EXTENDED ABSTRACT

Dynamic analysis of elastic solids by the finite element method

Vítor Hugo Amaral Carreiro

Supervisor: Professor Fernando Manuel Fernandes Simões

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Summary

The finite element method (FEM) is a numerical method that allows the attainment of finite dimension approximations for boundary value problems and is now a highly useful tool in the analysis of many linear or non-linear problems in continuum mechanics.

This paper provides a brief overview of the fundamental theoretical concepts behind the formulation and the integration of the dynamic equilibrium equations in plane elasticity. A description of the two methods for integrating in time the dynamic equations (explicit integration and implicit integration) is provided.

The two finite element programs written in Fortran (one with recourse to implicit integration and the other with explicit integration) are applied in the dynamic analysis of three structures with different loads and boundary conditions. The first such example (solid), where the results obtained are compared with results generated by the Abaqus commercial program, serves to validate the written program. In the other two more complex examples (cantilever beam and doubly-clamped beam), the solutions obtained with two different types of numerical time integration are compared. This article then summarily sets out and discusses the results of the cantilever beam example.

While the rule for explicit integration is that the time increment is limited for reasons of stability, the rule for implicit integration is that the time increment is limited only for reasons of precision. The utilisation of the same time increment in the two cases generally leads to a better approximation of the results obtained with the implicit integration rule at the cost of an increased time of calculation. Incorporating an implicit integration rule further enables, through the introduction of numerical damping, a reduction of the influence of the higher energy mode levels on the results.

Keywords: Finite element method, Explicit integration, Implicit integration, Dynamic analysis, Linear elastic solids.
1. Introduction

Almost all natural phenomena may be described in accordance with the laws of physics, through algebraic, differential or integral equations that mutually inter-relate the diverse important quantities making up the problems under study. Getting the equations to deal with such situations on the one hand represents a major challenge and on the other hand, their resolution through exact methods may be difficult or even impossible. This happens in continuum mechanics problems when, for example, data of the problem (domain geometry, material properties, boundary conditions, loads, etc.) are highly irregular. In such cases, there is a need of resorting to methods of approximate analysis, hence, to numerical techniques.

The finite element method (FEM) is one method that allows the attainment of finite dimension approximations of boundary value problems. A boundary value problem consists of ascertaining one or more unknown functions, called dependent variables, satisfying a given set of differential equations within a specific domain or region and which themselves, and possibly their derivatives, take known values on the boundary of this domain [1.2].

This project has as its core objective the development of a finite element program in the Fortran [3.4] language able to analyse plane dynamic problems with infinitesimal deformation involving linear elastic solids and where the integration of the dynamic equations may be carried out through either explicit or implicit methods. The programs developed are subsequently applied to civil engineering problems.

2. The linear elastic solids problem

2.1 Global dynamic equilibrium equations

The initial and boundary value problem consists in determining the displacements $u_i$, strains $\varepsilon_{ij}$ and stresses $\sigma_{ij}$ in the domain $\Omega$ occupied by a linear elastic body in its non-deformed configuration over the period of time $t \in [0, T]$. The body has volume mass $\rho$ and is subject to a field of mass forces $(\mathbf{X}_i)$, imposed displacements $\bar{u}_i$ annulled on the part of its surface $\partial\Omega_u$, and experiencing forces per unit surface $(t_j)$ applied in the remaining part of the surface $\partial\Omega_t$ (Figure 1) [5].
Figure 1 – Linear elastic body in its non-deformed configuration

So as to determine the aforementioned unknowns, the following equations are applied:

- **Dynamic Equilibrium in** $\Omega \times ]0, T[$:

  $\sigma_{ij,i} + X_j = \rho \ddot{u}_j \Leftrightarrow \frac{\partial \sigma_{ij}}{\partial x_i} + X_j = \rho \dddot{u}_j \quad (2.1)$

- **Linear cinematic relationship in** $\Omega \times ]0, T[$:

  $\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (2.2)$

- **Linear elastic constitutive relationship (stress - strain) in** $\Omega \times ]0, T[$:

  $\sigma_{ij} = E_{ijkl} \varepsilon_{kl} \quad (2.3)$

The boundary conditions under consideration are:

$u_j = \bar{u}_j (= 0)$ in $\partial \Omega_u \times ]0, T[ \quad ($essential conditions$) \quad (2.4)$
\[ t_j = \sigma_{ij} n_i \text{ in } \partial \Omega \times [0,T[ \text{ (natural conditions)} \] (2.5)

in which \( n_i \) are the components of the normal versor exterior to \( \partial \Omega \).

Regarding the initial displacement and velocity conditions, there comes:

\[ u_j(x, 0) = u_{0,j}(x) \quad x \in \Omega \] (2.6)

\[ \dot{u}_j(x, 0) = \dot{u}_{0,j}(x) \quad x \in \Omega . \] (2.7)

Using normal finite element methodological procedures, we reach the global dynamic equilibrium equation:

\[ K \, q + M \, \ddot{q} = F \Leftrightarrow p + M \ddot{q} = F \] (2.8)

in which \( p = Kq \) represents the internal forces, \( F \) the external forces, \( M \) the global mass matrix, \( K \) the global stiffness matrix, \( q \) the global vector of nodal displacements and \( \ddot{q} \) the global vector of nodal accelerations at each instant of time. Equation 2.8 has to be satisfied at each instant on the time range for which the analysis is carried out.

The dynamic equilibrium equation system (2.8) is complemented by the initial conditions:

\[ q(0) = q_0 \text{ and } \dot{q}(0) = \dot{q}_0 \] (2.9)

in which \( q_0 \) and \( \dot{q}_0 \) are the global initial nodal displacements and initial nodal velocities vectors, respectively.

The mass matrix \( M \) is designated as the consistent mass matrix. However, a diagonal mass matrix may also be calculated through recourse to the Hinton [6] technique. This technique consists of the following: for diagonal elementary mass matrix components take on values proportional to those corresponding to the consistent mass matrix with the proportional constant chosen so that the total mass of the element is preserved. The diagonalised mass matrix is of particular worth when the integration in time of the dynamic equations is carried out explicitly.
2.2 Temporal integration of dynamic equations

The temporal discretization of the dynamic equations (2.8) is achieved by approximating the velocities and accelerations through finite differences. Two different means of time integration are adopted for this project: an explicit integration, using central finite differences and an implicit integration adopting the α method of Hilber [7].

2.2.1 Explicit integration

In this scheme, the accelerations and velocities are approximated by central finite differences [5]. The accelerations are given by:

\[
\ddot{q}_n = \frac{1}{(\Delta t)^2} \left( q_{n+1} - 2q_n + q_{n-1} \right) \tag{2.10}
\]

and the velocities by:

\[
q_n = \frac{1}{2\Delta t} \left( q_{n+1} - q_{n-1} \right) \tag{2.11}
\]

in which \( \Delta t \) is the time increment. Substituting (2.10) in (2.8) one explicitly obtains the displacements at the instant \( t_{n+1} \) in terms of the displacements at the instants \( t_n \) and \( t_{n-1} \):

\[
q_{n+1} = (\Delta t)^2 M^{-1} (F_n - p_n) + 2q_n - q_{n-1} \tag{2.12}
\]

In order to start the algorithm, it is necessary to obtain the displacement \( q(-\Delta t) \) based on the initial conditions \( q(0) \) and \( \dot{q}(0) \), thereby obtaining the first increase:

\[
q_1 = \frac{(\Delta t)^2}{2} M^{-1} (F_0 - p_0) + q_0 + \Delta t \dot{q}_0 \tag{2.13}
\]

So that this explicit algorithm is stable, the time increment needs to be limited by:

\[
\Delta t \leq \frac{2}{\omega_{m,\infty}} \tag{2.14}
\]
in which \( \omega_{\text{max}} \) is the greatest frequency of the problem corresponding to the finite elements mesh used [8].

### 2.2.2 Implicit integration

The implicit integration approach is also known as the \( \alpha \) method or the Hilber-Hughes-Taylor [7] method and represents a modification of the Newmark method. In accordance with this method, temporal discretization of equation (2.8) is reached by

\[
M \ddot{q}_{n+1} + (1 + \alpha) \ p_{n+1} - \alpha \ p_n = F(t_{n+\alpha})
\]  
(2.15)

in which \( t_{n+\alpha} = (1 + \alpha) \ t_{n+1} - \alpha \ t_n = t_{n+1} + \alpha \ \Delta t \), and the evolution in time of the approximate solutions is provided by the following expressions in terms of finite differences:

\[
q_{n+1} = \ddot{q}_{n+1} + (\Delta t)^2 \ \beta \ \dddot{q}_{n+1},
\]  
(2.16)

\[
\dot{q}_{n+1} = \ddot{q}_{n+1} + \Delta t \ \gamma \ \dddot{q}_{n+1},
\]  
(2.17)

in which

\[
\ddot{q}_{n+1} = q_n + \Delta t \ \dddot{q}_n + \frac{(\Delta t)^2}{2} (1 - 2 \beta) \ \dddot{q}_n,
\]  
(2.18)

\[
\dddot{q}_{n+1} = q_n + \Delta t \ (1 - \gamma) \ \dddot{q}_n.
\]  
(2.19)

The values \( \ddot{q}_{n+1} \) and \( \dddot{q}_{n+1} \) are predictor values for the nodal displacements and velocities while \( q_{n+1} \) and \( q_{n+1} \) are the respective corrected values. The parameters \( \alpha \) and \( \beta \) control the precision and stability of the method. Parameter \( \alpha \) provides the damper for the effect of the highest system frequencies without however impacting on the convergence rate of the method. The Newmark method corresponds to the situation \( \alpha = 0 \), and is also termed the “Trapezoidal Rule”, where \( \beta = \frac{1}{4} \) and \( \gamma = \frac{1}{2} \).

To launch the algorithm, the accelerations \( \dddot{q}_0 \) are obtained based on the initial conditions \( q_0 \) and \( \dddot{q}_0 \) by resolving
\[ M \dot{q}_0 = F_0 - p_0 . \]  

(2.20)

3. Application of the finite element method

This project applies the finite element in the dynamic analysis of three different problems: solid, cantiliver beam and doubly-clamped beam. The displacements and stresses calculations are carried out with the two written programs in Fortran language, one adopting the explicit method and the other using the implicit method of integration of the dynamic equations.

For each structure, the explicit and implicit integration schemes are applied with different values for parameter \( \alpha \) and for meshes with elements of 4 and 8 nodes (Gauss integration order 2 and 3, respectively), with the objective of ensuring a basis of comparison.

In the example of the solid, where the results were compared with those obtained by the Abaqus commercial program, the objective was to validate the programs written. In the examples of the cantiliver beam and the doubly-clamped beam, a more complex discretization was carried out not only enabling the validation of the programs but also to learn about the behaviour of the structures over long periods of time when subject to certain types of load. This paper provides the space only for a brief overview of the cantiliver beam example.

3.1 Cantiliver Beam

The cantiliver beam under study is in steel, of 2 metres in lengths, 0.5 metres in height and 0.1 metres in thickness. The beam is subject to an initial axial displacement of 0.5 millimetres (Figure 2). A homogenous mesh of 100 elements and 126 global nodes was defined in which node 126, located on the furthest upper point of the free end section was chosen for analysis of the horizontal displacement.
The constitutive material properties of the beam are shown in Table 1.

For the explicit integration, $\Delta t = 1.75\times10^{-5}$ s is used respecting the $\Delta t \leq \frac{2}{\omega_{\text{max}}}$ relationship, in which $\omega_{\text{max}} = 1.09007\times10^5$ rad/s. The implicit integration uses the same value for $\Delta t$ in order to compare the results.

<table>
<thead>
<tr>
<th>Material</th>
<th>Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elasticity Module ($E$)</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Poisson Coefficient ($\nu$)</td>
<td>0.3</td>
</tr>
<tr>
<td>Specific Mass ($\rho$)</td>
<td>7,860 Kg/m$^3$</td>
</tr>
</tbody>
</table>

Table 1 – Cantilever beam properties

Figures 3 and 4 represent the evolution in time of the horizontal displacement of node 126 for the interval $[0 - 0.02\text{s}]$, when the dynamic equations are integrated explicitly or implicitly with $\alpha = 0$ and $\alpha = -1/3$. 
Figure 3 – Explicit Integration / Implicit Integration ($\alpha = 0$) – Horizontal displacement for the interval [0 – 0.02s]

Figure 4 - Explicit Integration / Implicit Integration ($\alpha = -1/3$) – Horizontal displacement for the interval [0 – 0.02s]

Figures 5 and 6 show the same evolution in time but in this case for the interval [10 – 10.02s].
Analysis of the graphics verifies that, subsequent to the imposition of the initial 0.5 mm displacement to the beam end, the horizontal displacement oscillates around its point of equilibrium ($\delta h = 0$) over the course of time.

In the interval [0 – 0.02s], the explicit integration curves, implicit with $\alpha = 0$ and implicit with $\alpha = -1/3$, are overlapping over the 0.02 s (Figures 3 and 4).
In the interval \([10 - 10.02s]\), there is a decoupling of the displacement between the explicit integration curve and the implicit integration curve with \(\alpha = 0\). This decoupling results from the fact that the precision obtained by the implicit integration equations in time is greater than that by explicit integration given that the same increase in time was utilised.

Figure 6 clearly shows the numeric damper on the results of the implicit integration with \(\alpha = -1/3\). The horizontal displacement oscillates around the point of equilibrium but with lesser amplitude than in the case of explicit integration. In Figure 5, there is no such damping on the implicit integration with \(\alpha = 0\).

The fact that there is numerical damping only in the implicit integration with \(\alpha = -1/3\) is due to the value of \(\alpha\), that is, the greater the value of \(\alpha\) (in absolute value) the greater the numerical damping over the course of time.

Figures 7 and 8 portray the evolution over time for the interval \([10 – 10.02s]\) and at five distinct instants of time, of the axial stress \(\sigma_{xx}\) in the clamped section of the beam when the dynamic equations are integrated explicitly and implicitly with \(\alpha = -1/3\), respectively.

**Figure 7** - Distribution of stresses in the interval \([10 – 10.02s]\) – Explicit Integration (\(T1 = 10.0000150s\); \(T5 = 10.0003650s\); \(T9 = 10.0007150s\); \(T13 = 10.0010650s\); \(T17 = 10.0014150s\))
The tension diagrams approximate the uniform distribution of the case of the quasi-static extension while differences are notable between the explicit and implicit integrations. The implicit integration diagrams are closer to attaining uniform distribution.

In each figure, signal differences are visible in the diagram. The negative signal means that the beam section is under compression and the positive signal means that there is traction. This variation in signal, over the course of time is due to the horizontal oscillations (lengthening/shortening) of the beam around its point of equilibrium.

4. Conclusions and suggestions for future development

The main results of this project were the two functional FEM calculation programs for the dynamic analysis of plane elastic solids with one program using explicitly integration of the dynamic equations with the other implicitly integrating these equations through the Hilber $\alpha$ method [7].

The programs were applied to the analysis of three different structures. In the example of the cantilever beam with axial initial displacement, it was found that the displacements oscillate around the quasi-static value and that the stress distributions at each instant of time approximate those corresponding to the quasi-static axial extension problem.

In the explicit integration, the time increment $\Delta t$ is limited for reasons of solution stability while this is not the case for the implicit integration where the time increment in linear problems is only
restricted by the precision of results. Using an implicit integration scheme and the same time increment as that in the case of explicit integration enabled more precise results to be attained, hence the differences found in the results.

In the implicit integration, value differences were encountered between $\alpha = -1/3$ and $\alpha = 0$, enabling the conclusion that with $\alpha = -1/3$ there is numeric damping, that is, consideration of a negative value for $\alpha$ brings about a reduction in the influence of the higher (spurious) modes of energy. In many structural applications, only responses involving the lower level modes are of practical interest.

It was also experienced that the calculation time for explicit analyses, carried out with a diagonal mass matrix, is always lesser than the corresponding calculation time for implicit analyses due to their more straight-forward nature.

The programs developed are designed to be used by researchers at subunit 2 of the Instituto de Engenharia de Estruturas, Território e Construção (ICIST – Institute of Structural, Territorial and Construction Engineering) both for the dynamic analysis of elastic solids and as the foundation for the future development of other programs that involve, for example, more complex material behaviours. We plan to develop and adapt these programs to deal with elastoplastic, viscoelastic or viscoplastic material behaviours. These programs may also serve as the point of departure for the dynamic analysis of porous media to be used either in Civil Engineering (for example, in consolidation problems) or in Biomechanics (for example, in analysis of the behaviour of articular cartilage).

5. References


