Frequency-dependent underground cable model for electromagnetic transient simulation

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Abstract – A computation method for transients in a three-phase underground power-transmission system is presented in this paper. Two methods will be used for this purpose: The first using the Fourier Transform, allowing the transient analysis in linear time-invariant systems. The second is an equivalent network with lumped parameters whose behaviour, within a given frequency band, is similar to the transmission line itself. Transient waveforms are evaluated using a software for mathematical applications, MATLAB, and in particular one of its tools, SIMULINK.

The main interest in the use of a method to make a time analysis is the introduction possibility of non-linear elements in the network.

Nomenclature

- \( V \) Voltage vector
- \( I \) Current vector
- \( Z \) Impedance matrix
- \( Y \) Admittance matrix
- \( J_m \) Bessel Function
- \( Y_m \) Bessel functions
- \( H_m \) Hankel function
- \( r \) Radius
- \( r_c \) Radius of central conductor
- \( r_{hi} \) Radius over main insulation
- \( r_{be} \) Radius over conducting sheath
- \( r_e \) Outer radius cable
- \( \omega \) Angular frequency
- \( \rho \) Resistivity or charge density
- \( \sigma \) Conductivity
- \( \epsilon \) Permittivity
- \( \mu \) Permeability
- \( \delta \) Soil penetration

I – Introduction

The social and economic development that occurred in the past years led to an urban and industrial centre growth, increasing the electrical power demand and leading to the use of relatively long cable circuits operating at high voltage. In these conditions it is expected to overcome transient overvoltages induced in the conductors of the underground system. For the stated reasons and the recent interest in underground transmission systems, researches on the viability of the underground cable models became necessary.

The magnetic field based on Maxwell’s equations is calculated for the underground cable model system. The general solution for the magnetic field on the soil is developed using arbitrary boundary conditions in a cylindrical surface, at a finite depth under the plane earth/air surface. The general Polaczek solution is developed. Finally the system constitutive parameters are evaluated.

Two methods for the electromagnetic transient simulation on underground cable systems are studied in this paper. The first is the Fourier Transform that will be implemented using the Fast Fourier Transform (FFT) and the Inverse Fast Fourier Transform (IFFT). The second is the equivalent network with lumped parameters. Transient regimes obtained by both methods are compared showing an excellent result accuracy.

II – Magnetic field in underground power-transmission systems

The problem of an infinitely long cylindrical conductor can be treated as a 2D problem which is easier to analyze. When the conductors are displayed with an axial symmetry the field also satisfies this symmetry and the solution becomes considerably easier.

The calculation of the magnetic field due to an underground cable of finite radius with cylindrical boundary will be made taking into account several assumptions:

1. The earth is a semi-infinite surface where the Earth / Air is a plane.
2. The geometry is considered infinitely long in the z coordinate (axial).
3. The cable is cylindrical and it is buried at a constant depth.
4. Earth and air are considered homogeneous, the air with a permeability \( \mu_0 \) and earth with a permeability \( \mu_s \) and conductivity \( \sigma_s \).
5. The hypothesis of a quasi-static regime is considered, neglecting the capacitive effects, which for the case of the earth is an adequate approximation for frequencies up to 1 MHz.
The formulation of the electromagnetic field in a power transmission system is based on the magnetic vector potential, $A$, which satisfies in the frequency domain:

$$\begin{align*}
\text{Lap} \ A - j\omega \mu \nabla A &= -\sigma \nabla \eta, \quad \text{in the conductors} \\
\text{Lap} \ A - j\omega \sigma \mu \nabla A &= 0, \quad \text{in the soil}
\end{align*}$$

(1)

Where $\eta$ represents the voltage drop per unit of axial length.

### Magnetic vector potential inside earth

The general solution for the magnetic field in the soil is developed satisfying arbitrary boundary conditions in a cylindrical surface buried at a finite depth under a plain Earth/Air surface. It is considered a generalization of the Pollaczek solution for cylindrical underground cables with circular sheath and finite radius, taking into account the proximity of the magnetic field. The corrections for the longitudinal impedance due to the return path of the earth are determined at the expense of an approximation equivalent to the solution of Pollaczek.

The solution of the field in the soil can be made by the linear combination of two linearly independent terms. The first, $A'$, to consider the boundary conditions on the earth/air surface, $S_e$. The second, $A''$, in turn, allows the boundary conditions to be considered on the surface $S_c$. The solution can then be written in Cartesian coordinates $(x,y)$:

$$ \vec{A}(x,y) = \int_{-\infty}^{+\infty} N(a,y) e^{jka} \, da $$

(2)

Where

$$ N(a,y) = F(a) e^{\sqrt{a^2 - q_s^2}} e^{-y \sqrt{a^2 - q_s^2}} \, da, \quad y < 0 $$

(3)

$$ \text{Re}(\sqrt{a^2 - q_s^2}) > 0 $$

(4)

$$ q_s = \sqrt{2e^{-\gamma} \delta_s}, \quad \delta_s = \sqrt{\frac{2}{\omega_0 \sigma_0}} $$

(5)

$F(a)$ is a function to be determined by the boundary conditions of the problem.

The $A''$ solution is written in Fourier series:

$$ \vec{A}'' = \sum_{m=-\infty}^{+\infty} R_m(r) e^{j\mu \phi} $$

(6)

Considering the Bessel functions we obtain:

$$ R_m = F_m J_m(\bar{q}_e r) + F_m Y_m(\bar{q}_e r) $$

(7)

Or:

$$ R_m = G_m H^{(1)}_m(\bar{q}_e r) + G_m H^{(2)}_m(\bar{q}_e r) $$

(8)

Where $F_m(\bar{q}_e r)$ is a Bessel function of the first kind of order $m$ and argument $\bar{q}_e r$. $Y_m(\bar{q}_e r)$ is a Bessel function of the second kind of order $m$ and argument $\bar{q}_e r$. $H^{(1)}_m$ and $H^{(2)}_m$ are the Hankel functions of the first and second kind respectively of order $m$ and argument $\bar{q}_e r$.

For a hollow conductor, the case in study, it is used the Hankel equation of the second kind that is regular for $r \to \infty$.

$$ R_m = G_m H^{(2)}_m(\bar{q}_e r) $$

(9)

Thus, the term $A''$ can be written in cylindrical coordinates $(r, \phi)$ around $O'$ by:

$$ \vec{A}''(r,\phi) = \sum_{m=0}^{+\infty} G_m H^{(2)}_m(\bar{q}_e r) e^{j\mu \phi}, r > r_e $$

(10)

Where $H^{(2)}_m$ is the Hankel function of the second kind and order $m$ with argument $(\bar{q}_e r)$ and $G_m, m = 0, \pm 1, \pm 2, \ldots$ are coefficients to be determined.

In order to impose boundary conditions on the surface $S_e$ is convenient to write (12) in Cartesian coordinates.

Since the Hankel function of the second type is defined by [2]:

$$ H^{(2)}_m(\bar{q}_e r) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(w) e^{-j\mu \phi} dw $$

(11)

Considering the integration paths defined in [3] we obtain for the case study:

$$ \int_{-\infty}^{+\infty} W_m(a) e^{-y\sqrt{a^2 - q_s^2}} e^{jka} \, da $$

(12)

Where:

$$ W_m(a) = \frac{a - \sqrt{a^2 - q_s^2}}{j\bar{q}_e s} e^{\frac{m}{\sqrt{a^2 - q_s^2}}}, \quad y > -h $$

(13)
Thus the solution of the vector potential is:

\[
\vec{A}(x,y) = \vec{A}^I + \vec{A}^O = \int_{-\infty}^{+\infty} \vec{G}(a,y) e^{iakx} da, -h < y < 0
\]

(14)

With:

\[
G(a,y) = F(a) e^{i\sqrt{\sigma_{a}\mu_{a}}y} + \frac{1}{\mu_{a}} \sum_{n=-\infty}^{\infty} G_{m} W_m(a)
\]

(15)

**Potential vector of the magnetic field in air**

Assuming that the solution is:

\[
A = \int_{-\infty}^{+\infty} M(a,y) e^{i\alpha y} da
\]

And:

\[
\frac{\partial^2 M(a,y)}{\partial y^2} - a^2 M(a,y) = 0
\]

(17)

Where:

\[
M(a,y) = U(a) e^{-|a|y}, y > 0
\]

(18)

The following result in:

\[
A(x,y) = \int_{-\infty}^{+\infty} U(a) e^{-|a|y} e^{i\alpha x} da, y > 0
\]

(19)

**Boundary conditions on the earth/air surface**

The boundary conditions are:

\[
\begin{align*}
\left\{ \frac{1}{\mu_{a}} \frac{\partial A(x,y)}{\partial y} \right|_{y=0^-} & - \frac{1}{\mu} \frac{\partial A(x,y)}{\partial y} \right|_{y=0^+} \\
A(x,0^-) & = A(x,0^+) 
\end{align*}
\]

(20)

In the earth, taking into account the earth/air and earth/conductor boundaries:

\[
U(a) = \frac{1}{\pi \mu |a|} \sqrt{\frac{|a^2 - q_e^2|}{|a^2 - q_e^2|}} \sum_{m=-\infty}^{\infty} G_m W_m(a)
\]

(21)

\[
F(a) = -\frac{\mu}{\pi |a|} \sqrt{\frac{|a^2 - q_e^2|}{|a^2 - q_e^2|}} e^{-i\alpha x} \sum_{m=-\infty}^{\infty} G_m W_m(a)
\]

(22)

Magnetic vector potential in the air, taking into account the two boundary conditions:

\[
\vec{A}(x,y) = \int_{-\infty}^{+\infty} \frac{1}{\pi \mu |a|} \sqrt{\frac{|a^2 - q_e^2|}{|a^2 - q_e^2|}} e^{-i\alpha x} \sum_{m=-\infty}^{\infty} G_m W_m(a) da
\]

(23)

Under certain conditions it is possible to use the Pollaczec solution where:

\[
G_m \approx 0, \forall m \in Z, m \neq 0
\]

valid for \(q_a r_e \ll 1, q_a = |\vec{q}_a|\) and \(q_a h \ll 1, q_a = |\vec{q}_a|\)

\[
\vec{A}^O = \sum_{m=-\infty}^{\infty} G_m H_m^{(2)}(q_m r) e^{im\phi}, r > r_e
\]

(24)

In the surface of the conductor and taking into consideration the boundaries:

\[
\vec{A}^O \approx G_0 H_0^{(2)}(q_0 r) , r = r_e
\]

(25)

**Boundary conditions on the surface cable/earth**

Due to the geometric shape of the surface of the cable, the solution is written in cylindrical coordinates.

[Figure 2. Representation of the earth/cable surface with indication of the cable axis and the exterior radius \(r_e\)]

Given the Pollaczec approximation:

\[
|\vec{A}|_{r=r_e} \approx J_0(q_m r_e) G_0 b_{0,0} + G_0 H_0^{(2)}(q_0 r_e)
\]

(26)

Now taking into account the boundary conditions at \(r = r_e\) (earth/conductor):

\[
\left\{ \frac{1}{\mu_a} \frac{\partial A(r,\phi)}{\partial \phi} \right|_{r=r_e^+} = \frac{1}{\mu} \frac{\partial A(r,\phi)}{\partial \phi} \right|_{r=r_e^-}
\]

(27)

\[
A(r_e^+, \phi) = A(r_e^-, \phi)
\]

Inside the cable sheath with dielectric characteristics, being the radius shown in Figure 3:

\[
A = C_1 \ln \left( \frac{r}{r_e} \right) + C_2 , r_e < r < r_e
\]

(28)

Then

\[
\left\{ \begin{align*}
G_0 \left[ H_0^{(2)}(q_m r_e) + J_0(q_m r_e) b_{0,0} \right] & = C_1 \ln \left( \frac{r}{r_e} \right) + C_2 \\
\frac{1}{\mu_a} G_0 \left[ H_0^{(2)}(q_0 r_e) + J_0(q_0 r_e) b_{0,0} \right] & = C_1 \frac{1}{q_0 r_e}
\end{align*} \right.
\]

(29)
the phase conductor and the conducting sheath of the cables are in flat configuration.

The mutual impedance between the cable and the earth, $Z_3$, is the impedance related to the earth, $Z_3$ is due to the variation of the flux in the outer insulation, and $Z_5$ is the outer sheath internal impedance given from the voltage drop along the outer surface of the sheath [4].

$$Z_3 = j \frac{\omega \mu_b}{2\pi} 1_{b0}$$

$$Z_5 = j \frac{\omega \mu_b}{2\pi} 1_{b0}$$

With:

$$D = \frac{1}{\epsilon_r} [H^{(1)}(q_{bc})H^{(2)}(q_{bc}) - H^{(1)}(q_{bc})H^{(2)}(q_{bc})]$$

The mutual impedance between the cable and the sheath is:

$$Z_{cb} = Z_{bc} = Z_{bb} - Z_4$$

$Z_4$ is the sheath mutual impedance given by [4]:

$$Z_4 = -j \frac{\mu_b}{\pi \epsilon_r b0 b0}$$

The impedance of the conductor itself is given by:

$$Z_1 = j \frac{\omega \mu_b}{2\pi} 1_{b0}$$

$$Z_2 = -j \frac{\omega \mu_b}{2\pi} 1_{bc}$$

$$Z_3 = \frac{\mu_b}{4\pi} \frac{[H^{(1)}(q_{bc})H^{(2)}(q_{bc}) - H^{(1)}(q_{bc})H^{(2)}(q_{bc})]}{q_{bc}}$$

In the calculation of the impedance for elements outside the diagonal, the provision of the different cables must be taken into account. In this case the cables are in flat configuration.

### III – Constitutive parameters of underground power-transmission systems

The three phase underground cables can be monopolar or tripolar. The conductors are isolated and surrounded by a sheath, with mechanical and chemical protection function, connected to the earth. Each cable has two metallic conductors, one is the central conductor and the other the conductor sheath. This is the basic configuration normally used for high voltage cables. In this work it was considered monopolar cables.

**Figure 3. Cross section of an underground cable.**

**Longitudinal Impedance**

Assuming a three-phase system consisting of three equal cables:

$$\begin{pmatrix} (q_{1b}) \\ (q_{2b}) \\ (q_{3b}) \end{pmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} 1_1 \\ 1_2 \\ 1_3 \end{bmatrix}$$

The matrices of the diagonal are then:

$$[Z_{11}] = \begin{bmatrix} Z_{cc} & Z_{cb} & Z_{cb} \\ Z_{bc} & Z_{bb} & Z_{cb} \\ Z_{bc} & Z_{cb} & Z_{bb} \end{bmatrix}$$

The impedance of the sheath itself is then:

$$Z_{bb} = Z_3 + Z_6 + Z_5$$

Where $Z_6$ is the impedance related to the earth, $Z_6$ is due to the variation of the flux in the outer insulation,
Thus the impedance between cable $i$ and $j$ is given by:

$$
[Z_{ij}] = \begin{bmatrix}
Z_{cci} & Z_{cbi} \\
Z_{bci} & Z_{bbi}
\end{bmatrix}
$$

(44)

$Z_{cci}$ is the impedance between the phase conductor of cable $i$ and the phase conductor of cable $j$. $Z_{bbi}$ is the impedance between the sheath of cable $i$ and the sheath of cable $j$.

$$
Z_{cci} = Z_{bbi} = Zm_{ij}
$$

(45)

The field outside the cable is taken into account for the calculation of $Zm_{ij}$. For the mutual impedance between the cable sheath and the conductor, the cable is seen as driven by a conductor running in a homogeneous soil. $Zm_{ij}$ may be evaluated by using a closed form with appropriate Bessel and Hankel functions:

$$
Zm_{ij} = j\omega [G_0 H_0^{(2)} (q_{xij}) + J_0 (q_{xij}) G_0 b_{0,0}]
$$

(46)

**Longitudinal Admittance**

The complex amplitude $U$ and $I$ are now just function of the longitudinal coordinate $x$.

$$
\begin{bmatrix}
I_{c1} \\
I_{c2} \\
I_{c3} \\
I_{b1} \\
I_{b2} \\
I_{b3}
\end{bmatrix} =
\begin{bmatrix}
[Y_{cc}] & [Y_{cb}] \\
[Y_{bc}] & [Y_{bb}]
\end{bmatrix}
\begin{bmatrix}
U_{c1} \\
U_{c2} \\
U_{c3} \\
U_{b1} \\
U_{b2} \\
U_{b3}
\end{bmatrix}
$$

(47)

**Capacity coefficients matrix**

The capacitance matrix allows the conductor to hold the potential throughout the insulation. In order for this matrix to be determined, it is considered its inverse, the potential coefficients matrix. It is only considered the potential vector inside the cables, being the non diagonal elements, $S_{ij}$, null. For the main diagonal elements, $S_{ii}$, we have:

$$
S_{ii} = \begin{bmatrix}
[S_{cci}] & [S_{cbi}] \\
[S_{bci}] & [S_{bbi}]
\end{bmatrix}
$$

(48)

With

$$
S_{cci} = \frac{1}{2\pi \epsilon_{ci}} \ln \frac{r_{ci}}{r_{ce}}
$$

(49)

$$
S_{cbi} = \frac{1}{2\pi \epsilon_{ci} \epsilon_{bi}} \ln \frac{r_{ci}}{r_{ce}} = S_{bci}
$$

(50)

$$
S_{bbi} = \frac{1}{2\pi \epsilon_{bi} \epsilon_{bi}} \ln \frac{r_{bi}}{r_{be}}
$$

(51)

In matrix form:

$$
S_{ii} = \frac{1}{j\omega} \begin{bmatrix}
\ln \frac{r_{ci}^2}{r_{ce}^2} & \ln \frac{r_{ci}}{r_{ce}} \\
\ln \frac{r_{ci}}{r_{ce}} & \ln \frac{r_{ci}^2}{r_{ce}^2}
\end{bmatrix}
$$

(52)

Finally resulting in the longitudinal admittance matrix, inverting the $S$ matrix in order to obtain $[C]$.

$$
Y_T = j\omega [C] =
\begin{bmatrix}
y_{cc11} & 0 & 0 & y_{cb11} & 0 & 0 \\
0 & y_{cc22} & 0 & 0 & y_{cb12} & 0 \\
0 & 0 & y_{cc33} & 0 & 0 & y_{cb13} \\
y_{bc11} & 0 & 0 & y_{bb11} & 0 & 0 \\
0 & y_{bc22} & 0 & 0 & y_{bb22} & 0 \\
0 & 0 & y_{bc33} & 0 & 0 & y_{bb33}
\end{bmatrix}
$$

(60)

**Frequency domain propagation**

According to the linearity problem, the propagation problem is entirely formulated in the frequency domain. So the following systems can be written:

$$
\begin{align*}
\frac{d[U]}{dx} &= -[Z_L][U] \\
\frac{d[U]}{dx} &= -[Y_T][U]
\end{align*}
$$

(53)

$$
\begin{align*}
\frac{d^2[U]}{dx^2} &= [Z_L][Y_L][U] \\
\frac{d^2[U]}{dx^2} &= [Y_T][Z_L][U]
\end{align*}
$$

(54)

The $([Z_L][Y_T])$ product can be transformed in a diagonal matrix using the transformation matrix $[T]$. The transformation matrix $[W]$ can transform the $([Y_T][Z_L])$ product in a diagonal matrix. Matrix $[T]$ is obtained by the eigenvectors of $([Z_L][Y_T])$ and matrix $[W]$ by the eigenvectors of $([Y_T][Z_L])$.

The diagonal matrix of the eigenvalues of $([Z_L][Y_T])$ is $[\gamma]^2$ and is obtained by:

$$
[\gamma]^2 = [T]^{-1}([Z_L][Y_T])[T]
$$

(55)

$$
[\gamma]^2 = [W]^{-1}([Y_T][Z_L])[W]
$$

(56)

Can be decomposed into a product of two diagonal matrices $[Z_L]$ and $[Y_T]$ by:

$$
\begin{align*}
[Z_L] &= [T]^{-1}([Z_L][W]) \\
[Y_T] &= [W]^{-1}([Y_T][T])
\end{align*}
$$

(57)

For the calculation of the matrix $[W]$, it is necessary to build a matrix $[Y_T]$ so it verifies the equation $[Z_L][Y_T] = [\gamma]^2$, then:
\[
\begin{align*}
[W] &= [Y_T][T][\mathcal{P}_T] \\
[W]^{-1} &= [\mathcal{P}_T][T]^{-1}[T][\mathcal{P}_T]^{-1}
\end{align*}
\] (58)

Introducing \([T]\) and \([W]\) to the system and applying the necessary simplifications:

\[
\begin{align*}
[\mathcal{U}] &= [T]^{-1}[U] \\
[I] &= [W]^{-1}[I]
\end{align*}
\] (59)

There is now a system of equations, which can be written in the following matrix form:

\[
\begin{align*}
[\mathcal{U}] &= \exp(-[\mathcal{G}]z)[U_1] + \exp([\mathcal{G}]z)[U_2] \\
[I] &= \exp(-[\mathcal{G}]z)[I_1] + \exp([\mathcal{G}]z)[I_2]
\end{align*}
\] (60)

\([\mathcal{U}_1], [\mathcal{U}_2], [I_1] \) and \([I_2]\) are column vectors for modal quantities. For the voltage the expression is:

\[
[U] = \exp(-[\Gamma]z)[U_1] + \exp([\Gamma]z)[U_2]
\] (61)

And for the current:

\[
[I] = [Z_c]^{-1}[\Gamma]\exp(-[\Gamma]z)[U_1] - [Z_c]^{-1}[\Gamma]\exp([\Gamma]z)[U_2]
\] (62)

Assuming:

\[
[I] = [Y_T][\Gamma]^{-1}\exp(-[\Gamma]z)[U_1] - [Y_T][\Gamma]^{-1}\exp([\Gamma]z)[U_2]
\] (68)

Where:

\[
\begin{align*}
[\Gamma]^{-1} &= [T][\mathcal{G}]^{-1}[T]^{-1} \\
\exp(\pm[\Gamma]z) &= [T]\exp(\pm[\mathcal{G}]z)[T]^{-1}
\end{align*}
\] (63)

Note that \([\Gamma]\) is a non-diagonal matrix.

**IV-Transient analysis of underground power-transmission systems**

The tools used in this work are the Fast Fourier Transform (FFT) and the Inverse Fast Fourier Transform (IFFT). These are fast algorithms for implementing a number of samples where the input signal is transformed in the same number of frequency points. The calculations performed by these algorithms are \(\frac{1}{2}\log_2 N\) multiplications and \(\log_2 N\) additions for \(2^N\) samples.\(^7\)

**Equations of the line ended with a three-phase load**

Considering a three-phase generator and the frequency propagation equations:

\[
[U(0)] = [U_0] - [Z_g][I(0)]
\] (64)

Where:

\([U(0)]\) - Column vector of complex amplitude voltages at the beginning of the line, \(z = 0\).

\([U_0]\) - Column vector of complex amplitude voltages for the three-phase generator

\([Z_g]\) - 3x3 matrix with each element being an impedance

\([I(0)]\) - Column vector of complex amplitude currents at the beginning of the line, \(z = 0\).

Now considering the three-phase load at the end of the line:

\[
[U(I)] = [Z_c][I(I)]
\] (65)

Where:

\([U(I)]\) - Column vector of complex amplitude voltages at the end of the line, \(z = l\).

\([Z_c]\) - 3x3 matrix with each element being an impedance

\([I(I)]\) - Column vector of complex amplitude currents at the end of the line, \(z = l\).

It will be established matrix transfer functions in order to relate the voltage and current complex amplitudes on a determined place of the line. The generator is considered to be of 230 kV amplitude.

**Transfer Functions**

For \(z = 0\) and \(z = l\):

\[
\begin{align*}
[U(0)] &= [U_1] + [U_2] \\
[I(0)] &= [Y_T][\Gamma]^{-1}[U_1] - [Y_T][\Gamma]^{-1}[U_2] \\
[U(I)] &= \exp(-[\Gamma]z)[U_1] + \exp([\Gamma]z)[U_2] \\
[I(I)] &= [Y_T][\Gamma]^{-1}\exp(-[\Gamma]z)[U_1] - [Y_T][\Gamma]^{-1}\exp([\Gamma]z)[U_2]
\end{align*}
\] (66)

Where:

\[
[U_1] = BA^{-1}[U_0]
\] (67)

\[
[U_2] = A^{-1}[U_0]
\] (68)

And the transfer functions that allow the calculation of voltages and currents at a generic point \(z\):

\[
\begin{align*}
[U_z] &= \{\exp(-[\Gamma]z)BA^{-1} + \exp([\Gamma]z)A^{-1}\}[U_0] \\
[U_z] &= Y_T[\Gamma]^{-1}\{\exp(-[\Gamma]z)BA^{-1} - \exp([\Gamma]z)A^{-1}\}[U_0]
\end{align*}
\] (69)
With:

\[
A = \left( (Z_0)[Y_2][\Gamma]^{-1} + [E] \right) e^{\Gamma_0 (Z_0)[Y_2][\Gamma]^{-1} - [E]}^{-1} (Z_0)[Y_2][\Gamma]^{-1} + [E] e^{\Gamma_0 [E] + (1 - [E])}
\]

(70)

\[
B = e^{\Gamma_0 ([E])^{-1} (Z_0)[Y_2][\Gamma]^{-1} + [E])}
\]

(71)

Frequency domain

To analyze the electromagnetic transient in a power-transmission system, it is necessary to pass the time domain voltages of the generator to the frequency domain. The FFT algorithm was used, the real part is represented in blue with the imaginary in red.

Transient analysis of a three-phase line with load

To study the electromagnetic transients it is necessary to know the parameters of the three-phase line:

![Figure 5. Fourier transform of the phase generator voltages.](image)

**Table 1. Parameters of the system.**

<table>
<thead>
<tr>
<th>Phase</th>
<th>E0/m</th>
<th>( Z_0 )</th>
<th>( Y_0 )</th>
<th>( \rho )</th>
<th>( \rho' )</th>
<th>( \rho'' )</th>
<th>( Y_{0real} )</th>
<th>( Y_{0imag} )</th>
<th>( Y_0 )</th>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<td>0.00</td>
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<td>1.20E+07</td>
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<td>0.00</td>
<td>1.20E+07</td>
<td>1.20E+07</td>
<td>1.20E+07</td>
</tr>
</tbody>
</table>

The permittivity and permeability represented in the above table refers to the insulating layer surrounding the phase conductor and conducting sheath.

No Load

Phase impedance \( Z_c = \infty \), sheath impedance \( Z_s = 0 \Omega \). The generator phases are in short-circuit and the sheaths with a 1 MΩ impedance. The system is 5km in length. Real part is represented in blue with the imaginary in red.

![Figure 6. Phase Voltage at z=5km.](image)

![Figure 7. Sheath current at z=5km.](image)

The other voltages and currents are not represented because the results gave a null amplitude.

Short-Circuit Load

Phase impedance \( Z_c = 0 \Omega \), sheath impedance \( Z_s = 0 \Omega \). The generator phases are in short-circuit and the sheaths with a 1 MΩ impedance. The system is 5km in length. Real part is represented in blue with the imaginary in red.

![Figure 8. Phase current at z=5km.](image)

![Figure 9. Sheath current at z=5km.](image)
The other voltages and currents are not represented because the results gave a null amplitude.

**Adapted Load**

Phase impedance $[Z_{c1}] = [Z_{c2}] = 66.4\Omega$, $[Z_{c3}] = 63.14\Omega$, sheath impedance $[Z_s] = 0\Omega$. The generator phases are in short-circuit and the sheaths with a $1\, M\Omega$ impedance. The system is 5km in length. Real part is represented in blue with the imaginary in red.

![Figure 10. Phase voltage at z=5km.](image1)

![Figure 11. Phase current at z=5km.](image2)

![Figure 12. Sheath current at z=5km.](image3)

The other voltages and currents are not represented because the results gave a null amplitude.

**V – Equivalent network with lumped parameters**

An equivalent network with lumped parameters is a system that is close to the behavior of a power transmission line in a determined frequency band. In this work the conducting sheaths are connected to the earth. This allows the reduction of the matrix dimensions from 6x6 to 3x3, obtaining three independent modes.

![Figure 13. Real part of the longitudinal impedance.](image4)

**Propagation mode parameters**

From the equivalent quadripole of a mono-phase line section:

$$\begin{align*}
\dot{I}_k(0) &= \frac{\ddot{u}_k(0) - \ddot{u}_k(l)}{Z_l} \\
\dot{I}_k(l) &= -\frac{\ddot{u}_k(l) - \ddot{u}_k(0)}{Z_l}
\end{align*}$$

(72)

Where $\ddot{z}_l$ and $\ddot{y}_k$ are the longitudinal modal impedance and transversal modal admittance:

$$\begin{align*}
\ddot{z}_l &= \frac{L}{\gamma_k} \sinh(\gamma_k l) \\
\ddot{y}_k &= \frac{2 \cosh(\gamma_k l) - 2}{\gamma_k \sinh(\gamma_k l)} = 2 \frac{2_k}{\gamma_k} \tgh \left( \frac{\gamma_k l}{2} \right)
\end{align*}$$

(73)

If $\gamma_k l \ll 1$, then $\sinh(\gamma_k l) \approx \gamma_k l, \tgh \left( \frac{\gamma_k l}{2} \right) \approx \frac{\gamma_k l}{2}$ and:

$$\begin{align*}
\ddot{x}_l &= \ddot{z}_k l \\
\ddot{y}_k &= \ddot{y}_k l
\end{align*}$$

(74)

Where $\ddot{z}_k$ is the $k$ mode longitudinal impedance in modal coordinates. $\ddot{x}_l$ and $\ddot{y}_k$ are the diagonal elements of $[\ddot{Z}]$ and $[\ddot{Y}]$. In order to simplify the problem is is considered that the sheaths are connected to the earth, this way only the phases are considered.

$$[\ddot{Y}] = \begin{bmatrix}
\gamma_a + 2\gamma_m & 0 & 0 \\
0 & \gamma_a - \gamma_m & 0 \\
0 & 0 & \gamma_a - \gamma_m
\end{bmatrix}$$

(75)

$$\gamma_a = \frac{(\gamma_{11} + \gamma_{22} + \gamma_{33})}{3}$$

(76)

$$\gamma_m = \frac{(\gamma_{12} + \gamma_{23} + \gamma_{31})}{3}$$

(77)

Transformation matrix $[T]$ is the eigenvectors matrix that diagonalizes the $([Z_s][Y_T])$ product. Transformation matrix $[W]$ is:

$$[W] = [Y][T][Y]^{-1}$$

(78)

Modal impedance matrix is obtained by:

$$[\ddot{Z}] = [\gamma][Y]^{-1}$$

(79)
Longitudinal impedance synthesis

Each element of $[Z]$ will be approximate by the 5 parallel $RL$ branches of the circuit impedance.

$$Z_i(\omega_k) = \frac{1}{\sum_{j=1,j\neq k}^{n} \frac{1}{r_{ij} + j\omega L_{ij}}}$$  \hspace{1cm} (80)

For $n$ frequencies (86) represents a system of $n$ equations. It is resolved by the following interactive process:

$$R_k + j\omega L_k = \frac{1}{\sum_{j=1,j\neq k}^{n} \frac{1}{r_{ij} + j\omega L_{ij}}}$$  \hspace{1cm} (81)

The process is initiated with $R_k$ and $L_k$ values that correspond to 5 times the longitudinal impedance. This values are changed at each interaction, and it is applied to all $k$. The process is repeated until there is no significant change in the parameters values. The adjustment frequencies where chosen in geometric progression so no negative elements would occur. The following figures represent the results for 3 propagation modes at a frequency band, the equivalent system is represented by lines and the equivalent network by dots. The line has 5 km divided by 20 sections 250 m each.

Transformation matrices of voltages and currents

It is implemented in this section a circuit with the objective to simulate the behavior of an underground system with a equivalent network. With the 20 section circuit there exist two adaptation nets, one at the generator and the other at the load that transform modal magnitudes into real ones and vice-versa. It is considered the normalized $[T]$ and $[W]$ matrices, and their inverse. No frequency variation is considered.

No Load

Figure 17. Current by the Fourier Transform for $z=0$ km

Figure 18. Current by the equivalent network for $z=0$ km

Figure 19. Voltage by the Fourier Transform for $z=5$ km

Figure 20. Voltage by the equivalent network for $z=5$ km
VI – Conclusions

The equivalent network with lumped parameters is a more complete method for the system analysis, but requires more memory and processor capacity. After applying some simplifications, such as the sheath connection to the earth, the phase modes have similar attenuations, converging at high frequencies. The modal method and the longitudinal impedance synthesis converge at both low and high frequencies. The results are similar, obtaining a small difference in the oscillations of the transients for both voltage and current. For a possible investigation topic in this subject, it is recommended a development of a program able to solve the set of differential equations to treat this transmission system, as well as the problem of the complete set of six modes.

VII – References