. This time, the covering condition ensures the existence of disjoint open neighbourhoods U_i and U_j separating $f(z)$ and $g(z)$. By continuity, there is an open neighbourhood W of z where f and g are also distinct. □

In particular, if a covering automorphism of a connected cover has a fixed point, that is, if it is equal to the identity map at some point, then it has to be the identity map.

4.2.4 Action of $Aut_{\mathbf{Cov(X)}}(Y, p)$ on Y

$Aut_{\mathbf{Cov(X)}}(Y, p)$ naturally acts on the points of Y. Each automorphism $\phi : Y \to Y$ sends a point y to its image $\phi(y)$ preserving the fiber structure. Therefore, this action on Y restricts to an action $\phi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x)$, on the fiber of each point $x \in X$. Since ϕ is invertible, these maps are bijections, and in the particular case of finite coverings, are permutations of the fiber sets.

Proposition 4.2.4.1. *For a connected cover $p : Y \to X$ the action of $Aut_{\mathbf{Cov(X)}}(Y, p)$ on Y is even.*

Proof. The definition of a covering space ensures the existence of a neighbourhood V for each point $x \in X$, such $p^{-1}(V)$ is a disjoint union of open sets U_i homeomorphic to V. Each point $y \in X$ is contained on a subset of U_i of this kind. Each automorphism ϕ maps U_i to some other U_j. For Y connected, the previous lemma implies that for $\phi \neq I_Y$ we have $i \neq j$. □

Therefore, the quotient map $Y \to Y/Aut_{\mathbf{Cov(X)}}(Y, p)$ is a covering and it is an easy task to verify that its group of automorphisms is precisely $Aut_{\mathbf{Cov(X)}}(Y, p)$. In general:

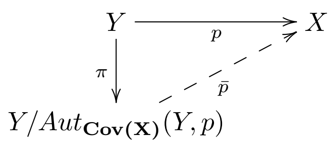
Proposition 4.2.4.2. *If G acts evenly on Y, then the group of covering automorphism of $Y \to Y/G$ is precisely G*

4.2.5 Galois Coverings and Classification of their subcoverings

In this section we describe a theorem classifying all the intermediate covers of a certain kind of cover- A Galois Cover.

Definition 4.2.5.1. *A covering $p : Y \to X$ is said to be Galois (or normal) if the action of $Aut_{\mathbf{Cov(X)}}(Y, p)$ on each fiber is transitive. In other words, if for any $x \in X$, given two points y and y' on its fiber, there is a covering automorphism taking y to y'. For Galois coverings we call $Aut_{\mathbf{Cov(X)}}(Y, p)$ the Galois group of the covering.*

Another characterization is possible: Considering the action of $Aut_{\mathbf{Cov(X)}}(Y, p)$ on Y consider the quotient $\pi : Y \to Y/Aut_{\mathbf{Cov(X)}}(Y, p)$, which we now know to be a covering map because the action is even. Under this conditions, p factors as



(4.2)

We prove the following:

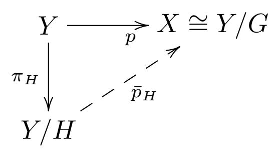
Proposition 4.2.5.2. *A covering map $p : Y \to X$ is Galois if and only if the map $\bar{p}$ is an homeomorphism.*

Proof. $\bar{p}$ is continuous. This follows from the construction properties of quotient spaces in the category of topological spaces. See 1.9.3. Morever, $\bar{p}$ is a bijection. Notice that each point in $Y/Aut_{\mathbf{Cov(X)}}(Y, p)$ represents an orbit of a point $y \in Y$ under the action of $Aut_{\mathbf{Cov(X)}}(Y, p)$. If the covering is Galois, this action is transitive on the fibers and therefore, the whole fiber of $p(y)$ equals the orbit of y. This way, the map $\bar{p}$, sending the orbit of y to $p(y)$ is a bijection. Since it is continuous, is a homeomorphism. Conversely, if $\bar{p}$ is a bijection, the whole orbits equals the fibers and the action has to be transitive. □

In particular, this implies that a covering map constructed through an even action of a group G on a space Y, by forming the quotient $Y \to Y/G$ is a Galois covering.

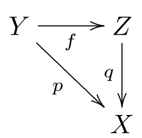
Let $p : Y \to X$ be a covering map with automorphism group G. The action of a subgroup $H \subseteq G$ on Y is still even, and the quotient $Y \to Y/H$ is a covering map that is also Galois, as a corollary to the last propositon. Morever, we know that its automorphism group is precisely H.

Furthermore, if (Y, p) is itself a Galois covering, p is a G-invariant morphism and therefore also H-invariant, so that it factors through a unique continuous map, $\bar{p}_H$:



which also turns out to be a covering map. Summarizing this, for each subgroup $H \subseteq Aut_{\mathbf{Cov(X)}}(Y, p)$ for $p : Y \to X$ Galois, we constructed an intermediate cover $\bar{p}_H : Y/H \to X$. We now address the inverse problem.

For an intermediate cover $Z \to X$ of a cover $p : Y \to X$, which we can always assume as connected



by 4.2.1.3, the map $f : Y \to Z$ is also a covering. Each automorphism of Y over Z is naturally an automorphism of Y over X and $H := Aut_{\mathbf{Cov(Z)}}(Y, f)$ is a subgroup of G. We prove that $f : Y \to Z$ is Galois by showing that the action of H on the fibers is transitive. Given $z \in Z$ and two points in $f^{-1}(z)$, z_1 and z_2, they are also points in the fiber $p^{-1}(q(z))$ and since $Y \to X$ is Galois the action of G on this fiber is transitive. This way we have some $\phi \in G$ with $\phi(z_1) = z_2$. In fact, for z_1 we have $f(z_1) = (f \circ \phi)(z_1)$. By lemma 4.2.3.1 the maps have to be equal and ϕ belongs to H. Moreover, we have $Z \cong Y/H$.

This whole correspondence has the following property:

Theorem 4.2.5.3. *Let $p : Y \to X$ be a Galois cover. The correspondences assigning to each subgroup $H \subseteq G$ the cover $(\bar{p}_H : Y/H \to X)$ and to each subcover $(q \circ f : Y \to Z \to X)$ the subgroup $Aut_{\mathbf{Cov(Z)}}(Y, f)$, defines a bijection between subgroups of G and intermediate covers of $Y \to X$.*

In particular,

Proposition 4.2.5.4. *Let $Y \to X$ be a Galois cover with Galois group G. An intermediate cover ($q \circ f : Y \to Z \to X$) is Galois if and only if the corresponding subgroup $Aut_{\mathbf{Cov(Z)}}(Y, f)$ is a normal subgroup of G.*

Proof. Given a normal subgroup$H \subseteq G$ let $Z \to X$ be the corresponding subcovering of $Y \to X$. Since H is normal, G/H is a group and acts on $Z \to X$ which we known to be isomorphic to $Y/H \to X$. This action is injective and we have$Z/(G/H) \cong Y/G \cong X$, and therefore is Galois. For $Z \to X$ a Galois subcovering, each automorphism ϕ of Y over X induces a unique automorphism of Z over X: Given $y \in Y$ we known $f(y)$ and $f(\phi(y))$ to be on the same fiber in Z, over $(q \circ f)(y)$. Since $Z \to X$ is galois, there is some automorphism $\tilde{\phi}$ of Z over X with $\tilde{\phi}(f(y)) = f(\phi(y))$. There is only one $\tilde{\phi}$ with this property: if λ was another one then the composition $\tilde{\phi} \circ \lambda^{-1}$ would be the identify map, from lemma 4.2.3.1. This correspondence defines a surjective map $G \to Aut_{\mathbf{Cov(X)}}(Z, q)$ and its kernel is precisely $H = Aut_{\mathbf{Cov(Z)}}(Y, f)$. □

4.3 The Fiber Functor and The Algebraic Fundamental Group

4.3.1 Definition

We proceed now to a different characterization of covering spaces by focusing on the fiber set $p^{-1}(\{x\})$ of each point $x \in X$. We prove the following:

Proposition 4.3.1.1. *The correspondence mapping each covering $Y \to X$ to the fiber of x, $p^{-1}(\{x\})$ is a functor from the category of coverings to the category of sets.*

Proof. As already seen in section 4.2.1, a morphism of coverings $f : (Y, p) \to (Z, p')$ preserves the fiber structure by sending points on the fiber of x to points in the same fiber of x, inducing a map $f|_{p^{-1}(x)}$ on each fiber. Morever, the identity morphism restricts to the identity map on the fibers. The composition of these maps between fibers is by definition inherited from the composition of covering maps restricted to them. □

We call it the fiber functor at the point $x \in X$ and denote it by Fib_x. Moreover, this functor is covariant.

Definition 4.3.1.2. *The algebraic fundamental group of a connected space X is the group of automorphisms of Fib_x. We denote it by $\pi_1^{alg}(X, x)$.*

4.3.2 Topological Properties of Coverings - Homotopy Lifting Property

Let $p : Y \to X$ be a covering and consider a continuous map $f : Z \to X$, from another topological space Z to X. A lifting of f is a map $\tilde{f} : Z \to Y$ such that $p \circ \tilde{f} = f$.

We will not prove the following fundamental lemma.

Lemma 4.3.2.1. *Let $p : Y \to X$ be a covering. Let Z be another topological space and consider maps $f, g : Z \to X$ and $H : I \times Z \to X$ one homotopy between them. Choosing a lift for f, the whole homotopy can be lifted on an unique way and its lifting is an homotopy between the lifting of f and a lifting of g.*

This lemma implies the possibility of lifting paths on X to paths on Y: Consider $Z = \{p\}$ as a one-point set endowed with the discrete topology. Seen this way, an homotopy H between maps $f, g : Z \to X$ may be interpreted as a path on X starting on $f(p)$ and ending at $g(p)$. A lifting of f is simply a choice of a point y on the fiber of $f(p)$. This choice is always possible. Therefore, by the lemma, the whole homotopy (in this case the whole path) can by lifted and it connects y to a certain point over $g(p)$. So, for each point in Y over the starting point we have a unique lift of the whole path, ending at a certain point on the fiber of the final point of the path. Also by the lemma if two paths are homotopic on X their lifts will also be homotopic on Y.

4.3.3 Dependence on the base point

As a functor, Fib_x depends on the base point $x \in X$. Let $p : Y \to X$ be any cover and consider x and x', two points in X. For any path γ joining x to x' we define a map $\phi_\gamma^Y : Fib_x(Y,p) \to Fib_{x'}(Y,p)$ by sending each point z over x to the final point of the unique lift of γ starting at z, denoted by $\tilde{\gamma}_z(1)$. As seen in the previous section, such point is on the fiber of x': any lifting $\tilde{\gamma}$ of a path γ, has initial and final points lying over the respective initial and final points of γ. Also by the previous lemma, this map is injective, since the lifting is unique. It is also surjective: Given any point z' over x' one can consider a path on Y connecting z' to some z on the fiber of x. If we project this path on X we get a path joining x and x' and by applying the construction above to this path we get z' as image of z. We conclude ϕ_γ^Y is a bijection.

Observe now that if γ' is another path between x and x', homotopic to γ, the induced maps $\phi_{\gamma'}^Y$ and ϕ_γ^Y are precisely the same. This follows also from lemma 4.3.2.1: if two paths are homotopic on X, their lifts starting on the a same point are also homotopic and therefore they have the same endpoints.

One can also easily see that for three points x, x' and $\tilde{x}$ in X and homotopy classes of paths $[\gamma]$ and $[\beta]$ connecting them, the composition map $\phi_{[\beta]}^Y \circ \phi_{[\alpha]}^Y$ is precisely $\phi_{[\beta\circ\alpha]}^Y$.

$\phi_{[\gamma]}^Y$ has functorial properties: If $f : (Y,p) \to (Z,q)$ is a morphism of covers, the commutativity of the diagram

$$\begin{array}{ccc}
Fib_x(Y,p) & \xrightarrow[Fib_x(f)]{} & Fib_{x'}(Z,q) \\
{\scriptstyle \phi_{[\gamma]}^Y}\downarrow & & {\scriptstyle \phi_{[\gamma]}^Y}\downarrow \\
Fib_x(Y,p) & \xrightarrow[Fib_{x'}(f)]{} & Fib_{x'}(Z,q)
\end{array}$$

follows from the uniqueness of liftings fixed an initial point.

The collection (ϕ_γ^Y), one for each covering $Y \to X$, defines a natural isomorphism $\phi_{[\gamma]}$ between Fib_x and $Fib_{x'}$ and we conclude

Proposition 4.3.3.1. *Let X be a path-connected topological space. For every two points x and x' in X:*

- *The fiber functors Fib_x and $Fib_{x'}$ are isomorphic*
- *The algebraic fundamental groups $\pi_1^{alg}(X,x)$ and $\pi_1^{alg}(X,x')$ are isomorphic.*

In particular, if the functors Fib_x and Fib_x' are representable, we conclude that their representing objects are isomorphic.

4.3.4 The Monodromy Action of $\pi_1(X, x)$

As a particular case of the previous construction, for any homotopy class $[\gamma] \in \pi_1(X, x)$ we have the corresponding natural isomorphism $\phi_{[\gamma]} : Fib_x \to Fib_x$. This defines a group homomorphism

$$\phi : \pi_1(X, x) \to \pi_1^{alg}(X, x)$$

and in particular, an action of $\pi_1(X, x)$ on the fibers of each covering map. This action is called the monodromy action.

4.4 Galois Theory of Covering Spaces - Representable Case

4.4.1 Representation of the Fiber Functor

The main theorem of this section concerns the representability of the fiber functor. We introduce some standart terminoloy before giving the main result:

We say that a topological space is locally path-connected if every point admits an open neighbourhood U, path-connected by paths whose image lie in U. Also, we say that a space is semilocally simply connected it every point x admits a neighbourhood U such that the inclusion homomorphism $\pi_1(U, x) \to \pi_1(X, x)$ is the null map. Until the end of section 4.4, we will assume X to verify both this properties.

Theorem 4.4.1.1. *If X is a connected, path connected, locally path connected and semilocally simply connected topological space, then the functor Fib_x is representable, $\forall x \in X$.*

Before proving this result we discuss some consequences and properties of this representation. As seen in section 1.6.4, the representation of a functor is given by a pair (A, ζ), where A is an object, in this case, a certain cover $\pi : \tilde{X}_x \to X$ and ξ is a natural transformation $Fib_x \to h_{(\tilde{X}_x,\pi)}$[1]. As Fib_x depends on the point x we also emphasize the dependence of the object representing it.

By Yoneda's lemma, this natural transformation is induced by a unique element $\zeta \in Fib_x(\tilde{X}_x, \pi)$, that is, a certain point in the fiber of x.

Following the remark 1.6.4.2, the representability property ensures for every covering map $p : Y \to X$ the existence of a functorial bijection between $Hom_{\mathbf{Cov(X)}}((\tilde{X}_x, \pi), (Y, p))$ and $Fib_x(Y, p)$. Any morphism of coverings $(\tilde{X}_x, \pi) \to (Y, p)$ is uniquely determined by a point $\tilde{y} \in p^{-1}(x)$ and vice-versa, through $\tilde{y} = Fib_x(u : \tilde{X}_x \to Y)(\zeta)$.

We will now briefly indicate how to construct the space $\tilde{X}_x$ and the map $\pi : \tilde{X}_x \to X$ and introduce a topology that turns this map into a cover.

Construction 4.4.1.2. *Recall the construction of the fundamental groupoid $\Pi_1(X)$ of a path connected space X as a category whose objects were the points of X and between two points x, y we defined $Hom(x, y) =$"Set of homotopy equivalence classes of paths joining x and y".*

We define $\tilde{X}_x$ as the set of homotopy classes of paths starting at x. In other words, the set of all morphisms in $\Pi_1(X)$ starting at x. Each path γ representing a certain class has a final point in X and we

[1]Remember that Fib_x is a covariant functor

define a set map $\pi : \tilde{X}_x \to X$ *mapping* $[\gamma] \mapsto \gamma(1)$. *Remember that we are working with path-homotopy equivalence. If two paths are in the same class, they both end at the same point, and this map is well-defined.*

We add a topology to $\tilde{X}_x$*, introducing a basis of open neighbourhoods of a point* $[\gamma]$*,* $\tilde{U}_{[\gamma]}$*. Here we need some conditions on the space* X*, precisely the ones defined in the beginning of this section - locally path-connected and semi-locally simply connected. For the open sets* $\tilde{U}_{[\gamma]}$ *we consider the set of all homotopy classes of paths obtained by composing* $[\gamma]$ *with some homotopy class of a path* α *with its image contained on a semi-simply connnected neighbourhood of* $\gamma(1)$*. Such neighbourhood of* $\gamma(1)$ *exists because of those topological conditions we are imposing on* X*.*

$$\tilde{U}_{[\gamma]} = \{[\alpha \circ \gamma] : \alpha(I) \subset U\}$$

Semi-simply connectedness of U *ensures that any two paths on* U *when included on* X *with the same endpoints are homotopic.*

The sets $\tilde{U}_{[\gamma]}$ *are a basis for a topology on* $\tilde{X}_x$*: given any two such neighbourhoods of* $[\gamma]$*,* $\tilde{U}_{[\gamma]}$ *and* $\tilde{V}_{[\gamma]}$ *they are constructed from semi-simply connected and locally path connected neighbourhoods* U *and* V *of* $\gamma(1)$ *in* X*, and we can always consider some subset of* $U \cap V$ *with these same properties and repeat the above construction for that subset obtaining a new neighbourhood contained in both neighbourhoods we started with.*

This topology on $\tilde{X}_x$ *makes the above map* $\pi : \tilde{X}_x \to X$ *continuous. Moreover, it is a covering map. Every point* $y \in X$ *is the endpoint of some path* γ *connecting* x *to* y *and admits an open path-connected and semi-simply connected neighbourhood. Its preimage under* π *is precisely the union of all the sets* $\tilde{U}_{[\gamma]}$*. It follows easily that for different equivalence classes, those neighbourhoods are disjoint.*

The fiber of each point $y \in X$ *is precisely the set* $Hom_{\Pi_1(X)}(x, y)$*. In particular, the fiber of* x *is the set* $\pi_1(X, x)$*.*

Finally, we notice a canonical element in $\tilde{X}_x$*, the equivalence class of the constant path on* x*, from now on denoted by* $\tilde{x}$*.*

We will not prove the following properties

Proposition 4.4.1.3. $\tilde{X}_x$ *is a connected and simply-connected topological space.*

We are now in conditions to prove the representability property, constructing a natural transformation $\eta : Fib_x \to h_{(\tilde{X}_x, \pi)}$. First we define a bijection between $Fib_x(Y, p)$ and $Hom_{\mathbf{Cov(X)}}((\tilde{X}_x, \pi), (Y, p))$, valid for any covering $p : Y \to X$ and then we prove that is has functorial properties, inducing a natural transformation.

Proof. (Theorem 4.4.1.1) For each point $\tilde{y} \in Fib_x((Y, p))$ we must find an unique morphism $u_{\tilde{y}} : \tilde{X}_x \to Y$ over X. Remember that points on $\tilde{X}_x$ are equivalence classes of paths on X starting at x. The homotopy lifting property of the covering $p : Y \to X$ ensures that each point $[\gamma] \in \tilde{X}_x$ (joining x and $\gamma(1)$ in X) can be lifted to a path $\tilde{\gamma}$ on Y joining some point over x and some point over $\gamma(1)$. For each choice of a point over x this lifting is known to be unique. Given a point $\tilde{y} \in Fib_x((Y, p))$ we define a map $u_{\tilde{y}} : \tilde{X}_x \to Y$ by sending a class $[\gamma]$ to the final point of the lifting which starts at $\tilde{y}$. This map is well-defined: if γ' is another path homotopic to γ, both their lifts starting at the same point, have the same endpoint, as seen when constructing the monodromy action.

The correspondence $\tilde{y} \to (u_{\tilde{y}} : \tilde{X}_x \to Y)$ is a bijection: $\tilde{y}$ is recovered as the image of the constant path $\tilde{x}$ on x: By construction, this paths lifts to a constant path starting on $\tilde{y}$ and we have $\pi_{\tilde{y}}(\tilde{x}) = \tilde{y}$. The constant path $\tilde{x}$ seen as a point on $\tilde{X}_x$ over x, is the element ζ in the pair representing the functor, inducing the natural transformation.

Denote the correspondence $(u : \tilde{X}_x \to Y) \to \pi(\tilde{x})$ by ξ_x^Y. This collection induces a natural transformation: A morphism of coverings $\phi : (Y, p) \to (Z, q)$ taking y to $\phi(y) = z$, induces morphisms of sets $Fib_x(Y, p) \to Fib_x(Z, q)$ and $Hom_{\mathbf{Cov(X)}}((\tilde{X}_x, \pi), (Y, p)) \to Hom_{\mathbf{Cov(X)}}((\tilde{X}_x, \pi), (Z, q))$. The commutativity of the diagram

$$\begin{array}{ccc} Hom_{\mathbf{Cov(X)}}((\tilde{X}_x, \pi), (Y, p)) & \xrightarrow[\xi_x^Y]{} & Fib_x(Y, p) \\ \downarrow & & \downarrow \\ Hom_{\mathbf{Cov(X)}}((\tilde{X}_x, \pi), (Z, q)) & \xrightarrow[\xi_x^Z]{} & Fib_x(Z, q) \end{array}$$

follows from the fact that each $\tilde{y} \in Fib_x(Y, p)$, for any cover, can be obtained as the image of $\tilde{x}$ through the map $\xi_Y^{-1}(\tilde{y}) : \tilde{X}_x \to Y$.

□

4.4.2 Algebraic Fundamental Group - Representable Case

Denote by U_x the covering representing Fib_x. As seen in section 4.3.3, for any two points x and x' in X, the functors Fib_x and $Fib_{x'}$ are isomorphic and whenever they are representable, so are their representing objects. We say that a covering space $\tilde{X} \to X$ is universal, if it is isomorphic to some U_x representing Fib_x for some $x \in X$. We summarize some important results:

Lemma 4.4.2.1. *Let X be a connected topological space*

- *A covering space $\tilde{X} \to X$ is universal if and only if $\tilde{X}$ is simply-connected;*
- *Any universal cover is Galois.*

Remember the construction of $U_x = (\tilde{X}_x \to X)$ in the last section as the set of homotopy classes of paths starting at x. Each homotopy class of loops $[\gamma] \in \pi_1(X, x)$ defines a map $\psi_{[\gamma]} : \tilde{X}_x \to \tilde{X}_x$ by sending each class $[\alpha]$ to the composition $[\alpha \circ \gamma]$. This map is continuous and it is also a covering morphism because the final point of $\alpha \circ \gamma$ is the final point of α. In additions, $\psi_{[\gamma]}$ is an invertible map, since we could repeat the construction for an inverse path of γ. Therefore, we have a group homomorphism

$$\psi : \pi_1(X, x) \to Aut_{\mathbf{Cov(X)}}(\tilde{X}_x, \pi) \tag{4.3}$$

This map is injective since any path γ that is not homotopic to the constant path $\tilde{x}$, happens to move $\tilde{x}$. Given any automorphism $\phi : \tilde{X}_x \to \tilde{X}_x$ over X, a point $[\alpha]$ is taken to $\phi([\alpha])$ which writable as $[\alpha']$, for some path $\alpha' : I \to X$ starting in x and with the same endpoint as α. The composition $\alpha^{-1} \circ \alpha'$ is a loop in x and $\phi \circ \psi^{-1}_{[\alpha^{-1} \circ \alpha']}$ fixes α. Since the cover $\tilde{X}_x$ is connected, by lemma 4.2.3.1 it has to be the identify map. This proves $\phi = \psi_{[\alpha^{-1} \circ \alpha']}$. We conclude:

Proposition 4.4.2.2. *The group of automorphisms of a universal cover of X is isomorphic to $\pi_1(X, x)$.*

This result is the key to understant why the representable case is so important. Recall section 1.6.4.4: if a functor Fib_x is representable by a pair (U_x, ξ_x), where $\xi_x : h_{U_x} \to Fib_x$ is now assumed to be the natural isomorphism constructed in the previous section, then any automorphism of Fib_x is induced by an automorphism ϕ of U_x:

$$\begin{array}{ccc} h_{U_x} & \xrightarrow{\xi_x} & Fib_x \\ {\scriptstyle (-\circ\phi)}\downarrow & & \downarrow \\ h_{U_x} & \xrightarrow{\xi_x} & Fib_x \end{array}$$

The map $\chi : Aut_{\mathbf{Cov(X)}}(U_x) \to Aut_{\mathbf{Fun(C,Sets)}}(Fib_x) = \pi_1^{alg}(X, x)$ mapping each automorphism ϕ of U_x to $\xi_x \circ (- \circ \phi) \circ \xi_x^{-1}$, is a group isomorphism. Composing the two isomorphisms

$$\pi_1(X, x) \xrightarrow{\psi} Aut_{\mathbf{Cov(X)}}(\tilde{X}_x, \pi) \xrightarrow{\chi} Aut_{\mathbf{Fun(C,Sets)}}(Fib_x) = \pi_1^{alg}(X, x)$$

we conclude that the topological and the algebraic fundamental groups are isomorphic. Moreover, this composition gives precisely the monodromy action: Given a homotopy class $[\gamma] \in \pi_1(X, x)$, we consider $\psi_{[\gamma]}$. Let us see how the action of this automorphism manifests itself on the fibers. Let $Y \to X$ be a covering map and start with a point $y \in Fib_x(Y)$. Let us chase the diagram above: To y we assign the covering map $(\xi_x^Y)^{-1}(y) = (u_y : \tilde{X}_x \to Y)$, as defined in the previous section while proving Fib_x is representable. We act with $\psi_{[\gamma]}$ and get a new covering map, $u_y \circ \psi_{[\gamma]}$. The corresponding fiber point is $\xi_x^Y(u_y \circ \psi_{[\gamma]}) = u_y \circ \psi_{[\gamma]}(\tilde{x}) = u_y(\tilde{x} \circ [\gamma]) = u_y([\gamma])$. Following the construction of u_y, this is precisely the result of the monodromy action of $[\gamma]$ on y.

We summarize all the results above:

Proposition 4.4.2.3. *For a connected, path-connected, locally path connected and semi-locally simply connected topological space X, the monodromy action induces an isomorphism between the topological and the algebraic fundamental groups.*

4.4.3 Categorical Formulation of Galois Theory of $Cov(X)$

We present the main classification theorem for coverings, viewed as an equivalence of categories

Theorem 4.4.3.1. *Let X be a connected, path-connected, locally path-connected and semilocally simply connected topological space and x a point in X. The category **Cov(X)** and the category of $\pi_1^{alg}(X, x)$-sets with a finite number of orbits, are equivalent*

Proof. We define a functor H from **Cov(X)** to $\mathbf{Sets}(\pi_1^{alg}(X, x))$ mapping each cover to the respective fiber set and each covering map to its restriction to the fibers. However, we consider not only the fiber set, but also the action of $\pi_1^{alg}(X, x)$ on this set given by monodromy. This correspondence is a well-defined functor to $\pi_1^{alg}(X, x) - \mathbf{Sets}$ because the restriction of each covering map to the fibers is compatible with the monodromy action.

To prove that this functor H defines an equivalence of categories, we use lemma 1.4.2.3, proving it is fully faithfull and essentially surjective.

For the surjective part, we show that any set S with an action of $\pi_1(X,x)$ producing a finite number of orbits, is isomorphic as a set to the fiber of a certain cover of X. First we consider sets S where the action is transitive. Fixed a point $y \in S$ we consider its subgroup of stabilizers, H_y. Since we are considering the action to be transitive, S is isomorphic as a set to the quotient $\pi_1(X,x)/H_y$ through the map sending each point z of S to the element g_z taking y to z. To construct a covering with fiber S, we use the main theorem of Galois Theory for Galois covering maps. We already known $\tilde{X}_x \to X$ to be Galois and so, the subgroup H_y corresponds to an intermediate cover $\tilde{X}_x/H_y \to X$, obtained by taking a quotient of $\tilde{X}_x$ under the action of H_y.

For full faithfulness, we show that given two covers $Y \to X$ and $Z \to X$, each map $\phi : Fib_x(Y|X) \to Fib_x(Z|X)$ of $\pi_1^{alg}(X,x)$-sets comes from a unique map $Y|X \to Z|X$ of covers. For this, we may assume that Y and Z are connected and consider the map $\pi_y : \tilde{X}_x \to Y$ corresponding to a fixed $y \in Fib_x(Y|X)$. By theorem 4.2.5.3, the map π_y realizes Y as a quotient of $\tilde{X}_x$ by the stabilizer $U_y = Aut_{\mathbf{Cov(Y)}}(\tilde{X}_x|Y)$ of y; Let $\psi_y : Y \to \tilde{X}_x/U_y$ be the inverse map. U_y injects into the stabilizer of $\phi(y)$ via ϕ, the natural map $\pi_{\phi(y)} : \tilde{X}_x \to Y$ corresponding to $\phi(y)$ induces a map $\tilde{X}_x/U_y \to Z$ by passing to the quotient; composing it with ψ_y gives the required map $Y \to Z$.

□

Connected covers are mapped to transitive $\pi_1^{alg}(X,x)$-sets. This induces a bijection between isomorphism classes of connected covers and subgroups of $\pi_1^{alg}(X,x)$. From A.3.3.2,

Corollary 4.4.3.2. *Let X be a connected, path-connected, locally path-connected and semilocally simply connected topological space and x a point in X. The category of finite coverings of X, **Fin-Cov(X)** is equivalent to the category of $\widehat{\pi_1(X,x)}$-sets, where $\widehat{\pi_1(X,x)}$ is the profinite completion of the topological fundamental group.*

This result extends to a larger class of topological spaces

Theorem 4.4.3.3. *(Grothendieck) Let X be a connected topological space and x a point in X. There is an equivalence between the category of finite covering maps of X and the category of finite sets endowed with a continuous action of the algebraic fundamental group, which in this case, is a profinite group,i.e. the limit of the projective system of the finite automorphism groups of all finite Galois coverings.*

This result will be proved in the next chapter, following Grothendieck's approach of *Galois Categories.*

Chapter 5

Galois Categories and Fundamental Groups

The main reference for this chapter is *[4]- Exposé V Section 4 - "Conditions axiomatiques d'une thérie de Galois"* and [2].

5.1 Pro-Representable Functors

This section uses the results of the first chapter and follows Grothendieck's paper [2].

5.1.1 Pro-Objects

Let $\mathbf{C}$ be a category. Any projective system $\beta : \mathbf{I}^\circ \longrightarrow \mathbf{C}$ has an associated limit functor (see 1.7.7)

$$\underleftarrow{lim}_I \beta : \mathbf{C} \longrightarrow \mathbf{Sets}$$

We present another construction assigned to β: The opposite functor $\beta^\circ : \mathbf{I} \longrightarrow \mathbf{C}^\circ$ defines an inductive system in $\mathbf{C}^\circ$: Following from the Yoneda's lemma, the functor $h : \mathbf{C}^\circ \longrightarrow \mathbf{Fun(C,Sets)}$ establishes an equivalence between $\mathbf{C}^\circ$ and its image and the composition

$$\mathbf{I} \xrightarrow{\beta^\circ} \mathbf{C} \xrightarrow{h} \mathbf{Fun(C,Sets)}$$

defines an inductive system of functors $h \circ \beta^\circ$. As an inductive system, it has is own limit functor (see 1.7.7) $\underrightarrow{lim}_I (h \circ \beta^\circ) : \mathbf{Fun(C,Sets)} \longrightarrow \mathbf{Sets}$ given by

$$F \mapsto \underleftarrow{lim}_I (Hom_{\mathbf{Fun(C,Sets)}}(h_{\beta(i)}, F)) = CoCone(Diagram(h \circ \beta^\circ), F) = \underleftarrow{lim}_I (F(\beta(i))) \qquad (5.1)$$

where we use the fact that all projective systems of sets have a limit set and the last equality follows from the Yoneda's Lemma.

Proposition 5.1.1.1. *For any projective system* $\beta : \boldsymbol{I}^{\circ} \longrightarrow \boldsymbol{C}$ *, the limit functor* $\underrightarrow{lim}_I(h \circ \beta^{\circ})$ *is representable in* ***Fun(C,Sets)****.*

Proof. Starting with β, we construct a functor $L(\beta) : \mathbf{C} \longrightarrow \mathbf{Sets}$ and prove that it represents $\underrightarrow{lim}_I(h \circ \beta^{\circ})$. Firstly, we notice that given an object X in $\mathbf{C}$, the collection of sets $\{Hom_{\mathbf{C}}(\beta(i), X\}_{i \in I}$, together with the natural composition maps induced by β, defines an inductive system of sets and after the results of 1.7.7.3 we know that such a limit always exists. The assignment

$$X \to \underrightarrow{lim}_I(Hom_{\mathbf{C}}(\beta(i), X)$$

defines a functor $L(\beta) : \mathbf{C} \longrightarrow \mathbf{Sets}$. The fact that $L(\beta)$ represents $\underrightarrow{lim}_I(h \circ \beta^{\circ})$ follows directly from the construction of inductive limits of sets. Given another functor $F : \mathbf{C} \longrightarrow \mathbf{Sets}$, for each object X in $\mathbf{C}$, by construction (see 1.7.7.6) we have

$$Hom_{\mathbf{Sets}}(L(\beta)(X), F(X)) \cong \underleftarrow{lim}_I(Hom_{\mathbf{Sets}}(Hom_{\mathbf{C}}(\beta(i), X), F(X)))$$

and therefore

$$Hom_{\mathbf{Fun(C,Sets)}}(L(\beta), F) \cong (\underrightarrow{lim}_I(h \circ \beta^{\circ}))(F) = \underleftarrow{lim}_I(Hom_{\mathbf{Fun(C,Sets)}}(h_{\beta(i)}, F))$$

□

We conclude that any projective system in **C** can be seen as an inductive system in **Fun(C,Sets)** by means of the Yoneda's embedding, and more importantly, even if β does not have a limit in **C**, the assigned inductive system $h \circ \beta^{\circ}$ always has a colimit object in **Fun(C,Sets)**.

Definition 5.1.1.2. *Let* ***C*** *be a category. The category of pro-objects in* ***C****, denoted* ***Pro(C)****, is such that:*

- *the objects are projective systems* $\beta : \boldsymbol{I}^{\circ} \longrightarrow \boldsymbol{C}$ *;*
- *given two projective systems* $\beta : \boldsymbol{I}^{\circ} \longrightarrow \boldsymbol{C}$ *and* $\beta' : \boldsymbol{J}^{\circ} \longrightarrow \boldsymbol{C}$ *we define*

$$Hom_{\boldsymbol{Pro(C)}}(\beta, \beta') := Hom_{\boldsymbol{Fun(C,Sets)}}(L(\beta'), L(\beta)) \quad (5.2)$$

Indeed, the fact that **Fun(C,Sets)** is a category, ensures that **Pro(C)** as defined above, is also a category.

Notice that, as a result of 5.1.1.1, the formula 5.2 can be rewritten as

$$\underleftarrow{lim}_J(Hom_{\mathbf{Fun(C,Sets)}}(h_{\beta'(j)}, L(\beta)))$$

and using Yoneda's Lemma

$$= \underleftarrow{lim}_J(L(\beta)(\beta'(j)) = \underleftarrow{lim}_J \underrightarrow{lim}_I Hom_{\mathbf{C}}(\beta(i), \beta'(j)) \quad (5.3)$$

Proposition 5.1.1.3. *Let $\boldsymbol{C}$ be a category, $\beta : \boldsymbol{I}^\circ \longrightarrow \boldsymbol{C}$ a projective system and $\varphi : \boldsymbol{J} \longrightarrow \boldsymbol{I}$ a functor, with both $\boldsymbol{I}$ and $\boldsymbol{J}$ small categories. If φ is cofinal then β and $\beta \circ \varphi^\circ$ are isomorphic in **Pro(C)**.*

Proof. By 1.7.8.2, if φ is cofinal, both limit functors $\varprojlim_I \beta : \mathbf{C} \longrightarrow \mathbf{Sets}$ and $\varprojlim_I (\beta \circ \varphi^\circ) : \mathbf{C} \longrightarrow \mathbf{Sets}$ are isomorphic and therefore, the objects representing them, respectively, $L(\beta)$ and $L(\beta \circ \varphi^\circ)$ are also isomorphic in **Fun(C,Sets)**. By the construction of **Pro(C)**, β and $\beta \circ \varphi^\circ$ are isomorphic. □

Note that **C** can be seen as a subcategory of **Pro(C)** by identifying each object X in **C** with the constant functor $\beta_X : \mathbf{I}^\circ \longrightarrow \mathbf{C}$ assigning $i \mapsto X$ and $(i \to j) \mapsto I_X$. Under this circunstances, we trivially have $L(\beta_X) = h_X$. If X and Y are two objects in **C**, and β_X and β_Y their, respective, trivially assigned constant projective systems, we have (after the Yoneda's Lemma)

$$Hom_{\mathbf{Pro(C)}}(\beta_X, \beta_Y) = Hom_{\mathbf{Fun(C,Sets)}}(h_X, h_Y) \cong Hom_{\mathbf{C}}(Y, X)$$

The correspondence $\beta_{(-)} : X \mapsto \beta_X$ defines a functor that, after this result, is fully faithfull, allowing us understand **C** as a subcategory of **Pro(C)**.

Remark 5.1.1.4. *With the previous embedding and using the formula 5.3 above, $L(\beta)$ can be rewriten as*

$$L(\beta) = Hom_{\boldsymbol{Pro(C)}}(\beta, \beta_{(-)}) : \boldsymbol{C} \longrightarrow \boldsymbol{Sets}$$

*or, in other words, $L(\beta)$ is simply the restriction of $Hom_{\boldsymbol{Pro(C)}}(\beta, -)$ to $\boldsymbol{C}$ viewed as a subcategory of **Pro(C)**.*

Notation 5.1.1.5. *When viewing $\boldsymbol{C}$ as a subcategory of **Pro(C)**, sometimes we simply write X to denote β_X, whenever there is no ambiguity.*

Moreover, the assignment $\beta \mapsto L(\beta)$ defines a functor

$$L : \mathbf{Pro(C)} \longrightarrow \mathbf{Fun(C,Sets)}$$

By definition (5.1.1.2) this functor is fully faithfull and therefore we may also understand **Pro(C)** as a subcategory of **Fun(C,Sets)**. This construction yields an extension of the Yoneda's embedding.

5.1.2 Pro-Representable Functors

Definition 5.1.2.1. *A functor F in **Fun(C,Sets)** is called pro-representable if it is isomorphic to some functor in the image of L.*

Every functor $F : \mathbf{C} \longrightarrow \mathbf{Sets}$ can be naturally extended to a functor $\bar{F} : \mathbf{Pro(C)} \longrightarrow \mathbf{Sets}$, given by

$$(\ \beta : \mathbf{I}^\circ \longrightarrow \mathbf{C}\ ) \mapsto \varprojlim_I (F(\beta(i)))$$

By applying the formula 5.1 and Prop. 5.1.1.1 we find that

$$\bar{F} = Hom_{\mathbf{Fun(C,Sets)}}(L(-), F)$$

Proposition 5.1.2.2. *A functor* $F : \boldsymbol{C} \longrightarrow \boldsymbol{Sets}$ *is pro-representable in* $\boldsymbol{C}$ *if and only if the extension* $\bar{F} : \boldsymbol{Pro(C)} \longrightarrow \boldsymbol{Sets}$ *is representable in* $\boldsymbol{Pro(C)}$*.*

Proof. If $F \cong L(\beta)$, we have

$$\bar{F} = Hom_{\mathbf{Fun(C,Sets)}}(L(-), F) \cong Hom_{\mathbf{Fun(C,Sets)}}(L(-), L(\beta)) = Hom_{\mathbf{Pro(C)}}(\beta, -)$$

The converse also follows immediately from the above equations.

□

Let $F : \mathbf{C} \longrightarrow \mathbf{Sets}$ be pro-representable by β. After the previous result, the whole theory concerning representable functors in 1.6.4 can be applied to $\bar{F}$:

1. We have

$$Aut_{\mathbf{Fun(C,Sets)}}(F) \cong Aut_{\mathbf{Fun(C,Sets)}}(L(\beta)) = Aut_{\mathbf{Pro(C)}}(\beta) \cong Aut_{\mathbf{Fun(Pro(C),\ Sets)}}(\bar{F})$$

   where the equality in the middle follows from the definition of $\mathbf{Pro(C)}$.

2. The isomorphism

$$\xi : Hom_{\mathbf{Pro(C)}}(\beta, -) \to \bar{F}$$

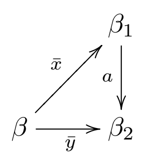
   is determined by a unique element $\zeta = (\zeta_i)_{\in I} \in \bar{F}(\beta) = \varprojlim_I F(\beta(i))$, related to ξ through

$$\zeta = (\zeta_i) = \xi_\beta(I_\beta)$$

   Given any other β' in $\mathbf{Pro(C)}$, each element $x \in \bar{F}(\beta')$ is identified with a unique morphism $\bar{x} : \beta \to \beta'$ in $\mathbf{Pro(C)}$ through the relation

$$x = \bar{F}(\bar{x})(\zeta)$$

   Moreover, given any morphism $a : \beta_1 \to \beta_2$ between pro-objects, the induced map $\bar{F}(a) : \bar{F}(\beta_1) \to \bar{F}(\beta_2)$ sending $x \mapsto \bar{F}(a)(x)$ is identified with the composition operation



$$\bar{x} \longmapsto a \circ \bar{x}$$

5.1.3 Conditions for Pro-Representability

Let **C** be a category where all limits exist. Given a functor $F : \mathbf{C} \longrightarrow \mathbf{Sets}$ we define (1.3)

$$\mathbf{I}_F = \text{Subcategory of } \mathbf{Fun(C,Sets)}/F \text{ of all } (\mathbf{C}) \text{ representable functors, over } F$$

Remark 5.1.3.1. *Using the Yoneda's lemma we can identify $\mathbf{I}_F$ with the opposite category of pairs (X, x) where X is an object in $\mathbf{C}$ and x is an element in $F(X)$, where we define a morphism $(X, x) \to (Y, y)$ as a morphism $f : X \to Y$ in $\mathbf{C}$ with $F(f)(x) = y$.*

There is a natural source functor

$$source_F : \mathbf{I}_F \longrightarrow \mathbf{Fun(C,Sets)}$$

assigning $(h_X \to F) \mapsto h_X$ and sending each natural transformation $h_X \to h_Y$ over F, to itself. The image of this functor spams a diagram, denoted $Diagram(source)$, in **Fun(C,Sets)**.

Proposition 5.1.3.2. *The colimit of $Diagram(source_F)$ in **Fun(C,Sets)** is isomorphic to F.*

Proof. Consider the colimit functor of $source_F$

$$\varinjlim_{\mathbf{I}_F}(source_F) : \mathbf{Fun(C,Sets)} \longrightarrow \mathbf{Sets}$$

assigning

$$A \mapsto \varprojlim_{\mathbf{I}_F} Hom_{\mathbf{Fun(C,Sets)}}(source(h_X \to F), A) = \varprojlim_{\mathbf{I}_F} Hom_{\mathbf{Fun(C,Sets)}}(h_X, A) = \varprojlim_{\mathbf{I}_F} A(X)$$

We notice that each natural transformation $\phi : F \to A$ corresponds to a unique element in $\varprojlim_{\mathbf{I}_F} A(X)$ and vice-versa. This follows immediately, since we can simply identify each natural transformation $\phi : F \to A$ with the family $(\phi(x))_{x \in F(X)}$. This identification is well-defined because, for every morphism $v : X \to Y$ in **C**, the diagrams

$$\begin{array}{ccc} F(X) & \longrightarrow & A(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & A(Y) \end{array}$$

□

Notice that the source functor can be written as the composition

$$\mathbf{I}_F \longrightarrow \mathbf{C}^\circ \xrightarrow{h} \mathbf{Fun(C,\ Sets)}$$

where the first functor sends $(h_X \to F) \mapsto X$ and using the Yoneda's Lemma, each morphism $h_X \to h_Y$ over F to the unique correspondent morphism $Y \to X$ in $\mathbf{C}$. Its opposite functor can be understood as a projective system in $\mathbf{C}$

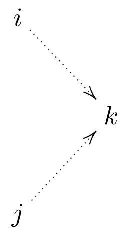
$$\beta_F : \mathbf{I}_F^\circ \longrightarrow \mathbf{C} \tag{5.4}$$

and by Prop. 5.1.3.2 we immediately have $F \cong L(\beta_F)$ and therefore, F is pro-representable by β_F.

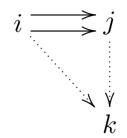
Proposition 5.1.3.3. *Under the hypothesis that $\boldsymbol{C}$ has all limits, if F commutes with limits then $\boldsymbol{I}_F$ is filtrant.*

Proof.

- Since all limits exist in $\mathbf{C}$, in particular, $\mathbf{C}$ has a terminal object $1_{\mathbf{C}}$ and since F commutes with limits, the set $F(1_{\mathbf{C}}) = \{\star\}$ is non-empty and therefore, by the Yoneda's Lemma, there is at least one functor $h_{1_{\mathbf{C}}} \to F$ over F and $\mathbf{I}_F$ is non-empty.
- Writing $i = (h_X \to F)$ and $j = (h_Y \to F)$, since $\mathbf{C}$ has all products and F commutes with products, we have $F(X \times Y) = F(X) \times F(Y)$ and writting $k = (h_{X\times Y} \to F)$, the canonical projections $X \times Y \to X$ and $X \times Y \to Y$ induce morphisms



- Again, using the same notation for i and j and the fact that all equalizers exist in $\mathbf{C}$, we have



with $k = (h_Z \to F)$ where $Z = Equalizer(Y \rightrightarrows X)$

□

5.1.4 Minimal Pairs

Before ending this section and working with the result that F is pro-representable by β_F 5.4, remember the remark 5.1.3.1. We introduce minimal pairs

Definition 5.1.4.1. *A pair (X, x) is called minimal if every morphism of pairs $f : (Y, y) \to (X, x)$ with $f : Y \to X$ a strict monomorphism in $\boldsymbol{C}$ is an isomorphism.*

As seen in 1.8 Prop. 1.8.0.11, $f : Y \to X$ is a strict monomorphism if and only if $Y \cong Im(f)$.

Proposition 5.1.4.2. *If for every pair (X, x) there is a morphism of pairs $(X, x) \to (Y, y)$ with (Y, y) a minimal pair then:*

1. *The subcategory $\boldsymbol{I}_F^M$ identified with the opposite subcategory of all minimal pairs, is filtrant;*

2. *The inclusion functor $\varphi : \boldsymbol{I}_F^M \longrightarrow \boldsymbol{I}_F$ is cofinal and F is pro-represented by $\beta_F \circ \varphi^\circ$*

Proof. This follows directly from 1.7.9.3. □

Minimal pairs have the following property:

Proposition 5.1.4.3. *Under the hypothesis that $\boldsymbol{C}$ as all limits and F commutes with them, for every minimal pair (A, a), the natural map $\tilde{a} : h_A(X) \to F(X)$ is injective, for every object X in $\boldsymbol{C}$.*

Proof. Recall that this map is defined by $(v : A \to X) \mapsto F(v)(a)$. Suppose that $F(v)(a)$ is equal to $F(u)(a)$ for two different maps v and u, $A \to X$. Since all equalizers exist in **C** and in **Sets** and F commutes with equalizers, also the common equalizer of all pairs $A \underset{u}{\overset{v}{\rightrightarrows}} X$ exists in **C**, given by an object C and a strict monomorphism $f : C \to A$. (1.7.1-3). In this case, for each object X, the diagram

$$C \xrightarrow[f]{} A \underset{u}{\overset{g}{\rightrightarrows}} X$$

is mapped to

$$F(C) \xrightarrow[F(f)]{} F(A) \underset{F(u)}{\overset{F(g)}{\rightrightarrows}} F(X)$$

and a is an element in $F(C)$ with $F(f)(a) = a$. Since $C \to A$ is a strict mono and A is minimal we conclude that f is an isomorphism and so $f = g$. □

5.2 Grothendieck-Galois Theory

Let $\mathbf{C}$ be a category and $F : \mathbf{C} \longrightarrow \mathbf{Sets}$ a covariant functor with values on finite sets. We will consider pairs $(\mathbf{C}, F)$ satisfying the following conditions:

On $\mathbf{C}$:

- *(G1) $\boldsymbol{C}$ has a final object $1_{\boldsymbol{C}}$ and the fiber products of any two objects over a third one exist.*

  After the results in the first chapter, this is equivalent to the existence of all limits in $\mathbf{C}$.

- *(G2) $\boldsymbol{C}$ admits finite sums (in particular has an initial object $\emptyset_{\boldsymbol{C}}$) and quotients (1.9.3) by finite group actions exist in $\boldsymbol{C}$*;

- *(G3) Every morphism $u : Y \to X$ in $\boldsymbol{C}$ factors as $Y \xrightarrow{u'} X' \xrightarrow{u''} X$ with u' a strict epimorphism and u'' is a monomorphism that is an isomorphism with a direct summand of X*;

On F:

- *(G4) F maps terminal objects to terminal objects (one-element sets) and commutes with fibre products;*

  After 1.7.6 this is indeed equivalent to say that F commutes with all limits in $\mathbf{C}$ whenever they exist. In particular, with all equalizers and finite products.

- *(G5) F commutes with finite sums, transforms strict epimorphisms in epimorphisms and commutes with passage to the quotient by an action of a finite group of automorphisms;*

  This means F takes sums of objects in $\mathbf{C}$ to finite disjoint unions of sets. Moreover, we have $F(X/G) = F(X)/G$ where the action of G on $F(X)$ happens according to 1.9.3.

- *(G6) Given some morphism $u : Y \to X$ in $\boldsymbol{C}$, if $F(u) : F(Y) \to F(X)$ is an isomorphism then u is also an isomorphism.*

Following Grothendieck's seminar [4] Exposé V, we have two goals in mind:

- Describe the group $\pi := Aut_{\mathbf{Fun(C,Sets)}}(F)$;
- Construct an equivalence of categories between $\mathbf{C}$ and the category of finite sets endowed with a continuous action of π.

5.2.1 Subobjects of an object X and Subsets of $F(X)$

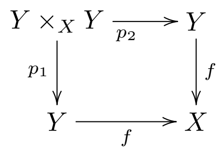
1.

Proposition 5.2.1.1. *A morphism u in $\boldsymbol{C}$ is a monomorphism if and only if $F(u) : F(Y) \to F(X)$ is a monomorphism.*

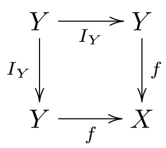
To prove this we use the following lemma:

Lemma 5.2.1.2. *Consider $u : Y \to X$ in $\boldsymbol{C}$. u is a monomorphism if and only if the first projection $p_1 : Y \times_X Y \to Y$ obtained together with the fiber product of the diagram $Y \xrightarrow{u} X \xleftarrow{u} Y$ is an isomorphism.*

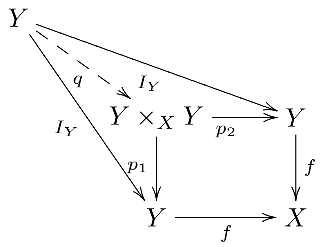
Proof. Supposing that all fibre products exist in $\mathbf{C}$, we have a commutative diagram



($\Rightarrow$): If f is a monomorphism, the fact that $f \circ p_1 = f \circ p_2$ implies that $p_1 = p_2 =: p$. Moreover, the identity morphism $I_Y : Y \to Y$ induces another commutative diagram

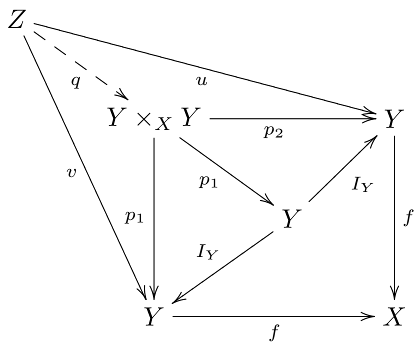


and therefore, by the properties of the fiber product, factors in a unique way



we have $p \circ q = I_Y$ and by the universal property of the fiber product, $q \circ p$ has to be identity morphism.

($\Leftarrow$) Given any object Z in $\mathbf{C}$ and any two morphisms $u, v : Z \to Y$ with $f \circ u = f \circ v$, we know that this pair factors in a unique way through the canonical pair (p_1, p_2). We have a commutative diagram



Since p is an isomorphism, by the universal property of the fiber product, Y together with the pair (I_Y, I_Y) is also a fiber product and therefore $u = v$.

$\square$

We are now able to prove 5.2.1.1:

Proof. This follows directly from (G1), (G4) and (G6), together with the previous lemma: If $Y \to X$ is mono, by (G1) the fiber product $Y \times_X Y \to Y$ exists which is, by the lemma, an isomorphism.

Therefore, $F(Y \times_X Y) \to F(Y)$ is also an isomorphism. By (G4), we have $F(Y) \times_{F(X)} F(Y) \to F(Y)$ also an isomorphism and again by the lemma, $F(Y) \to F(X)$ is a monomorphism.

Conversely, if $F(Y) \to F(X)$ is a monomorphism, since all fibre products exist in **Sets**, we have $F(Y) \times_{F(X)} F(Y) \to F(Y)$ also an isomorphism. Again, by (G4), $F(Y \times_X Y) \to F(Y)$ is an isomorphism and using (G6), we conclude that $Y \times_X Y \to Y$ is also an isomorphism and finally $Y \to X$ is mono. □

From this proposition, we conclude that the subobjects of a given object X are in bijection with subsets $F(Y) \subseteq F(X)$.

2. Consider now a sequence of monomorphisms $X' \xrightarrow{f} X'' \xrightarrow{g} X$. According to the last item, this induces a sequence of inclusions $F(X') \xrightarrow{F(f)} F(X'') \xrightarrow{F(g)} F(X)$.

   If the images of $F(X')$ and $F(X'')$ are equal in $F(X)$ we conclude that $F(f)$ is a surjective map. Since it was already injective, it is a bijection. By (G6) we conclude that f is an isomorphism.

   Proposition 5.2.1.3. *If two subobjects of X are mapped into the same subset of $F(X)$ then they are isomorphic.*

3. Under the hypothesis (G2), there is an initial object $\emptyset_{\mathbf{C}}$ in $\mathbf{C}$. Moreover, in **Sets** we known that the empty set plays a similar role. We say an object I in $\mathbf{C}$ is a right unit in if it is isomorphic to an initial object. After 1.7.4, any right unit plays a neutral role with respect to the sum operation on objects.

   If I is a right unit, condition (G5) implies that $F(I) = \emptyset$, a right unit in **Sets**. Conversely, if $F(I)$ is the empty set, after (G5) and (G6) we conclude that I has to be a right unit. So

   Proposition 5.2.1.4. *$F(X)$ is empty if and only if X is isomorphic to $\emptyset_{\mathbf{C}}$.*

   In particular, $F(\emptyset_{\mathbf{C}}) = \emptyset$. We also find that $F(X)$ is non-empty if and only if X is not isomorphic to $\emptyset_{\mathbf{C}}$. Moreover, there are no morphism in $\mathbf{C}$ from a non-right unit object X to a right-unit object I: Indeed, there are no set maps from a non-empty set to the empty set.

4. (Subobjects and Summands) If an object X in $\mathbf{C}$ is written as $Y \coprod Z$, since F preserves finite sums (G5), we have $F(X) = F(Y \coprod Z) = F(Y) \coprod F(Z)$. The canonical map $Y \to X$ induces an inclusion $F(Y) \to F(X)$ and by the first item above, we conclude that the canonical morphism is a monomorphism.

   Conversely, if $u : Y \to X$ is a monomorphism, by (G3), Y is naturally identified with a direct summand of X and u is the canonical inclusion.

   Proposition 5.2.1.5. *$Y \to X$ is a monomorphism if and only if Y is a direct summand of X.*

   As a corollary, for any object X in $\mathbf{C}$, the unique morphism $\emptyset_{\mathbf{C}} \to X$ is a monomorphism, since $\emptyset_{\mathbf{C}}$ is a direct summand of any object.

5.2.2 Connected Objects

1.

   Definition 5.2.2.1. *Let $\mathbf{C}$ be a category and X some object. We say that X is connected if it is not isomorphic to a direct sum of other two objects in $\mathbf{C}$ not isomorphic to the initial object $\emptyset_{\mathbf{C}}$.*

With all the information gathered from the last section, we conclude that an object X is connected if and only if its only subobjects are $\emptyset_{\mathbf{C}} \to X$ and $I_X : X \to X$. Equivalently, the subsets of $F(X)$ corresponding to some direct summand of X are precisely $F(X)$ and $\emptyset$. If X is not connected we can write it as $Y \coprod Z$ and (G5) implies $F(X)$ to be a disjoint union of $F(Y)$ and $F(Z)$. Proceeding by induction, we conclude that every object X is isomorphic to a finite direct sum of connected subobjects of X, called the connected components of X.

Remark 5.2.2.2. *Notice that an initial object is not connected.*

2.

Proposition 5.2.2.3. *Every morphism $u : Y \to X$ in $\boldsymbol{C}$ with Y not isomorphic to $\emptyset_{\boldsymbol{C}}$ and X connected is a strict-epimorphism.*

Proof. Consider the factorization of u as in (G3):

$$Y \xrightarrow{u'} X' \xrightarrow{u''} X$$

Since Y is not isomorphic to $\emptyset_{\mathbf{C}}$, by 5.2.1.4 we conclude that $F(Y) \neq \emptyset$. Therefore $F(X') \neq \emptyset$. By the same result we have that $X' \ncong \emptyset_{\mathbf{C}}$. Since X is connected, $X' \to X$ is a monomorphism and $X' \ncong \emptyset_{\mathbf{C}}$, we conclude that $X' = X$ and so $u = u'$, the first term in the factorization. ☐

We immediately conclude that any endomorphism $u : X \to X$ of a connected object X is a strict epimorphism. After (G5), F(u) is surjective and since F(X) is finite, it is a bijection. By (G6) we conclude that u is an isomorphism:

Corollary 5.2.2.4. *Every endomorphism of a connected object is an automorphism.*

3. After the description in 5.1.3, under (G1), (G2) and (G3) we shall prove that:

Proposition 5.2.2.5. *If X is a connected object then any pair (X, x) is minimal*

Proof. Every strict monomorphism is a monomorphism and so, if X is connected, for every morphism of pairs $(C, c) \to (X, x)$ induced by a strict monomorphism we have $c \in F(C)$ and $F(C) \subseteq F(X)$ and $C \ncong \emptyset_{\mathbf{C}}$. Since X is connected, we conclude that f is an isomorphism. ☐

Indeed, under the conditions worked in 5.1.3, (G1) and (G4) are equivalent to (C1) and (C2), respectively. It remains to prove that connected objects verify condition (C3). In fact:

Proposition 5.2.2.6. *Every pair (A, a) is bounded by a pair (X, x) with X connected.*

Proof. This follows immediately from the fact that any object A is a direct sum of connected objects. The natural inclusion is the desired bound. ☐

From the results in 1.7.9.3 and 5.1.4.2, and denoting by $\mathbf{I}_F^c$ the subcategory of $\mathbf{I}_F$, of all pairs (X, x) with X a connected object and $x \in F(X)$, we immediately conclude that the inclusion functor $\mathbf{I}_F^c \longrightarrow \mathbf{I}_F$ is cofinal with respect to P, the system of all minimal objects.

Corollary 5.2.2.7. *Under the conditions (G4), (G5) and (G6), F is pro-representable by the pro-object $P^c : \boldsymbol{I}_F^c \longrightarrow \boldsymbol{C}$, where $\boldsymbol{I}_F^c$ is the dual category of the subcategory of $\boldsymbol{C/F}$, of all pairs $i = (X, x)$ with $P^c(i) = X$ a connected object in $\boldsymbol{X}$ and $x \in F(X)$.*

From 5.2.2.3 we conclude that all transition morphisms defining P^c are strict epimorphisms. In this case, we say F is *strictly pro-representable.*

5.2.3 Galois Objects

1. In the end of the last section we were able to restrict our attention to the system of connected objects, simpler and more familiar than minimal objects. In the present section we will show that there is another possible simplification: We define *Galois Objects* as a special type of connected objects and prove that they still pro-represent F. By representing F using all connected objects there is no canonical way to describe the group π. The definition of a Galois object gives us a simpler description:

   Definition 5.2.3.1. *We say that a connected object X is* Galois with respect to F *if the natural map $\tilde{x}_X : h_X(X) \to F(X)$ is surjective, for all $x \in F(X)$.*

   Notice that if X is Galois, since it is connected, the sets $h_X(X) = Hom_{\mathbf{C}}(X, X)$ and $Aut_{\mathbf{C}}(X)$ are equal. Moreover, from 5.1.4.3 the above map $h_X(X) = Aut_{\mathbf{C}}(X) \to F(X)$ is injective and so a connected object is Galois if and only if the above maps

   $$\tilde{x} : Aut_C(X) \longrightarrow F(X) \qquad\qquad \sigma \longmapsto F(\sigma)(x)$$

   are bijections, for all $x \in F(X)$. In particular, we find that

   Corollary 5.2.3.2. *The group of automorphisms of a Galois object is finite.*

   We give another description of Galois Objects: Notice that the above maps naturally define an action of $Aut_C(X)$ on the set $F(X)$. Each automorphism $\sigma : X \to X$ induces a permutation $F(\sigma) : F(X) \to F(X)$ mapping $x \mapsto F(\sigma)(x)$. Surjectivity of the above maps is equivalent to transitivity of this action. In other words, the above map is surjective if and only if the quotient $F(X)/Aut_{\mathbf{C}}(X)$ is a terminal object in **Sets** (a one-element set). By applying (G5), (G2) and (G6) we conclude that

   Proposition 5.2.3.3. *Let X be an object in $\boldsymbol{C}$. Then X is Galois if and only if the quotient $X/Aut_{\boldsymbol{C}}(X)$ is isomorphic to a terminal object $1_{\boldsymbol{C}}$. In particular, the definition of a Galois object does not depend on F.*

2. We introduce an important property of Galois objects, concerning the description of the group π: Given any morphism $f : (X, x) \to (Y, y)$ between a pair of Galois objects, there is a canonical way to define a morphism of groups $Aut_{\mathbf{C}}(X) \to Aut_{\mathbf{C}}(Y)$, induced by means of the following diagram and using the fact that both horizontal arrows are isomorphisms:

   $$\begin{array}{ccc} (X, x) & \qquad & Aut_{\mathbf{C}}(X) \xrightarrow[\tilde{x}]{\sim} F(X) \\ \downarrow f & & \bar{f} \downarrow \qquad\qquad \downarrow F(f) \\ (Y, y) & & Aut_{\mathbf{C}}(Y) \xrightarrow[\tilde{y}]{\sim} F(Y) \end{array}$$

   mapping each automorphism $\sigma : X \to X$ to the unique automorphism λ of Y with $F(f \circ \sigma)(x) = F(\lambda)(y)$. Since f is a morphism between connected objects, by 5.2.2.3, f is a strict epimorphism and by (G5) the map $F(f)$ is surjective. and so is $\bar{f}$.

3. Consider the category $\mathbf{I}_F^G$ of all Galois pairs (X, x) where X a Galois object in $\mathbf{C}$ and $x \in F(X)$. We have a natural sequence of inclusions

   $$\mathbf{I}_F^G \longrightarrow \mathbf{I}_F^c \longrightarrow \mathbf{I}_F \qquad (5.5)$$

Every Galois pair is a connected pair and, therefore, also a minimal pair. To prove that $\mathbf{I}_F^G$ is cofinal with respect to $\mathbf{I}_F^c$, by **??**, it suffices to prove that every connected pair is bounded by a Galois pair:

Let X be some connected object. From (G1) we know that the finite product $\Pi_{x\in F(X)}X$ exists in $\mathbf{C}$ and by (G4) we have $F(\Pi_{x\in F(X)}X) = \Pi_{x\in F(X)}F(X)$ where each element is a family $\bar{x} = (\bar{x}_x)_{x\in X}$. After 5.2.2, we consider the connected component A of this product, containing the family $\bar{x} = (\bar{x}_x)_{x\in X}$ with $\bar{x}_x = x$ for all $x \in F(X)$. This means that the coordinates of $\bar{x}$ are precisely all the elements of $F(X)$, each occuring once. We prove that A is Galois. Indeed, since A is connected the map $\tilde{\bar{x}} : h_A(X) \to F(X)$ is injective. It is also surjective, because it factors through the canonical projections $\Pi_{x\in F(X)}X \to X$. Therefore, we have a bijection $h_A(X) \cong F(X)$ and $h_A(X)$ is finite. If we consider another element $\bar{a}$ in $F(A)$, the induced map $\tilde{a} : h_A(X) \to F(X)$ is again injective. Since both sets have the same cardinality, this map is also a bijection. We conclude that the coordinates of any element $\bar{a} \in F(A)$ are precisely all the elements of $F(X)$, each occuring once. By permuting the factors, $\Pi_{x\in F(X)}X \to \Pi_{x\in F(X)}X$ it is possible to construct an automorphism of the product mapping $\bar{x}$ to any other element $\bar{a}$ in $F(A)$. Since A is connected, the restriction of this automorphism to A is an automorphism of A. This way we proved that the action of $Aut_C(A)$ on $F(A)$ is transitive and by 5.2.3.3 we conclude that A is Galois.

Remark 5.2.3.4. *To find A we could consider instead a connected pair (C, c) bounding the pair $(\Pi_{x\in F(X)}, \bar{x})$. By (G3), this bounding factors as $C \to A \to \Pi_{x\in F(X)}$.*

We summarize these results:

Proposition 5.2.3.5. *Let X be a connected object in $\boldsymbol{C}$. Then, there is a Galois pair (A, a) such that the induced morphism $\tilde{a} : h_A(X) \to F(X)$ is an isomorphism.*

Proposition 5.2.3.6. *The inclusion functor $\boldsymbol{I}_F^G \longrightarrow \boldsymbol{I}_F^c$ is cofinal with respect to the pro-object $P^c : \boldsymbol{I}_F^c \longrightarrow \boldsymbol{C}$ of all connected objects in $\boldsymbol{C}$.*

and finally,

Theorem 5.2.3.7. *Let $\boldsymbol{C}$ be a category under the conditions (G1), (G2) and (G3) and F covariant functor with values in the category of finite sets, satisfying conditions (G4), (G5) and (G6). Then, F is pro-represented by $P^G : \boldsymbol{I}_F^G \longrightarrow \boldsymbol{I}_F^c \longrightarrow \boldsymbol{C}$, the system of all Galois pairs.*

Corollary 5.2.3.8. *The groups π and $Aut_{\boldsymbol{Pro(C)}}(P^G)$ are anti-isomorphic*

4. After the considerations in 5.2.3-2, P^G naturally induces a projective system of finite groups where every transition morphism is a surjective map. Moreover,

$$Hom_{\mathbf{Pro(C)}}(P^G, P^G) = \bar{F}(P^G) = \varprojlim_{\mathbf{I}^G} F(P_i) \cong \varprojlim_{\mathbf{I}^G} Aut_{\mathbf{C}}(P_i)$$

Therefore, every endomorphism of P^G is an automorphism. We finally conclude that

$$\pi \cong Aut_{\mathbf{Pro(C)}}(P^G) = \varprojlim_{\mathbf{I}_F^G} Aut_{\mathbf{C}}(P_i)$$

and π is described as a profinite group, the limit of a projective system of automorphism's groups of all galois objects, $Aut_{\mathbf{C}}(P_i^G)$.

5.2.4 Equivalence of Categories

Now that we have a description of π as a profinite group, we construct an equivalence between $\mathbf{C}$ and the category of finite sets with a continuous action of π.

As in 1.9.2.1, the action of π on the sets $F(X)$ is equivalent to the natural action of $Aut_{\mathbf{Pro(C)}}(P^G)$ on $Hom_{\mathbf{Pro(C)}}(P^G, X)$. This action is continuous. After the results in A.3.3.1 this is equivalent to say that it factors through a finite quotient $Aut_{\mathbf{C}}(P_i^G)$. From 5.2.3.5, and the fact that $\mathbf{I}_F^G$ is filtrant, we conclude that for every X in $\mathbf{C}$ there is a Galois pair $(P_i^G = A, \zeta_i = a)$ such that every morphism $P \to X$ factors (in a unique way) through $P \to^{\bar{a}} A \to X$. This implies that the action of $Aut_{\mathbf{Pro(C)}}(P^G)$ factors through $Aut_{\mathbf{C}}(A)$.

Proposition 5.2.4.1. *The action of π on the sets $F(X)$ is continuous.*

Let $\pi - \mathbf{FSets}$ be the category of finite discrete topological spaces equipped with a continuos action of π and consider those morphisms compatible with the action. By definition, any morphism $F(u) : F(X) \to F(Y)$ is compatible with the action of π. Therefore, F can be seen as a functor

$$\mathbf{C} \longrightarrow \pi - \mathbf{FSets}$$

We can define a quasi-inverse functor: Notice the fact that both $\pi - \mathbf{FSets}$ and $\mathbf{C}$ have all finite sums and every object in both categories can be decomposed as a sum of connected objects. We define the values of a functor

$$G : \pi - \mathbf{FSets} \longrightarrow \mathbf{C}$$

first on connected objects and then on all objects. Notice that E is connected if and only if the action of π is transitive. In this case, and since the action is continuous, the stabilizer of an element $e \in E$ is an open subgroup $\pi_e \subseteq \pi$ (A.3.2.1). From the results in A.3.3.1, the fact that the action is continuous and π is profinite, also implies that π_e contains some kernel $p_i : \pi \to \pi_i$, where $\pi_i = Aut_{\mathbf{C}}(P_i^G)^\circ$ is one of the finite groups in the projective system whose limit is π and P_i^G is a Galois object. Therefore E is isomorphic (as a π-set) to π/π_e. From (G2) we can define

$$G(E) := P_i^G / p_i(\pi_e)$$

If E is not connected, we have a decomposition of $E = \coprod_{[e] \in E/\pi} \mathcal{O}_{[e]}$, and the action of π is transitive on each orbit $\mathcal{O}_{[e]}$. Again by (G2), we define

$$G(E) := \coprod_{[e] \in E/\pi} G(\mathcal{O}_{[e]})$$

Let E be a transitive π-set. From G(5) we have $F(G(E)) = F(P_i^G/p_i(\pi_e)) = F(P_i^G)/p_i(\pi_e)$ and since P_i^G is Galois, $F(G(E))$ is isomorphic to $Aut_{\mathbf{C}}(P_i^G)/p_i(\pi_e) = \pi_i/p_i(\pi_e) \cong E$. Hence, there is an isomorphism of π-sets

$$\alpha_E : E \to F(G(E))$$

Proposition 5.2.4.2. *The correspondence*

$$Hom_{\boldsymbol{C}}(G(E), X) \to Hom_{\pi-\boldsymbol{FSets}}(E, F(X))$$

mapping $u : G(E) \to X$ *to* $F(u) \circ \alpha_E : E \to F(G(E)) \to F(X)$*, is a bijection.*

Proof.

- Injectivity: Given two morphisms $u, v : G(E) \to X$, since α_E is an isomorphism, we find that if $F(u) \circ \alpha_E = F(v) \circ \alpha_E$, we have $F(u) = F(v)$ and the equalizer of this pair is isomorphic to F(G(E)). By (G5) and (G6), we find that $u = v$;
- Surjectivity: Given any morphism $f : E \to F(X) = Hom_{\mathbf{Pro(C)}}(P^G, X)$, the image of an element $e \in E$ is identified with a morphism $u : P^G \to X$. This morphism factors through a unique Galois object $P^G \to_{\xi_i} \to P_i^G \to_{\tilde{u}} X$. Moreover, the action of π on E factors through $\pi_i = Aut_{\mathbf{C}}(P_i^G)^{\circ}$, and $\tilde{u}$ factors through $P_i^G \to P_i^G/Est(\tilde{u}) \to X$, where $Est(\tilde{u})$ is the stabilizer of $\tilde{u}$ in π_i. This gives a canonical morphism

$$G(E) \cong P_i^G/p_i(\pi_e) \to P_i^G/Est(\tilde{u}) \to X$$

□

This result allows us to prove that G is a functor. Indeed, given a morphism of π-sets $f : E \to E'$, the composition

$$E \to_f E' \to_{\alpha_{E'}} F(G(E'))$$

is also a morphism of π-sets. Since the above map is a bijection, for $X = G(E')$ we find a morphism $G(E) \to G(E')$ that we define as $G(f)$. This assignment preserves the composition of morphisms between π-sets and therefore G is a functor.

We easily conclude that the collection of all α_E establishes an isomorphism of functors between the identity functor in π-**FSets** and the composition $F \circ G$. Similary, we conclude that $G \circ F$ is isomorphic to the identity functor in **C**.

Theorem 5.2.4.3. *Let* $\boldsymbol{C}$ *be a category under the conditions (G1), (G2) and (G3) and* F *a covariant functor with values in the category of finite sets, satisfying conditions (G4), (G5) and (G6). Then* F *induces an equivalence between* $\boldsymbol{C}$ *and the category of finite sets with a continuous action of the profinite group* $\pi = Aut_{\boldsymbol{Fun(C,Sets)}}(F)$.

5.3 Galois Categories, Fundamental Functors and Fundamental Groups

5.3.1 Galois Categories

Let π be a profinite group and let $\mathbf{C} = \pi$-**FSets** the category of finite sets with a continuous action of π. This category, equipped with the forgetful functor F, assigning to each π-set its underlying set verifies all conditions (G1) to (G6) and there is a canonical isomorphism between π and group of automorphisms of F.

Clearly, $\mathbf{C}$ has a terminal object given by a one-element set and all fiber products are built upon the fiber products of the underlying sets and their canonical morphisms. (G2) follows from the fact that the sum of π-sets is given by the disjoint union of the underlying sets equipped with a canonical action of π on each component. Also, there is an initial object given by the empty set and the quotient of any π-set by a finite group action exists in $\mathbf{C}$; Every morphism of sets factors as a surjective map followed by an inclusion. The same for π-sets; Conditions (G4), (G5) and (G6) follow by definition. According with the remarks in section A.1, a π-set is connected if and only if the action of π is transitive. Any transitive π-set S is isomorphic to a quotient space π/H, where H is a subgroup of π acting by left-multiplications, identified with a stabilizer of some point in S. In this case, the projective system $P^c : \mathbf{I}_F^c \longrightarrow \mathbf{C}$ is precisely the system of finite quotients π/H indexed by finite index subgroups $H \subseteq \pi$.

We introduce *Galois* Categories:

Definition 5.3.1.1. *(Galois Categories) We say that a category* ***C*** *is* Galois *if* ***C*** *is equivalent to a category of the form* π-***FSets****, with* π *a profinite group.*

From the previous results, a category $\mathbf{C}$ is *Galois* if and only if all conditions (G1), (G2) and (G3) are verified and $\mathbf{C}$ admits a covariant functor F with values in the category of finite sets, satisfying conditions (G4), (G5) and (G6). Such a functor will be called a *fundamental functor* of the Galois category $\mathbf{C}$. In this case, we proved that F is pro-representable by means of a pro-object, which we denote as P_F, called a *fundamental pro-object of* ***C***. We denote by Γ the *category of fundamental functors over the Galois category* ***C***. There is an equivalence between Γ and the category of fundamental pro-objects.

Definition 5.3.1.2. *The group* $\pi_F = Aut_{\boldsymbol{Fun(C,Sets)}}(F)$ *is called the* fundamental group of the Galois Category C with fundamental functor F. π *is anti-isomorphic to* $Aut_{\boldsymbol{Pro(C)}}(P_F)$.

5.3.2 Fundamental Groupoid

Let $\mathbf{C}$ be a Galois category and consider F and F' two fundamental functors on $\mathbf{C}$. From the previous results, both F and F' are pro-represented by pro-objects $P_F : \mathbf{I}_F^G \longrightarrow \mathbf{C}$ and $P_{F'} : \mathbf{I}_{F'}^G \longrightarrow \mathbf{C}$ where $\mathbf{I}_F^G$ (resp. $\mathbf{I}_{F'}^G$) is the category of pairs (X, x) with X Galois in $\mathbf{C}$ (5.2.3.3) and x is an element in $F(X)$ (resp. $F'(X)$). If X is a Galois object, each choice (x, x') with x in $F(X)$ and x' in $F'(X)$ induces a sequence of isomorphisms

$$F(X) \xleftarrow{\tilde{x}} Aut_{\mathbf{C}}(X) \xrightarrow{\tilde{x'}} F'(X)$$

The choice of such a pair for each galois object in $\mathbf{C}$ induces an equivalence of categories

$$\mathbf{I}_F^G \longrightarrow \mathbf{I}_{F'}^G$$

Therefore, both fundamental pro-objects P_F and $P_{F'}$ are isomorphic and so are the fundamental functors.

Proposition 5.3.2.1. *The category* Γ *of fundamental functors on a Galois Category* ***C*** *is a connected groupoid. We call it the* Fundamental groupoid of the Galois category $\mathbf{C}$. *In particular, if* F *and* F' *are two fundamental functors over a Galois category* ***C*** *then both fundamental groups of* (C, F) *and* (C, F') *are isomorphic. Sometimes we simply write the fundamental group of a Galois category without any reference to the choice of a fundamental functor.*

5.3.3 Exact Functors and Morphisms of Fundamental Groups

Proposition 5.3.3.1. *([4] - Exposé V - 6) Consider **C** and **C'** two Galois categories and let F' be a fundamental functor on **C'**. Consider also a functor*

$$H : \boldsymbol{C} \longrightarrow \boldsymbol{C'}$$

*Then, the composition $F = F' \circ H$ is a fundamental functor in **C** if and only if H commutes with all limits and colimits.*

In this case, denoting by Γ and Γ the fundamental groupoids of $\mathbf{C}$ and $\mathbf{C}'$, if H is exact, the composition $F' \mapsto F' \circ H$ defines a functor

$$\tilde{H} : \Gamma' \longrightarrow \Gamma$$

In particular, with F' fixed, $\tilde{H}$ induces a morphism of fundamental groups

$$\pi_{F'} \to \pi_{F' \circ H}$$

5.4 Examples of Galois Categories and Fundamental Groups

From now on, we follow more informal style, presenting the logical sequence of ideas towards Grothendieck's theory of dessin's enfants. First, we recover the results in the end of the fourth chapter:

5.4.1 Example: Fundamental Groups of Topological Spaces

Let X be a connected topological space and consider

$$\boldsymbol{C} = \textit{Category of Finite Covering Spaces of } X$$

and Fib_x, the fiber functor over a point $x \in X$, assigning to each covering $p : Y \to X$ the finite set $p^{-1}(x)$ over x. We prove that this category together with the fiber functor is Galois, verifying all conditions (G1) to (G6):

- (G1) - The trivial cover $id_X : X \to X$ is clearly a terminal object in $\mathbf{C}$; The fiber product of any two covers $Y \to X$ and $W \to X$ over a third cover $Z \to X$ exists, given by the fiber product of topological spaces $Y \times_Z W$ and the canonical morphism to X;
- (G2) - The sum of two covers $Y \to X$ and $W \to X$ is given by the disjoint sum of topological spaces $Y \coprod Z$ and the canonical morphism to X. The empty set is an initial object and, as seen in chapter 4, if $Y \to X$ is finite a cover the group $Aut_{\mathbf{Cov(X)}}(Y|X)$ is finite and the natural action on Y is even and there is a quotient cover $Y/Aut_{\mathbf{Cov(X)}}(Y|X) \to X$ (4.2.4);

- (G3) and (G6) This follows from the fact that the image of a morphism of covers $h : Y \to Z$, over X, is open and closed in Z, as seen in the proof of 4.2.1.3.
- (G4), (G5) These properties follow by definition.

We conclude that

Theorem 5.4.1.1. *The category of finite covering spaces of a connected space X is Galois. The fiber functor over a point $x \in X$ is a fundamental functor. The fundamental group of C.*

5.4.2 Example: Fundamental Groups of Schemes

1. Let S be a scheme.

   Definition 5.4.2.1. *A morphism of schemes $X \to S$ is said to be* finite étale *if there exists a covering of X by open affine sets, U_i with structure ring A_i, such that for each i, the open scheme $f^{-1}(U_i)$ of Y is affine, and equal to $Spec(B_i)$, where B_i is a free separable A_i-algebra.*

   A finite étale cover of a scheme S is another scheme X together with a finite étale morphism $X \to S$. This class of objects is a subcategory of **Schemes/S**. We have the following result:

   Theorem 5.4.2.2. *([4] - Exposé V. Sec. 7) Let S be a connected and locally noetherien scheme and $s : Spec(\Omega) \to S$ a Ω-geometric point of S, with Ω an algebraically closed field. The category*

   $$\mathbf{C} = \text{category of finite étale coverings of } S$$

   *together with the functor F, for each object X in **C** defined by*

   $$F(X) = \text{set of geometric points of } X \text{ over } s$$

   is a Galois Category.

   Definition 5.4.2.3. *The fundamental group of a locally noetherien and connected scheme S at a geometric point s, denoted as $\pi_1(S, s)$, is the fundamental group of the Galois category of finite étale coverings over S, together with the fiber functor over s.*

   As a particular case, we recover the classical Galois Theory of Fields:

   Proposition 5.4.2.4. *([4] - Exposé V. Prop. 8.1) Let k be a field and $S = Spec(k)$. Let $s : Spec(\bar{k}) \to S$ be a geometric point of S with values on $\bar{k}$. Denote by k_S the separable closure of k in Ω. In this, case, there is a canonical isomorphism between the absolute topological Galois group of $k_S|k$ and the fundamental group of S at the point s*

   $$Gal(k) \cong \pi_1(Spec(k), s)$$

Chapter 6

Dessins d'Enfants

This chapter is a brief survey of the main ideas leading to Grothendieck's Theory of Dessins d'Enfants as presented in [5].

6.1 Geometric Galois Actions

1. Let X be a scheme over a field k, $(X \to Spec(k))$ and consider $s : Spec(\bar{k}) \to Spec(k)$ a geometric point of $Spec(k)$. In this case, by 2.3.5.2, the fiber product $\bar{X} = X \times_{Spec(k)} Spec(\bar{k})$ exists and through the canonical morphism $\bar{X} \to Spec(\bar{k})$, is a scheme over $\bar{k}$.

   Theorem 6.1.0.5. *([4]- Exposé X) Let X be a quasi-compact and geometrically integral scheme over a field k. Fix an algebraic closure $\bar{k}$ of k and let $k_S|k$ be the corresponding separable closure. Let $\bar{x}$ be a geometric point of $\bar{X} = X \times_{Spec(k)} Spec(k_S)$ with values in k_S. In this case, there is an exact sequence of profinite groups:*

   $$1 \to \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x) \to \pi_1(Spec(k), s) \to 1$$

   induced by the canonical maps $\bar{X} \to X$ and $X \to Spec(k)$. The group $\pi_1(\bar{X}, \bar{x})$ is called the geometric fundamental *group of X.*

   Notice that, by 5.4.2.4, the last group $\pi_1(Spec(k), s)$ is isomorphic to the Galois group $Aut_{\mathbf{E(k)}}(k_S|k) = Gal(k)$

2. Given an exact sequence of topological groups

   $$1 \to N \to G \to P \to 1$$

   there is a natural continuous action of G on the normal subgroup N via conjugation:

   $$G \to Aut_{\mathbf{TopGroups}}(N)$$

   The image of N is the normal subgroup $Inn(N) \subseteq Aut_{\mathbf{TopGroups}}(N)$ of inner automorphisms, i.e. those that come from conjugation by an element of N. The quotient group $Aut_{\mathbf{TopGroups}}(N)/Inn(N)$, denoted by $Out(N)$, is called the group of *outer automorphisms* of N. By passing to the quotient we obtain an action of P:

$$P \cong G/N \to Aut_{\mathbf{TopGroups}}(N)/Inn(N) =: Out(N)$$

3. Applying the last result to the exact sequence of fundamental groups above, we find that if X is a quasi- compact and geometrically integral scheme over a field k, there is a canonical action of the absolute Galois group of k on the geometrical fundamental group of X.

6.2 Complex Analytic Space

Let $U \subseteq \mathbb{C}^n$ be an open disc and consider $f_1, ..., f_q$ holomorphic functions on U. Consider the closed subspace $Y \subseteq U$ of all common zeros of these functions. Take $\mathcal{O}_Y$ to be the sheaf $\mathcal{O}_U/(f_1, ..., f_n)$, where $\mathcal{O}_U$ is the sheaf of holomorphic functions on U.

Definition 6.2.0.6. *A complex analytic space is a locally ringed space, locally isomorphic to one of the kind* $(Y, \mathcal{O}_Y)$ *as above.*

First introduced by Serre in [9], there is a natural way to assign to each finite scheme X over $\mathbb{C}$, a complex analytic space X^{an}, over the same base topological space of X. This assignment is functorial. For a detailed exposition on this matter, we recommend the reader to follow not only the original source [9], but also [13]- Appendix B and [4]- "Exposé XII - Géométrie Algébrique et Géométrie Analytique", where Grothendieck proves the following remarkable theorem [1]

Theorem 6.2.0.7. *([4] - Exposé XII Cor. 5.2) Let X be a connected scheme of finite type over $\mathbb{C}$. The functor $(Y \to X) \mapsto (Y^{an} \to X^{an})$ induces an equivalence between the category of finite étale covers of X and the category of finite topological covers of X^{an}. As a result, for every $\mathbb{C}$-point, $\bar{x} : Spec(\mathbb{C}) \to X$, this functor induces an isomorphism*

$$\pi_1^{top}(\widehat{X^{an}, \bar{x}(Spec}(\mathbb{C})) \cong \pi_1(X, \bar{x})$$

where the first group is the profinite completion (A.2.5-3) of the topological fundamental group of X^{an} and the second group is the fundamental group of X at the geometric point $\bar{x}$.

6.3 Belyi's Theorem

We introduce the general definition of an algebraic variety over a field k.

Definition 6.3.0.8. *Let k be a field. A variety over k is a scheme over k*

$$\begin{array}{c} X \\ \downarrow{\scriptstyle p_X} \\ Spec(k) \end{array}$$

[1](also using the result proved 4.4.3.2 in chapter 4)

where X is an integral scheme, p_X is a separated morphism of finite type that does not factor through any scheme of the form Spec(L), with L a finite extension of k, and the fiber product $X \times_k k_S$ is still irreducible.

A one-dimensional variety over k is called a curve over k. A morphism of varieties over k is a morphism of k-schemes as in 1.3.

Example 6.3.0.9. *(Projective Line over k) Consider $X = Spec(k)$, where k is a field. We define the projective line over k by gluing the affine schemes $Spec(k[x])$ and $Spec(k[x^{-1}])$, together along their isomorphic subschemes $Spec(A[x, x^{-1}])$. For a detailed construction please follow [13]- p.103.*

Let $L|k$ be an extension of fields. From the remarks in 2.3.5.2-3, the inclusion morphism of affine schemes $Spec(L) \to Spec(k)$ induced by the inclusion morphism $k \to L$ establishes a pullback functor

$$\phi^* : \textbf{Schemes}/Spec(k) \longrightarrow \textbf{Schemes}/Spec(L)$$

A curve X over L is said to be defined over k [2] if there exists a curve X_k over k such that $X_k \times_k L$ is isomorphic to X as a scheme over $Spec(L)$. In a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\sim} & X_k \times_k L & \longrightarrow & X_k \\ & \searrow & \downarrow & & \downarrow \\ & & Spec(L) & \longrightarrow & Spec(k) \end{array}$$

Again by 2.3.5.2-3, a morphism of curves $f : X \to X'$ over L is said to be defined over k, if both curves X and X' are defined over k and there exists a morphisms $f_K : X_k \to X'_k$ of curves over k such that the following diagram commutes

$$\begin{array}{ccc} X_k \times_k L & \xrightarrow[f_k]{} & X'_k \times_k L \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & X' \end{array}$$

Finally, we introduce the key result towards Grothendieck Theory of Dessins d'enfants

Theorem 6.3.0.10. *(Belyi's Theorem) An algebraic curve X over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if there exists an étale morphism of curves over $\mathbb{C}$, $f : X \to P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. In this case, this morphism is also defined over $\overline{\mathbb{Q}}$.*

Proof. See [16] - Theorem 4.7.6. □

This theorem implies the following one

Corollary 6.3.0.11. *The category of finite etale covers of $P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ is equivalent to the the category of finite etale covers of $P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ and therefore, the (algebraic) fundamental groups of these curves are isomorphic.*

[2] Sometimes also said that X has a model over k

Together with the result of 6.2.0.7, we conclude the following remarkable result:

Corollary 6.3.0.12. *The category of finite étale covers of* $P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ *is equivalent to the category of finite topological covering maps of* $(P^1_{\mathbb{C}})^{an} \setminus \{0, 1, \infty\}$*, where* $(P^1_{\mathbb{C}})^{an} \cong \mathbb{C}_\infty$ *is the Riemann's Sphere.*

At last, by gathering all the main results presented in this work,

Corollary 6.3.0.13. *The fundamental group of* $P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ *is isomorphic to the profinite completion of topological fundamental group of* $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$ *and the following categories are equivalent*

- *finite étale covers of* $P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$*;*
- *finite étale covers of* $P^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$
- *finite topological covers of* $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$*;*
- *finite sets with a continuous action of the fundamental group of* $P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$
- *finite sets with a continuous action of the profinite completion of the topological fundamental group of* $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$.

and finally, from A.3.3.2

- *finite sets with an action of the topological fundamental group of* $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$

Now we present another key result

Proposition 6.3.0.14. *There is an isomorphism of curves over* $\overline{\mathbb{Q}}$*, between* $P^1_{\overline{\mathbb{Q}}}$ *and* $P^1_{\mathbb{Q}} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$.

And finally, as corollary after the results in 6.1

Corollary 6.3.0.15. *There is an exact sequence of profinite groups:*

$$1 \to \pi_1(P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}) \to \pi_1(P^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}) \to Gal(\mathbb{Q}) \to 1$$

and a natural action

$$Gal(\mathbb{Q}) \to Out(\pi_1(P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})) \cong Out(\pi_1^{top}(\widehat{\mathbb{C}_\infty \setminus \{0, 1, \infty\}})) \tag{6.1}$$

Before ending this section we notice the fact that the group $\pi_1^{top}(\mathbb{C}_\infty \setminus \{0, 1, \infty\})$ is generated by three elements: γ_0, γ_1 and γ_∞, respectively, equivalence classes of loops around the points 0, 1 and ∞, with the relation $\gamma_0\gamma_1\gamma_\infty = 1$.

6.4 Dessins d'Enfants

Definition 6.4.0.16. *(Graph) A graph* Γ *is a triple* (V, E, I)*, where* V *is a finite set, whose elements are called* vertices*,*E *is another finite set, the* edges *and* I *is an incidence relation, that is, a subset of* $V \times E$*, such that* $(v, e) \in I$ *iff the edge* e *is incident with the vertice* v*. Each edge should always be incident to two vertices. If they are the same, we call the edge a* loop.

Two graphs $\Gamma = (V, E, I)$ and $\Gamma' = (V', E', I')$ are said to be isomorphic if there are isomorphisms $\phi : V \to V'$ and $\psi : E \to E'$ presenving the incidence relation.

Graphs admit many possible representations. Their structure can be identified as collections of objects and morphisms on general categories, or they can be simply drawn on surfaces, each vertice being identified by a point on the surface and each edge by a curve, the image of a path.

Definition 6.4.0.17. *A map (also called a topological graph) $M_X = (X_0 \subseteq X_1 \subseteq X)$ is a graph Γ, embedded in an orientable connected and compact surface X, in such a way that*

- *vertices are represented by a set $X_0 \subset X$ of distinct points on the surface;*
- *the edges are represented as curves only intersecting at points representing vertices, notation $X_1 \in X$;*
- *the set $X \setminus X_1$ is a disjoint union of connected components, each one homeomorphic to an open disk $\mathbb{D}$ in $\mathbb{R}^2$. We call each one of these connected components a* face *of the map.*

We say that an edge is incident to a face if it belongs to the boundary of this face. The degree of a face is the number of edges incident to it. Let M be a map on a surface X, and denote by F the set of its faces. Then, the number $\chi(M) := |V| - |E| + |F|$ does not depend on the map, only on the surface's genus g and it is equal $2 - 2g$. We introduce morphisms between maps:

Definition 6.4.0.18. *A morphism between two maps $X_0 \subseteq X_1 \subseteq X$ and $Y_0 \subseteq Y_1 \subseteq Y$ is an orientation-preserving continuous map $X \to Y$ mapping X_0 to Y_0 and X_1 to Y_1.*

Definition 6.4.0.19. *(Dessin d'enfant) A* marking *on a map $M_X = (X_0 \subseteq X_1 \subseteq X)$ is a selection of a fixed point on each component of $X_1 - X_0$, and one point in each open cell of $X - X_1$. We will denote points on X_0 by $\bullet$, points on $X_1 - X_0$ by $\circ$ and points on $X - X_1$ by $\star$. A map with a marking is called a* dessin d'enfant.

The category of dessins d'enfant is defined considering dessins as objects and by taking morphisms as morphisms of maps that preserve markings. Finally,

Theorem 6.4.0.20. *The category of dessins d'enfant is equivalent to the category of finite connected topological covers of $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$*

Proof. For a detailed proof see [11] -Chapter 1. □

Example 6.4.0.21. *The following example is directly understood by following the proof of 6.4.0.20 in [11]. The map $\mathbb{C} \to \mathbb{C}$ assigning $z \mapsto z^8$ can be seen as an holomorphic morphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$, unramified in $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$. The associated dessin is*

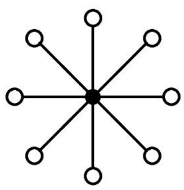


Figure 6.1: Dessin corresponding to $z \mapsto z^8$

obtained as $f^{-1}(]0, 1[)$.

From 6.4.0.20, we also conclude that there is an equivalence between the category of dessins d'enfant and the category of finite sets with a continuous transitive action of $\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})$.

The action in 6.1 gives rise to an action of $Gal(\mathbb{Q})$ on the dessins: Each dessin corresponds to a $\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})$ finite set. Each element σ of $Gal(\mathbb{Q})$ defines an outer isomorphism $\phi(\sigma) : \pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\}) \to \pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})$ and therefore, if D is a dessin corresponding to a finite $\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\})$- set S with an action

$$\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\}) \longrightarrow Aut_{\mathbf{Sets}}(S)$$

then $\sigma(D)$ is the same set S, equipped with the action

$$\pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\}) \xrightarrow[\phi(\sigma)]{} \pi_1^{top}(\mathbb{C}_\infty \setminus \{0,1,\infty\}) \longrightarrow Aut_{\mathbf{Sets}}(S)$$

In [1] - L.Schneps - "Dessins d'enfants on the Riemann Sphere", it is proved that this action is faithful on dessins on curves of genus 0.

Appendix A

Group Actions and Profinite Groups

A.1 G-Sets

Let G be a group and consider the category G-**Sets** as introduced in 1.9.4. On the following, we denote by $G - \mathbf{FSets}$ the category of finite G-sets.

We briefly describe some notations and concepts related to the action of a group on a set.

1. On any G-set S, there is a natural equivalence relation between the elements of S: s and $s' \in S$ are said to be equivalent if there is an element $g \in G$ such that $g(s) = s'$. Under this relation, an equivalence class $[s]$ is called the orbit of s and also denoted by $\mathcal{O}_{[s]}$ when seen as a subset of S. The quotient set S/G is naturally identified with the set of orbits in S produced by the action of G. We say the action is transitive if S equals the orbit of a single element or, equivalently, if S/G is a one-element set

2. A one-element G-set is a terminal object in G-**Sets** and the empty set is an initial object.

3. G is a natural G-set. There are canonical actions of G on itself by left multiplication $L_g : h \mapsto g \circ h$, right multiplication $R_g : h \mapsto h \circ g$ and conjugacy $C_g : h \mapsto g \circ h \circ g^{-1}$;

4. Each action above induces an action of every subgroup $H \subseteq G$ on G. We restrict ourselves to the left-coset space, the quotient set $G/H = \{xH\}$ (where we write $xH = \{y \in G : \exists h \in H : y = xh\}$) produced under the action of H through right multiplications. There is a natural action of G on this set, induced by left-multiplications $g : xH \mapsto (gx)H$. This is a well-defined map: if $yH = xH$ then $y = xh$ for some $h \in H$ and so $gyH = gxhH = gxH$; We say H has finite index in G if the quotient set G/H is finite.

5. Two subgroups H and H' are said to be conjugated if there is an element $a \in G$ such that $H = aH'a^{-1}$. In this case there is a natural map on the right coset spaces $u : G/(H') \to G/H$ mapping $xH' \mapsto xa^{-1}H$. This map is well-defined: if $xH' = yH'$, there is some element $h' \in H'$ such that $y = xh'$. Since H and H' are conjugated, we have $h' = a^{-1}ha$ for some $h \in H$ and so $ya^{-1}H = xh'H = xa^{-1}haa^{-1}H = xa^{-1}H$. This map is a bijection since we easily identify an inverse , and the diagram commutes with an action of G by left multiplications:

$$\begin{array}{ccc} G/H' & \xrightarrow[u]{} & G/H \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ G/H' & \xrightarrow[u]{} & G/H \end{array}$$

for all $g \in G$. Therefore, both coset spaces are isomorphic G-sets.

6. Given two subgroups H and H', if the sets G/H and G/H' are isomorphic as G-sets then H and H' are conjugate. To check this, simply chase the commutative diagram associated to this isomorphism, following the class eH', where e is the identity in G.

7. Let S be a G-set. The stabilizer of a point $s \in S$ is by definition, the subset of elements in G fixing s. This subset is a subgroup $H_s \subseteq G$. The stabilizers of two elements s and s' on the same orbit are easily seen to be conjugated subgroups. Suppose the action of G is transitive on S and fix an element $s \in S$. We define a map $\phi : S \to G/H_s$ sending each element $x \in S$ to the element $g_x \in G$ which maps s to x. This map is well-defined and is a bijection. Morever it commutes with the left-action of G on G/H. We conclude that any transitive $G - set$ is isomorphic to a quotient set G/H with a left G action.

   Theorem A.1.0.22. *There is a bijection between isomorphism classes of G-sets with a transitive action of G and conjugacy classes of subgroups of G, Moreover, every transite G-sets is isomorphic to a class space G/H (formed from the left-multiplication action of H on G) with a action of G by left multiplications. Clearly, isomorphism classes of transitive finite G-sets are in bijection with conjugacy classes of finite index subgroups.*

8. Given two G-Sets, S and S', we construct their directed sum using the disjoint union S and S'. This union comes equipped with a natural action of G obtained from the actions on S and S' and natural inclusion morphism $S \to S \coprod S'$ and $S' \to S \coprod S'$. For each $g \in G$ we have

$$\begin{array}{ccccc} S & \longrightarrow & S \coprod S' & \longleftarrow & S' \\ {\scriptstyle g}\downarrow & & \downarrow & & {\scriptstyle g}\downarrow \\ S & \longrightarrow & S \coprod S' & \longleftarrow & S' \end{array}$$

   inducing a canonial action $G \to Aut_{G-\mathbf{Sets}}(S \coprod S')$, turning $S \coprod S'$ into a G-Set. This construction indeed satisfies all the proporties of a directed sum, as defined in 1. We denote this sum by $S \coprod_G S'$.

9.

   Definition A.1.0.23. *A G-Set E is said to be connected if everytime we write $E = S \coprod_G S'$, we have $S = \emptyset$ or $S' = \emptyset$*

   In fact,

   Proposition A.1.0.24. *A G-set S is connected if and only if the action of G on S is transitive.*

   and any G-set S has a decomposition on its orbits

$$S = \coprod_{[s] \in S/G} \mathcal{O}_{[s]}$$

   and each one is a connected G-set. In particular, if S is finite, this decomposition is finite.

A.2 Profinite Groups

A.2.1 Topological Groups and Continuous Actions

1. A topological group is a group G whose set of elements is a topological space and the group operation $m : G \times G \to G$ is a continuous and inverse $inv : G \to G$ is an homeomorphism. For each $g \in G$ the maps $x \mapsto gx$ and $x \mapsto xg$ are also homeomorphisms. A morphism of topological groups $G \to G'$ is a group homomorphism that is also continuous. As an example, every finite group becomes a topological group by considering the discrete topology;

2. If $H \subset G$ is an open (resp. closed) subgroup then all cosets gH and Hg are open (resp. closed) subsets of G. This follows from the fact that the maps $x \mapsto gx$ are homeomorphisms.

   Every open subgroup $H \subseteq G$ is also closed. Indeed, we can always write both $G = \coprod_{[g] \in G/H} gH$, and $G - H = \coprod_{[g] \in G/H, g \notin H} gH$ as an union of open subsets and so, H is closed;

   Every finite index closed subgroup is open. If H is a closed subgroup, the collection of all gH are a finite cover of G by closed subsets. Therefore its complement $G - H = \coprod_{[g] \in G/H, g \notin H} gH$ is closed (a finite union of closed subsets is closed) and H is open;

   Proposition A.2.1.1. *Let G be a topological group. If a subgroup $H \subseteq G$ contains an open subset then H is open*

   After the two previous results we conclude that

   Proposition A.2.1.2. *Let G be a compact topological group and $H \subseteq G$ a subgroup. Then, H is open if and only if it is closed and of finite index;*

3. If H is a subgroup containing an open subset $V \subseteq H$ then H is also open. Indeed, we have $H = \cup_{h \in H} V_h$ where $V_h = \{vh, v \in V\}$;

4. If H is a subgroup of a topological group G then H, with the subspace topology is also a topological group; If K is a normal subgroup of G, then G/K is also a topological group, by considering the quotient topology. In this case, the canonical map $G \to G/K$ is continuous.

5. We present some results concerning compact and totally disconnected topological groups:

   Lemma A.2.1.3. *([8] - Lemma 0.3.1) If G is a totally disconnected topological group (the only connected subsets are one-point sets) then G is Hausdorff.*

   This lemma implies that all finite groups, endowed with the discrete topology, are Hausdorff.

   Lemma A.2.1.4. *([8] - Lemma 0.3.2) Let G be a compact topological group. If U is both open and closed and contains the identity of G then, there is an open normal subgroup N of G contained in U.*

   As a corollary, we find that any subgroup $H \subseteq G$, being open and closed and containing 1_G, contains also an open normal subgroup of G. Moreover, we conclude that

   Corollary A.2.1.5. *If G is a compact and totally disconnected topological group then every open subset of G is the union of left cosets of open normal subgroups of G*

A.2.2 Projective Systems of Topological Spaces

1. Let $X : \mathbf{I}^{\circ} \longrightarrow \mathbf{Top}$ be a projective system of topological spaces. After the construction presented in 1.7.7.3, we notice that the limit X of the projective system of all the underlying spaces in the system $\tilde{X}$, exists as a subset of the product of all sets X_i. Therefore, there is a natural topology on X by considering the subspace topology of the product topology. The restrictions of the canonical projections $\phi_i = p_i|X : X \to X_i$ to this subspace are continuous maps by definition.

   Proposition A.2.2.1. *([8] - Prop. 1.1.5) Under the notations above*

   - *If each X_i is Hausdorff then X is Hausdorff;*
   - *If each X_i is totally disconnected then so is X;*
   - *If each X_i is Hausdorff then X is a closed subspace of $\Pi_{i\in I}X_i$;*
   - *If each X_i is compact and Hausdorff then so is X (this follows from Tychonovs' Theorem);*
   - *If each X_i is non-empty, compact and Hausdorff then X is non-empty;*

2. Let X be a projective limit of non-empty compact and Hausdorff topological spaces indexed by a set I. After the last item, we describe a basis for the topology on X. A basis for the product topology in $\Pi_{i\in I}X_i$ is given by open sets of the form

$$p_{i_1}^{-1}(U_1) \cap ... \cap p_{i_n}^{-1}(U_n)$$

   where each U_r is an open subset in X_{i_r}. Since X has the subspace topology, every open set P of X is of the form

$$P = X \cap p_{i_1}^{-1}(U_1) \cap ... \cap p_{i_n}^{-1}(U_n)$$

   Let $a = (a_i)$ be a point in P. Assuming that $\mathbf{I}$ is filtrant, there is some k with $i_1, ..., i_n \leqslant k$ and we can consider the respective transition morphisms $\rho_{k,i_r} : X_k \to X_{i_r}$, which are continuous maps. Therefore, we have a collection of open sets $\rho_{k,i_r}^{-1}(U_r)$ in X_k, all containing a_k. Their finite intersection, U is also open and contains a_k and therefore $\phi_k^{-1}(U)$ is an open neighbourhood of a in X and is contained in P. We conclude that

   Proposition A.2.2.2. *Let X be a projective limit of non-empty compact and Hausdorff topological spaces indexed by a set I. Then, the subsets $\phi_i^{-1}(U)$ with $i \in I$ and U some open set in X_i are a basis of the topology in X.*

A.2.3 Projective Systems of Groups

As seen in 1.7.7.3, the projective limit of a projective system of sets always exists in **Sets**. Also, we notice the fact that all products exist in **Groups**. We use this result to prove the following:

Proposition A.2.3.1. *Any projective system of groups $\mathcal{G} : \boldsymbol{I}^{\circ} \longrightarrow$ **Groups** has a limit in **Groups**.*

Proof. Seen as a projective system of sets, $\mathcal{G}$ has a limit in **Sets** given by the set $S = \{(g_i)_{i\in I} \in \Pi_{i\in I} : \mathcal{G}(i,j)(g_i) = g_j \text{ for all } i,j \text{ with } j \leqslant i\} \subseteq \Pi_{i\in I}\mathcal{G}(i)$ and a family of canonical projections $p_j : S \to G(j)$ with $p_j((g_i)_{i\in I}) = g_j$. The product $\Pi_{i\in I}\mathcal{G}(i)$ has a natural group structure, coordinate by coordinate, and $\varprojlim_I G$ inherits this structure. We easily conclude this set to be the projective limit of $\mathcal{G}$ in **Groups** using its properties as a projective limit in **Sets**. $\square$

Since $\varprojlim_I G$ is a subgroup of $\Pi_{i\in I}\mathcal{G}(i)$, its identity is precisely the element $1 = (1_i)$, where each 1_i is the identity in $\mathcal{G}(i)$. Moreover, the canonical projections $p_i : \varprojlim_I G \to \mathcal{G}(i)$ are group homomorphisms and everytime $i \leqslant j$ we have $kernel p_i \subseteq kernel p_j$.

A.2.4 Projective Systems of Topological Groups

1. Let $G : \mathbf{I}^{\circ} \longrightarrow \mathbf{TopGroups}$ be a projective system of topological groups. By the results in A.2.3, the limit G of this system exists as a subgroup of product group of all the groups in the system and all the canonical projections restricted to this subgroup are group homomorphisms. By A.2.2, if each G_i is a topological group, there is a natural topology on G, as a subspace of the product. All the canonical projections and transition morphisms are continuous and the kernels of the canonical projections are open normal subgroups of G.

2. By A.2.2.1, the limit of a projective system of compact spaces is compact. In this case, by A.2.1.2, a subgroup L is open if and only if it is closed and has finite index.

   As seen in A.2.2.2, the open sets $\phi_i^{-1}(U)$ form a basis of the topology in G. Let L be an open subgroup. In this case, 1_G is contained in some $\phi_i^{-1}(U) \subseteq L$ for some $i \in I$ and some U open in G_i. Therefore, U contains 1_{G_i} and we have $kernel(\phi_i) \subseteq \phi_i^{-1}(U) \subseteq L$.

   Corollary A.2.4.1. *Let G be a topological group, the limit of a projective system of compact and groups G_i. For every open normal subgroup L of G there is an $i \in I$ such that kernel $\phi_i \subseteq L$.*

A.2.5 Profinite Groups

1. We introduce profinite groups:

   Definition A.2.5.1. *A profinite group is a topological group isomorphic (in the category of topological groups) to a limit of a projective system of finite groups.*

   Let G be a profinite group, the limit of a projective system of finite groups $\mathcal{G} : \mathbf{I}^{\circ} \longrightarrow \mathbf{TopGroups}$. As described in the last section, the canonical projections $p_i : G \to \mathcal{G}(i)$ are continuous and since each $\mathcal{G}(i)$ is finite and discrete, the kernel of each projection p_i is open in G.

2. By A.2.2.1, since each $\mathcal{G}(i)$ is compact, Hausdorff and totally disconnected, the limit G is also compact and totally disconnected. In fact, there is an equivalence:

   Proposition A.2.5.2. *A group is profinite if and only if is compact and totally disconnected.*

   Proof. See [10] Proposition 0 □

   Definition A.2.5.3. *A morphism of profinite groups is a morphism in the category of topological groups.*

3. (Profinite Completion of a Group)

   Let G be group. There is a canonical method to construct a profinite group starting from G. The set I of normal subgroups of G of finite index has a natural order relation given by the inclusion:$H \leqslant H'$ if and only if $H' \subseteq H$. Following the procedure indicated in 1.7.7, we define a category $\mathbf{I}$ associated to I. The correspondence $\mathcal{G}$ assigning to each normal subgroup H, the quotient group $\mathcal{G}(H) := G/H$ and to each morphism $H \leqslant H'$ the canonical projection $\rho_{H,H'} : G/H' \to G/H$ defined by $xH' \mapsto xH$ (this is well-defined since $H' \subseteq H$) has functorial properties and defines a projective system of finite groups (since all subgroups H have finite index):

$$\mathcal{G} : \mathbf{I}^{\circ} \longrightarrow \mathbf{Groups}$$

We define the profinite completion of G, denoted by $\widehat{G}$, as the limit of this projective system. Obviously, this limit comes equipped with a family of canonical projections $\rho_H : \widehat{G} \to G/H$ commuting with all composable transition morphisms $\rho_{H,H'}$.

A.3 Continuous Actions on Topological Spaces

A.3.1 Generalities

An action of a topological group on a topological space X is a morphism of groups $G \to Aut_{Top}(X)$. We say that the action is continuous if the induced map $G \times X \to X$ is continuous with respect to the product topology on $G \times X$. We call it a continuous G-action. A space X equipped with a continuous action of a topological group is called a G-space A morphism of G-spaces X and Y with continuous G actions is a continuous map $X \to Y$ compatible with the action.

If G is a topological group, we denote by $G - \mathbf{Top}$ the category of topological spaces with a continuous G-action.

A.3.2 Continuous Actions on finite discrete spaces

1. Let S be a discrete topological space equipped with an action of a topological group $\phi : G \times S \to S$. A basis for the topology in S is given by the sets $\{s\}$, for each $s \in S$. The preimage under ϕ of these sets are of the form

$$\phi^{-1}(\{s\}) = \coprod_{y \in S} \{(g, y) \in G \times \{y\} : gy = x\}$$

   and we have two possibilities: Either y is on the same orbit of x and in this case each set in the union above is homeomorphic to G_s, the stabilizer of s, or, each set in the union is empty.

   Therefore, if all the stabilizers of an element $s \in S$ in G are open subgroups, the set $\phi^{-1}(\{s\})$ is open and the map ϕ is continuous. Conversely, if ϕ is continuous, G_s is open, since it is the preimage of s under the the composition $G \to G \times S \to S$, where the first arrow is the map $g \to (g, s)$.

   Proposition A.3.2.1. *The action of a topological group G on a discrete topological space S is continuous if and only if for each $s \in S$, the stabilizer $H_s \subseteq G$ is open in G.*

2. Suppose now that S is a finite set endowed with an action of a topological group G. If the action is continuous, then every stabilizer G_x is a open subgroup of G. The interesection of all the stabilizers equals the kernel of the action and since the set is finite, this intersection is finite and so the kernel is an open subgroup of G. Conversely, if the kernel is an subgroup of G, since every stabilizer contains the kernel, by A.2.1.1 we conclude that every stabilizer is an open subgroup and again by the last proposition, the action is continuous. Therefore

Proposition A.3.2.2. *Let S be a finite set equipped with an action of a topological group G. Then, the action is continuous if and only if its kernel is an open subgroup.*

A.3.3 Continuous Action of a Profinite Group on a finite space

1. Let S be a finite set with an action of a profinite group G, the limit of a projective system of groups $(G_i)_{i \in I}$:

$$G \to Aut_{\mathbf{FSets}}(S)$$

After the last section, we know that the action is continuous if and only if the kernel N of the above morphism is an open subgroup. In this case, and since the kernel of any morphism of groups is a normal subgroup, by A.2.4.1, we conclude that there is some $i \in I$ such that $kernel(\phi_i : G \to G_i) \subseteq N$. In addition, the action factors as

$$G \to G_i \to G/N \to Aut_{\mathbf{FSets}}(S)$$

where the first arrow is the canonical projection.

Conversely, if the action of a profinite group factors through some finite quotient, the action's kernel contains the kernel of the canonical projection $G \to G_i$ and since this kernel is open in G, by A.2.1.1, we conclude that the action is continuous.

Proposition A.3.3.1. *The action of a profinite group $G = \varprojlim_I G_i$ on finite set S is continuous if and only if it factors through a finite quotient $G \to G_i \to Aut_{\mathbf{FSets}}(S)$.*

2. As a corollary we obtain the following result:

Corollary A.3.3.2. *Let G be a group and $\widehat{G}$ its profinite completion. Then there is an equivalence between the category of finite sets with an action of G and the category of finite sets equipped with the discrete topology, with a continuous action of the profinite completion of G, $\widehat{G}$.*

Proof. This follows immediately from the fact that the action of G on any finite set factors through the quotient of G by the action's kernel. ☐

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