INSTITUTO SUPERIOR TÉCNICO
Universidade Técnica de Lisboa

# Discrete Family Symmetries and Tri-Bimaximal Leptonic Mixing

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Dissertação para a obtenção de Grau de Mestre em

**Engenharia Física Tecnológica**

**Júri**

| | |
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**Setembro 2008**

# Acknowledgements

I am grateful to CFTP members and in particular to my supervisor, co-supervisor, and David Emmanuel Costa for the help given. The support of my colleagues and family was also of the utmost importance to me.

# Resumo

Recentemente foi descoberto que os neutrinos oscilam pelo que têm massa. Contrariamente ao que se passa com os quarks, os ângulos de mistura dos leptões são grandes. De facto a mistura leptónica aproxima-se dum limite conhecido por *tri-bimaximal mixing*. O Modelo Padrão da física de partículas tem tido um enorme sucesso a descrever o comportamento da Natureza a um nível microscópico e pode facilmente ser estendido de forma a ter em conta estes novos factos. No entanto, muitos dos seus parâmetros não se encontram constrangidos do ponto de vista teórico. Uma forma de introduzir relações em alguns dos seus parâmetros - massas e matrizes de mistura dos fermiões - consiste em postular a existência de simetrias discretas de sabor na teoria.

Numa primeira parte deste trabalho, são revistas as bases da física teórica de partículas. Segue-se o estudo de uma classe de extensões do Modelo Padrão que contêm simetrias discretas de sabor, neutrinos com massa e um ou mais dubletos de Higgs. O objectivo é deduzir propriedade genéricas destes modelos e daí extrair conclusões acerca da possibilidade de se ter *tri-bimaximal mixing* leptónico. Concluo que tal não é possível para a classe de modelos considerados com apenas um dubleto de Higgs invariante.



**Palavras-chave:** Extensões do Modelo Padrão, Mistura Leptónica, Simetrias Discretas, Simetrias Discretas de Família, Simetrias Discretas de Sabor, Tri-Bimaximal Mixing

# Abstract

It was recently discovered that neutrinos oscillate and therefore they must have mass. In contrast with the quark sector, leptonic mixing angles are large. In fact leptonic mixing is said to be approximately tri-bimaximal. The Standard Model of particle physics has been very successful at describing Nature at the microscopic level and it can easily be extended to take into account these new facts. However, there are no theoretical constraints on many of its parameters. One way to introduce relations between some of its parameters - the fermions' masses and mixing matrices - is to postulate the existence of discrete family symmetries in the theory.

In the first part of this work, the basics of particle physics theory are reviewed. This is followed by the study of a class of extensions of the Standard Model which contain discrete family symmetries, massive neutrinos, and one or more Higgs doublets. The aim is to deduce general properties of these models and find out if they can lead to leptonic tri-bimaximal mixing. I conclude that for the class of models considered, with a single invariant Higgs doublet, it is impossible to obtain such a mixing.



**Keywords:** Discrete Symmetries, Discrete Family Symmetries, Discrete Flavour Symmetries, Leptonic Mixing, Extensions of the Standard Model, Tri-Bimaximal Mixing

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# Abbreviations

| | |
|---|---|
| **CKM** | Cabibbo-Kobayashi-Maskawa |
| **CP** | Charge-Parity |
| **CW** | Cabibbo-Wolfenstein |
| **DOF** | Degrees of freedom |
| **GIM** | Glashow-Iliopoulos-Maiani |
| **GWS** | Glashow-Weinberg-Salam |
| **HPS** | Harrison-Perkins-Scott |
| **IRREP** | Irreducible representation |
| **NEMO** | Neutrino Ettore Majorana Observatory |
| **PMNS** | Pontecorvo-Maki-Nakagawa-Sakata |
| **QCD** | Quantum chromodynamics |
| **QFT** | Quantum field theory |
| **SDSS** | Sloan Digital Sky Survey |
| **SM** | Standard Model |
| **SSB** | Spontaneous symmetry breaking |
| **TBM** | Tri-bimaximal mixing |
| **VEV** | Vacuum expectation value |
| **WMAP** | Wilkinson Microwave Anisotropy Probe |

# Part I

# The Standard Model

# Chapter 1

# Generalities

From both theoretical as well as experimental arguments, a model was established that describes the vast majority of observed particle physics events with great precision - the Standard Model of particle physics (SM). This is a relativistic quantum field theory which means that:

- It provides a description of nature at a quantum level. Fields are the fundamental entities, whose excitations are interpreted as being the particles we observe;
- It is compatible with special relativity so that the laws of physics prescribed by the SM are invariant for the proper orthochronous Lorentz group of transformations.

As with any theory, it is important to know to what the SM applies to and how do we extract information from it. Specifically, what is the field content and the dynamics of the theory.

## 1.1 Field content

The SM is a quantum field theory. As such, the excitations of these fields are quantified as particles with characteristic masses and charges. We may speak of three types of interactions each mediated by different types of particles:

| Interaction | Mediator |
|---|---|
| strong | gluons |
| weak | W and Z bosons |
| electromagnetic | photons |

Quantum chromodynamics (QCD) is the part of the model that describes the strong interaction. The weak and the electromagnetic interactions are presented in an unified way by the electroweak theory. In this unified framework, the SM has 12 true force mediators; eight gluons and the $W^-, W^+, Z$, and $B$ bosons. These are the gauge (spin 1) bosons which result from the fact that the SM is a gauge theory based on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ group, an issue that will be explored latter. It should also be pointed out that there is at least one extra interaction in the Universe, gravitation, of which the SM says nothing about.

Knowing the forces, we have to specify the matter fields which interact through these gauge bosons: the up and down quarks; and the charged leptons and neutrinos. Each of these fields is replicated in three families/generations:

| Quarks | Leptons |
|---|---|
| Up ($u$) | Electron neutrino($\nu_e$) |
| Down ($d$) | Electron ($e^-$) |
| Charm ($c$) | Muon Neutrino ($\nu_\mu$) |
| Strange ($s$) | Muon ($\mu^-$) |
| Top ($t$) | Tau neutrino ($\nu_\tau$) |
| Bottom ($b$) | Tau ($\tau^-$) |

The six quark flavors indicated must additionally be multiplied by three to obtain the number of quarks present in the theory, since there are three strong interaction charges (colors). An important point is that these are all spin 1/2 particles, meaning that they are all fermions.

Fermions are split in two chiralities - right and left handed fields - which transform differently under the gauge group. This makes two complex scalar fields (spin 0) necessary for the consistency of the theory - the two scalar fields form the Higgs doublet. Gauge symmetry breaking and the Higgs mechanism are important aspects of the SM which will be reviewed shortly. In the end though, one extra particle is added to the energy spectrum of the SM - the Higgs boson.

## 1.2 Dynamics of the fields

In classical mechanics, in order to obtain the equations of motion of a given system with a set of configuration variables $q_i$, we specify a Lagrangian $L$ dependent on the $q_i$'s and their time derivatives $\dot{q}_i$. From it, an action

$$I = \int_{t_1}^{t_2} L\left(q, \dot{q}\right) dt \tag{1.1}$$

is built [1, 2]. The principle of least action then states that the physical path $q_i\left(t\right)$ from time $t_1$ to $t_2$ is the one for which small variations $\delta q_i$ produce no chance in the action ($\delta I = 0$). In terms of the Lagrangian, this implies that the equations of motion are given by

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \tag{1.2}$$

This is the Lagrangian formalism. Another formulation of classical mechanics is the Hamiltonian one. For each of the configuration variables $q_i$ we define the conjugate momenta $p_i = {}^{\partial L}/_{\partial \dot{q}_i}$. We may then build the Hamiltonian

$$H\left(q, p\right) = \sum_i p_i \dot{q}_i - L\left(q, p\right) \tag{1.3}$$

The set of variables $(q, p)$ form the phase space and from the principle of least action follows that any function $f$ of this space has a time evolution given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \tag{1.4}$$

where

$$\{f_1, f_2\} = \sum_i \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q_i} - \frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i} \tag{1.5}$$

is the Poisson brackets. In particular,

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \tag{1.6}$$
$$\{q_i, p_j\} = -\delta_{ij} \tag{1.7}$$
$$\{H, q_i\} = \frac{dq_i}{dt} \tag{1.8}$$
$$\{H, p_i\} = \frac{dp_i}{dt} \tag{1.9}$$

The obvious generalization is to consider a system with infinite degrees of freedom. These are expressed as fields $\phi_i$ which are now not just time dependent but also functions of the space coordinates. It is then possible to build a Lagrangian density $\mathscr{L}$ dependent on these fields and their derivatives and from it define an action

$$I = \int \mathscr{L}(x)\, d^4x \tag{1.10}$$

where the integration is to be taken over all spacetime. Again, if we require that the action is stationary for the true physical configuration of the system, we get the following equations ( the Generalized Euler-Lagrange Equations ):

$$\frac{\partial \mathscr{L}}{\partial \phi_i} = \partial_\mu \left[ \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_i)} \right] \tag{1.11}$$

with the Einstein summation convention. An important point is that the $\phi_i$'s are not necessarily observables. In fact, we just require that any measurable quantity of the system is a function of the fields. One notable example of this are gauge theories, where the change of gauge is irrelevant for the physics of the system. Another relevant matter is that for a relativistic theory, the Lagrangian density is invariant for transformations of the Lorentz group. In particular, the above expression shows that the Lagrangian formulation is manifestly covariant. It should come as no surprise that the same can't be said of the Hamiltonian approach since it requires the discrimination of time, i.e., there is a need to foliate spacetime in hypersurfaces with some time label. If we do so, we may define conjugate fields

$$\pi_i = \frac{\delta L}{\delta (\partial_0 \phi_i)} \tag{1.12}$$

where $\delta/\delta\varphi$ is the functional derivative and the Lagrangian $L$ is the space integral of $\mathscr{L}$. With the Hamiltonian function

$$H = \int d^3x \left( \sum_i \pi_i \partial_0 \phi_i - \mathscr{L} \right) \tag{1.13}$$

, Hamilton's equations of motions may be written as

$$\partial_0 \phi_i = \{H, \phi_i\} \tag{1.14}$$
$$\partial_0 \pi_i = \{H, \pi_i\} \tag{1.15}$$

with the Poisson brackets defined as

$$\{f_1, f_2\} = \int d^3x \sum_i \left[ \frac{\delta f_1}{\delta \pi_i(x)} \frac{\delta f_2}{\delta \phi_i(x)} - \frac{\delta f_1}{\delta \phi_i(x)} \frac{\delta f_2}{\delta \pi_i(x)} \right] \tag{1.16}$$

for some time $t$. Again, similarly to a discrete system,

$$\{\phi_i(\mathbf{x}), \phi_j(\mathbf{x}')\} = \{\pi_i(\mathbf{x}), \pi_j(\mathbf{x}')\} = 0 \tag{1.17}$$
$$\{\phi_i(\mathbf{x}), \pi_j(\mathbf{x}')\} = -\delta^3(\mathbf{x} - \mathbf{x}')\, \delta_{ij} \tag{1.18}$$

This is the classical field theory. However, where are the photons of the electromagnetic field? Where is the quantization? Things seem to interact as packets like for instance an electron absorbing a photon. This problem was addressed by many physicists and a general prescription was found that allows for the transformation of a classical field theory into a quantum one - the canonical quantization formalism [3]. Qualitatively, the main step in canonical quantization is the promotion of the canonical variable $\phi_i$ and $\pi_i$ to operators acting on a Fock space and the substitution of the Poisson bracket by commutators.

To be specific, for bosons we have

$$[\phi_i(\mathbf{x}), \phi_j(\mathbf{x}')] = [\pi_i(\mathbf{x}), \pi_j(\mathbf{x}')] = 0 \tag{1.19}$$
$$[\phi_i(t, \mathbf{x}), \pi_j(t, \mathbf{x}')] = i\delta_{ij}\delta^3(\mathbf{x} - \mathbf{x}') \tag{1.20}$$
$$\partial_0 \phi_i(t, \mathbf{x}) = i[H, \phi_i(t, \mathbf{x})] \tag{1.21}$$
$$\partial_0 \pi_i(t, \mathbf{x}) = i[H, \pi_i(t, \mathbf{x})] \tag{1.22}$$

with $[A,B]=AB-BA$. For fermions, the commutators in the first two equations are to be replaced by anticommutators. From 1.21 and 1.22 ,the time evolution of $\phi_i$ and $\pi_i$ may be written as

$$\begin{aligned}\phi_i(t,\mathbf{x}) &= e^{iH(t-t_0)}\phi_i(t_0,\mathbf{x})\,e^{-iH(t-t_0)} &(1.23)\\ \pi_i(t,\mathbf{x}) &= e^{iH(t-t_0)}\pi_i(t_0,\mathbf{x})\,e^{-iH(t-t_0)} &(1.24)\end{aligned}$$

For simple Lagrangian densities like free field theories, the Hamiltonian $H$ may have a form that allows to solve these equations exactly. For the rest of the cases, $H$ may be regarded as the sum of an unperturbed $H_0$ and a perturbation $H'$,

$$H = H_0 + H' \quad (1.25)$$

we may then switch to the interaction picture and use perturbation theory. Then

$$\begin{aligned}\phi_i^I(t,\mathbf{x}) &= U(t,0)\,\phi_i(t,\mathbf{x})\,U^{-1}(t,0) &(1.26)\\ \pi_i^I(t,\mathbf{x}) &= U(t,0)\,\pi_i(t,\mathbf{x})\,U^{-1}(t,0) &(1.27)\end{aligned}$$

with the time evolution operator given by

$$U(t,t_0) = e^{iH_0t}e^{-iHt}e^{iHt_0}e^{-iH_0t_0} \quad (1.28)$$

Notice that the time evolution of $\phi_i^I$ and $\pi_i^I$ is soluble since we only have to deal with the unperturbed $H_0$

$$\begin{aligned}\partial_0\phi_i^I(t,\mathbf{x}) &= i\left[H_0,\phi_i^I(t,\mathbf{x})\right] &(1.29)\\ \partial_0\pi_i^I(t,\mathbf{x}) &= i\left[H_0,\pi_i^I(t,\mathbf{x})\right] &(1.30)\end{aligned}$$

The influence of $H'$ is felt through $U(t,t_0)$, whose time evolution is given by

$$i\frac{\partial}{\partial t}U(t,t_0) = H'^I(t)\,U(t,t_0) \quad (1.31)$$

with

$$H'^I(t) = e^{iH_0t}H'e^{-iH_0t} \quad (1.32)$$

For the explicit solution of $U(t,t_0)$, one needs to make the following time integration:

$$U(t,t_0) = \exp\left[-i\int_{t_0}^{t}dt'H'^I(t')\right] \quad (1.33)$$

This expression can be expanded in a power series of operators. The application of these operators to the vacuum state $|0\rangle$ can be uniquely associated to Feynman diagrams.

As will be discussed in the next section, in gauge theories such as the Standard Model, the fields may not be observables of the theory since there is gauge freedom. In other words, there are less degrees of freedom than one would first assume. In fact, if we take the fields as variables then these must be viewed as being constrained. Now, since the canonical quantization formalism is not particularly suited to deal with constrained systems, another approach to the quantization problem is commonly used - path integration. The basic ideas of this approach are now presented for the simple case where there is only one degree of freedom $q$ and the the Hamiltonian is given by

$$H = \frac{P^2}{2m}+V(Q) \quad (1.34)$$

where $P$ and $Q$ are the momentum and position operators. The transition probability from state $|a(t)\rangle$ to $|b(t')\rangle$ is given by the amplitude

$$\langle b(t') \mid a(t)\rangle = \langle b(0) \mid e^{-iH\left(t'-t\right)}\mid a(0)\rangle \tag{1.35}$$

We can introduce a partition of the identity $1 = \int dq_1 \, |q_1(t_1)\rangle \langle q_1(t_1)|$ on the left side of this expression for an intermediate time $t_1$ yielding

$$\langle b(t') \mid a(t)\rangle = \int dq_1 \, \langle b(t') \mid q_1(t_1)\rangle \langle q_1(t_1) \mid a(t)\rangle \tag{1.36}$$

If we do this for $n$ equal spaced times, particularly for $n \to \infty$,

$$\langle b(t') \mid a(t)\rangle = \lim_{n\to\infty} \int \left(\prod_{p=1}^{n-1} dq_p\right) \prod_{p=0}^{n-1} \langle q_{p+1}(t_{p+1}) \mid q_p(t_p)\rangle \tag{1.37}$$

with

$$t_p = \frac{1}{n}\left[(n-p)\,t + pt'\right] \tag{1.38}$$

$$q_0(t_0) = a(t) \tag{1.39}$$

$$q_n(t_n) = b(t') \tag{1.40}$$

In this limit ($\Delta t = \frac{t'-t}{n} \to 0$), under reasonable assumptions for the potential $V$,

$$\langle q_{p+1}(t_p + \Delta t) \mid q_p(t_p)\rangle \to \sqrt{\frac{me^{-i\frac{\pi}{2}}}{2\pi\Delta t}}\, e^{iI[q_{p+1}(t_p+\Delta t), q_p(t_p)]} \tag{1.41}$$

with $I$ being the action evaluated for a linear path:

$$I\left[q_{p+1}(t_p+\Delta t), q_p(t_p)\right] = \int_{q_p(t_p)}^{q_{p+1}(t_p+\Delta t)} dt L(q,\dot{q}) \tag{1.42}$$

$$q(t) = \left(1 - \frac{t-t_p}{\Delta t}\right) q_p + \frac{t-t_p}{\Delta t} q_{p+1} \tag{1.43}$$

Then,

$$\begin{aligned}\langle b(t') \mid a(t)\rangle &= \lim_{n\to\infty} \int \left(\prod_{p=1}^{n-1} dq_p\right) \left[\frac{n\,m\,e^{-i\frac{\pi}{2}}}{2\pi\,(t'-t)}\right]^{\frac{n}{2}} e^{iI\left[b\left(t'\right),a(t)\right]} \\ &\equiv \int \mathscr{D}(q)\, e^{iI\left[b\left(t'\right),a(t)\right]}\end{aligned} \tag{1.44}$$

where $\mathscr{D}(q)$ is the integration measure.

For the purpose of this short description of the dynamics of the fields in a QFT, it suffices to note that in gauge theories (see next section) the integration measure in the path integrals must be restricted in order not to overcount field configurations that correspond to the same physical configuration (i.e., that are related by a gauge transformation). This leads to the introduction of Faddeev-Popov ghost fields, which are unphysical but necessary for the consistency of gauge QFTs.

## 1.3 Gauge invariance

There is a deep connection between symmetry and invariant quantities. This connection is made by Noether's Theorem which is reviewed below. It predicts the conservation of $n$ local currents just from the condition that the Lagrangian density $\mathscr{L}$ remains unchanged for any $n$-parameter continuous set of transformations. In the context of gauge theories, symmetry goes further than just predicting these conservations laws and in fact determines in part the dynamics of the theory. We shall now see this for classical physics.

### 1.3.1 Noether Theorem - Symmetry and conservation laws

Let us assume that that the Lagrangian density is dependent on a field $\phi(x)$ and is invariant for the following transformation:

$$\phi(x) \rightarrow \phi'(x) + \delta\phi(x) \tag{1.45}$$

with

$$\delta\phi(x) = \varepsilon^a \sigma^a(x) \quad , a = 1, \cdots, n \tag{1.46}$$

This is a $n$-parameter continuous transformation, since this is the number of (constant) $\varepsilon$'s. $\sigma^a(x)$ are some functions that characterize the transformation. We shall now study the analytical variation induced by this transformation:

$$\begin{aligned}
\delta\mathscr{L} &= \frac{\delta\mathscr{L}}{\delta\phi}\delta\phi + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\
&= \partial_\mu\left[\frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\right]\delta\phi + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\delta\phi) \\
&= \partial_\mu\left[\frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\delta\phi\right] \\
&= \partial_\mu\left[\frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\sigma^a(x)\right]\varepsilon^a
\end{aligned} \tag{1.47}$$

The Euler-Lagrange equation was used in addition to the fact that the usual spacetime derivatives commute with the functional derivative. Now, $\delta\mathscr{L}$ is null by assumption and since the $n$ parameters $\varepsilon^a$ are arbitrary we are lead to the conclusion that the $n$ currents

$$J^{a\mu} \equiv \frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\sigma^a(x) \tag{1.48}$$

are conserved ($\partial_\mu J^{a\mu} = 0$). Using the three dimensional notation $J^{a\mu} = \left(J^{a0}, \mathbf{J^a}\right)$,

$$\frac{\partial J^{a0}}{\partial t} = \nabla \cdot \mathbf{J^a} \tag{1.49}$$

, the time derivative of the charge $Q^a$ defined by

$$Q^a = \int d^3x J_0^a(x) \tag{1.50}$$

is

$$\begin{aligned}
\frac{dQ^a}{dt} &= \int d^3x \frac{\partial J_0^a(x)}{\partial t} \\
&= \int d^3x \nabla \cdot \mathbf{J^a} \\
&= \int_S \mathbf{dS} \cdot \mathbf{J^a} \\
&= 0
\end{aligned} \tag{1.51}$$

showing that $Q^a$ is a constant of motion. The last equality results from the fact that we have considered an infinite volume of integration, which implies that, on its surface $S$, $\phi = 0$. Another interesting property of these charges is the fact that they can be seen as generators of the symmetry, i.e., if in 1.46 $\sigma^a(x)$ is replaced with $iT^a\phi(x)$, where $T^a$ are the generators of a Lie Algebra so that

$$\left[T^a, T^b\right] = iC_{abc}T^c \tag{1.52}$$

and $C_{abc}$ are the structure constants of the Lie Group, then it can be shown using the canonical commutation relations that the same relation holds for the charges $Q^a$.

It is important to note that the $\varepsilon^a$ parameters of the transformation were assumed to be independent of the spacetime point under consideration. Such transformation is called global. Otherwise it would be local. This latter type of transformations are at the core of gauge theories.

### 1.3.2 Gauge theories

The word "gauge" appeals directly to a geometric context. In fact, the origin of what we now call gauge theories is a model proposed by Weyl that intended to make a geometric description of electromagnetism. The idea was that a field - the electromagnetic potential $A_\mu$ - was required if the free Lagrangian was to be invariant for arbitrary local changes of gauge (i.e. for local changes of the scale ). Although initially Weyl did not arrive at a successful theory, it was soon found that all that was needed was to identify the scale field $S_\mu$ with $iA_\mu$ instead of simply $A_\mu$. This extra $i$ factor in an exponent transformed Weyl's gauge/scale invariant theory into a phase invariant theory. The use of original name remains with us today though.

We shall now part from the historical perspective. Starting from the kinetic Lagrangian term for a fermion $\psi$,

$$\mathscr{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{1.53}$$

, shall see what it takes for it to be invariant under the transformation

$$\psi(x) \rightarrow \psi'(x) = U(x)\,\psi(x) \tag{1.54}$$

where the matrix $U$ is a spacetime dependent, unitary representation of an element $g$ from a Lie Group $G$. The initial Lagrangian density is transformed as follows

$$\begin{aligned}\mathscr{L}_0 &\rightarrow i\bar{\psi}U^\dagger\gamma^\mu\partial_\mu U\psi \\ &= i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\bar{\psi}\gamma^\mu U^\dagger\left(\partial_\mu U\right)\psi \\ &= \mathscr{L}_0 + i\bar{\psi}\gamma^\mu U^\dagger\left(\partial_\mu U\right)\psi\end{aligned} \tag{1.55}$$

making it clear that for global transformations, $\mathscr{L}_0$ would be invariant. But not so for local transformations. In order to compensate for the extra term, the introduction of new fields $A^a_\mu$ is required. That may be done introducing the covariant derivative

$$D_\mu = \partial_\mu - igT^aA^a_\mu \tag{1.56}$$

where $g$ is a constant and $T^a$ are the generators of the Lie algebra of group $G$. So the numbers of fields introduced is equal to the number of generators of the group algebra. We need the covariant derivative to transform in such a way that $D'_\mu U = UD_\mu$. Let's elaborate on this:

$$\begin{aligned}\left(\partial_\mu - igT^aA'^a_\mu\right)U &= U\left(\partial_\mu - igT^aA^a_\mu\right) \\ &\Leftrightarrow \\ \partial_\mu - igT^aA'^a_\mu &= U\left(\partial_\mu - igT^aA^a_\mu\right)U^\dagger \\ &\Leftrightarrow \\ T^aA'^a_\mu &= UT^aU^\dagger A^a_\mu + \frac{i}{g}U\left(\partial_\mu U^\dagger\right)\end{aligned} \tag{1.57}$$

The matrix $U$ has the form $\exp\left(-i\theta^iT^i\right)$. We are just interested in transformations close to the identity so in the following calculations, we will discard all terms of order in the $\theta^i$'s greater than one. In this case, notice that

$$U(\theta) = 1 - i\theta^i T^i \tag{1.58}$$

so

$$\begin{aligned} UT^a &= T^a U - i\theta^b \left[T^b, T^a\right] \\ &= T^a U - C_{abc}\theta^b T^c \end{aligned} \tag{1.59}$$

where the $C_{abc}$'s are the structure constants of the Lie Group ($\left[T^a, T^b\right] = iC_{abc}T^c$). Inserting this in 1.57 and assuming that the normalization of generators is $\mathrm{tr}\left(T^a T^b\right) = 1/2\delta^{ab}$, the necessary transformation rule for the fields $A^a_\mu$ is as follows:

$$\begin{aligned} T^a A'^a_\mu &= T^a A^a_\mu - C_{cba}\theta^b T^a A^c_\mu - \frac{1}{g}\left(\partial_\mu \theta^a\right) T^a \\ &\Rightarrow \\ A'^a_\mu &= A^a_\mu + C_{abc}\theta^b A^c_\mu - \frac{1}{g}\left(\partial_\mu \theta^a\right) \end{aligned} \tag{1.60}$$

Now, the adjoint representation of the Lie algebra is given by the choice of generators $\left(T^a\right)^c_b = iC_{abc}$ so an infinitesimal transformation $U_{adj}(\theta)$ for a vector $v$ in such a representation would be given by

$$\begin{aligned} v^a &\rightarrow \left(1 - iT^b\theta^b\right)^a_c v^c \\ &= v^a - i\left(T^b\right)^a_c \theta^b v^c \\ &= v^a + C_{abc}\theta^b v^c \end{aligned} \tag{1.61}$$

Comparing this to 1.60 and ignoring the derivative term in it (which is equivalent to considering a global transformation), one concludes that the gauge bosons transform according to the adjoint representation.

So far, it was shown that the invariance of a fermionic kinetic term requires that these spin one (bosonic) gauge fields exist and that they have a definite transformation rule. Next, the kinetic term of a zero spin field $\phi$ is presented. To make it gauge invariant, the ordinary derivative is replaced by a covariant one:

$$\begin{aligned} \left(D_\mu\phi\right)^\dagger D^\mu\phi &\rightarrow \left(D'_\mu U\phi\right)^\dagger D'^\mu U\phi \\ &= \left(UD_\mu\phi\right)^\dagger UD_\mu\phi \\ &= \left(D_\mu\phi\right)^\dagger U^\dagger UD^\mu\phi \\ &= \left(D_\mu\phi\right)^\dagger D^\mu\phi \end{aligned} \tag{1.62}$$

Now that the gauge field were introduced in the theory, the following question arises: are there new terms involving these fields that are invariant under the gauge symmetry? The answer is yes - we may have a kinetic Lagrangian term for the gauge bosons of the form $-1/4F^a_{\mu\nu}F^{a\mu\nu}$ with

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC_{abc}A^b_\mu A^c_\nu \\ &= \frac{2i}{g}\mathrm{tr}\left(T^a\left[D_\mu, D_\nu\right]\right) \end{aligned} \tag{1.63}$$

In this last form it is simple to derive the transformation rule for this rank 2 tensor (in the infinitesimal approximation):

$$
\begin{aligned}
F'^{a}_{\mu\nu} &= \frac{2i}{g}\mathrm{tr}\left(T^a\left[D'_\mu, D'_\nu\right]\right) \\
&= \frac{2i}{g}\mathrm{tr}\left(T^a\left[D'_\mu, D'_\nu\right]UU^\dagger\right) \\
&= \frac{2i}{g}\mathrm{tr}\left(U^\dagger T^a U\left[D_\mu, D_\nu\right]\right) \\
&= \frac{2i}{g}\mathrm{tr}\left\{\left(T^a + C_{abc}\theta^b T^c\right)\left[D_\mu, D_\nu\right]\right\} \\
&= F^a_{\mu\nu} + C_{abc}\theta^b F^c_{\mu\nu} \qquad (1.64)
\end{aligned}
$$

so

$$
\begin{aligned}
F'^{a}_{\mu\nu}F'^{a\mu\nu} &= F^a_{\mu\nu}F^{a\mu\nu} + 2C_{abc}\theta^b F^c_{\mu\nu}F^{a\mu\nu} \\
&= F^a_{\mu\nu}F^{a\mu\nu} \qquad (1.65)
\end{aligned}
$$

since $F^c_{\mu\nu}F^{a\mu\nu}$ is symmetric and $C_{abc}$ antisymmetric is $ac$.

In conclusion, the imposition of gauge invariance forces the introduction of gauge bosons which couple to the existing fields of the model. For each simple Lie group $G_i$ there is a $g_i$ constant which is correlated to the coupling strength of the fields with the gauge bosons. There is a relevant difference between abelian (e.g. $U(1)$) and non-abelian groups (e.g. $SU(n), n > 1$) though. In the first case, the structure constants $C_{abc}$ are null so, for fields transforming with $U(\theta) = \exp\left(-i\theta^i T^i\right)$, from 1.60 we have

$$
A'^{a}_{\mu} = A^a_\mu - \frac{1}{g}\left(\partial_\mu\theta^a\right) \qquad (1.66)
$$

with $D_\mu = \partial_\mu - igT^aA^a_\mu$. Suppose we wanted to couple another field multiplet to the same gauge bosons but with a different coupling constant $g \to g' = \alpha g$, i.e., with a covariant derivative $D_\mu = \partial_\mu - i\alpha g T^a A^a_\mu$. That would be possible since equation 1.66 would be unchanged for $(g, \theta^a) \to (\alpha g, \alpha\theta^a)$. So all that is needed is to state that this new field multiplet actually transforms according to $U(\alpha\theta)$ . For non-abelian groups this is not possible since, with non-null structure constants, the gauge bosons transform according to the full equation 1.60. Indeed, with the transformation $(g, \theta^a) \to (\alpha g, \alpha\theta^a)$ one obtains $A'^{a}_{\mu} = A^a_\mu + \alpha C_{abc}\theta^b A^c_\mu - \frac{1}{g}\left(\partial_\mu\theta^a\right)$ and there is no way of absorbing the $\alpha$ factor.

What interests us at this point is that no mass term for the gauge bosons is allowed because a bilinear term $A^a_\mu A^{a\mu}$ is not gauge invariant. This poses a problem since a particle with no mass should easily be detected. But, in the electroweak sector, the only massless boson that we know of is the photon. To solve this issue, things must get worse before they get better - the phenomenon of spontaneous symmetry breaking will be explored next and, as we shall see, it could lead to even more massless particles.

### 1.3.3 Spontaneous symmetry breaking (SSB)

When the Lagrangian or the Hamiltonian is invariant for a symmetry transformation and the vacuum of the theory is not, there is spontaneous symmetry breakdown. Assume that there is a symmetry transformation

$$
|A\rangle \to |A_U\rangle \equiv U|A\rangle \qquad (1.67)
$$

where $U$ is a unitary representation of an element of the symmetry group. With a Hamiltonian $H$ the energy of these two states is

$$
\begin{aligned}
E &\equiv \langle A|H|A\rangle \qquad (1.68) \\
E' &\equiv \langle A_U|H|A_U\rangle \\
&= \left\langle A\left|U^\dagger HU\right|A\right\rangle \qquad (1.69)
\end{aligned}
$$

so that for

$$UHU^{\dagger} = H \tag{1.70}$$

$E = E'$. This reasoning fails to take into account that there is a ground state $|0\rangle$, the vacuum, to which all other states are related by some operator. For instance,

$$|A\rangle \equiv \phi_A |0\rangle \tag{1.71}$$
$$|A_U\rangle \equiv \phi_{A_U} |0\rangle \tag{1.72}$$

Since an operator $\mathcal{O}$ transforms to $U\mathcal{O}U^{\dagger}$ we know that

$$\phi_{A_U} = U\phi_A U^{\dagger} \tag{1.73}$$

Equation 1.67 is satisfied only if $U^{\dagger}|0\rangle = |0\rangle$ or equivalently $U|0\rangle = |0\rangle$. This means that, in order to have the above energy degeneracy between states connected by a symmetry, it is necessary to have an invariant Hamiltonian as well as an invariant ground state. If this latter condition is not met, there is spontaneous symmetry breakdown (SSB).

From the theoretical point of view, SSB is an appealing way to break a symmetry mainly for two reasons. One is the fact that there is no need to introduce explicit breaking terms in the Hamiltonian / Lagrangian which would bring new parameters into the theory. The second is that this mechanism preserves the renormalizability properties of the model.

There is a potential problem though; SSB gives rise to massless scalar particles: the Goldstone bosons. This result is known (appropriately) as the Goldstone theorem. Its rigorous proof requires naturally a quantum treatment of the problem. Suppose that the Lagrangian has a continuous symmetry. By Noether's theorem, there is a conserved current from which we may define a charge $Q$. Special care is necessary here since $Q|0\rangle$ does not have a finite norm. In particular, we may not use the relation $^{dQ}/_{dt} = 0$ but that is not a problem since all we need is its commutator with some local scalar field operator $\phi(x)$:

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) \\ &= U(Q)\,\phi(x)\,U^{\dagger}(Q) \\ &= \exp(i\varepsilon Q)\,\phi(x)\exp(-i\varepsilon Q) \\ &= \phi(x) + i\varepsilon[Q,\phi(x)] + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{1.74}$$

Notice then that the temporal variation of the following quantity is null:

$$\begin{aligned} \frac{d}{dt}[Q,\phi(0)] &= \int d^3x\,[\partial_0 J^0(x,t),\phi(0)] \\ &= \int d^3x\,[\partial_\mu J^\mu(x,t),\phi(0)] - \int_S \mathbf{dS}\cdot[\mathbf{J}(x,t),\phi(0)] \\ &= 0 \end{aligned} \tag{1.75}$$

since the current is conserved and the volume of integration may be taken to infinity. Also, that due to causality $[\mathbf{J}(x,t),\phi(0)]_{x\in S} \rightarrow 0$. So,

$$\frac{d}{dt}\langle 0|[Q,\phi(0)]|0\rangle = 0 \tag{1.76}$$

On the other hand if the symmetry is spontaneously broken, there must be an observable - $\phi(0)$ for simplicity - for which

$$\langle 0|[Q,\phi(0)]|0\rangle \neq 0 \tag{1.77}$$

Using the translation invariance property of $J^0(x)$,

$$J^0(x) = e^{iP\cdot x} J^0(0) e^{-iP\cdot x} \tag{1.78}$$

and introducing a complete set of states $|n\rangle$ with definite four-momentum we have

$$\begin{aligned} \langle n| J^0(x) |0\rangle &= \langle n| e^{ip_n\cdot x} J^0(0x) e^0 |0\rangle \\ &= \langle n| J^0(0) |0\rangle e^{ip_n\cdot x} \end{aligned} \tag{1.79}$$

$$\begin{aligned} \langle 0| J^0(x) |n\rangle &= \langle 0| e^0 J^0(0) e^{-ip_n\cdot x} |n\rangle \\ &= \langle 0| J^0(0) |n\rangle e^{-ip_n\cdot x} \end{aligned} \tag{1.80}$$

Then, from equation 1.77 it follows that

$$\begin{aligned} 0 &\neq \sum_n \int d^3x \left[\langle 0| J^0(x) |n\rangle \langle n| \phi(0) |0\rangle - \langle 0| \phi(0) |n\rangle \langle n| J^0(x) |0\rangle\right] \\ &= \sum_n \int d^3x \left[\langle 0| J^0(0) |n\rangle \langle n| \phi(0) |0\rangle e^{-ip_n\cdot x} - \langle 0| \phi(0) |n\rangle \langle n| J^0(0) |0\rangle e^{ip_n\cdot x}\right] \\ &= \sum_n (2\pi)^3 \delta(\mathbf{p_n}) \left[\langle 0| J^0(0) |n\rangle \langle n| \phi(0) |0\rangle e^{-iE_n t} - \langle 0| \phi(0) |n\rangle \langle n| J^0(0) |0\rangle e^{iE_n t}\right] \end{aligned} \tag{1.81}$$

As explained, this expression must be non-null and constant with time. That can only happen if there is a state $|n\rangle$ with $E_n = 0$ for $\mathbf{p_n} = \mathbf{0}$ and $\langle n| \phi(0) |0\rangle \neq 0$, $\langle 0| J^0(0) |n\rangle \neq 0$. This massless state is the Nambu-Goldstone boson.

Let us examine a physically relevant example in the classical formulation. Assume that there is a multiplet of spinless real fields with the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - V(\phi) \tag{1.82}$$

which is invariant for the global gauge transformation

$$\phi_i \rightarrow \phi_i' = \phi_i + i\varepsilon^a T^a_{ij}\phi_j \tag{1.83}$$

$T^a_{ij}$ being the component $(ij)$ of the $n$ generators $T^a$ and $V(\phi)$ the potential of the fields. The vevs $v_i$ of the $\phi_i$'s are calculated by finding the value for which the potential is minimal:

$$\left.\frac{\partial V}{\partial\phi_i}\right|_{\phi_i=v_i} = 0 \tag{1.84}$$

Since the potential in itself is gauge invariant,

$$\begin{aligned} 0 &= \delta V \\ &= \frac{\partial V}{\partial\phi_i}\delta\phi_i \\ &= \frac{\partial V}{\partial\phi_i} i\varepsilon^a T^a_{ij}\phi_j \end{aligned} \tag{1.85}$$

The $\varepsilon^a$'s are arbitrary, so $\frac{\partial V}{\partial\phi_i}T^a_{ij}\phi_j$ must be null for all $a = 1, \cdots, n$. Differentiating this relation in respect to the fields and evaluating it for the minimum yields

$$0 = \left[\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}T^a_{ij}\phi_j + \frac{\partial V}{\partial\phi_i}T^a_{ij}\delta_{jk}\right]_{\phi_i=v_i} \tag{1.86}$$

$$= \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi_i=v_i} T^a_{ij}v_j \tag{1.87}$$

Having an invariant vacuum means that

$$\begin{aligned} 0 &= \delta\phi_i|_{\phi_i=v_i} \\ &= i\varepsilon^a T^a_{ij} v_j \\ &\Rightarrow \\ 0 &= T^a_{ij} v_j \quad , a = 1, \cdots, n \end{aligned} \tag{1.88}$$

So $\left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_k}\right|_{\phi_i=v_i}$ can take any value. But was is this quantity? From classical mechanics we know that it is related to mass. Indeed, for real scalar fields it is a square mass, as can be seen from the Taylor expansion of the potential around its minimum:

$$\begin{aligned} V(\phi_i) &= V(v_i) + \left.\frac{\partial V}{\partial\phi_i}\right|_{\phi_i=v_i} (\phi_i - v_i) + \frac{1}{2} \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_k}\right|_{\phi_i=v_i} (\phi_i - v_i)(\phi_j - v_j) + \mathcal{O}\left(\phi_i^3\right) \\ &= V(v_i) + \frac{1}{2} \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_k}\right|_{\phi_i=v_i} (\phi_i - v_i)(\phi_j - v_j) + \mathcal{O}\left(\phi_i^3\right) \end{aligned} \tag{1.89}$$

The conclusion then is that, in the absence of SSB, no restriction is imposed on the field masses. On the other hand, if there are some generators $T^a$ $(a = 1, \cdots, m)$ that do break the vacuum $(T^a_{ij} v_j \neq 0, a = 1, \cdots, m)$, then from equation 1.86 this means that there are $m$ linearly independent combinations of the $\left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_k}\right|_{\phi_i=v_i}$ coefficients that are null. To see this, we shall cast the problem in vectorial form. With

$$\left(M^2\right)_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_k}\right|_{\phi_i=v_i} \tag{1.90}$$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \tag{1.91}$$

if

$$T^a v \neq 0 \quad , a = 1, \cdots, m \tag{1.92}$$

$$T^a v = 0 \quad , a = m+1, \cdots, n \tag{1.93}$$

then, by equation 1.86, $M^2$ has $m$ zero eigenvalues, meaning that there are $m$ Goldstone bosons.

### 1.3.4 The Higgs mechanism

Local gauge symmetry and spontaneous symmetry breakdown, taken separately, generate massless bosons. But taken together one may avoid this. Take for instance the previous case where we had a multiplet of real scalar fields $\phi$ and let us make the usual substitution

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - igT^a_{ij} A^a_\mu \tag{1.94}$$

in order to obtain invariance for local gauge transformations. The evaluation of this Lagrangian term for the vev $v \equiv \langle\phi\rangle_0$ yields

$$\begin{aligned} \mathscr{L} &= \cdots - \frac{1}{2} g^2 \left(T^a v\right)^T T^b v A^a_\mu A^{b\mu} \\ &\equiv \cdots - \frac{1}{2} A^T_\mu M^2_A A^\mu \end{aligned} \tag{1.95}$$

In the last line, $A_\mu \equiv \begin{pmatrix} A^1_\mu & \cdots & A^n_\mu \end{pmatrix}^T$ and $\left(M^2_A\right)_{ab} = g^2 \left(T^a v\right)^T T^b v$. So some gauge bosons will pick up mass. The exact number is given by the non-null eigenvalues of $M^2_A$ which from 1.92 we know to be $m$ - the same as the number of Goldstone bosons.

But do we really get these $m$ massless Goldstone fields, as described above for a global gauge invariant theory? Yes, but one must remember that in field theory the fields are not necessarily directly observable quantities. That is the case with the $m$ "would-be-Goldstone bosons". Why? That's because the theory is now invariant for local gauge transformations so that we can successfully make a transformation which absorbs these massless fields. That is the unitary gauge, which is convenient for the analysis of the particle spectrum of the theory. First, we re-parametrize the fields in the following way:

$$\phi = \exp\left(\frac{iT^a \cdot \xi^a(x)}{\|v\|}\right)[v + \phi'(x)] \quad , a = 1, \cdots, m \tag{1.96}$$

Note that the sum in $a$ is over the broken generators so that the $\xi^a(x)$'s are to be viewed as the $m$ degrees of freedom that correspond to the would-be-Goldstone bosons and $\phi'(x)$ is left only with the remaining degrees of freedom. Clearly, with a local gauge transformation

$$\phi \rightarrow \exp\left(-\frac{iT^a \cdot \xi^a(x)}{\|v\|}\right)\phi \tag{1.97}$$

the Goldstone bosons vanish. Where have these degrees of freedom (dof) gone? Well, we've started with $n_\phi$ real fields (1 dof each) and $n_g$ massless gauge bosons (2 dofs each since there are two possible polarizations for massless vectorial field). If $n_{bg}$ generators break the vacuum, after SSB $n_{bg}$ of the Goldstone bosons will disappear (1 dof lost for each) due to the gauge change that can be made; only $n_g - n_{bg}$ massive $\phi$'s survive. But we must remember that this will affect the gauge fields too. In fact, as we've seen, $n_{bg}$ of them gain mass - 1 dof won for each since a massive vectorial field may be longitudinal polarized. For this reason, it is said that the Goldstone bosons are absorbed by the gauge bosons associated with the broken symmetry generators.

## 1.4 Renormalization

What appears in the Lagrangian density as a mass $m_0$ or a coupling constant $g_0$ does not coincide with what experimentally is called mass $m$ and coupling constant $g$. For that reason, it is crucial to know $m(m_0, g_0)$ and $g(m_0, g_0)$ in order to set the bare quantities - $m_0$ and $g_0$ - so that we end up with the right physical values for $m$ and $g$. This is the renormalization procedure. To complicate this picture, relevant theories, starting with finite $m_0$ and $g_0$ yields infinities for $m$ and $g$ due to unbound momentum integration in loops. Since it is impossible to work with infinite quantities, there is need to regulate these infinities with some parameter $\varepsilon$ called the regulator which gives a finite form to the divergences. These only turn infinite for some limit of the regulator, for instance $\varepsilon \rightarrow 0$. This can be done according to some regularization scheme - dimensional regularization, Pauli-Villars regularization, momentum cutoff - to name the most common ones. Then, we have $m(m_0, g_0, \varepsilon)$ and $g(m_0, g_0, \varepsilon)$ and the challenge is to write the bare quantities in such a way that, when $\varepsilon \rightarrow 0$, we still get the correct physics. In other words, the divergences of the theory are absorbed into the bare parameters. This procedure is possible in renormalizable theories such as the Standard Model described in the next chapter.

One interesting feature of renormalization is that it is not a uniquely determined process - exactly what is absorbed into $m_0$ and $g_0$ has some arbitrariness and yet the resulting physics must be the same. This must be viewed as a symmetry of the theory. In fact, it turns out that it is a scale symmetry: if we scale all the momenta of the particles participating in a process by a factor $\alpha$, for appropriate renormalized quantities $m(\alpha)$ and $g(\alpha)$ the physics stay the same, i.e., the theory is invariant for the group transformation

$$p_1, ..., p_n \rightarrow \alpha p_1, ..., \alpha p_n \tag{1.98}$$

$$m \rightarrow m(\alpha) \tag{1.99}$$

$$g \rightarrow g(\alpha) \tag{1.100}$$

- the renormalization group. The relevant consequence is this: 'constants' of a theory such as mass and coupling constants are not constant at all; varying the scale of energies (via $\alpha$) we get an exact replica of the theory in the original scale ($\alpha = 1$) provided we consider running masses $m(\alpha)$ and running coupling constants $g(\alpha)$.

In particular, any theory that dictates some relations between these parameters either has to be valid (for some reason) for a particular scale or it must hold for all scales $\alpha$ without conflicting with the renormalization group equations. This observation is specially relevant in this work since the fermions' mixing matrices are energy-scale dependent.

# Chapter 2

# The Standard Model

The basic principles of the SM have been reviewed. I shall now move on to describe the various aspects of the model.

## 2.1 Quantum Chromodynamics

After the discovery of the proton and neutron in 1918 and 1932, a zoo of particles were discovered which interacted via the strong interaction - hadrons. These particles were later classified as baryons ("heavy-weight" particles) and mesons ("middle-weight" particles), as opposed to the leptons ("light-weight" particles). The number of mesons in a reaction was not preserved but the number of baryons was. Then, beginning in 1947 heavier than usual baryons and mesons were discovered, making them strange particles [4]. Pais, Gell-Mann and Nishijima latter provided a scheme in which a new quantum number called strangeness was given to particles. Its sum was postulated to be conserved in strong interactions but not so in weak interactions.

With the continued increase of the number of what was then considered as elementary particles, there was urgent need of something analogous to the periodic table for elementary particles. That was provided via Gell-Mann's eightfold way in 1961 and by the quark model proposed three years latter by Gell-Mann and Zweig. According to the quark model, the hadrons are not elementary particles - they are made of three types (flavors) of quarks. The baryons are in fact three quarks and the mesons simply a pair quark-antiquark. It is assumed that there is a $SU(3)_{flavor}$ symmetry so that the three quark flavors - $u$ (up), $d$ (down) and $s$ (strange) - are in the fundamental representation ($\mathbf{3}$) and their antiparticles in $\overline{\mathbf{3}}$. Since

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \tag{2.1}$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \tag{2.2}$$

it would be expected that mesons had an octet and a singlet of degenerate particles. As for baryons there should be one decuplet, two octets and one singlet. That's essentially what Gell-Mann noted in 1961: when baryons and mesons with the same parity and spin are plotted in a grid according to their strangeness and isospin, these multiplets show up clearly as geometric figures.

A striking problem of the quark model is that no particles in the $SU(3)_{flavor}$ fundamental representation are observed. In particular, single quarks have not been detected. But that's not all - clearly the $SU(3)_{flavor}$ is not an exact symmetry since different particles in the same multiplet have significant mass differences. Additionally, we now know of the existence of six quarks. In any case, the proposal of $SU(3)_{flavor}$ as an approximate symmetry was extremely useful since it pointed in the correct way: hadrons are composite particles made of quarks ($qqq$ for baryons and $q\overline{q}$ for mesons).

The explanation for only encountering $qqq$ and $q\overline{q}$ finally came with QCD - a $SU(3)$ local gauge invariant theory of color, which is a new quantum number. The non abelian nature of the symmetry brings some complications to the theory which delayed its acceptance as the theory of the strong interaction. However, after significant theoretical developments, that has finally happened.

With the introduction of color, it is easy to understand the reason why free quarks can't be observed: bounded states of quarks must be colorless (color singlets) due to the phenomenon of color confinement which is related to the non abelian nature of the gauge theory. Quarks are in the fundamental representation ($\mathbf{3}$) and antiquarks in $\overline{\mathbf{3}}$. From group theory, it is easily seen that the direct product of $n_q$ $\mathbf{3}$

representations and $n_{\overline{q}}$ $\overline{\mathbf{3}}$ representations will only contain a singlet if $n_q + 2n_{\overline{q}}$ is a multiple of 3. Up to bound states of three quarks/antiquarks, that means that color singlets are only possible for $q\overline{q}$, $qqq$, and $\overline{qqq}$ (mesons, baryons and antibaryons).

In line with the previous discussion of gauge theories, we may write the QCD Lagrangian density as follows:

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{2}\mathrm{tr}\left(G_{\mu\nu}G^{\mu\nu}\right) + \sum_{k=1}^{6}\overline{q}_k\left(i\gamma^\mu D_\mu - m_k\right)q_k && (2.3)\\
G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] && (2.4)\\
G^a_{\mu\nu} &= 2\mathrm{tr}\left(T^a G_{\mu\nu}\right) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC_{abc}A^b_\mu A^c_\nu && (2.5)\\
D_\mu &= \partial_\mu - igA_\mu && (2.6)\\
A_\mu &= A^a_\mu T^a && (2.7)
\end{aligned}$$

The $k$ index runs over the six quark flavors and the $a$ index is due to the $3^2 - 1 = 8$ QCD gauge bosons - the gluons. Note that the first term of the Lagrangian is equivalent to $-1/4 G^a_{\mu\nu}G^a_{\mu\nu}$:

$$\begin{aligned}
-\frac{1}{2}\mathrm{tr}\left(G_{\mu\nu}G^{\mu\nu}\right) &= -\frac{1}{2}G^a_{\mu\nu}G^{b\mu\nu}\mathrm{tr}\left(T^aT^b\right)\\
&= -\frac{1}{2}G^a_{\mu\nu}G^{b\mu\nu}\frac{\delta^{ab}}{2}\\
&= -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} && (2.8)
\end{aligned}$$

In $SU(3)$ , we may choose the following representation for the group's algebra:

$$T^a = \frac{1}{2}\lambda_a \qquad (2.9)$$

The $\lambda$'s are the traceless Gell-Mann matrices:

$$\begin{aligned}
\lambda_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\\
\lambda_4 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\\
\lambda_7 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} && (2.10)
\end{aligned}$$

The totally antisymmetric structure constants $f_{abc}$ are non-vanishing only for the following combinations (and their permutations):

$$\begin{aligned}
f_{123} &= 1 \quad & f_{147} &= \tfrac{1}{2} \quad & f_{156} &= -\frac{1}{2}\\
f_{246} &= \frac{1}{2} \quad & f_{257} &= \tfrac{1}{2} \quad & f_{345} &= \frac{1}{2}\\
f_{367} &= -\frac{1}{2} \quad & f_{458} &= \tfrac{\sqrt{3}}{2} \quad & f_{678} &= \frac{\sqrt{3}}{2} && (2.11)
\end{aligned}$$

No description of QCD would be complete without a reference to the fact that it is asymptotically free. This property means that in the regime of high energy/short distances, the strength of the interaction between two quarks goes to zero. This can be traced directly to the non-abelian nature of $SU(3)$. In my brief reference to the renormalization program, it was said that masses and coupling constants cease to be constant. Indeed, for a renormalized coupling constant $g$, if one changes the momentum scale by a factor $\lambda$ ($p_i \to \lambda p_i$) then

$$\frac{dg}{dt} = \beta_g \tag{2.12}$$
$$t = \ln \lambda \tag{2.13}$$

For a non-abelian gauge theory with the gauge field coupling with fermions $F$ and scalar mesons $S$, this $\beta_g$ parameter is given by

$$\beta_g = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}t_2(V) + \frac{4}{3}t_2(F) + \frac{1}{3}t_2(S)\right] \tag{2.14}$$

where $t_2(V)$,$t_2(F)$ and $t_2(S)$ are the normalization factors of the representations of the gauge bosons $V$, fermions and scalar mesons, respectively. More precisely, if $T^a(V)$, $T^a(F)$ and $T^a(S)$ are the representation matrices of $V$, $F$ and $S$,

$$t_2(V)\,\delta^{ab} = \mathrm{tr}\left[T^a(V)\,T^b(V)\right] \tag{2.15}$$
$$t_2(F)\,\delta^{ab} = \mathrm{tr}\left[T^a(F)\,T^b(F)\right] \tag{2.16}$$
$$t_2(S)\,\delta^{ab} = \mathrm{tr}\left[T^a(S)\,T^b(S)\right] \tag{2.17}$$

For the $SU(n)$'s adjoint representation one has $t_2(\mathrm{Adj}) = n$ so that $t_2(V) = n$. If we assume no coupling to scalar mesons, we have additionally $t_2(S) = 0$. As for the fermions, if there are $n_f$ gauge multiplets of them in the fundamental representation of the group ($t_2(\mathrm{Fund}) = 1/2$), then $t_2(F) = n_f/2$. Inserting all this in equation 2.14,

$$\begin{aligned}\beta_g &= \frac{g^3}{16\pi^2}\left[-\frac{11}{3}n + \frac{2}{3}n_f\right] \\ &\equiv -bg^3\end{aligned} \tag{2.18}$$

In QCD's case, $n = 3$ and $n_f = 6$ which makes $\beta_g$ negative. The effect of this is immediately clear from relation 2.12: with the increase of energy, the coupling strength decreases. In fact, it is possible to make the integration in $t$, yielding

$$g^2(t) = \frac{g^2(0)}{1 + 2bg^2t} \tag{2.19}$$

Changing the energy scale factor $t$ to $1/2 \ln Q^2/\mu^2$ ($Q$ is the momentum of interest and $\mu$ is the subtraction scale) and introducing the parameter $\Lambda$ (QCD scale parameter) such that

$$\ln \Lambda^2 = \ln \mu^2 - \frac{1}{g^2 b}$$

we may express the variation with $Q^2$ of the strong gauge coupling constant $\alpha_s\left(Q^2\right) \equiv g^2(t)/4\pi$ as

$$\alpha_s\left(Q^2\right) = \frac{4\pi}{\left(\frac{11}{3}n - \frac{2}{3}n_f\right)\ln\frac{Q^2}{\Lambda^2}} \tag{2.20}$$

In QCD when $Q \to +\infty$, $\alpha_s \to 0$ (asymptotic freedom of the quarks). On the other hand, this expression gives us some understanding of quark confinement since as $Q$ decreases and approaches $\Lambda$, $\alpha_s$ diverges, signaling that as distances increase ( momentum transverse $Q$ decrease) it is expected that the inter-quark force intensifies.

## 2.2 Electroweak theory

### 2.2.1 A bit of history

At the heart of the development of the electroweak theory was the discovery of the neutrino. In 1930 it was discovered that in a beta decay

$$ {}^{A}_{Z}X \quad \rightarrow \quad {}^{A}_{Z+1}X + e^{-} \tag{2.21} $$

the spectrum of the emitted electron was continuous. That is not what is to be expected in a decay into two bodies since the energy of final bodies, $E_e$ and $E_X$, is completely fixed by energy and momentum conservation. Add to this the fact that the maximum energy observed for $e^-$ is the energy predicted for a two body decay, and there is a strong case for suggesting the existence of a third particle that is produced in the decay. This would account for the missing energy which would vary, explaining the continuous spectrum. Pauli postulated its existence and Fermi named it the neutrino. In fact, we now know that beta radiation results from the decay of a neutron into a proton, an electron and an antineutrino:

$$ n \quad \rightarrow \quad p + e^{-} + \overline{\nu} \tag{2.22} $$

Fermi went on to build a theory for beta decay. He proposed a contact interaction, occurring at a single point:

$$ \mathscr{L}_F \quad = \quad -\frac{G_F}{\sqrt{2}} \left(\overline{p}\gamma_\mu n\right) \left(\overline{e}\gamma^\mu \nu\right) + \text{h.c.} \tag{2.23} $$

$G_F$ is Fermi's coupling constant and the particle field operators are $p$ for the proton, $n$ for the neutron, $e$ for the electron, and $\nu$ for the neutrino. The size of $G_F$ clearly indicated the presence of a new interaction.

With the discovery of parity non conservation the vectorial Fermi theory was modified to a $V - A$ theory with a vectorial part ($\gamma^\mu$) minus an axial one ($\gamma^\mu\gamma_5$). So, with 3 quark flavors and two leptonic families one has the following effective Lagrangian for the interaction of weak currents:

$$ \mathscr{L}_F \quad = \quad -\frac{G_F}{\sqrt{2}} J^\dagger_\mu J^\mu + h.c. \tag{2.24} $$

$$ J^\mu \quad = \quad J^\mu_{\text{leptonic}} + J^\mu_{\text{hadronic}} \tag{2.25} $$

$$ J^\mu_{\text{leptonic}} \quad = \quad \left[ \begin{array}{cc} \overline{\nu}_e & \overline{\nu}_\mu \end{array} \right] \gamma^\mu \left(1 - \gamma_5\right) \left[ \begin{array}{c} e \\ \mu \end{array} \right] \tag{2.26} $$

$$ J^\mu_{\text{hadronic}} \quad = \quad \overline{u}\gamma^\mu \left(1 - \gamma_5\right) d' \tag{2.27} $$

The problem here is that $d'$ appears to be a mixture of the $d$ and $s$,

$$ d' \quad = \quad A\, d + B\, s \tag{2.28} $$

so that the quarks' transitions $u \leftrightarrow d$ and $u \leftrightarrow s$ do not have the same coupling constants as the leptons' equivalents. In this sense, there is no weak universality. Cabibbo solved this issue by realizing that $A^2 + B^2 \approx 1$. He introduced the angle $\theta_c$ so that

$$ d' \quad = \quad \cos\theta_c d + \sin\theta_c s \tag{2.29} $$

With three quarks, the predicted decay rate of particles such as $K^0 = d\overline{s}$ would be greater that the values obtained experimentally. Glashow, Iliopoulos and Maiani (GIM) realized that the introduction of a new quark $c$ would suppress these events. For two quark families, the hadronic current then becomes

$$ J^\mu_{\text{hadronic}} \quad = \quad \left[ \begin{array}{cc} \overline{u} & \overline{c} \end{array} \right] \gamma^\mu \left(1 - \gamma_5\right) \left[ \begin{array}{c} d' \\ s' \end{array} \right] $$

where $\begin{bmatrix} d' \\ s' \end{bmatrix}$ is a mere rotation of the quark mass eigenstates $\begin{bmatrix} d \\ s \end{bmatrix}$:

$$\begin{bmatrix} d' \\ s' \end{bmatrix} = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} \tag{2.30}$$

From experiments, $\theta_c \approx 13^o$. In order to have universality, this mixing matrix needs to be unitary. That is true for the one above. A $2 \times 2$ unitary matrix could be more general than the one shown in 2.30. But as Kobayashi and Maskawa realized, only for three or more families is the general form of this matrix complex (see part II of this work). That could explain the CP violation that was being observed. We now know of the existence of three families of quarks and the $3 \times 3$ complex matrix for the quark mixing is known as the Cabibbo-Kobayashi-Maskawa matrix (CKM).

Fermi's four fermion interaction has two fatal flaws - it is not renormalizable and it gives infinite cross sections for certain processes (violation of unitarity). These two problems are simultaneously solved if instead of considering a direct interaction of the currents $J^\mu$ we introduce intermediate bosons that mediate the interaction. If these weak bosons were massive, it would account for the short range nature of the interaction too.

### 2.2.2 The Glashow-Weinberg-Salam (GWS) model

Introducing the chirality projectors $P_R$ and $P_L$,

$$P_{R/L} = \frac{1}{2}(1 \pm \gamma_5) \tag{2.31}$$
$$P_{R/L}^2 = P_{R/L} \tag{2.32}$$
$$P_{R/L} P_{L/R} = 0 \tag{2.33}$$
$$P_R + P_L = 1 \tag{2.34}$$

it is possible to decompose spin 1/2 fields $\psi$ in their chiral components:

$$\psi = \psi_R + \psi_L \tag{2.35}$$
$$\psi_{R/L} = P_{R/L}\psi \tag{2.36}$$

The GWS model is a gauge theory based on the $SU(2)_L \times U(1)_Y$ where $Y$ is the hypercharge. Segmenting the Lagrangian density in four parts,

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3 + \mathscr{L}_4 \tag{2.37}$$

I will explore each of them separately.

#### $\mathscr{L}_1$ - Lagrangian of the free gauge fields

In accordance with the gauge field theory formalism the kinetic terms for the three $SU(2)_L$ plus one $U(1)_Y$ gauge bosons is as follows:

$$\mathscr{L}_1 = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \ , i = 1,2,3 \tag{2.38}$$
$$F^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\varepsilon^{ijk}W^j_\mu W^k_\nu \ , i = 1,2,3 \tag{2.39}$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.40}$$

#### $\mathscr{L}_2$ - Gauge invariant Lagrangian for the fermions

The first task is to place the fields of the SM fermions in representations of the $SU(2)_L \times U(1)_Y$ group. As shown in table 2.1 that is done using only the trivial or the fundamental representations. For the $U(1)_Y$ the indicated values refer to the hypercharge $y$ that specifies the $U(1)$ transformation of a particular field $\psi$:

| Fields | $SU(2)_L$ | $U(1)_Y$ |
|---|---|---|
| $\begin{bmatrix} u \\ d' \end{bmatrix}_L, \begin{bmatrix} c \\ s' \end{bmatrix}_L, \begin{bmatrix} t \\ b' \end{bmatrix}_L$ | **2** | $+\frac{1}{6}$ |
| $u_R, c_R, t_R$ | **1** | $+\frac{2}{3}$ |
| $d'_R, s'_R, b'_R$ | **1** | $-\frac{1}{3}$ |
| $\begin{bmatrix} \nu_e \\ e \end{bmatrix}_L, \begin{bmatrix} \nu_\mu \\ \mu \end{bmatrix}_L, \begin{bmatrix} \nu_\tau \\ \tau \end{bmatrix}_L$ | **2** | $-\frac{1}{2}$ |
| $e_R, \mu_R, \tau_R$ | **1** | $-1$ |

Table 2.1: $SU(2)_L \times U(1)_Y$ representations of the fields

$$\psi(x) \overset{U(1)_Y}{\longrightarrow} e^{-iy\mathbb{1}\alpha(x)}\psi(x) \tag{2.41}$$

or simply

$$\psi(x) \overset{U(1)_Y}{\longrightarrow} e^{-iY\alpha(x)}\psi(x) \tag{2.42}$$

using $Y \equiv y\mathbb{1}$.

The gauge invariant Lagrangian for the fermions is given by

$$\mathscr{L}_2 = \overline{\psi_i} i\gamma^\mu D_\mu \psi_i \tag{2.43}$$
$$D_\mu = \partial_\mu - igT^j W_\mu^j - ig'YB_\mu \quad , j = 1,2,3 \tag{2.44}$$

The sum in $i$ is over all the $SU(2)_L \times U(1)_Y$ fermion multiplets. As for the covariant derivative, since the fermions are either in the trivial representation of $SU(2)_L$ (right handed spinors) or in the fundamental representation (left handed spinors) we can be more specific. For a right singlet $\Psi_R$ and a left doublet $\Psi_L$,

$$D_\mu \Psi_R = (\partial_\mu - ig'YB_\mu)\Psi_R \tag{2.45}$$
$$D_\mu \Psi_L = \left(\partial_\mu - ig\frac{\tau^j}{2}W_\mu^j - ig'YB_\mu\right)\Psi_L \tag{2.46}$$

where $\tau^1, \tau^2, \tau^3$ are the three Pauli matrices.

### $\mathscr{L}_3$ - Higgs sector

Only one gauge boson from the electroweak sector may be massless - the photon - leading to the exact conservation of electric charge $Q$. It is then necessary to account for the breakdown of $SU(2)_L \times U(1)_Y$ since otherwise there would exist four massless gauge bosons. That is done in the GWS model via the Higgs Mechanism, for which one needs a Higgs field which is a doublet under $SU(2)_L$ and has half a unit of hypercharge. Using the nomenclature

$$\Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \tag{2.47}$$

, $\mathscr{L}_3$ is written as

$$\mathscr{L}_3 = (D_\mu \Phi)^\dagger D_\mu \Phi - V(\Phi) \tag{2.48}$$
$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda\left(\Phi^\dagger \Phi\right)^2 \tag{2.49}$$

with $\mu^2, \lambda > 0$. This potential, whose shape is frequently compared to a Mexican hat, has a minimum at

$$\langle\Phi\rangle^\dagger \langle\Phi\rangle \;=\; \frac{\mu^2}{2\lambda} \equiv \frac{1}{2}v^2 \tag{2.50}$$

Since it is possible to "rotate away" the first component of $\langle\Phi\rangle$, we may settle for a vev

$$\langle\Phi\rangle \;=\; \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} \tag{2.51}$$

Substituting this in $\mathscr{L}_3$,

$$\begin{aligned}
\langle\mathscr{L}_3\rangle &= \mathscr{L}_3\left(\langle\Phi\rangle\right) \\
&= \frac{v^2}{2}\begin{bmatrix} 0 & 1 \end{bmatrix}\left(ig\frac{\tau^i}{2}W^i_\mu + ig'\frac{1}{2}B_\mu\right)\left(-ig\frac{\tau^j}{2}W^{j\mu} - ig'\frac{1}{2}B^\mu\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{v^2}{8}\begin{bmatrix} 0 & 1 \end{bmatrix}\left(g^2\tau^i\tau^j W^i_\mu W^{j\mu} + g'^2 B_\mu B^\mu + 2gg'\tau^i W^i_\mu B^\mu\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{v^2}{8}\left(g^2 W^i_\mu W^{i\mu} + g'^2 B_\mu B^\mu - 2gg' W^3_\mu B^\mu\right) \\
&= \frac{v^2 g^2}{8}\left(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu}\right) + \frac{v^2}{8}\begin{bmatrix} W^3_\mu & B_\mu \end{bmatrix}\begin{bmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{bmatrix}\begin{bmatrix} W^{3\mu} \\ B^\mu \end{bmatrix}
\end{aligned} \tag{2.52}$$

From the second to the third line, use was made of the relation

$$\sum_{i\neq j} \tau^i\tau^j S^{ij} \;=\; \sum_{i<j} \left\{\tau^i, \tau^j\right\} S^{ij} = 0 \tag{2.53}$$

for an arbitrary symmetric tensor $S^{ij}$. Now, we redefine the fields introducing new ones:

$$\begin{aligned}
W^\pm_\mu &= \frac{W^1_\mu \mp iW^2_\mu}{\sqrt{2}} \\
\begin{bmatrix} Z^\mu \\ A^\mu \end{bmatrix} &= \frac{1}{\sqrt{g^2+g'^2}}\begin{bmatrix} g & -g' \\ g' & g \end{bmatrix}\begin{bmatrix} W^{3\mu} \\ B^\mu \end{bmatrix} \\
&\equiv \begin{bmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{bmatrix}\begin{bmatrix} W^{3\mu} \\ B^\mu \end{bmatrix}
\end{aligned} \tag{2.54}$$

where $\theta_w$ is the Weinberg angle. As a function of the coupling constants $g$ and $g'$,

$$\cos\theta_w \;=\; \frac{g}{\sqrt{g^2+g'^2}} \tag{2.55}$$

$$\sin\theta_w \;=\; \frac{g'}{\sqrt{g^2+g'^2}} \tag{2.56}$$

Using these fields,

$$\begin{aligned}
\langle\mathscr{L}_3\rangle &= \frac{v^2 g^2}{4}W^+_\mu W^{-\mu} + \frac{v^2}{8}\begin{bmatrix} Z_\mu & A_\mu \end{bmatrix}\begin{bmatrix} g^2+g'^2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z^\mu \\ A^\mu \end{bmatrix} \\
&= \frac{v^2 g^2}{4}W^+_\mu W^{-\mu} + \frac{v^2}{8}\left(g^2+g'^2\right)Z_\mu Z^\mu \\
&= M^2_W W^+_\mu W^{-\mu} + \frac{1}{2}M^2_Z Z_\mu Z^\mu + \frac{1}{2}M^2_A A_\mu A^\mu
\end{aligned} \tag{2.57}$$

which implies that we have identified the correct mass eigenstates: $W^\pm_\mu$,$Z_\mu$ and $A_\mu$ (the photon). Their masses are

$$M_W = \frac{vg}{2} \tag{2.58}$$

$$M_Z = \frac{v\sqrt{g^2+g'^2}}{2} = \frac{vg}{2}\frac{1}{\cos\theta_w} \tag{2.59}$$

$$M_A = 0 \tag{2.60}$$

Let's now go back to $\mathscr{L}_2$ and replace $W_\mu^i$ and $B_\mu$ by these mass eigenstates. For $SU(2)_L$ doublets,

$$\begin{aligned}\mathscr{L}_2 &= \sum_{\psi_L}\overline{\psi}_L i\gamma^\mu\left\{\partial_\mu - ig\frac{1}{2}\left[\begin{array}{cc}(\cos\theta_w Z_\mu+\sin\theta_w A_\mu) & \sqrt{2}W_\mu^+\\ \sqrt{2}W_\mu^- & -(\cos\theta_w Z_\mu+\sin\theta_w A_\mu)\end{array}\right]\right.\\ &- \left. igy\tan\theta_w\left[\begin{array}{cc}(-\sin\theta_w Z_\mu+\cos\theta_w A_\mu) & 0\\ 0 & (-\sin\theta_w Z_\mu+\cos\theta_w A_\mu)\end{array}\right]\right\}\psi_L+\cdots\end{aligned} \tag{2.61}$$

and for singlets,

$$\mathscr{L}_2 = \cdots+\sum_{\psi_R}\overline{\psi}_R i\gamma^\mu\left[\partial_\mu - igy\tan\theta_w\left(-\sin\theta_w Z_\mu+\cos\theta_w A_\mu\right)\right]\psi_R \tag{2.62}$$

Neutral and charged currents are read off in the diagonal and off diagonal entries of the above matrices, respectively. Let's start with the charged currents. For just one family,

$$\mathscr{L}_{CC} = \frac{g}{\sqrt{2}}\left(J_\mu^+ W^{+\mu}+J_\mu^- W^{-\mu}\right) \tag{2.63}$$

$$\begin{aligned}J_\mu^+ &= \left(J_\mu^-\right)^\dagger\\ &= \overline{\nu}_L\gamma_\mu l_L+\overline{u}_L\gamma_\mu d'_L\end{aligned} \tag{2.64}$$

As for neutral currents, using $f$ to name a fermion,

$$\mathscr{L}_{NC} = g\sin\theta_w J_\mu^{em}A^\mu+\frac{g}{\cos\theta_w}J_\mu^0 Z^\mu \tag{2.65}$$

$$\begin{aligned}J_\mu^{em} &= \sum_f\left[t_3(f)+y(f)\right]\overline{f}\gamma_\mu f\\ &\equiv \sum_f q(f)\,\overline{f}\gamma_\mu f\end{aligned} \tag{2.66}$$

$$\begin{aligned}J_\mu^0 &= \sum_{f_L}\left[t_3(f_L)-q(f_L)\sin^2\theta_w\right]\overline{f}_L\gamma_\mu f_L+\sum_{f_R}\left[-y(f)\sin^2\theta_w\right]\overline{f}_R\gamma_\mu f_R\\ &= \sum_f\left[t_3(f)-q(f)\sin^2\theta_w\right]\overline{f}\gamma_\mu f\end{aligned} \tag{2.67}$$

Notice the equation $Q=T_3+Y$, also known as the Gell-Mann–Nishijima relation.

### $\mathscr{L}_4$ - Yukawa couplings

Mass terms of the type $m\overline{\psi}\psi$ are not $SU(2)_L$ invariant since

$$m\overline{\psi}\psi = m\left(\overline{\psi}_R\psi_L+\overline{\psi}_L\psi_R\right) \tag{2.68}$$

Instead, there is need to couple a Higgs field to the fermions. Before showing how this is done, it pays off if we change the notation to take into account the three families of the SM:

$$Q_L = \left[ \begin{array}{c} \left[ \begin{array}{c} u_1 \\ d_1 \end{array} \right]_L \\ \left[ \begin{array}{c} u_2 \\ d_2 \end{array} \right]_L \\ \left[ \begin{array}{c} u_3 \\ d_3 \end{array} \right]_L \end{array} \right] \quad u_R = \left[ \begin{array}{c} (u_1)_R \\ (u_2)_R \\ (u_3)_R \end{array} \right] \quad d_R = \left[ \begin{array}{c} (d_1)_R \\ (d_2)_R \\ (d_3)_R \end{array} \right] \quad L_L = \left[ \begin{array}{c} \left[ \begin{array}{c} \nu_{l_1} \\ l_1 \end{array} \right]_L \\ \left[ \begin{array}{c} \nu_{l_2} \\ l_2 \end{array} \right]_L \\ \left[ \begin{array}{c} \nu_{l_3} \\ l_3 \end{array} \right]_L \end{array} \right] \quad l_R = \left[ \begin{array}{c} (l_1)_R \\ (l_2)_R \\ (l_3)_R \end{array} \right] \tag{2.69}$$

Then,

$$\mathscr{L}_4 \;=\; \overline{L}_L Y_l \Phi l_R + \overline{Q}_L Y_d \Phi d_R + \overline{Q}_L Y_u \widetilde{\Phi} u_R + \text{h.c.} \tag{2.70}$$

where the doublet

$$\widetilde{\Phi} \;=\; i\tau_2 \Phi^* \tag{2.71}$$

has hypercharge opposite to $\Phi$. $Y_l, Y_u$, and $Y_d$ are $3 \times 3$ Yukawa matrices which are closely related to the fermions' mass matrices. Indeed, after SSB

$$\begin{aligned} \langle \mathscr{L}_4 \rangle \;&=\; \frac{v}{\sqrt{2}} \left\{ \left[ \begin{array}{ccc} \bar{l}_1 & \bar{l}_2 & \bar{l}_3 \end{array} \right]_L Y_l \left[ \begin{array}{c} l_1 \\ l_2 \\ l_3 \end{array} \right]_R + \left[ \begin{array}{ccc} \overline{d}_1 & \overline{d}_2 & \overline{d}_3 \end{array} \right]_L Y_d \left[ \begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array} \right]_R \right. \\ &+\; \left. \left[ \begin{array}{ccc} \overline{u}_1 & \overline{u}_2 & \overline{u}_3 \end{array} \right]_L Y_u \left[ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right]_R + \text{h.c.} \right\} \end{aligned} \tag{2.72}$$

so that $M_f = -\frac{v}{\sqrt{2}} Y_f, \;\; f = l, u, d.$

# Part II

# Mass and Mixing Matrices

Next, a brief review of the fermion family structure is made. Although in the SM neutrinos don't have mass, there is unambigous experimental evidence that points to massive neutrinos. This motivates the search of extensions of the model that account for this fact. The simplest of such extensions are discussed here.

# Chapter 3

# Quarks

We've seen that the quark's mass appears in the Lagrangian in the form

$$-\langle \mathscr{L} \rangle \quad = \quad ... + \overline{Q}_L M_d d_R + \overline{Q}_L M_u u_R + \text{h.c.} \tag{3.1}$$

in the weak eigenstate basis. Mixing occurs when $M_u$ and $M_d$ are diagonalized, due to the off-diagonal nature of the covariant derivative $D_\mu$ of $Q_L$ responsible for the weak charged currents. So, while in this basis

$$\langle \mathscr{L} \rangle \quad = \quad ... + \frac{g}{\sqrt{2}} W^{\mu +} \overline{u_L} \gamma_\mu \mathbb{1} d_L + \text{h.c.} \tag{3.2}$$

with

$$u_L = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_L \quad ; \quad d_L = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_L \tag{3.3}$$

, when we change basis of the fields to the mass eigenstates basis by diagonalizing $M_d$ and $M_u$ with the bi-unitary transformations

$$\begin{aligned} D_{u/d} \quad &= \quad U_L^{u/d\dagger} M_{u/d} U_R^{u/d} \\ &= \quad \text{diag}\left(m_{u/d}, m_{c/s}, m_{b/t}\right) \end{aligned} \tag{3.4}$$

there is clearly a mixing between $u_L^{mass} \equiv U_L^{u\dagger} u_L$ and $d_L^{mass} \equiv U_L^{d\dagger} d_L$:

$$\langle \mathscr{L} \rangle \quad = \quad ... + \frac{g}{\sqrt{2}} W^{\mu +} \overline{u_L^{mass}} \gamma_\mu U_L^{u^\dagger} U_L^d d_L^{mass} + h.c. \tag{3.5}$$

The quark mixing matrix

$$V_{ckm} \quad = \quad U_L^{u^\dagger} U_L^d = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \tag{3.6}$$

is the Cabibbo–Kobayashi–Maskawa (CKM) matrix. In the SM, it is clearly a unitary matrix. Notice that it does not depend on the basis of the right handed quarks. To eliminate this redundancy, we may define

$$H_{u/d} \quad \equiv \quad M_{u/d} M_{u/d}^\dagger \tag{3.7}$$

Since

$$D^2_{u/d} = U_L^{u/d\dagger} H_{u/d} U_L^{u/d} \tag{3.8}$$

, $U_L^{u/d}$ are just the transformation matrices that diagonalize $H_q^{u/d}$. The eigenvalues of $H_q^{u/d}$ are the square masses of the quarks.

## 3.1 CKM matrix parametrizations

The work of Kobayashi and Maskawa was very important to the understanding of quark mixing. A $n \times n$ matrix has the equivalent to $2n^2$ real parameters. But unitarity enforces $n^2$ constraints as can be clearly seen by looking at the generators of $U(n)$ (the $n \times n$ skew-hermitian matrices). Of these $n^2$ parameters, $2n-2$ can be absorbed by the fields as relative phases (using some phase convention) while 1 parameter is just a global phase. Of the remaining $(n-1)^2$ parameters of the $n \times n$ mixing matrix, $n(n-1)/2$ are real (mixing angles) and $(n-1)(n-2)/2$ are complex (CP violation phases). For 3 families, that's 3 mixing angles $\theta_{ij}$ plus one complex phase $\delta$:

$$V_{ckm} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \tag{3.9}$$

($s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$). The fact that $s_{13} \ll s_{23} \ll s_{12} \ll 1$ made Wolfenstein propose a parametrization based on an expansion of $\lambda \approx |V_{us}|$ which is similar to the following one [8, 9]:

$$s_{12} \equiv \lambda \tag{3.10}$$
$$s_{23} \equiv A\lambda^2 \tag{3.11}$$
$$s_{13}e^{-i\delta} \equiv A\lambda^3(\rho - i\eta) \tag{3.12}$$

leading to

$$V_{ckm} = \begin{bmatrix} 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1 + 4A^2) & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 + \frac{1}{2}A\lambda^4[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}A^2\lambda^4 \end{bmatrix} + \mathcal{O}(\lambda^5) \tag{3.13}$$

## 3.2 Unitarity constraints on the CKM matrix

Unitarity implies orthonormality of row and column vectors:

$$\sum_i V_{ij}V_{ik}^* = \delta_{jk} \tag{3.14}$$
$$\sum_j V_{ij}V_{kj}^* = \delta_{ik} \tag{3.15}$$

where $V$ is the mixing matrix. Each of these six constraints can be viewed as a triangle. For instance, from

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \tag{3.16}$$

we get

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0 \tag{3.17}$$

which represents a unitarity triangle in the Argand plane (figure 3.1).

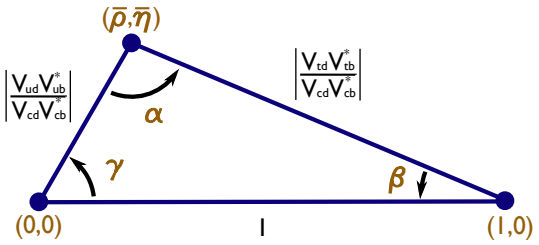



Figure 3.1: Unitarity triangle in the complex plane (adapted from [7]).

The angles of this triangle are given by

$$\alpha = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) \tag{3.18}$$

$$\beta = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) \tag{3.19}$$

$$\gamma = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) \tag{3.20}$$

## 3.3 CP violating phase

In the first parametrization presented above, CP violation is related to the complex phase $\delta$. But one does not want to be tied to a specific phase convention to express this CP violation phase. That kind of approach can be achieved as follows. An invariant measure of CP violation is given by

$$\mathrm{Im}\left(\det\left[H_q^u, H_q^d\right]\right) = 2J\Delta \tag{3.21}$$

where $J$, the Jarlskog invariant, and $\Delta$ are

$$J = \mathrm{Im}\left(V_{ij}V_{kl}V_{il}^*V_{kj}^*\right)\sum_{m,n}\varepsilon_{ikm}\varepsilon_{jln} \tag{3.22}$$

$$\Delta = \left(m_t^2 - m_c^2\right)\left(m_t^2 - m_u^2\right)\left(m_c^2 - m_u^2\right)\left(m_b^2 - m_s^2\right)\left(m_b^2 - m_d^2\right)\left(m_s^2 - m_d^2\right) \tag{3.23}$$

, $\varepsilon_{ijk}$ being the Levi-Civita tensor. Notice that picking any $(ijkl)$ produces the same result except for a possible minus sign. In the two parametrizations shown previously,

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin\delta \tag{3.24}$$

$$= A^2\lambda^6\eta\left(1 - \frac{1}{2}\lambda^2\right) + \mathcal{O}\left(\lambda^{10}\right) \tag{3.25}$$

Notice also that from 3.24 $J_{max} = 1/6\sqrt{3} \approx 0.1$. The Jarlskog invariant has a simple geometric meaning in terms of the unitary triangle. If we use the non normalized triangle, for instance the one with sides of magnitude $|V_{ud}V_{ub}^*|$,$|V_{cd}V_{cb}^*|$ and $|V_{td}V_{tb}^*|$ , its area is $|J|/2$. Phase redefinitions of the fields only rotate the triangle, leaving its area (and $J$) invariant.

## 3.4 $V_{ckm}$'s experimental values

Not everything in $V_{ckm}$ is physically meaningful due to the arbitrariness of phase conventions. In fact only the moduli of its entries and the quadri-products $V_{ij}V_{kl}V_{il}^*V_{kj}^*$ are observables. Listed bellow are these and other parameters mentioned above, obtained from fits of experimental values, at a confidence level correspondent to one standard deviation. The data is taken from [10].

$$|V_{ckm}| = \begin{bmatrix} 0.97400^{+0.00054}_{-0.00058} & 0.2265^{+0.0025}_{-0.0023} & 3.87^{+0.35}_{-0.30} \times 10^{-3} \\ 0.2264^{+0.0025}_{-0.0023} & 0.97317^{+0.00053}_{-0.00059} & 41.13^{+1.36}_{-0.58} \times 10^{-3} \\ 8.26^{+0.72}_{-0.86} \times 10^{-3} & 40.47^{+1.39}_{-0.62} \times 10^{-3} & 0.999146^{+0.000024}_{-0.000058} \end{bmatrix} \tag{3.26}$$

$$\sin 2\alpha = -0.14^{+0.37}_{-0.41} \tag{3.27}$$

$$\sin 2\beta = 0.739^{+0.048}_{-0.048} \tag{3.28}$$

$$\gamma = 62^{+10}_{-12} \ (^{\circ}) \tag{3.29}$$

$$J = 3.10^{+0.43}_{-0.37} \times 10^{-5} \tag{3.30}$$

$$\lambda = 0.2265^{+0.0025}_{-0.0023} \tag{3.31}$$

$$A = 0.801^{+0.029}_{-0.020} \tag{3.32}$$

$$\overline{\rho} = 0.189^{+0.088}_{-0.070} \tag{3.33}$$

$$\overline{\eta} = 0.358^{+0.046}_{-0.042} \tag{3.34}$$

$$\sin\theta_{12} = 0.2266^{+0.0025}_{-0.0023} \tag{3.35}$$

$$\sin\theta_{13} = 3.87^{+0.35}_{-0.30} \times 10^{-3} \tag{3.36}$$

$$\sin\theta_{23} = 41.13^{+1.37}_{-0.58} \times 10^{-3} \tag{3.37}$$

$$\delta \approx 62^{+10}_{-12} \ (^{\circ}) \tag{3.38}$$

# Chapter 4

# Leptons

In the SM there are no right handed neutrinos so it is not possible to have Dirac mass terms similar to the ones for quarks. Since neutrinos are known to have mass, the SM has to be modified.

## 4.1 Accounting for neutrinos' mass

There are multiple ways to account for neutrinos' mass. On general grounds, the presence of the effective (non renormalizable) dimension five operator $\lambda \overline{L}_{i\,L} \tilde{\Phi} \overline{L}_{i\,L} \tilde{\Phi}$ (figure 4.1) in the Lagrangian gives rise to neutrino mass [11]. The predicted mass for $\nu_{i\,L}$ would be

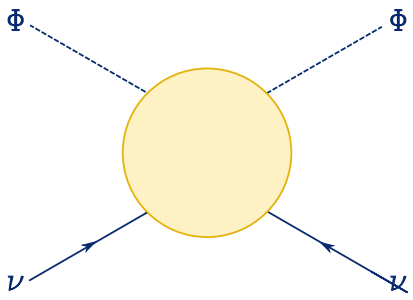



Figure 4.1: Dimension five operator that gives mass to neutrinos

$$m_{\nu_{iL}} = \lambda \left| \langle \Phi \rangle \right|^2 \tag{4.1}$$

The constant $\lambda$ is an inverse of a mass,

$$\lambda = \frac{\lambda_0}{M_x} \tag{4.2}$$

where $M_x$ may be several orders of magnitude larger than $|\langle \Phi \rangle|$ if for instance the dimension five operator comes from a higher energy theory, while $\lambda_0 = \mathcal{O}(1)$. That would account for the smallness of the neutrinos masses.

One can obtain such an effective operator in various ways (two examples are shown in figure 4.2). One such way is with the type I seesaw mechanism, which shall now be described.

## 4.2 The seesaw mechanism

For the three $SU(2)_L$ singlets $\nu_{i\,R}$ in the SM, with no electric charge, it is possible to add a Majorana mass term, which violates the leptonic number by two units. First, let's introduce the charge conjugation matrix $C$ which acts on the elements of the Clifford algebra in the following way:

$$C \gamma_\mu C^{-1} = -\gamma_\mu^T \tag{4.3}$$

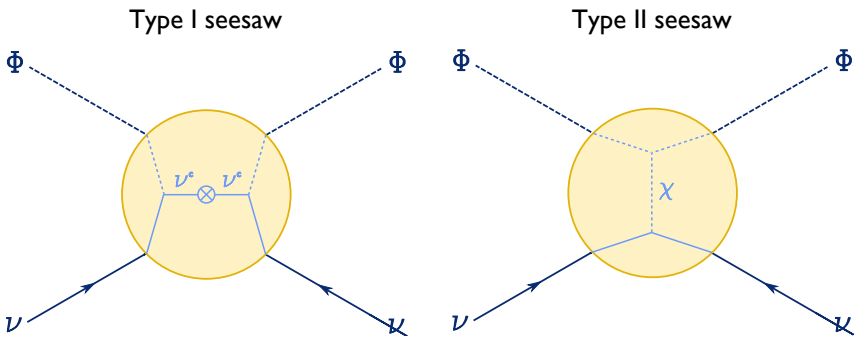



Figure 4.2: Seesaw type I (with the exchange of right handed neutrinos) and type II (with the exchange of a scalar particle).

In the Dirac representation,

$$C = i\gamma_2\gamma_0 \tag{4.4}$$

with $C^\dagger = C^T = -C$ and $C^\dagger C = \mathbb{1}$. The charge conjugate of a field $\psi$ is

$$\psi^c = C\overline{\psi}^T = C\gamma^0\psi^* \tag{4.5}$$

It is easy to check that charge conjugation changes chirality,

$$\left(\psi_{R/L}\right)^c = \left(\psi^c\right)_{L/R} \tag{4.6}$$

which allows us to write for $\psi_R$ a $SU\left(2\right)_L$ invariant Majorana mass term:

$$\psi_R^T C\psi_R + h.c. = \overline{\chi}\chi \tag{4.7}$$

with

$$\chi = \psi_R + \left(\psi_R\right)^c \tag{4.8}$$

One sees that the Majorana mass for $\psi_R$ is equivalent to a Dirac mass for $\chi$. Extending this reasoning to three families of neutrinos and including a Dirac mass for them [12],

$$-\mathscr{L} = ... + \overline{L}_L\tilde{\Phi}Y_D\nu_R + \frac{1}{2}\nu_R^T C M_R\nu_R + \text{h.c.} \tag{4.9}$$

After SSB,

$$\begin{aligned} -\langle\mathscr{L}\rangle &= ... + \overline{\nu}_L M_D\nu_R + \frac{1}{2}\nu_R^T C M_R\nu_R + \text{h.c.} \\ &= ... + \frac{1}{2}\begin{bmatrix}\nu_L \\ \nu_R^c\end{bmatrix}^T C\begin{bmatrix}0 & M_D \\ M_D^T & M_R\end{bmatrix}^*\begin{bmatrix}\nu_L \\ \nu_R^c\end{bmatrix} + \text{h.c.} \end{aligned} \tag{4.10}$$

The diagonalization of this $6\times 6$ matrix $M$ is done by

$$V = \begin{bmatrix}P & Q \\ S & T\end{bmatrix} \tag{4.11}$$

such that

$$V^T M^* V = \begin{bmatrix} D_\nu & 0 \\ 0 & D_R \end{bmatrix} \tag{4.12}$$

The $3 \times 3$ matrices $D_\nu$ and $D_R$ are diagonal, real, and positive. Provided that the charged leptons' Dirac mass matrix $M_l$ satisfies

$$U_L^{l\,\dagger} M_l U_R^l = D_l \tag{4.13}$$

with $D_l$ diagonal, real, and positive, then the weak charged current interactions are given by

$$\begin{aligned} \mathscr{L}_{CC} &= \frac{g}{\sqrt{2}} W^{\mu-} \bar{l}_L \gamma_\mu \nu_L + \text{h.c.} \\ &= \frac{g}{\sqrt{2}} W^{\mu-} \left( \bar{l}_L^{mass} \gamma_\mu U_L^{l\,\dagger} P \nu_L^{mass} + \bar{l}_L^{mass} \gamma_\mu U_L^{l\,\dagger} Q N_L^{mass} \right) + \text{h.c.} \end{aligned} \tag{4.14}$$

where $\nu^{mass}$ and $N^{mass}$ are the light and heavy neutrino eigenstates. In fact, if we assume that $\det M_R \neq 0$,

$$\begin{aligned} M &= \begin{bmatrix} 0 & M_D \\ M_D^T & M_R \end{bmatrix} \\ &= \begin{bmatrix} 1 & M_D M_R^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -M_D M_R^{-1} M_D^T & 0 \\ 0 & M_R \end{bmatrix} \begin{bmatrix} 1 & 0 \\ M_R^{-1} M_D^T & 1 \end{bmatrix} \end{aligned} \tag{4.15}$$

When the entries of $M_D.M_R^{-1}$ are much smaller than 1, a redefinition of the neutrino fields yields the following neutrino mass matrices:

$$\tilde{M} = \begin{bmatrix} -M_D M_R^{-1} M_D^T & 0 \\ 0 & M_R \end{bmatrix} \tag{4.16}$$

$$\equiv \begin{bmatrix} M_\nu & 0 \\ 0 & M_R \end{bmatrix} \tag{4.17}$$

In this limit, $Q = S = 0$ and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is defined as

$$V_{pmns} = U_L^{l\,\dagger} P \tag{4.18}$$

The mass and weak charged current Lagrangian is

$$\begin{aligned} -\langle \mathscr{L} \rangle &= \ldots + \bar{l}_L^{mass} D_l l_R^{mass} + \nu_L^{mass\,T} D_\nu \nu_L^{mass} + N_L^{mass\,T} D_R N_L^{mass} \\ &+ \frac{g}{\sqrt{2}} W_\mu^- \bar{l}_L^{mass} \gamma_\mu U_L^{l\,\dagger} P \nu_L^{mass} + \text{h.c.} \end{aligned} \tag{4.19}$$

In contrast with the pure Dirac masses situation, it is not possible to extract two phases of $V_{pmns}$ to $\nu_L^{mass}$ so that in general the PMNS matrix is parametrized as

$$V_{pmns} = U_\delta K \tag{4.20}$$

where $U_\delta$ has only one complex phase (as $V_{ckm}$),

$$U_\delta = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \tag{4.21}$$

, and

$$K = \text{diag}\left(e^{i\alpha}, e^{i\beta}, 1\right) \tag{4.22}$$

contains two Majorana phases.

## 4.3 Neutrino oscillations

If neutrinos interact according to their flavour states and propagate according to their mass eigenstates, one would expect to see neutrino flavour oscillation in detectors. Next I give a brief and simplified description of this phenomenon in the vacuum without resorting to a fully rigorous treatment which deals with wave packets. The simplification process can be tricky though [13].

First, we need to extract from 4.19 the relation between the neutrino flavour states and the mass states:

$$\nu^F = V\nu \tag{4.23}$$

where I have started the one way process of (over)simplifying notation - $V$ will stand for the leptonic mixing matrix, $\nu^F$ is the neutrino flavour mass eigenstates and $\nu$ is the mass eigenstates. Conforming to the usual notation when working componentwise, latin indexes will refer to mass eigenstates and the greek ones to flavour eigenstates. Now turning to the production of a neutrino flavour state $|\nu_\alpha\rangle$, it is important to note that $|\nu_\alpha\rangle \neq \sum_i V_{\alpha i} |\nu_i\rangle$ because the vertex for this process is $W^+ \overline{\nu} V^\dagger l$ , not $W^- \bar{l} V \nu$. The correct relations are then

$$|\nu_\alpha\rangle = \sum_i V^*_{\alpha i} |\nu_i\rangle \tag{4.24}$$

and the inverse relations

$$|\nu_i\rangle = \sum_\alpha V_{\alpha i} |\nu_\alpha\rangle \tag{4.25}$$

The probability of transition from a flavour $\alpha$ to $\beta$ is given by

$$P(\nu_\alpha \to \nu_\beta) = |\mathcal{A}(\nu_\alpha \to \nu_\beta)|^2 \tag{4.26}$$

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_i U^*_{\alpha i} \mathrm{Prop}(\nu_i) U_{\beta i} \tag{4.27}$$

with $\mathrm{Prop}(\nu_i)$ being the amplitude for the eigenstate $\nu_i$ to propagate from the source to the detector. In the rest frame of the neutrino, application of Schrödinger's equation yields

$$\frac{\partial}{\partial \tau_i} |\nu_i\rangle = -iH |\nu_i\rangle \tag{4.28}$$

or

$$\mathrm{Prop}(\nu_i) = e^{-im_i\tau_i} = e^{-i(E_i t - p_i x)} \tag{4.29}$$

for the laboratory referential where $x$ is the distance measured in the direction of $\nu_i$ 's momentum, $t$ is time, and $E_i$ is the neutrino's energy. Things get tricky here, if we start taking ultra-relativistic limits due to the smallness of the mass of neutrinos compared to their energy. First we set $x$ to some distance $L$ and then convert the unobserved time $t$ to $L$. Here comes a surprise for someone knowing the (correct) result: using $p_i = v_i E_i$ and two distinct times $t_i = {}^{L}/_{v_i}$,

$$\begin{aligned} E_i t - p_i x &= \frac{E_i^2 - p_i^2}{p_i} L \\ &= \frac{m_i^2}{p_i} L \end{aligned} \tag{4.30}$$

Computing phase differences $\Delta\phi_{ij}$ and using the reasonable approximation $p_i \approx E_i \approx E$ for all $i$, we get a factor of 2 wrong in comparison to the correct result. This goes to show the subtleties of neutrino

oscillations. To get the correct result one must view pairs of neutrinos $\nu_i$ and $\nu_j$ in a beam having momentum $p_i + p_j$ and $E_i + E_j$ having a group velocity

$$\overline{v_{ij}} = \frac{p_i + p_j}{E_i + E_j} \tag{4.31}$$

that in is fact very close to $\frac{1}{2}\left({}^{p_i}/_{E_i} + {}^{p_j}/_{E_j}\right)$. Using one common time $t = {}^{L}/_{\overline{v_{ij}}}$ to calculate $\Delta\phi_{ij}$,

$$\begin{aligned} \Delta\phi_{ij} &= (E_j - E_i)\, t - (p_j - p_i)\, L \\ &= \left[\frac{E_j^2 - E_i^2}{p_i + p_j} - (p_j - p_i)\right] L \\ &\approx \frac{m_j^2 - m_i^2}{2E} L \end{aligned} \tag{4.32}$$

having used $p_i \approx E$ in the denominator of the fraction. Plugging this into $P(\nu_\alpha \to \nu_\beta)$ and using $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$,

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) \sin^2\left(\Delta m_{ij}^2 \frac{L}{E}\right) \\ &\quad + 2\sum_{i>j} \mathrm{Im}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) \sin\left(\Delta m_{ij}^2 \frac{L}{E}\right) \end{aligned} \tag{4.33}$$

## 4.4 Experimental values of the neutrinos' masses and mixing

The relevant experimental data in the leptonic sector amount to knowing:

1. the $V_{pmns}$ matrix parameters and the neutrino masses;
2. if neutrinos have Majorana masses;
3. how many neutrinos are there.

As for the last question, we know of only 3 neutrinos with mass $< {}^{M_z}/_2$. It is known that the number of light neutrinos that are sensitive to the weak interaction is $2.994 \pm 0.012$ [7]. The existence of more neutrinos that are $SU(3)_c \times SU(2)_L \times U(1)_Y$ singlets (sterile neutrinos) is possible though.

A summary of the known data on neutrino masses and mixing from neutrino oscillation experiments is indicated on table 4.1.

| Parameter | central value | 99% C.L. range |
|---|---|---|
| $\Delta m_{12}^2 \quad (10^{-5}\mathrm{eV})$ | $8.0 \pm 0.3$ | $7.2 < \Delta m_{12}^2 < 8.9$ |
| $\lvert\Delta m_{23}^2\rvert \quad (10^{-3}\mathrm{eV})$ | $2.5 \pm 0.2$ | $2.1 < \lvert\Delta m_{23}^2\rvert < 3.1$ |
| $\tan^2\theta_{12}$ | $0.45 \pm 0.05$ | $30^\mathrm{o} < \theta_{12} < 38^\mathrm{o}$ |
| $\sin^2 2\theta_{23}$ | $1.02 \pm 0.04$ | $36^\mathrm{o} < \theta_{23} < 54^\mathrm{o}$ |
| $\sin^2 2\theta_{13}$ | $0 \pm 0.05$ | $\theta_{13} < 10^\mathrm{o}$ |

Table 4.1: Experimental data from neutrino oscillations [15]

The numbering of the neutrino eigenstates is not necessarily from lowest to highest mass. In fact, it is only known that $m_2 > m_1$; the relative sign between $m_{1/2}$ and $m_3$ is unknown giving rise to two main possibilities for the neutrino spectrum - a normal hierarchy or an inverted hierarchy (see figure 4.3).

On the other hand, little is known about the neutrinos' absolute mass scale. There are just some upper limits which are shown in table 4.2, where

$$\begin{aligned} m_{\alpha\beta} &= \left(V D_\nu V^T\right)_{\alpha\beta} \\ &= \sum_i m_i V_{\alpha i} V_{\beta i} \end{aligned} \tag{4.34}$$

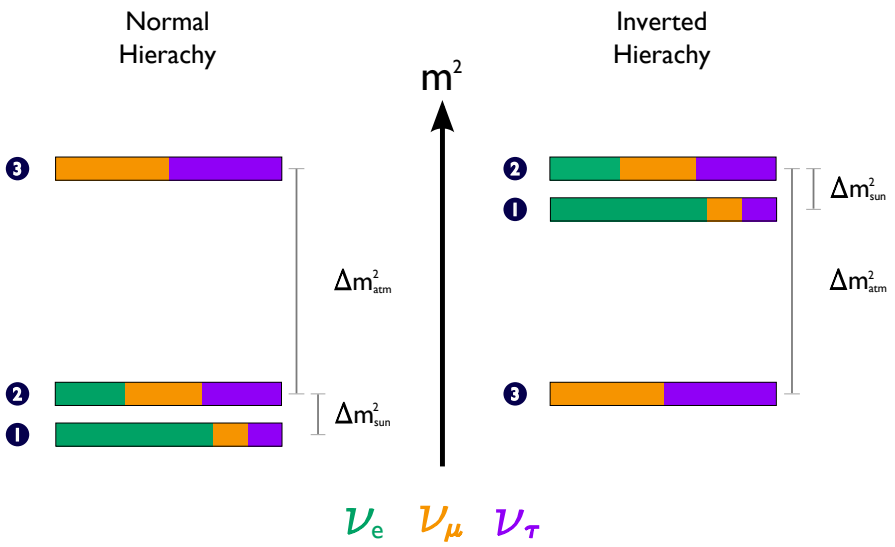



Figure 4.3: Neutrino mass hierarchy (adapted from [7])

| Source | 90% C.L. limit |
|---|---|
| NEMO-3($^{100}$Mo) | $\lvert m_{ee}\rvert < 0.60 - 2.40$ eV |
| Cuoricino($^{130}$Te) | $\lvert m_{ee}\rvert < 0.16 - 0.84$ eV |
| WMAP | $\sum_i m_i < 2.11$ (95%C.L.) eV |
| SDSS+WMAP | $\sum_i m_i < 0.16 - 0.84$ eV |

Table 4.2: Upper limits to neutrino masses [19]

The fact that the absolute mass scale is unknown leaves open the possibility that the three neutrino spectrum might be approximately degenerate ($m_1 \approx m_2 \approx m_3$).

As for the Majorana nature of neutrinos, expectations are that neutrinoless double beta decay ($0\nu2\beta$) experiments may shed some light on it. Here one uses the fact that some nuclei (e.g.: $^{76}$Ge,$^{130}$Te, $^{100}$Mo) have ground states such that the energy of the configurations $(A, Z)$ and $(A, Z+2)$ are lower that that of $(A, Z+1)$. In this case single $\beta$ decays are kinetically forbidden. Only double beta decays with the release of two neutrinos ($2\nu2\beta$) are possible if neutrinos are not Majorana particles. However, if they are Majorana particles it is also possible to have $0\nu2\beta$ decays which violates leptonic number by two units (figure 4.4). Distinguishing the two processes is a matter of checking if the two electrons get away with all available kinetic energy ($0\nu2\beta$) or not ($2\nu2\beta$).

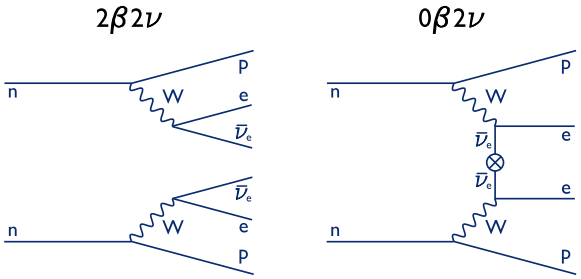



Figure 4.4: $2\nu2\beta$ vs $0\nu2\beta$

## 4.5 Tri-bimaximal mixing

With the progressive increase in experimental data coming from neutrino oscillations, various special forms were proposed for the leptonic mixing matrix. Trimaximal mixing,

$$|V|^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{(trimaximal)} \tag{4.35}$$

($|V|^2_{ij} \equiv |V_{ij}|^2$) implies 'total symmetry' between the three generations of neutrinos and charged leptons. It also maximizes CP violation ($J = 1/6\sqrt{3}$). Bimaximal mixing arises from the imposition of $V_{e3} = 0$ on trimaximal mixing. That can be done while still preserving $\mu \leftrightarrow \tau$ and $\nu_1 \leftrightarrow \nu_2$ symmetry:

$$|V|^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad \text{(bimaximal)} \tag{4.36}$$

Experimental data eventually excluded both of these forms, paving the way to "optimized bimaximal mixing" [20] or tri-bimaximal mixing [21] which is also often called HPS mixing after its first proponents:

$$|V|^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \quad (\text{tri} - \text{bimaximal}) \tag{4.37}$$

Notice that $\nu_2$ is trimaximally mixed (second column) and $\nu_3$ is bimaximally mixed (third column). Using a particular choice of phases, I will settle on the form

$$V_{hps} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{4.38}$$

HPS mixing has the following angles:

$$\theta_{12} = \arcsin \frac{1}{\sqrt{3}} \approx 35.3^{\text{o}} \tag{4.39}$$

$$\theta_{13} = 0 \tag{4.40}$$

$$\theta_{23} = \frac{\pi}{4} = 45^{\text{o}} \tag{4.41}$$

These values are well inside the 99% confidence level ranges shown in table 4.1. In fact HPS's values

$$\tan^2 \theta_{12} = \frac{1}{2} \tag{4.42}$$

$$\sin^2 2\theta_{13} = 0 \tag{4.43}$$

$$\sin^2 2\theta_{23} = 1 \tag{4.44}$$

are inside the $1\sigma$ ranges presented above. One should add with regard to CP violation that $\theta_{13} = 0$ makes the $V_{e3}$ entry null which in turn implies that there is no CP violation.

If neutrinos are Majorana particles, in the flavour basis ($M_l$ diagonal and no mixing)

$$\begin{aligned} M_\nu &= V^*_{hps}.\text{diag}\,(m_1, m_2, m_3)\,.V^\dagger_{hps} \\ &= \begin{bmatrix} a & b & b \\ b & a+c & b-c \\ b & b-c & a+c \end{bmatrix} \equiv M^{hps}_\nu \end{aligned} \tag{4.45}$$

with

$$a = \frac{1}{3}(2m_1 + m_2) \tag{4.46}$$

$$b = \frac{1}{3}(-m_1 + m_2) \tag{4.47}$$

$$c = \frac{1}{2}(-m_1 + m_3) \tag{4.48}$$

The inverse relations are

$$m_1 = a - b \tag{4.49}$$

$$m_2 = a + 2b \tag{4.50}$$

$$m_3 = a - b + 2c \tag{4.51}$$

$V_{hps}$ can be viewed as a special case of $\theta_{13} = 0, \theta_{23} = \frac{\pi}{4}$ and arbitrary $\theta_{12}$. In such a scenario, in the flavour basis,

$$M_\nu = \begin{bmatrix} a & b & b \\ b & c & d \\ b & d & c \end{bmatrix} \tag{4.52}$$

with $\theta_{12}$ given by

$$\tan 2\theta_{12} = \frac{2\sqrt{2}b}{c + d - a} \tag{4.53}$$

HPS mixing follows when $a + b = c + d$ .

# Part III

# Models in literature with tri-bimaximal mixing

Since the 2002 publication of the HPS ansatz for the leptonic mixing matrix, various models that attempt to explain this type of leptonic family structure have been presented. One frequently used method to do so is by imposing family symmetries on the various field - i.e., is there a set of transformations $S = \{O_i\}$ that when applied to the family of fields $\psi$ leaves the theory invariant, $O_i \mathscr{L}(\psi) \equiv \mathscr{L}(O_i \psi) = \mathscr{L}(\psi)$, up to unphysical changes in $\mathscr{L}$ such as a total derivative? This set must form a group under operator composition since

- the composition of two of these operators, $O_i O_j$, should also leave the Lagrangian invariant for all $i, j$ $(O_i O_j \in S)$;
- the identity operator $E$ on $\psi$ is certainly in the set $S$ so that, if $O_i$ leaves $\mathscr{L}$ invariant, so must $O_i^{-1}$ for any $i$ $(O_i^{-1} \in S)$.

Continuous groups such as $SU(3)_f$ are one possibility and have in fact already been studied [32]. This work focuses instead on the use of discrete groups. What follows in this part is an account of some of the attempts made to justify the leptons' tri-bimaximal mixing in such a way. A brief mention is made on the main features of some of the most commonly used groups.

# Chapter 5

# Frequently used groups

## 5.1 $C_n$

The cyclic group $C_n$ (or $Z_n$) contains the elements $\left\{g^0 = g^n, g^1, g^2, \cdots, g^{n-1}\right\}$. It is:

- Generated by a single element $g$;
- Abelian. For this reason, it has $n$ 1-dimensional irreps;

In the $j$ representation, the generator $g$ takes the value $e(n)^j$ where $e(n) \equiv \exp\left(2\pi i/n\right)$. Take $C_3 = \left\{g^0 = g^3, g^1, g^2\right\}$ as an example. In its three 1-dimensional irreps, $g$ takes the values 1, $e(3)$ and $e(3)^2$.

| Irrep \ Class | $\left\{g^0\right\}$ | $\left\{g^1\right\}$ | $\left\{g^2\right\}$ |
|---|---|---|---|
| **1** | 1 | 1 | 1 |
| **1'** | 1 | $w$ | $w^2$ |
| **1"** | 1 | $w^2$ | $w$ |

Table 5.1: $Z_3$'s character map, using $w \equiv e(3)$.

## 5.2 $S_n$

This is the symmetric or permutation group family. One can view the elements of $S_n$ as all the possible permutations of $n$ objects. The group is non abelian for $n > 2$ and its order is given by $n!$. The number of classes/irreps of $S_n$ is given by the number of partitions of the integer $n$ which is given by the partition function $p(n)$ (table 5.2).

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(n)$ | 1 | 2 | 3 | 5 | 7 |

Table 5.2: Some values of the partition function $p(n)$, which gives the number of irreps of $S_n$

For $n = 3$, in the cycle notation $S_3 = \{e, (12), (13), (23), (123), (132)\}$ and there are three irreps - $\mathbf{1}, \mathbf{1}'$ and $\mathbf{2}$. Picking a 2-cycle and a 3-cycle as generators (for instance $\{(12), (123)\}$) , their forms in the various irreps may be taken as the matrices shown in table 5.3. The irreducible representations of the rest of the elements of the group are obtainable using the relations $e = (12)^2 = (123)^3$, $(13) = (123)(12)$, $(23) = (12)(123)$ and $(132) = (123)(123)$.

Perhaps the only groups for which there is any usefulness in using a reducible representation are the $S_n$s. That is because there is an $n$-dimensional representation which has a very intuitive action on vectors - it permutes their components. This is the natural basis which I shall call $\mathbf{N}$. For $S_3$, $\mathbf{N}$ is equivalent to $\mathbf{1} \oplus \mathbf{2}$ and the $V$ matrix which does this block diagonalization ($\mathbf{N} = V(\mathbf{1} \oplus \mathbf{2})V^\dagger$) is

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & w^2 & w \\ 1 & w & w^2 \\ 1 & 1 & 1 \end{bmatrix} \tag{5.1}$$

| Irrep \ Generator | (12) | (123) |
|---|---|---|
| **1** | $1$ | $1$ |
| **1'** | $-1$ | $1$ |
| **2** | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}$ |
| **N** | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ |

Table 5.3: $S_3$'s two generators written in the group's irreps and in the natural representation.

$S_4$ with its 24 elements is also often used as a discrete family group due to the fact that it has 3-dimensional irreps. To be precise, its irreps are $\mathbf{1}, \mathbf{1'}, \mathbf{2}, \mathbf{3}, \mathbf{3'}$.

## 5.3 $A_n$

The alternating group $A_n$ is the group of even permutations of $n$ objects. For $n > 3$ it is a non abelian group and its order is given by $n!/2$ (for $n > 1$).

This set of groups includes one of the simplest and most promising group that is capable of explaining tri-bimaximal mixing under some assumptions [25, 26, 27, 28]. That's $A_4$, the first non-abelian group in the series. The appeal of $A_4$ is simple: it is the smallest group with a 3-dimensional irrep, which is able to account for the relationships that span across the three families in tri-bimaximal mixing. On the other hand, the existence of three 1-dimensional irreps allows for unrelated charged lepton masses.

Indeed from the fact that...

1. the dimension of the irreps divide the order of the group,
2. the order of the group is equal to the sum of the squares of the dimensions of its irreps,
3. all groups have the trivial 1-dimensional irrep,

one concludes that 3-dimensional irreps may only start to show up in groups of order 12 or bigger. That is indeed what happens with $A_4$. The next group with 3-dimensional irreps is of order 21[1].

In cycle notation, its elements are those of the forms $(x_1)(x_2)(x_3)(x_4)$ (the identity) ,$(x_1x_2)(x_3x_4)$ and $(x_1)(x_2x_3x_4)$ for any permutation $(x_1, x_2, x_3, x_4)$ of $(1, 2, 3, 4)$. In $S_n$ the cycle structure of the elements breaks the groups in its conjugacy classes. In $A_n$ things get a little trickier. Since we are mainly worried with $A_4$, it suffices to say that $(x_1)(x_2)(x_3)(x_4)$ and $(x_1x_2)(x_3x_4)$ represent one class each, while $(x_1)(x_2x_3x_4)$ represents two classes: $(x_1)(x_2x_3x_4)_{Odd}$ and $(x_1)(x_2x_3x_4)_{Even}$ depending on whether $(x_1, x_2, x_3, x_4)$ is an odd or an even permutation of $(1, 2, 3, 4)$. In all there are four conjugacy classes:

$$C_1 = \{(1)(2)(3)(4)\} = \{e\} \tag{5.2}$$
$$C_2 = \{(12)(34), (13)(24), (14)(23)\} \tag{5.3}$$
$$C_3 = \{(1)(234), (2)(143), (3)(124), (4)(132)\} \tag{5.4}$$
$$C_4 = \{(1)(243), (2)(134), (3)(142), (4)(123)\} \tag{5.5}$$

The entire group is generated by two elements $g_1$ and $g_2$ (for instance $(14)(23)$ and $(4)(123)$). In the groups' irreducible representations they take the forms shown in table 5.5.

The decomposition of Kronecker products of the four irreps is shown in table 5.6.

## 5.4 $\Delta\left(3n^2\right)$

Having two groups $G = (S_G, \cdot)$ and $H = (S_H, \circ)$ one can build its direct product $G \times H = (S_{G\times H}, \star)$. The cartesian product of sets $S_G$ and $S_H$ are the elements of group $G \times H$, meaning that $S_{G\times H} =$

[1] What about 4-dimensional irreps? The same reasoning indicates that they may only show up for groups with 20 or more elements. And there is indeed such a group with 20 elements.

| Irrep \ Class | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 |
| **1'** | 1 | 1 | $w$ | $w^2$ |
| **1"** | 1 | 1 | $w^2$ | $w$ |
| **3** | 3 | -1 | 0 | 0 |

Table 5.4: $A_4$'s character map

| Irrep \ Generator | $(14)(23)$ | $(4)(123)$ |
|---|---|---|
| **1** | 1 | 1 |
| **1'** | 1 | $w$ |
| **1"** | 1 | $w^2$ |
| **3** | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ |

Table 5.5: $A_4$'s two generators written in the group's irreps

$\{(g,h) : g \in S_G, h \in S_H\}$. The multiplication operation $\star$ for this new group is simple: $(g,h) \star (g',h') = (g \cdot g', h \circ h')$.

The direct product is actually a special case of the semi-direct product. For a semi-direct product $G \rtimes H = (S_{G\rtimes H}, \bullet)$ the set of elements $S_{G\rtimes H}$ is still the cartesian product $S_G \times S_H$. Intuitively, what changes relative to the direct product is structure of the group: $(g,h) \bullet (g',h')$ may not be equal to $(g \cdot g', h \circ h')$. Instead, $(g,h) \bullet (g',h') = (g \cdot \theta(h)[g'], h \circ h')$ where $\theta$ is a map from group $H$ to the group of automorphisms of $G$. The action of $\theta(h)$ on $g \in G$ can be defined by the conjugation transformation: $\theta(h)[g] = hgh^{-1}$. In a very crude way, for every $h \in H$, the function $\theta(h)$ when applied to group $G$ 'scrambles' its elements in some way. For instance $g' \in G$ gets sent to $g'' \in G$ so that in the multiplication $(g,h) \bullet (g',h') = (g \cdot X, h \circ h')$ the element $X \in G$ is not $g'$; it is its scrambled version $g''$. Notice however that this scrambling made by $\theta(h)$ has an important restriction; $\theta(h)$ must be an isomorphism, meaning that $\theta(h)$ when applied to $g \in G$ preserves the group's structure.

There is no single semi-direct product of $G$ and $H$ but a collection of them, depending on the isomorphism $\theta$ chosen. To avoid this ambiguity, one writes $G \rtimes_\theta H$. The direct product is the special case where $\theta(h)$ is taken to be the identity function, meaning that $hgh^{-1} = g$ for all $g \in G, h \in H$.

Take $G = H = Z_2$ as an example. $g$ generates $G$ and $h$ generates $H$ with $g^2 = e_G$ and $h^2 = e_H$, $e_G$ and $e_H$ being the identity elements of each group. The four elements of $G \rtimes_\theta H$ are $e_G e_H$,$e_G h$,$gh$,$g e_H$ . For the isomorphism, $\theta(h)[g] = hgh^{-1}$ is either $e_G$ or $g$ since these are the only two elements of $G$. The first choice leads to $Z_2 \rtimes_\theta Z_2 = Z_2 \times Z_2 \sim Z_4$ while the second choice leads to $Z_2 \rtimes_\theta Z_2 \sim D_2$, a dihedral group.

This is relevant for $\Delta(3n^2)$ since

$$\Delta(3n^2) \quad \sim \quad (Z_n \times Z_n) \rtimes_\theta Z_3 \tag{5.6}$$

with the following isomorphism. If $a$ generates the first $Z_n$ factor and $b$ the second, and $c$ generates $Z_3$, then $a^n = b^n = c^3 = e$. $\theta(c)$'s action on the generators of $Z_n \times Z_n$ is given by [34, 35]

$$\theta(c)[a] \quad \equiv \quad cac^{-1} = a^{-1}b^{-1} \tag{5.7}$$
$$\theta(c)[b] \quad \equiv \quad cbc^{-1} = a \tag{5.8}$$

| $\otimes$ | **1** | **1'** | **1"** | **3** |
|---|---|---|---|---|
| **1** | **1** | **1'** | **1"** | **3** |
| **1'** | **1'** | **1"** | **1** | **3** |
| **1"** | **1"** | **1** | **1'** | **3** |
| **3** | **3** | **3** | **3** | **1** $\oplus$ **1'** $\oplus$ **1"** $\oplus$ **3** $\oplus$ **3** |

Table 5.6: Kronecker product of $A_4$'s irreps

The elements of $g \in \Delta\left(3n^2\right)$ are written as

$$g \quad = \quad c^i a^j b^k \quad , i = 0, 1, 2 \; ; j, k = 0, 1, \cdots, n-1 \tag{5.9}$$

As shown in table 5.7, all groups of the family $\Delta\left(3n^2\right)$ have only 1 and 3-dimensional irreps. From the labels used to identify these irreducible representations, one should not jump to the conclusion that there are $n^2$ distinct 3-dimensional irreps. That's because the $(k,l)$ label in $\mathbf{3_{k,l}}$ is redundant; it is easily seen that label $(k,l)$ is equivalent to the labels $(-k-l,k)$ and $(l,-k-l)$. The result is that for $n = 3k \; , k \in \mathbb{N}$ there are nine 1-dimensional irreps plus $\frac{n^2-3}{3}$ 3-dimensional irreps while for $n \neq 3k \; , k \in \mathbb{N}$ the splitting is three 1-dimensional irreps plus $\frac{n^2-1}{3}$ 3-dimensional ones. To avoid the redundancy in the naming of the irreps $\mathbf{3_{k,l}}$ one can establish a mapping $\bar{}$ such that $\overline{(k,l)}$ takes one and only one of the values $(k,l)$ , $(-k-l,k)$ or $(l,-k-l)$ for each pair of values $k, l$.

| | $n = 3m \; , m \in \mathbb{N}$ | | |
|---|---|---|---|
| Irrep \ Generator | $a$ | $b$ | $c$ |
| $\mathbf{1_{r,s}}$ | $w^s$ | $w^s$ | $w^r$ |
| $\mathbf{3_{k,l}}$ | $\begin{bmatrix} e(n)^l & 0 & 0 \\ 0 & e(n)^k & 0 \\ 0 & 0 & e(n)^{-k-l} \end{bmatrix}$ | $\begin{bmatrix} e(n)^{-k-l} & 0 & 0 \\ 0 & e(n)^l & 0 \\ 0 & 0 & e(n)^k \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ |

| | $n \neq 3m \; , m \in \mathbb{N}$ | | |
|---|---|---|---|
| Irrep \ Generator | $a$ | $b$ | $c$ |
| $\mathbf{1_r}$ | $1$ | $1$ | $w^r$ |
| $\mathbf{3_{k,l}}$ | $\begin{bmatrix} e(n)^l & 0 & 0 \\ 0 & e(n)^k & 0 \\ 0 & 0 & e(n)^{-k-l} \end{bmatrix}$ | $\begin{bmatrix} e(n)^{-k-l} & 0 & 0 \\ 0 & e(n)^l & 0 \\ 0 & 0 & e(n)^k \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ |

Table 5.7: $\Delta\left(3n^2\right)$'s three generators written in the group's irreps. This family of groups can be separated in two sub-families, depending on whether $n$ is a multiple of 3 or not. Note that for $n = 3m$, $(k,l)$ must be different from $(0,0)$ , $\left(\frac{n}{3}, \frac{n}{3}\right)$ , $\left(\frac{2n}{3}, \frac{2n}{3}\right)$; and for $n \neq 3m$ $(k,l)$ is not allowed to be $(0,0)$.

The Kronecker product of representations for $n = 3m \; , m \in \mathbb{N}$ is given by

$$\mathbf{1_{r,s}} \otimes \mathbf{1_{r',s'}} \quad = \quad \mathbf{1_{r+r',s+s'}} \tag{5.10}$$

$$\mathbf{1_{r,s}} \otimes \mathbf{3_{(k,l)}} \quad = \quad \mathbf{3_{\overline{(k+n\,s/3,l+n\,s/3)}}} \tag{5.11}$$

$$\begin{aligned} \mathbf{3_{(k,l)}} \otimes \mathbf{3_{(k',l')}} \quad &= \quad \sum_{s=0}^{2} \delta_{(k',l'),\overline{(-k+n\,s/3,-l+n\,s/3)}} \left(\mathbf{1_{0,s}} \oplus \mathbf{1_{1,s}} \oplus \mathbf{1_{2,s}}\right) \\ &\oplus \quad \mathbf{3_{\overline{(k'+k,l'+l)}}} \oplus \mathbf{3_{\overline{(k'-k-l,l'+k)}}} \oplus \mathbf{3_{\overline{(k'+l,l'-k-l)}}} \end{aligned} \tag{5.12}$$

and for $n \neq 3m \; , m \in \mathbb{N}$

$$\mathbf{1_r} \otimes \mathbf{1_{r'}} \quad = \quad \mathbf{1_{r+r'}} \tag{5.13}$$

$$\mathbf{1_r} \otimes \mathbf{3_{(k,l)}} \quad = \quad \mathbf{3_{(k,l)}} \tag{5.14}$$

$$\begin{aligned} \mathbf{3_{(k,l)}} \otimes \mathbf{3_{(k',l')}} \quad &= \quad \delta_{(k',l'),\overline{(-k,-l)}} \left(\mathbf{1_0} \oplus \mathbf{1_1} \oplus \mathbf{1_2}\right) \\ &\oplus \quad \mathbf{3_{\overline{(k'+k,l'+l)}}} \oplus \mathbf{3_{\overline{(k'-k-l,l'+k)}}} \oplus \mathbf{3_{\overline{(k'+l,l'-k-l)}}} \end{aligned} \tag{5.15}$$

$A_4$ is the special case of $\Delta\left(3n^2\right)$ with $n = 2$. For $n = 3$, we get $\Delta\left(27\right)$ which has nine 1-dimensional irreps and two 3-dimensional irreps. Let us call these latter ones $\mathbf{3_a}$ and $\mathbf{3_b}$. From the above rules for the decomposition of the Kronecker product, $\mathbf{3_a} \otimes \mathbf{3_a} = \mathbf{3_b} \oplus \mathbf{3_b} \oplus \mathbf{3_b}$, $\mathbf{3_b} \otimes \mathbf{3_b} = \mathbf{3_a} \oplus \mathbf{3_a} \oplus \mathbf{3_a}$ and $\mathbf{3_a} \otimes \mathbf{3_b}$ is reducible into the direct sum of the nine 1-dimensional irreps (see [31] for a model using this group).

# Chapter 6

# Models with tri-bimaximal mixing

One method to obtain tri-bimaximal mixing explores the relation

$$V_{HPS} = V_{CW}^{\dagger}. \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{bmatrix} \tag{6.1}$$

where $V_{CW}$ is the matrix Cabibbo and Wolfenstein suggested back in 1978 as the leptonic mixing matrix itself:

$$V_{CW} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix} \tag{6.2}$$

For Majorana neutrinos, HPS mixing follows from the particular leptonic mass matrices

$$M_l = V_{CW}.\mathrm{diag}\left(m_e, m_\mu, m_\tau\right).\left(U_R^l\right)^{\dagger} \tag{6.3}$$

$$\begin{aligned} M_\nu &= V_{23}^{*}. \begin{bmatrix} a-b+c & 0 & 0 \\ 0 & a+2b & 0 \\ 0 & 0 & -a+b+c \end{bmatrix} .V_{23}^{\dagger} \\ &= \begin{bmatrix} a+2b & 0 & 0 \\ 0 & a-b & c \\ 0 & c & a-b \end{bmatrix} \end{aligned} \tag{6.4}$$

$V_{23}$ being the product of the last two matrices in 6.1. Let us see how $A_4$ can lead to this $M_l$ in a very natural way [22]. Consider the following assignment of irreducible representations to the fields:

$$L \sim \mathbf{3} \tag{6.5}$$

$$l_1 \sim \mathbf{1} \tag{6.6}$$

$$l_2 \sim \mathbf{1'} \tag{6.7}$$

$$l_3 \sim \mathbf{1''} \tag{6.8}$$

$$\Phi \sim \mathbf{3} \tag{6.9}$$

where $L$ and $\Phi$ should be understood as the vectors containing three $SU\left(2\right)_L$ doublets of leptons and Higgs fields respectively, while $l$ is the vector containing the three right handed charged leptons. For charged leptons we obtain the following Yukawa terms:

$$\begin{aligned} -\mathscr{L}_Y &= \overline{L}\left( \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Phi_1 + \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1 & w\lambda_2 & w^2\lambda_3 \\ 0 & 0 & 0 \end{bmatrix} \Phi_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_1 & w^2\lambda_2 & w\lambda_3 \end{bmatrix} \Phi_3 \right) l \\ &+ \text{h.c.} \end{aligned} \tag{6.10}$$

The $\lambda$'s are free parameters. Using $v_i$ for the vev of the charged component of $\Phi_i$ and setting $v_1 = v_2 = v_3 = v$,

$$\begin{aligned} M_l &= \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & w\lambda_2 & w^2\lambda_3 \\ \lambda_1 & w^2\lambda_2 & w\lambda_3 \end{bmatrix} v \\ &= \sqrt{3}v\, V_{CW}.\mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{6.11}$$

which is in the form above indicated. Another way to arrive at the same result is with the assignment

$$L \sim \mathbf{3} \tag{6.12}$$
$$l \sim \mathbf{3} \tag{6.13}$$
$$\Phi_0 \sim \mathbf{1} \tag{6.14}$$
$$\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \sim \mathbf{3} \tag{6.15}$$

leading to

$$\begin{aligned} -\mathscr{L}_Y &= \overline{L}\left( \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \Phi_0 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \Phi_1 \right. \\ &+ \left. \begin{bmatrix} 0 & 0 & \lambda_2 \\ 0 & 0 & 0 \\ \lambda_3 & 0 & 0 \end{bmatrix} \Phi_2 + \begin{bmatrix} 0 & \lambda_3 & 0 \\ \lambda_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Phi_3 \right) l + \text{h.c.} \end{aligned} \tag{6.16}$$

With the vevs of the charged components of $\Phi_1$, $\Phi_2$, and $\Phi_3$ all set to $v$ and $\left\langle \Phi_0^+ \right\rangle = v_0$,

$$\begin{aligned} M_l &= \begin{bmatrix} \lambda_1 v_0 & \lambda_3 v & \lambda_2 v \\ \lambda_2 v & \lambda_1 v_0 & \lambda_3 v \\ \lambda_3 v & \lambda_2 v & \lambda_1 v_0 \end{bmatrix} \\ &= V_{CW}.\begin{bmatrix} \lambda_1 v_0 + \lambda_2 v + \lambda_3 v & 0 & 0 \\ 0 & \lambda_1 v_0 + w^2\lambda_2 v + w\lambda_3 v & 0 \\ 0 & 0 & \lambda_1 v_0 + w\lambda_2 v + w^2\lambda_3 v \end{bmatrix}.V_{CW}^\dagger \end{aligned} \tag{6.17}$$

which is again of the form 6.3. As for the neutrino mass matrix, six Higgs triplets $\xi_i = \left(\xi_i^{++}, \xi_i^+, \xi_i^0\right)^T$ $(i = 1, \cdots, 6)$ are needed. With these, one can build the following $SU(2)_L$ invariants [23, 24]:

$$S_{ijk} = \xi_i^0 \nu_{L_j} \nu_{L_k} + \xi_i^+ \frac{\nu_{L_j} l_{L_k} + l_{L_i} \nu_{L_j}}{\sqrt{2}} + \xi_i^{++} l_{L_i} l_{L_j} \tag{6.18}$$

From the vevs of the six $\xi_i^0$s a neutrino mass matrix is generated. Assigning

$$\xi_1 \sim \mathbf{1} \tag{6.19}$$
$$\xi_2 \sim \mathbf{1}' \tag{6.20}$$
$$\xi_3 \sim \mathbf{1}'' \tag{6.21}$$
$$\begin{bmatrix} \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} \sim \mathbf{3} \tag{6.22}$$

one obtains for neutrinos[1]

$$\begin{aligned} M_\nu &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \left\langle \xi_1^0 \right\rangle + \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & w\lambda_2 & 0 \\ 0 & 0 & w^2\lambda_2 \end{bmatrix} \left\langle \xi_2^0 \right\rangle + \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & w^2\lambda_3 & 0 \\ 0 & 0 & w\lambda_3 \end{bmatrix} \left\langle \xi_3^0 \right\rangle \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_5 \\ 0 & \lambda_4 & 0 \end{bmatrix} \left\langle \xi_4^0 \right\rangle + \begin{bmatrix} 0 & 0 & \lambda_4 \\ 0 & 0 & 0 \\ \lambda_5 & 0 & 0 \end{bmatrix} \left\langle \xi_5^0 \right\rangle + \begin{bmatrix} 0 & \lambda_5 & 0 \\ \lambda_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left\langle \xi_6^0 \right\rangle + \text{transpose} \end{aligned} \tag{6.23}$$

[1] Notice that the $\lambda$s in $M_l$ and $M_\nu$ are unrelated.

Thus, $M_\nu$ has the following form:

$$M_\nu = \begin{bmatrix} a+b+c & f & e \\ f & a+wb+w^2c & d \\ e & d & a+w^2b+wc \end{bmatrix} \tag{6.24}$$

Imposing $b=c$ and $f=e=0$, $M_\nu$ adquires the form in equation 6.4. Notice that this last requirement is only achievable by getting a vev for $\left(\xi_4^0, \xi_5^0, \xi_6^0\right)$ in the direction $(1,0,0)$. On the other hand, $M_l$ and the diagonal entries of $M_\nu$ require spontaneous breakdown of the $A_4$ symmetry in the direction $(1,1,1)$ for other Higgs fields. Although such vacuum expectation values must be properly justified, I will not discuss that.

In [29] another $A_4$ model is described with

$$L \sim \mathbf{3} \tag{6.25}$$
$$l_1 \sim \mathbf{1} \tag{6.26}$$
$$l_2 \sim \mathbf{1'} \tag{6.27}$$
$$l_3 \sim \mathbf{1''} \tag{6.28}$$

for the leptons, and

$$\xi \sim \mathbf{1} \tag{6.29}$$
$$\varphi, \varphi' \sim \mathbf{3} \tag{6.30}$$
$$h_u, h_d \sim \mathbf{1} \tag{6.31}$$

for the scalar fields. $h_{u/d}$ are $SU(2)_L$ doublets with hypercharge $1/2$ while the seven remaining scalar fields are real and invariant under the SM gauge group. This gives rise to the following Yukawa interactions, where $\Lambda$ is a cut-off scale:

$$\begin{aligned} -\mathscr{L}_Y &= \frac{1}{\Lambda}\overline{L}h_d \left( \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varphi_1 + \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1 & w\lambda_2 & w^2\lambda_3 \\ 0 & 0 & 0 \end{bmatrix} \varphi_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_1 & w^2\lambda_2 & w\lambda_3 \end{bmatrix} \varphi_3 \right) l \\ &+ \frac{1}{\Lambda^2}\left(\overline{L}h_u\right) \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_5 \\ 0 & \lambda_4 & 0 \end{bmatrix} \varphi_1' + \begin{bmatrix} 0 & 0 & \lambda_4 \\ 0 & 0 & 0 \\ \lambda_5 & 0 & 0 \end{bmatrix} \varphi_2' + \begin{bmatrix} 0 & \lambda_5 & 0 \\ \lambda_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varphi_3' \right) \left(h_u^\dagger L\right) \\ &+ \frac{1}{\Lambda^2}\lambda_6 \xi \left(\overline{L}h_u\right) \mathbb{1} \left(h_u^\dagger L\right) + \text{h.c.} \end{aligned} \tag{6.32}$$

leading to

$$M_l = \frac{\sqrt{3}v\left\langle h_d^0 \right\rangle}{\Lambda} V_{CW}.\text{diag}\left(\lambda_1, \lambda_2, \lambda_3\right) \tag{6.33}$$

$$M_\nu = \frac{\left|\left\langle h_u^+ \right\rangle\right|^2}{\Lambda^2} \begin{bmatrix} \lambda_6\left\langle \xi \right\rangle & 0 & 0 \\ 0 & \lambda_6\left\langle \xi \right\rangle & \frac{1}{2}\left(\lambda_4+\lambda_5\right)v' \\ 0 & \frac{1}{2}\left(\lambda_4+\lambda_5\right)v' & \lambda_6\left\langle \xi \right\rangle \end{bmatrix} \tag{6.34}$$

if the fields $\varphi, \varphi'$ acquire vevs in the directions

$$\left\langle \varphi_i \right\rangle = v(1,1,1) \tag{6.35}$$
$$\left\langle \varphi_i' \right\rangle = v'(1,0,0) \tag{6.36}$$

These $M_l$ and $M_\nu$ matrices lead to tri-bimaximal mixing. Two problems subside though. One is the realization of the vevs of the scalar fields which the authors solve resorting to extra dimensions. The other is the need to suppress extra terms that could show up in the Yukawa Lagrangian. In fact the $A_4$

symmetry alone would allow the fifth order term $L^2 h_u^2$ as well as the a term of the form $L\varphi' h_d l$ analogous to the one with $\varphi$. These two undesirable terms are forbidden when the following symmetry is added:

$$\begin{aligned} L &\rightarrow iL && (6.37)\\ l &\rightarrow -il && (6.38)\\ \varphi &\rightarrow \varphi && (6.39)\\ \varphi' &\rightarrow -\varphi' && (6.40) \end{aligned}$$

A supersymmetric $A_4$ model leading to tri-bimaximal mixing is presented in [30]. An auxiliary symmetry $Z_4 \times Z_3$ is used; a field $\psi$ transforming according to the $i$-representation of $Z_n$ transforms to $e(n)^i \psi$. The assignment of $(A_4, Z_4, Z_3)$ irreps to the field content of the model is given in table 6.1. The resulting superpotential (which must transform as **2** under $Z_4$) is

| | $A_4$ | $Z_4$ | $Z_3$ | $SU(3)_c$ | $SU(2)_L$ | $U(1)_Y$ |
|---|---|---|---|---|---|---|
| $L$ | **3** | **1** | **0** | **1** | **2** | $-\frac{1}{2}$ |
| $l^c$ | $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ | **3** | **0** | **1** | **1** | 1 |
| $\nu^c$ | **3** | **0** | **1** | **1** | **1** | 0 |
| $E$ | **3** | **1** | **0** | **1** | **1** | $-1$ |
| $E^c$ | **3** | **1** | **0** | **1** | **1** | 1 |
| $h_u$ | **1** | **1** | **2** | **1** | **2** | $\frac{1}{2}$ |
| $h_d$ | **1** | **0** | **0** | **1** | **2** | $-\frac{1}{2}$ |
| $\chi$ | **3** | **2** | **0** | **1** | **1** | 0 |
| $\chi'$ | **3** | **2** | **1** | **1** | **1** | 0 |
| $S_{1,2}$ | **1** | **2** | **1** | **1** | **1** | 0 |

Table 6.1: Representations of the various fields under the symmetries of the model

$$\begin{aligned} W_Y &= \lambda_1 E^T \mathbb{1} E^c + \lambda_2 L^T \mathbb{1} E^c h_d + \frac{1}{2}\lambda_3 S_1 \nu^{cT} \mathbb{1} \nu^c + \frac{1}{2}\lambda_4 S_2 \nu^{cT} \mathbb{1} \nu^c + \lambda_5 L^T \mathbb{1} \nu^c h_u \\ &+ E^T \left( \begin{bmatrix} \lambda_6 & \lambda_7 & \lambda_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \chi_1 + \begin{bmatrix} 0 & 0 & 0 \\ \lambda_6 & w\lambda_7 & w^2\lambda_8 \\ 0 & 0 & 0 \end{bmatrix} \chi_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_6 & w^2\lambda_7 & w\lambda_8 \end{bmatrix} \chi_3 \right) l^c \\ &+ \nu^{cT} \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_{10} \\ 0 & \lambda_9 & 0 \end{bmatrix} \chi'_1 + \begin{bmatrix} 0 & 0 & \lambda_9 \\ 0 & 0 & 0 \\ \lambda_{10} & 0 & 0 \end{bmatrix} \chi'_2 + \begin{bmatrix} 0 & \lambda_{10} & 0 \\ \lambda_9 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \chi'_3 \right) \nu^c \end{aligned} \quad (6.41)$$

The desired vevs are

$$\begin{aligned} \langle S_1 \rangle &= 0 && (6.42)\\ \langle S_2 \rangle &= v_s && (6.43)\\ \langle \chi \rangle &= (v_\chi, v_\chi, v_\chi) && (6.44)\\ \langle \chi' \rangle &= (0, v_{\chi'}, 0) && (6.45) \end{aligned}$$

with $v_u$ and $v_d$ standing for the vevs of $h_u$ and $h_d$. To obtain these, three soft breaking terms of the $Z_4 \times Z_3$ symmetry are added to the Higgs superpotential. Being $M_{lE}$ and $M_{\nu\nu^c}$ $2 \times 2$ block matrices such that $-\mathscr{L} = \begin{bmatrix} l & E \end{bmatrix} M_{lE} \begin{bmatrix} l^c & E^c \end{bmatrix}^T + 1/2 \begin{bmatrix} \nu & \nu^c \end{bmatrix} M_{\nu\nu^c} \begin{bmatrix} \nu & \nu^c \end{bmatrix}^T + \cdots$ then

$$\begin{aligned} M_{lE} &= \begin{bmatrix} \mathbf{0} & \lambda_2 v_d \mathbb{1} \\ \sqrt{3} v_\chi V_{CW}.\mathrm{diag}(\lambda_6, \lambda_7, \lambda_8) & \lambda_1 \mathbb{1} \end{bmatrix} && (6.46)\\ M_{\nu\nu^c} &= \begin{bmatrix} \mathbf{0} & \lambda_5 v_u \mathbb{1} \\ \lambda_5 v_u \mathbb{1} & X \end{bmatrix} && (6.47) \end{aligned}$$

with the $3 \times 3$ matrix $X$ given by

$$X = \begin{bmatrix} \lambda_4 v_s & 0 & \frac{1}{2}(\lambda_9+\lambda_{10})\, v_{\chi'} \\ 0 & \lambda_4 v_s & 0 \\ \frac{1}{2}(\lambda_9+\lambda_{10})\, v_{\chi'} & 0 & \lambda_4 v_s \end{bmatrix} \tag{6.48}$$

After integrating out the $E$ and $E^c$ fields, which are assumed to have large masses,

$$M_l \propto V_{CW}.\mathrm{diag}\,(\lambda_6, \lambda_7, \lambda_8) \tag{6.49}$$

$$M_\nu^{light} \propto \begin{bmatrix} 1 & 0 & x \\ 0 & 1-x^2 & 0 \\ x & 0 & 1 \end{bmatrix} = V_\nu^*.D_\nu.V_\nu^\dagger \tag{6.50}$$

with

$$x = -\frac{\frac{1}{2}(\lambda_9+\lambda_{10})\, v_{\chi'}}{\alpha_4 v_s}$$

$$V_\nu = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{6.51}$$

and $D_\nu$ diagonal. As we've seen before, this leads to tri-bimaximal mixing.

A somewhat different approach is used in [33]. Instead of building $V_{HPS}$ with a $V_{CW}$ contribution from the charged leptons matrix and a rotation of $\pi/4$ in the $2-3$ sector from the neutrinos' mass matrix, the authors build $V_{HPS}$ entirely from $M_\nu$, with a diagonal $M_l$ from the start. Since $M_\nu$ is made up of a linear combination of three matrices in the flavor basis (4.45) they do a block diagonalization of a 2 by 2 block matrix where one of the diagonal blocks is dominant,

$$M = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \to \begin{bmatrix} A - BC^{-1}B^T & 0 \\ 0 & C \end{bmatrix} \tag{6.52}$$

In this way, if $A$ and $BC^{-1}B^T$ have any of the three matrices which makes up $M_\nu^{HPS}$, then the resulting light mass matrix $A - BC^{-1}B^T$ is a nice way to additively build $M_\nu^{HPS}$. The problem is that the $SU(2)_L$ symmetry makes it impossible to have a non null matrix $A$ in the standard seesaw mechanism. In this model, the authors get around that problem by doing the seesaw within the heavy neutrino matrix $M_R$.

The field content of this supersymmetric model is as follows:

- Fermions: $L_i$, $l_i$, $\nu_i$, $N$ with $i = 1, 2, 3$. Notice the presence of four right handed neutrinos ($\nu_i$ and $N$);
- Scalars: $\phi_i$, $\phi_\nu$, $\chi_i$, $S$ with $i = 1, 2, 3$. $\phi_i$ are ordinary Higgs doublets, while $\phi_\nu$ is an $SU(2)_L$ doublet with $y = -1/2$, to cancel anomalies. $\chi_i$ and $S$ are complex fields which are invariants under the SM gauge group.

There are four symmetry transformations that leave the Lagrangian invariant:

1. $z_i : L_i \to -L_i$; $l_i \to -l_i$ ; $\nu_i \to -\nu_i$; $\chi_i \to -\chi_i$ with $i = 1, 2, 3$. This symmetry imposes a diagonal form to the Yukawa matrices.
2. $z_i^h$: $l_i \to -l_i$ ; $\phi_i \to -\phi_i$ with $i = 1, 2, 3$. The coefficients $h_{ijk}$ in $h_{ijk}\overline{L}_i\phi_j l_k$ are null for $j \neq k$ due to this symmetry.
3. $z_\chi : N_i \to -N_i$; $\chi_i \to -\chi_i$ with $i = 1, 2, 3$. This symmetry distinguishes $N$ for the other right handed neutrinos $\nu_i$ and does the same between the three $\chi_i$s and $S$. As a consequence the terms $\chi NN$, $SN\nu$ and $\chi\nu\nu$ are forbidden.
4. $S_3$: $L \sim \mathbf{N}$; $l \sim \mathbf{N}$; $\nu \sim \mathbf{N}$; $\phi = (\phi_1, \phi_2, \phi_3)^T \sim \mathbf{N}$ ($\mathbf{N}$ is $S_3$'s natural representation). With this permutation symmetry the Yukawa matrices, which are diagonal due to the $z_i$ symmetries, have in fact only one independent parameter.

The resulting Yukawa interactions are

$$\begin{aligned}
-\mathscr{L}_Y &= \overline{L}\left(\begin{bmatrix}\lambda_1&0&0\\0&0&0\\0&0&0\end{bmatrix}\phi_1+\begin{bmatrix}0&0&0\\0&\lambda_1&0\\0&0&0\end{bmatrix}\phi_2+\begin{bmatrix}0&0&0\\0&0&0\\0&0&\lambda_1\end{bmatrix}\phi_3\right)l+\lambda_2\overline{L}\mathbb{1}\nu\phi_\nu \\
&+ \frac{1}{2}\nu^T\left(\begin{bmatrix}\lambda_3\\0\\0\end{bmatrix}\chi_1^*+\begin{bmatrix}0\\\lambda_3\\0\end{bmatrix}\chi_2^*+\begin{bmatrix}0\\0\\\lambda_3\end{bmatrix}\chi_3^*\right)C\,N \\
&+ \frac{1}{2}N^T\left(\begin{bmatrix}\lambda_3&0&0\end{bmatrix}\chi_1^*+\begin{bmatrix}0&\lambda_3&0\end{bmatrix}\chi_2^*+\begin{bmatrix}0&0&\lambda_3\end{bmatrix}\chi_3^*\right)C\,\nu \\
&+ \frac{1}{2}\lambda_4 S^*\nu^T\mathbb{1}C\,\nu+\frac{1}{2}\lambda_5 S^* N^T\mathbb{1}C\,N+\text{h.c.}
\end{aligned} \tag{6.53}$$

to which are added the following symmetry breaking terms:

$$-\mathscr{L}_M = \frac{1}{2}\lambda_6\nu^T\mathbb{1}C\,\nu+\frac{1}{2}\lambda_7\,\nu^T\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}C\,\nu+\frac{1}{2}\lambda_8 N^T\mathbb{1}C\,N+\text{h.c.} \tag{6.54}$$

Using

$$\langle\phi_\nu^0\rangle = v \tag{6.55}$$
$$\langle\chi_i\rangle = \omega_i \tag{6.56}$$
$$\langle S\rangle = s \tag{6.57}$$

, in the basis $\begin{bmatrix}\nu&\nu^c&N\end{bmatrix}^T$ we get the block matrix

$$M_{D+M} = \begin{bmatrix}\mathbf{0}&\lambda_2^*v^*\mathbb{1}&\mathbf{0}\\\lambda_2^*v^*\mathbb{1}&(\lambda_6^*+\lambda_4^*s)\,\mathbb{1}+\alpha_7^*A&M^T\\\mathbf{0}&M&(\lambda_8^*+\lambda_5^*s)\,\mathbb{1}\end{bmatrix} \tag{6.58}$$

with

$$M = \lambda_3^*\begin{bmatrix}\omega_1&\omega_2&\omega_3\end{bmatrix} \tag{6.59}$$
$$A = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} \tag{6.60}$$

The Majorana mass matrix of the right handed neutrinos is

$$\begin{aligned}
M_R &= \begin{bmatrix}(\lambda_6^*+\lambda_4^*s)\,\mathbb{1}+\lambda_7^*A&M^T\\M&(\lambda_8^*+\lambda_5^*s)\,\mathbb{1}\end{bmatrix} \\
&\to \begin{bmatrix}(\lambda_6^*+\lambda_4^*s)\,\mathbb{1}+\lambda_7^*A-(\lambda_8^*+\lambda_5^*s)\,M^TM&0\\0&(\lambda_8^*+\lambda_5^*s)\,\mathbb{1}\end{bmatrix}
\end{aligned} \tag{6.61}$$

in the assumption that the second diagonal block of $M_R$ is dominant, although the authors of this model show that HPS mixing is obtainable even without this assumption. In any case, it must be assumed that

$$\omega_1 = 0 \tag{6.62}$$
$$\omega_2 = -\omega_3 \equiv \omega \tag{6.63}$$

leading to $M$ of the form

$$M = \alpha_3^*\omega\begin{bmatrix}0&1&-1\end{bmatrix} \tag{6.64}$$

Since $M_\nu^{hps}$ is precisely a linear combination of $\mathbb{1}$, $A$ and $M^TM$, $(\alpha_6^*+\alpha_4^*s)\,\mathbb{1}+\alpha_7^*A-(\alpha_8^*+\alpha_5^*s)\,M^TM$ is of the desired form[2].

[2]Actually the light neutrino mass matrix $M_\nu$ is proportional to the inverse of this matrix which is also of the form of $M_\nu^{hps}$ which is what was needed since $M_l$ is diagonal.

# Part IV

# Theoretical considerations on models based on discrete family symmetries

After reviewing some of the models in literature that lead to tri-bimaximal mixing, I will now make some general theoretical considerations about such models, assuming that exact symmetry is preserved. We have just seen that obtaining tri-bimaximal mixing from the application of discrete symmetries to the fields frequently requires extending the SM. It may be the use of supersymmetry or extra dimensions. Other possibilities include introducing right-handed neutrinos or adding Higgs singlets, doublets, or triplets. In what follows I will only consider the possible existence of extra Higgs doublets analogous to the SM one and of three right-handed heavy neutrinos, which give rise to an effective left-handed neutrino mass matrix through the standard seesaw mechanism. In the first chapter, only one Higgs doublet will be considered though. That is a major simplification that allows us to make several general statements about such models. We will be using this relative simplicity of one Higgs models to explore different ways of applying a discrete symmetry to the fields. In the second chapter, some considerations are made on multiple Higgs models.

# Chapter 7

# Models with one invariant Higgs doublet

## 7.1 General Considerations

With one Higgs doublet, which will be taken to be invariant under the group's action, it is possible to predict the mixing matrix as well as the degeneracy of the fermions masses with simple considerations about the irreps assigned to the fields. An example shows the usefulness of the results to be derived. Consider the quark Yukawa couplings with one Higgs doublet, just as in the SM. Assume that the multiplets $Q_L$, $u_R$, and $d_R$ all transform through the natural representation of $S_3$ and that they do so in a synchronous fashion, i.e., the transformation is simultaneous for all fields. Knowing that this representation is equivalent to the direct sum of a 1-dimensional irrep and a 2-dimensional one, we can immediately conclude that there are two degenerate masses for the up quarks as well as for the down quarks and $V_{CKM} = \mathbb{1}$. Another possibility is to consider that $Q_L$, $u_R$, and $d_R$ transform independently. If that is the case, the $S_3$ symmetry imposes two null masses for the up and for the down quarks as well as $V_{CKM} = \mathbb{1}$ [18, 16, 17]. To arrive at this conclusion, one only needs to know that the natural representation of $S_3$ contains the trivial irrep once.

## 7.2 Some Definitions

Consider $\psi_L$ and $\psi_R$ as two multiplets of fields[1] transforming according to the unitary representations of groups $G$ and $H$ respectively. If we choose the nomenclature $L(g_i) \equiv L_i$ and $R(h_i) \equiv R_i$ $(g_i \in G, h_i \in H)$,

$$\psi_L \rightarrow \psi'_L = L_i \psi_L \tag{7.1}$$
$$\psi_R \rightarrow \psi'_R = R_j \psi_R \tag{7.2}$$

The relation between $i$ and $j$ may be non existent if we consider acting independently on the two fields. An alternative is to consider that $G$ and $H$ act in a synchronized form, in which case we must relate these two indexes.

We now turn our attention to the reduction of the representations $L$ and $R$. I will consider them reduced representations if they are block diagonalized, with each block being an irrep. In general

$$L_i = S_L D_{L\,i} S_L^\dagger \tag{7.3}$$
$$R_i = S_R D_{R\,i} S_R^\dagger \tag{7.4}$$

where $D_{L\,i}$ and $D_{R\,i}$ are in reduced form. $S_L$ and $S_R$ carry out a change of basis [2]. The important point here is that the physical relevant quantities are independent of the choice of basis. With this freedom,

[1] For most of this chapter, we will not be making assumptions on the dimension of these field multiplets, although the number we must have ultimately in mind is three.

[2] It is worth remembering that all the matrices in the both equations can always be chosen to be unitary matrices since $G$ and $H$ are discrete groups. In fact it is important that representations $A$ and $B$ are unitary so as to not to get in trouble with the propagators of the fields $\psi_L$ and $\psi_R$ which are for this discussion terms of the form $\psi^\dagger_{L/R}\psi_{L/R}$.

we will use representations $L$ and $R$ in their reduced form, meaning that for any $i$ the matrices $L_i$ and $R_i$ are block diagonal ($S_L = S_R = \mathbb{1}$). This choice does not exhaust all the freedom, since for each invariant subspace of $L$ and $R$ we are still free to choose a basis. So, to avoid unnecessary complications in subsequent results we require additionally that whenever there are equivalent irreps in the diagonals of $L$ and $R$, they must be in fact be identical. These conditions suffice for now, but attention should be brought to the fact that we still haven't determined completely the basis we will be using.

The following notation will often be used throughout this chapter: for an arbitrary matrix $X$ , $[X]^K_{\alpha\beta}$ will be $X$'s block $(\alpha\beta)$ where the block division is done according to a fully reduced representation $K$. For example, if we assume that the representation $K$ in the direct sum of a 1-dimensional irrep and a 2-dimensional irrep ($\mathbf{1} \oplus \mathbf{2}$) then

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \tag{7.5}$$

$$[X]^K_{11} = [x_{11}] \tag{7.6}$$

$$[X]^K_{12} = \begin{bmatrix} x_{12} & x_{13} \end{bmatrix} \tag{7.7}$$

$$[X]^K_{21} = \begin{bmatrix} x_{21} \\ x_{31} \end{bmatrix} \tag{7.8}$$

$$[X]^K_{22} = \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} \tag{7.9}$$

When no confusion arises, the label for the representation by which the block division is made will be suppressed.With such an understanding, we have for instance $L_i = \text{diag}\left([L_i]_{11}, [L_i]_{22}, ...\right)$ and similarly for the $R_i$ matrices.

## 7.3 Establishing the different scenarios

It is obvious that different results come as a consequence of different assumptions. With that in mind, we may wish to the establish various scenarios depending on the way the groups $G$ and $H$ act on the field multiplets. The following are possible differentiating factors:

1. $G$ and $H$ may act in a synchronized, depended fashion, or not. In the latter case $G$ and $H$ act independently on the multiplets;
2. $G$ and $H$ may be the same group or not. In particular, representations $L$ and $R$ may be equal;
3. Contrary to what we have done so far, we can entertain the possibility that each of the two multiplets - $\Psi_L$ and $\Psi_R$ - have different components transforming according to different groups instead of all of them transforming as a single 'super-block'.

## 7.4 Relevant Lagrangian Mass Terms

We are interested in Dirac mass terms ($\overline{\Psi_L} M_D \Psi_R$), Majorana mass terms ($\Psi_R^T M_R \Psi_R$), and effective mass terms ($M_{eff} = -M_D M_R^{-1} M_D^T$). Notice that since we are dealing with only one Higgs doublet, the mass matrices are proportional to the Yukawa interaction matrices. This legitimizes the application of the discrete symmetry directly to the mass matrices.

Let us start with $M_D$. In order to have an invariant Lagrangian when the two field multiplets transform according to 7.1 - 7.2 the mass matrix must obey

$$M_D = L_i^\dagger M_D R_j \tag{7.10}$$

We will often be interested in $H_D \equiv M_D M_D^\dagger$ and not on the mass matrix itself. It follows from the last equation that

$$H_D = L_i^\dagger H_D L_i \tag{7.11}$$

This equation does not express everything we know about $H_D$ since we are erasing some information on it when we ignore looking at $M_D$ and choose instead to study the condition imposed by the symmetry on $H_D$ directly. Nonetheless, in most of the relevant cases equation 7.11 is all that is needed to extract the information relative to mixing and masses. In those cases, the condition on $M_D$ imposes additional constraints on these quantities, such as dictating that some masses are null.

For $M_R$,

$$M_R = R_i^T M_R R_i \tag{7.12}$$

Notice that the group action on the Majorana mass matrix cannot by asynchronous in contrast to $M_D$. If we have in the Lagrangian density a Dirac mass term as well as a Majorana one and if we assume that the Majorana masses are must larger that the Dirac ones, the seesaw mechanism leads to an effective light mass matrix $M_{eff} = -M_D M_R^{-1} M_D^T$. From what is known of $M_D$ and $M_R$,

$$\begin{aligned} M_{eff} &= -L_i^\dagger M_D R_j R_j^\dagger M_R^{-1} R_j^* R_j^T M_D^T L_i^* \\ &= L_i^\dagger M_{eff} L_i^* \end{aligned} \tag{7.13}$$

which is to be expected since this effective mass matrix appear as $\mathscr{L} = \cdots + \Psi_L^T M_{eff}^* \Psi_L$. From the above equation $M_{eff}^*$ is qualitatively similar to $M_R$ requiring only the substitution of $R_i$ for $L_i$. Again, in this observation it must be taken into account that 7.13 isn't the only restriction on $M_{eff}$ - the compete set of constraints on this matrix comes from the restrictions on $M_D$ and $M_R$.

## 7.5 Synchronized action of the groups

We should pause a moment to think about the meaning of having $G$ and $H$ acting dependently (synchronized action). Intuitively this means that when $L_i$ is applied to $\Psi_L$, some $R_{j(i)}$ is applied to $\Psi_R$ and vice-versa. That is, there is a one to one map between the $j$ index in the $R_j$ and the $i$ index in $L_i$. Note that if $L$ and $R$ are representations of $G$ and $H$, then $L(g_x) L(g_y) = L(g_x g_y)$ as well as $R\left(h_{j(x)}\right) R\left(h_{j(y)}\right) = R\left(h_{j(x)} h_{j(y)}\right)$ for $g_x, g_y \in G, h_{j(x)}, h_{j(y)} \in H$. It makes sense to think of a new representation $R_i'$ (of a possible different group $H'$) such that $R_i' \equiv R_{j(i)}$ so that

$$\psi_L \rightarrow L_i \psi_L \tag{7.14}$$

$$\psi_R \rightarrow R_i' \psi_R \tag{7.15}$$

which in practice sets the function $j(i)$ to the identity map at the expense of considering a new group $H'$ and a new representation $R'$. Clearly the freedom to choose the function $j(i)$ is superfluous since it can be incorporated in the definition of group $H$ and its representation $R$. Whenever there is synchronized action of the groups, it is best then to always use $j(i) = i$ and define group $H$ and its representation $R$ such that

$$\psi_L \rightarrow L_i \psi_L \tag{7.16}$$

$$\psi_R \rightarrow R_i \psi_R \tag{7.17}$$

Let's analyze Dirac mass matrices in this scenario then. In terms of matrix blocks,

$$\sum_\gamma [L_i]_{\alpha\gamma} [M_D]_{\gamma\beta} = \sum_\gamma [M_D]_{\alpha\gamma} [R_i]_{\gamma\beta} \tag{7.18}$$

$$\Leftrightarrow$$

$$[L_i]_{\alpha\alpha} [M_D]_{\alpha\beta} = [M_D]_{\alpha\beta} [R_i]_{\beta\beta} \tag{7.19}$$

and

$$\sum_\gamma [L_i]_{\alpha\gamma} [H_D]_{\gamma\beta} = \sum_\gamma [H_D]_{\alpha\gamma} [L_i]_{\gamma\beta}$$

$$\Leftrightarrow$$

$$[L_i]_{\alpha\alpha} [H_D]_{\alpha\beta} = [H_D]_{\alpha\beta} [L_i]_{\beta\beta} \tag{7.20}$$

since

$$[R_i]_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta \quad (7.21)$$
$$[L_i]_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta \quad (7.22)$$

Looking at $H_D$, either $[L_i]_{\alpha\alpha}$ and $[L_i]_{\beta\beta}$ are equivalent or not. In the latter case, since for any $\mu$ $[L_i]_{\mu\mu}$ is an irrep of a group, Schur's second lemma states that the block $[H_D]_{\alpha\beta}$ is null. On the other hand if $[L_i]_{\alpha\alpha}$ and $[L_i]_{\beta\beta}$ are equivalent, by our choice of basis for the representations $R$ and $L$, they are in fact identical so that equation 7.20 states that

$$\left[[L_i]_{\alpha\alpha}, [H_D]_{\alpha\beta}\right] = 0 \quad (7.23)$$

which by Schur's first lemma implies that

$$[H_D]_{\alpha\beta} = \lambda_{\alpha\beta}\mathbb{1} \quad (7.24)$$

where $\lambda_{\alpha\beta}$ is an arbitrary complex value. Notice though that by construction $H_D$ is hermitian which means that $\lambda_{\beta\alpha} = \lambda^*_{\alpha\beta}$. Frequently the representation $L$ will contain distinct irreps with no repetitions ( e.g. $L = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ or $L = \mathbf{1} \oplus \mathbf{2}$ or $L = \mathbf{3}$). In that case, the previous analysis shows that $H_D$ is block diagonal, with each of the diagonal blocks proportional to the identity matrix. That means that $H_D$ is diagonal and the number of its degenerate diagonal entries (which are the fermions' square masses) is given by the dimension of the irreps. For instance $L = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ leads to $H_D = \text{diag}\,(a, b, c)$, $L = \mathbf{1} \oplus \mathbf{2}$ leads to $H_D = \text{diag}\,(a, b, b)$ and $L = \mathbf{3}$ leads to $H_D = \text{diag}\,(a, a, a)$.

The Majorana mass matrix $M_R$ is different since in 7.12 $M_R$ appears surrounded not by $R_i$ and its inverse but rather $R_i^T$ and $R_i$. The difference is significant. In fact, at first it would even seem to block us from applying the Schur lemmas since we now get

$$[R_i]^*_{\alpha\alpha}\,[M_R]_{\alpha\beta} = [M_R]_{\alpha\beta}\,[R_i]_{\beta\beta} \quad (7.25)$$

The key is to notice that the conjugate of the representation $R_i$ is also a representation of the group $G$. More importantly, the diagonal blocks of these matrices, $[R_i]^*_{\mu\mu}$, are also irreps of the group. We can then go ahead and apply Schur's lemmas and the end result is the same as for $H_D$ except that in this case we have to compare $[R_i]^*_{\alpha\alpha}$ to $[R_i]_{\beta\beta}$. But here there is something new: we know that if $[R_i]^*_{\alpha\alpha}$ is not equivalent to $[R_i]_{\beta\beta}$, meaning that there is no unitary matrix $X$ for which $[R_i]^*_{\alpha\alpha} = X\,[R_i]_{\beta\beta}\,X^\dagger$ holds for all $i$, then $[M_R]_{\alpha\beta} = 0$. But what if $[R_i]^*_{\alpha\alpha}$ and $[R_i]_{\beta\beta}$ are equivalent ($[R_i]^*_{\alpha\alpha} \sim [R_i]_{\beta\beta}$)? For $H_D$ we knew that in such case, the two representations would be identical. But that is not what happens here so we must consider $[R_i]^*_{\alpha\alpha} = \tau^R_{\alpha\beta}\,[R_i]_{\beta\beta}\,\tau^{R\dagger}_{\alpha\beta}$ with $\tau^R_{\alpha\beta}$ unitary but not necessarily equal to $\mathbb{1}$. As a consequence,

$$\tau^R_{\alpha\beta}\,[R_i]_{\beta\beta}\,\tau^{R\dagger}_{\alpha\beta}\,[M_R]_{\alpha\beta} = \left[M^R\right]_{\alpha\beta}[R_i]_{\beta\beta}$$
$$\Leftrightarrow \quad (7.26)$$
$$\left[[R_i]_{\beta\beta}, \tau^{R\dagger}_{\alpha\beta}\,[M_R]_{\alpha\beta}\right] = 0 \quad (7.27)$$

which by Schur's first lemma means that

$$[M_R]_{\alpha\beta} = \lambda_{\alpha\beta}\tau^R_{\alpha\beta} \quad (7.28)$$

with an arbitrary $\lambda_{\alpha\beta}$. In the last section of this chapter, I will elaborate on the relation of an irrep and its conjugate representation. That obviously related to the form of these $\tau$ matrices.

As for $M_{eff}$, it clearly can be computed directly from $M_R$ and $M_D$ once we know these two matrices. But it is nice to know outright the form of $M_{eff}$ due to symmetry constraints, just as was done to $M_D/H_D$ and $M_R$. Actually, since $M_{eff}$ is subject to a restraint similar to $M_R$'s, once the substitution $R_i \to L_i^*$ is made, the result for $M_R$ is applicable to $M_{eff}$ as well. To be precise,

$$[M_{eff}]_{\alpha\beta} = \begin{cases} 0 & \text{if } [L_i]^*_{\alpha\alpha} \nsim [L_i]_{\beta\beta} \\ \lambda_{\alpha\beta}\tau^{L*}_{\alpha\beta} & \text{if } [L_i]^*_{\alpha\alpha} \sim [L_i]_{\beta\beta} \end{cases} \tag{7.29}$$

with $\tau^L_{\alpha\beta}$ defined by the relation

$$[L_i]^*_{\alpha\alpha} = \tau^L_{\alpha\beta}\,[L_i]_{\beta\beta}\,\tau^{L\dagger}_{\alpha\beta} \tag{7.30}$$

None of these conclusions depends on any relation between groups $G$ and $H$ or their representations $L$ and $R$. So, is it inconsequential that the representations are chosen from different groups as opposed to $G = H$? And in this latter case, is the relation between $R$ and $L$ irrelevant? No, all this is relevant. The way it affects the above results is to possibly force the constants $\lambda_{\alpha\beta}$ to take some special values, such as zero. Take the following case, were $G = H = S_3$. We could have been lead to believe that $H_D$ only depends on the $L$ representation and not on $R$. But if $L = \mathbf{1} \oplus \mathbf{2}$ and $R = \mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$, since none of the irreps of $L$ matches those of $R$, from the application of Schur's lemmas to equation 7.19 it is clear that all blocks of $M_D$ are null, making $M_D = H_D = 0$ so that all the $\lambda_{\alpha\beta}$ in 7.24 are forced to zero. One the other hand, for $L = R = \mathbf{1} \oplus \mathbf{2}$ we get $H_D \neq 0$ and in fact no restrictions are placed on the $\lambda_{\alpha\beta}$'s except those coming from the condition $H_D = H_D^\dagger$.

But what if different groups $G$ and $H$ (with the same number of elements) are used? Since Majorana mass matrices only depends on $R$, they do not 'feel' that $G \neq H$; only Dirac mass matrices must be considered then. In this regard, we shall see that the constants $\lambda_{\alpha\beta}$ in $H_D$ can be restricted to zero in some situations. Where should one look for these restrictions? It is obvious they can only come from the different group structures of $G$ and $H$ . Let's introduce some simple notation to take on this subject; the numbers $ij$ and $\underline{ij}$ are defined as

$$g_{ij} \equiv g_i g_j \tag{7.31}$$
$$h_{\underline{ij}} \equiv h_i h_j \tag{7.32}$$

for any $g_i, g_j \in G$ and $h_i, h_j \in H$ meaning in particular that

$$L_{ij} = L_i L_j \tag{7.33}$$
$$R_{\underline{ij}} = R_i R_j \tag{7.34}$$

Applying twice equation 7.10 for any $i$ and $j$,

$$\begin{aligned} M_D &= L_j^\dagger L_i^\dagger M_D R_i R_j \\ &= L_{ij}^\dagger M_D R_{\underline{ij}} \end{aligned} \tag{7.35}$$

But since $ij$ and $\underline{ij}$ are just indexes, it is equally true that

$$M_D = L_{ij}^\dagger M_D R_{ij} \tag{7.36}$$
$$M_D = L_{\underline{ij}}^\dagger M_D R_{\underline{ij}} \tag{7.37}$$

We may rearrange these relations in the following way:

$$L_{ij} M_D = L_{\underline{ij}} M_D \;\; \forall_{i,j} \tag{7.38}$$
$$M_D R_{ij} = M_D R_{\underline{ij}} \;\; \forall_{i,j} \tag{7.39}$$
$$L_k M_D = M_D R_k \;\; \forall_k \tag{7.40}$$

Let us go from $M_D$ to $H_D$ and use block notation. We get from the first and third equation the following:

$$[L_{ij}]_{\alpha\alpha}\,[H_D]_{\alpha\beta} = \left[L_{\underline{ij}}\right]_{\alpha\alpha}[H_D]_{\alpha\beta} \quad \forall_{i,j} \tag{7.41}$$

$$[L_k]_{\alpha\alpha}\,[H_D]_{\alpha\beta} = [H_D]_{\alpha\beta}\,[L_k]_{\beta\beta} \quad \forall_k \tag{7.42}$$

where the second expression was already obtained in 7.20 but is here reproduced. The relevant question now is: in what conditions can the block $[H_D]_{\alpha\beta}$ be different from the null rectangular matrix? Equation 7.42 tells us that we must have at least the condition $[L_k]_{\alpha\alpha} = [L_k]_{\beta\beta}$ in which case $[H_D]_{\alpha\beta}$ is a multiple of the identity matrix ($[H_D]_{\alpha\beta} = \lambda_{\alpha\beta}\mathbf{1}$). This is not new to us. But in such a situation, provided that $\lambda_{\alpha\beta}$ is not zero, $[H_D]_{\alpha\beta}$ is an invertible matrix and by 7.41 we arrive at the expression $[L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha}$ for all $i$ and $j$. In general this is not a true statement though. We conclude then that in order to have a non null $[H_D]_{\alpha\beta}$ (i.e., $\lambda_{\alpha\beta} \neq 0$) we must additionally require that $[L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha}$. To summarize,

$$[H_D]_{\alpha\beta} = \lambda_{\alpha\beta}\mathbb{1} \text{ if } [L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha} = [L_{ij}]_{\beta\beta} = \left[L_{\underline{ij}}\right]_{\beta\beta} \quad \forall_{i,j} \tag{7.43}$$

$$[H_D]_{\alpha\beta} = 0 \text{ otherwise} \tag{7.44}$$

This is indeed a very strong condition on $H_D$ since the whole point of using two distinct groups would be to explore their different structures. In our notation that equates to having $ij \neq \underline{ij}$ for at least some values of $i$ and $j$. But we have just found out that unless the irreps of $L$ are degenerate, $ij \neq \underline{ij}$ will lead to $[L_{ij}]_{\alpha\alpha} \neq \left[L_{\underline{ij}}\right]_{\alpha\alpha}$. This in effect nullifies blocks of $H_D$ that otherwise (i.e., with equal groups) could be proportional to the identity matrix.

## 7.6 Independent action of the groups

Having the groups act on the right and left field multiplets independently unquestionably imposes stronger conditions on the mass matrices if the Lagrangian is to remain invariant under such transformation. Now the indexes $i$ and $j$ in equations 7.1 and 7.2 are unrelated and can take any values. The Majorana mass matrix is immune to such independent action on the multiplets since it only deals with $\psi_R$. We should expect changes relative to the previous section only on the Dirac mass matrices then.

$M_D$ is subject to the restriction

$$M_D = L_i^\dagger M_D R_j \quad \forall_{i,j} \tag{7.45}$$

If we pick $i$ such that $L_i = \mathbb{1}$, then

$$M_D = M_D R_j \quad \forall_j \tag{7.46}$$

and, on the other hand, if we admit an arbitrary $i$ but a $j$ such that $R_j = \mathbb{1}$, then

$$M_D = L_i^\dagger M_D \quad \forall_i \tag{7.47}$$

First, I would like to point out that the dagger in this last equation is irrelevant since $L_i^\dagger$ is just $L_k$ for some $k$. On the other hand, this two last equations contain all the information from the original condition on $M_D$, since

$$\begin{aligned} M_D &= L_i^\dagger M_D \quad \forall_i \\ &= L_i^\dagger (M_D R_j) \quad \forall_{i,j} \end{aligned} \tag{7.48}$$

Notice also that

$$M_D = L_i^\dagger M_D R_i \quad \forall_i \tag{7.49}$$

is a special case of equation 7.45, so that the results for $H_D$ derived in the previous section continues to be valid. Let us see if we can restrict the constants $\lambda_{\alpha\beta}$ introduced previously. Starting from the condition

$$\begin{aligned} M_D &= L_i M_D \\ &\Leftrightarrow \\ H_D &= L_i H_D \end{aligned} \tag{7.50}$$

in block notation we get

$$[H_D]_{\alpha\beta} = [L_i]_{\alpha\alpha} [H_D]_{\alpha\beta} \tag{7.51}$$

Now, from last section's results either $[H_D]_{\alpha\beta}$ is null or it is a multiple of the identity matrix. This latter case occurs only when $[L_i]_{\alpha\alpha} = [L_i]_{\beta\beta}$. But if that is so, from the last equation we get $[L_i]_{\alpha\alpha} = \mathbb{1}$ , $\forall_i$. We must then remember that we are dealing with irreps, so this only happens in the exceptional case where we have $[L_i]_{\alpha\alpha} = [L_i]_{\beta\beta} = 1$, the trivial irrep.

Here is an example. In light of this information, it seems a ratter curious coincidence that we do indeed get a non null Dirac mass matrix when using the natural representation of $S_3$, the permutation group of three elements. Indeed $(S_3)_L \times (S_3)_R$, as it is usually referred to, leads to $H_D \propto \Delta$, the democratic matrix which has $(\Delta)_{\alpha\beta} = 1$. This happens because the natural representation of $S_3$, which I've called $\mathbf{N}$ in the previous chapter, contains once the trivial representation (it is equivalent to $\mathbf{1} \oplus \mathbf{2}$). So we would expect from our analysis that there is only one non-null eigenvalue in $H_D$ which is precisely what get in $\Delta$.

How does $G = H$, $G \neq H$ affect $H_D$ in this scenario where the two groups act independently on $\Psi_L$ and $\Psi_R$? All that was said for the synchronous group action can be reused here. In addition, we know that only with $[L]_{\alpha\alpha}$ and $[L]_{\beta\beta}$ equal to the trivial representation can $[H_D]_{\alpha\beta}$ be non null. So with minimal effort,

$$[H_D]_{\alpha\beta} = \lambda_{\alpha\beta} \text{ if } [L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha} = [L_{ij}]_{\beta\beta} = \left[L_{\underline{ij}}\right]_{\beta\beta} = 1 \ \ \forall_{i,j} \tag{7.52}$$

$$[H_D]_{\alpha\beta} = 0 \text{ otherwise} \tag{7.53}$$

But this is a lazy way of stating the result. That's because if $[L]_{\alpha\alpha}([L]_{\beta\beta})$ is the trivial representation, obviously $[L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha}([L_{ij}]_{\beta\beta} = \left[L_{\underline{ij}}\right]_{\beta\beta})$ for all $i, j$. What's at stake here is then: what is the difference between the trivial representation of a group $G$ and a group $H$ with the same number of elements as $G$? The answer is none! It's just a list of 1s in both cases, so whether $G = H$ or $G \neq H$ is irrelevant.

## 7.7 More than one group acting on each multiplet

It could happen that each multiplet, $\Psi_L$ and $\Psi_R$, does not transform according to a single group. For instance, these multiplets can be slit in two parts,

$$\Psi_L = \begin{bmatrix} \Psi_{\dot{L}} \\ \Psi_{\ddot{L}} \end{bmatrix} \tag{7.54}$$

each transforming according to a different group:

$$\Psi_{\dot{L}} = \begin{bmatrix} \varphi_{L\,1} \\ ... \\ \varphi_{L\,m} \end{bmatrix} \ \rightarrow \ \dot{L}_i \Psi_{\dot{L}} \tag{7.55}$$

$$\Psi_{\ddot{L}} = \begin{bmatrix} \varphi_{L\,m+1} \\ ... \\ \varphi_{L\,n} \end{bmatrix} \ \rightarrow \ \ddot{L}_i \Psi_{\ddot{L}} \tag{7.56}$$

and similarly for $\Psi_R$. $\dot{L}$, $\ddot{L}$,$\dot{R}$, and $\ddot{R}$ are representations equal in every aspect to the $L$ and $R$ representations used so far. I shall call $\dot{G}$, $\ddot{G}$,$\dot{H}$, and $\ddot{H}$ to the corresponding groups. Looking at $\Psi_{L/R}$ as a whole, in the language of equations 7.1 - 7.2

$$L_k = \begin{bmatrix} \dot{L}_i & 0 \\ 0 & \ddot{L}_j \end{bmatrix} \tag{7.57}$$

$$R_k = \begin{bmatrix} \dot{R}_i & 0 \\ 0 & \ddot{R}_j \end{bmatrix} \tag{7.58}$$

with some relation between the indexes $i$, $j$ and $k$. For instance, if we allow $\dot{L}$ and $\ddot{L}$ to act independently on the fields, then $i$ and $j$ are independent and $k$ can be taken as the label $(i, j)$. In that case $L$ is easily seen to be a representation of the direct product of groups $\dot{G}$ and $\ddot{G}$. The other alternative would be for $\dot{L}$ and $\ddot{L}$ to act dependently, in which case $j\,(i)$ and $k\,(i)$ are some functions of $i$. In this situation $L$ may not be a representation of a group at all. For example, we could have a situation where $\dot{L}_1\dot{L}_2 = \dot{L}_3$ but $\ddot{L}_{j(1)}\ddot{L}_{j(2)} \neq \ddot{L}_{j(3)}$ so that there would not be any $k$ for which $L_k = L_1L_2$. Previously it was stated that it is natural for a set of transformations that leave the Lagrangian invariant to form a group. While being true, that has not stopped us from analyzing some paragraphs above the possibility of having $\Psi_L$ and $\Psi_R$ transforming dependently, according to different groups $G$ and $H$. That transformation as a whole may also violate the definitions of a group, but it was possible to do some analysis of that scenario nonetheless since equations 7.10 to 7.13 express matricial relations regardless of the origin of the intervening matrices. The reward of having looked into such a scenario was to get as a result, in simple terms, that the relevant mass matrices would be null unless some irreps of group $G$ present in $L$ were actually irreps of group $H$ in $R$. That is, for all practical purposes we could then have used $G = H$.

In the case in hand, it proves useful to use the letters $\alpha, \beta$ for the blocks associated with the 'single-dot part' of a vector or matrix and $\mu, \nu$ for the 'double-dot part'. Figure 7.1 graphically clarifies things. In this notation, $[L_k]_{\alpha\beta} = \left[\dot{L}_i\right]_{\alpha\beta}$, $[L_k]_{\mu\nu} = \left[\ddot{L}_j\right]_{\mu\nu}$ and $[L_k]_{\alpha\mu} = [L_k]_{\mu\alpha} = 0$. Here is an example: if $\dot{L}$ and $\ddot{L}$ contain 2 irreps and 3 irreps respectively, our convention so far would be to call them $\left[\dot{L}_i\right]_{11}$,$\left[\dot{L}_i\right]_{22}$ and $\left[\ddot{L}_i\right]_{11}$,$\left[\ddot{L}_i\right]_{22}$,$\left[\ddot{L}_i\right]_{33}$. For obvious reasons, it becomes more useful in this scenario to make the change of the indexing convention $\left\{\left[\ddot{L}_i\right]_{11}, \left[\ddot{L}_i\right]_{22}, \left[\ddot{L}_i\right]_{33}\right\} \rightarrow \left\{\left[\ddot{L}_i\right]_{33}, \left[\ddot{L}_i\right]_{44}, \left[\ddot{L}_i\right]_{55}\right\}$. And so in this example $\alpha, \beta \in \{1, 2\}$ and $\mu, \nu \in \{3, 4, 5\}$.

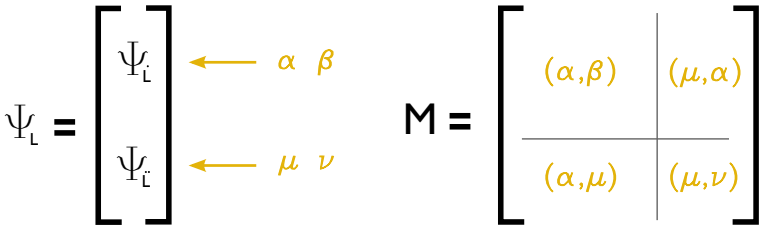



Figure 7.1: On the use of $\alpha, \beta$ for components related to one group and $\mu, \nu$ for the components related to the other group. For example, the $\alpha/\beta$ components of $\Psi_L(\Psi_R)$ are transformed according to the representation $\dot{L}(\dot{R})$ of group $\dot{G}(\dot{H})$ while the $\mu/\nu$ components transform according to the representation $\ddot{L}(\ddot{R})$ of group $\ddot{G}(\ddot{H})$.

Having put aside the nomenclature issues, the problem at hand is in fact of simple analysis. For Dirac masses, the relation

$$H_D = L_i^\dagger H_D L_i \tag{7.59}$$

breaks $H_D$ in four regions - precisely the $(\alpha, \beta)$, $(\alpha, \mu)$, $(\mu, \alpha)$, and $(\mu, \nu)$ regions. Explicitly,

$$[H_D]_{\alpha\beta} = \left[\dot{L}_i\right]^\dagger_{\alpha\alpha} [H_D]_{\alpha\beta} \left[\dot{L}_i\right]_{\beta\beta} \tag{7.60}$$

$$[H_D]_{\alpha\mu} = \left[\dot{L}_i\right]^\dagger_{\alpha\alpha} [H_D]_{\alpha\mu} \left[\ddot{L}_j\right]_{\mu\mu} \tag{7.61}$$

$$[H_D]_{\mu\alpha} = \left[\ddot{L}_j\right]^\dagger_{\mu\mu} [H_D]_{\mu\alpha} \left[\dot{L}_i\right]_{\alpha\alpha} \tag{7.62}$$

$$[H_D]_{\mu\nu} = \left[\ddot{L}_j\right]^\dagger_{\mu\mu} [H_D]_{\mu\nu} \left[\ddot{L}_j\right]_{\nu\nu} \tag{7.63}$$

The difference is that while blocks in $(\alpha, \beta)$ and $(\mu, \nu)$ suffer the influence of the one group only ($\dot{G}$ and $\ddot{G}$ respectively), blocks in the $(\alpha, \mu)$ and $(\mu, \alpha)$ regions are subject to the action of different groups. Both of these situations were contemplated before for whole matrices and in the present situation, one only has to apply those results to the four distinct sub-matrices of $H_D$.

Majorana masses are different in the sense that they were immune to our attempts to apply different groups to its left and right side. Here we finally get to do just that, in the four sub-matrices framework. The analysis is analogous to $H_D$ except that we are dealing with the $R$ representation, not $L$, and additionally irreps in $R$ are to be compared to those in $R^*$ in the manner indicated in the previous sections. The same is true of $M_{eff}$. To be specific, for a synchronous application of $\dot{L}$ and $\ddot{L}$, all blocks of $M_{eff}$ are null except the following:

$$[M_{eff}]_{\alpha\beta} = \lambda_{\alpha\beta}\tau^{L*}_{\alpha\beta} \quad \text{if} \quad [L_i]_{\alpha\alpha} = [L_i]^*_{\beta\beta} \ \ \forall_i \tag{7.64}$$

$$[M_{eff}]_{\mu\nu} = \lambda_{\mu\nu}\tau^{L*}_{\mu\nu} \quad \text{if} \quad [L_i]_{\mu\mu} = [L_i]^*_{\nu\nu} \ \ \forall_i \tag{7.65}$$

$$[M_{eff}]_{\alpha\mu} = \lambda_{\alpha\mu}\tau^{L*}_{\alpha\mu} \text{ and } [M_{eff}]_{\alpha\mu}\, \lambda_{\mu\alpha}\tau^{L*}_{\mu\alpha} \quad \text{if} \quad [L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha} = [L_{ij}]^*_{\mu\mu} = \left[L_{\underline{ij}}\right]^*_{\mu\mu} \quad \forall_{i,j} \tag{7.66}$$

In the asynchronous case, in the last equation, the irreps must be equal to the trivial one.

I will not pursue further this possibility of having a family of fields transforming according to different groups, only retaining the fact that this situation can be derived from the earlier ones. In the next section, a summary of these results is presented.

## 7.8 Summary

The following are the main results from the analysis we have made so far written in a fairly self contained manner.

The left and right family of fields transform according to

$$\psi_L \rightarrow \psi'_L = L_i \psi_L \tag{7.67}$$

$$\psi_R \rightarrow \psi'_R = R_j \psi_R \tag{7.68}$$

with

$$i = j \qquad \text{(syncronous group action)} \tag{7.69}$$

$$\text{any } i, j \qquad \text{(assyncronous group action)} \tag{7.70}$$

where $L$ and $R$ are representations of group $G$ an $H$ in a special base where all the matrices are block diagonal. These blocks of $L$ and $R$ are irreps of $G$ an $H$, which provide a natural way to partition in blocks not only the representation matrices $L_i$ and $R_j$ themselves but also the mass matrices. In the $[]_{\alpha\beta}$ block notation, $L_i = \text{diag}\left([L_i]_{11}, [L_i]_{22}, \cdots\right)$ and similarly for $R_j$. The indexes $ij$ and $\underline{ij}$ are defined as

$$g_{ij} \equiv g_i g_j \tag{7.71}$$

$$h_{\underline{ij}} \equiv h_i h_j \tag{7.72}$$

with $g_i, g_j \in G$, $h_i, h_j \in H$. If $G = H$, naturally $ij = \underline{ij}$ for all $i, j$.

All blocks of the relevant matrices ($H_D = M_D M_D^\dagger$, $M_R$, and $M_{eff}$) are null with the following exceptions:

| **R/L group action relation** | **Non-null blocks allowed** |
|---|---|
| synchronous | $[H_D]_{\alpha\beta} = \lambda_{\alpha\beta}\mathbb{1}$ if $[L_{ij}]_{\alpha\alpha} = \left[L_{\underline{ij}}\right]_{\alpha\alpha} = [L_{ij}]_{\beta\beta} = \left[L_{\underline{ij}}\right]_{\beta\beta}$ $\forall_{i,j}$ |
| synchronous | $[M_R]_{\alpha\beta} = \lambda_{\alpha\beta}\tau^R_{\alpha\beta}$ if $[R]_{\alpha\alpha} \sim [R]^*_{\beta\beta}$ |
| synchronous | $[M_{eff}]_{\alpha\beta} = \lambda_{\alpha\beta}\tau^{L*}_{\alpha\beta}$ if $[L]_{\alpha\alpha} \sim [L]^*_{\beta\beta}$ |
| asynchronous | $[H_D]_{\alpha\beta} = \lambda_{\alpha\beta}$ if $[L_i]_{\alpha\alpha} = [L_i]_{\beta\beta} = 1$ $\forall_i$ |
| asynchronous | $M_R$ (same as in the synchronous case) |
| asynchronous | $M_{eff}$ (same as in the synchronous case) |

Table 7.1: Summary of the main results. Differences occur depending on whether the discrete group acts on the right an left family multiplets in a dependent/synchronous fashion or not.

In addition, if $G = H$, the results in table 7.2 apply to $M_D$.

| **R/L group action relation** | **Non-null blocks allowed** |
|---|---|
| synchronous | $[M_D]_{\alpha\beta} = \lambda_{\alpha\beta}\mathbb{1}$ if $[L_i]_{\alpha\alpha} = [R_i]_{\beta\beta}$ $\forall_i$ |
| asynchronous | $[M_D]_{\alpha\beta} = \lambda_{\alpha\beta}$ if $[L_i]_{\alpha\alpha} = [R_i]_{\beta\beta} = 1$ $\forall_i$ |

Table 7.2: Summary of the main results for Dirac masses (valid if groups $G$ and $H$ are the same).

The similarity matrices $\tau^{L/R}_{\alpha\beta}$ for a given $\alpha$ and $\beta$ are defined by

$$[L_i]^*_{\alpha\alpha} = \tau^L_{\alpha\beta}\,[L_i]_{\beta\beta}\,\tau^{L\dagger}_{\alpha\beta}\ \forall_i \tag{7.73}$$

$$[R_i]^*_{\alpha\alpha} = \tau^R_{\alpha\beta}\,[R_i]_{\beta\beta}\,\tau^{R\dagger}_{\alpha\beta}\ \forall_i \tag{7.74}$$

Such matrices only exist if the irreps $[L]^*_{\alpha\alpha}$ and $[L]_{\beta\beta}$ ($[R]^*_{\alpha\alpha}$ and $[R]_{\beta\beta}$) are equivalent, $[L]_{\alpha\alpha} \sim [L]^*_{\beta\beta}$ ($[R]_{\alpha\alpha} \sim [R]^*_{\beta\beta}$). Next, we shall analyze the form of these $\tau$ matrices.

## 7.9 Relation between an irrep and its complex conjugate

I shall now say a few words about the $\tau_{\alpha\beta}$ matrices introduced previously. These are the change-of-basis matrices that relate the conjugate of an irrep $\alpha$ to an irrep $\beta$. If irreps $\alpha$ and $\beta$ are distinct, it is possible to put them in a basis such that irrep $\beta$ is equal to the conjugate of an irrep $\alpha$. Problem solved then: $\tau_{\alpha\beta} = \mathbb{1}$. If on the other hand irreps $\alpha$ and $\beta$ stand for the same (irreducible) representation, we are left with calculating the relation between an irrep and its conjugate.

To avoid confusion with the above notation let us introduce $A_i \equiv A(a_i)$, an irrep of some group $\mathcal{A}$. First and foremost we wish to know if irreps $A$ and $A^*$ are equivalent. The answer comes from the knowledge of the irrep's characters: $A$ and $A^*$ are equivalent if and only if the characters of representation

$A$ are all real; Since $A$ is an irrep, so is $A^*$ and from the orthogonality relations of a group's irreducible characters (see V ) irrep $A^*$ is equivalent to $A$ iff both share the same characters. Since $\chi(A_i^*) = \chi(A_i)^*$, that only happens when the characters are real.

Take $S_3$ as an example (table 7.3) . All rows of its character table are real. Thus all three irreps are equivalent to their complex conjugate representations.

| Irrep \ Class | $\{e\}$ | $\{(12),(13),(23)\}$ | $\{(123),(132)\}$ |
|---|---|---|---|
| **1** | 1 | 1 | 1 |
| **1′** | 1 | -1 | 1 |
| **2** | 2 | 0 | -1 |

Table 7.3: Character table of the $S_3$ group.

Assuming then that irrep $A$ is self conjugate (meaning $A \sim A^*$) let $\tau$ define the unitary change-of-basis matrix between $R$ and $R^*$,

$$A_i^* = \tau A_i \tau^\dagger \ \forall_i \tag{7.75}$$

What can we say about $\tau$? Notice that it is this sort of matrices that appear in blocks of the Majorana mass matrices $M_R$ or the effective mass matrices $M_{eff}$ in the see-saw mechanism. We shall return to this point, but for now it suffices to know that we want these matrices symmetric. If they are not, we need to symmetrize them. For $\tau$ in particular, knowledge of the relation between $\tau$ and $\tau^T$ is important.

We could approach this issue in the following way. Assume that there is an $i$ such that $A_i$ has non repeated eigenvalues. We can then change the basis of the representation to one in which this matrix becomes diagonal. Let's call this matrix $X$. Then

$$X = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \tag{7.76}$$

Note that the order of element $a_i$, which I'll call $m$, is such that $X^m = \mathbb{1}$ so that the eigenvalues $\lambda_i$ are $m$-roots of unity. Knowing that $X^* = \tau X \tau^\dagger$ means that either the $\lambda_i$'s are real or they appear in conjugate pairs ($\lambda_i = \lambda_j^*$ for some $i$ and $j$). It's clear then that $X^*$ is obtained from $X$ by a permutation $P$ of the eigenvalues in its diagonal entries. I.e., $X^* = PXP^T$ where $P^T = P^\dagger = P^{-1}$. We would then be tempted to say $\tau = P$. In other words, in this particular representation basis, matrix $\tau$ would be a permutation matrix. In the same spirit, we can conjugate 7.75 and we get for $X$ that $X = \tau^* X^* \tau^T$ . Inserting the expression for $X^* (= \tau X \tau^\dagger)$,

$$\begin{aligned} X &= \tau^* \tau X \tau^\dagger \tau^T \\ &\Leftrightarrow \\ [X, \tau^* \tau] &= 0 \end{aligned} \tag{7.77}$$

Knowing that $X$ is diagonal with distinct eigenvalues again could lead temptation to get the best of us and make believe that $\tau^* \tau = \mathbb{1}$. Since $\tau$ is unitary this would mean indeed that $\tau = \tau^T$. But we know better. In the first case, $X^* = PXP^T$ does not mean that $\tau = P$. That's because we can insert at will a unitary diagonal phase matrix $K$ in the following way,

$$X^* = PKXK^\dagger P^T \tag{7.78}$$

with $K = \mathrm{diag}\left(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}\right)$. In fact, since $X^*$ is, just like $X$, a diagonal matrix we may insert another phase matrix $K'$:

$$\begin{aligned} K'^\dagger X^* K' &= PKXK^\dagger P^T \\ &\Leftrightarrow \\ X^* &= K' PKXK^\dagger P^T K'^\dagger \\ &\Leftrightarrow \\ X^* &= P\left(P^\dagger K' P\right) KXK^\dagger \left(P^\dagger K' P\right)^\dagger P^T \end{aligned} \tag{7.79}$$

The last equation is meant to show that, since $P^\dagger K' P$ is just the phase matrix that results from permuting $K'$'s diagonal entries, $\left(P^\dagger K' P\right) K$ is also some phase matrix so that we return to what we had in equation 7.78. We conclude that $\tau = PK$ for some phase matrix $K$. And the same is true about setting $\tau^*\tau = \mathbb{1}$: in general this is false and the correct statement is $\tau^*\tau = K'$ where $K'$ is another phase matrix. How then can one determine this phase matrix and in so doing, also $\tau$? Condition 7.75 taken only for one particular value of $i$, as we've just shown with the $X$ matrix, is not enough; we need to consider the whole set of $A_i$'s that make up the representation when using equation 7.75 to determine $\tau$.

Let us try a different approach then. We know that we must somehow take into account the information we have for various $A_i$'s. The following analysis will lead to a relevant conclusion. Take

$$I_Y \equiv \sum_i A_i^\dagger Y A_i \tag{7.80}$$

where $Y$ is an arbitrary matrix. $I_Y$ is always proportional to the identity matrix. It is easy to see why. For all $j$,

$$\begin{aligned} A_j^\dagger I_Y A_j &= \sum_i A_j^\dagger A_i^\dagger Y A_i A_j \\ &= \sum_{i'} A_{i'}^\dagger Y A_{i'} \\ &= I_Y \end{aligned} \tag{7.81}$$

where use of the rearrangement lemma was made. Being $A$ an irrep of a group, we know by Schur's second lemma that the above equation leads to $I_Y = c\mathbb{1}$ where $c$ is some constant. We can in fact calculate the constant $c$ by taking the trace of both sides of this equality:

$$\begin{aligned} \operatorname{tr}(I_Y) &= \operatorname{tr}(c\mathbb{1}) \\ &\Leftrightarrow \\ \sum_i \operatorname{tr}\left(A_i^\dagger Y A_i\right) &= c \operatorname{tr}(\mathbb{1}) \\ &\Leftrightarrow \\ \sum_i \operatorname{tr}(Y) &= c \operatorname{tr}(\mathbb{1}) \\ &\Leftrightarrow \\ c &= n_g \frac{\operatorname{tr}(Y)}{\operatorname{tr}(\mathbb{1})} \end{aligned} \tag{7.82}$$

where $n_g$ is the order of group $G$. Choosing $Y = \tau^*\tau$,

$$\begin{aligned} I_{\tau^*\tau} &= \sum_i A_i^\dagger \tau^*\tau A_i \\ &= n_g \frac{\operatorname{tr}(\tau^*\tau)}{\operatorname{tr}(\mathbb{1})}\mathbb{1} \end{aligned} \tag{7.83}$$

and so, noting that equation 7.75 can be written as $\tau^T A_i^\dagger \tau^* = A_i^T$,

$$\tau^T I_{\tau^*\tau}\tau^\dagger = \sum_i \tau^T A_i^\dagger \tau^*\tau A_i \tau^\dagger = \sum_i A_i^T A_i^* = \sum_i \mathbb{1} = n_g\mathbb{1} \tag{7.84}$$

$$= \tau^T \left[ n_g \frac{\operatorname{tr}(\tau^*\tau)}{\operatorname{tr}(\mathbb{1})}\mathbb{1} \right] \tau^\dagger = n_g \frac{\operatorname{tr}(\tau^*\tau)}{\operatorname{tr}(\mathbb{1})} \tau^T\tau^\dagger \tag{7.85}$$

We conclude that

$$\mathbb{1} = \frac{\operatorname{tr}(\tau^*\tau)}{\operatorname{tr}(\mathbb{1})} \tau^T\tau^\dagger \tag{7.86}$$

$$\Leftrightarrow$$

$$\tau = \frac{\operatorname{tr}(\tau^*\tau)}{\operatorname{tr}(\mathbb{1})} \tau^T \tag{7.87}$$

Due to the unitarity of $\tau$, this factor between $\tau$ and $\tau^T$ can have only two values. To see that, we take the trace of 7.86:

$$\begin{aligned} \mathrm{tr}\,(\mathbb{1}) &= \mathrm{tr}\left[\frac{\mathrm{tr}\,(\tau^*\tau)}{\mathrm{tr}\,(\mathbb{1})}\tau^T\tau^\dagger\right] \\ &= \frac{\mathrm{tr}\,(\tau^*\tau)}{\mathrm{tr}\,(\mathbb{1})}\mathrm{tr}\,(\tau^*\tau)^T \\ &= \frac{\mathrm{tr}^2\,(\tau^*\tau)}{\mathrm{tr}\,(\mathbb{1})} \\ &\Rightarrow \\ \sigma &\equiv \frac{\mathrm{tr}\,(\tau^*\tau)}{\mathrm{tr}\,(\mathbb{1})} = \pm 1 \end{aligned} \tag{7.88}$$

So $\tau = \sigma\tau^T$ is either symmetric or anti-symmetric. Due to the symmetric nature of $M_R$ and $M_{eff}$ mass matrices, $\tau = \tau^T$ is just in the form we need, while $\tau = -\tau^T$ effectively has a null contribution to the mass matrices since $1/2\left(\tau + \tau^T\right)$ is zero.

Knowing that $\tau = \sigma\tau^T$ with $\sigma = \pm 1$, let's go back to our previous thought. If there is an $i$ such that $A_i$ has non repeated eigenvalues, matrix $X$, then we get

$$X^* = \tau X \tau^\dagger \tag{7.89}$$

with

$$\tau = PK \tag{7.90}$$

where $P$ is a permutation matrix and $K$ some phase matrix. Since $\tau^*\tau = \sigma\mathbb{1}$,

$$\begin{aligned} \sigma\mathbb{1} &= P^*K^*PK \\ &= P^*P\left(K^*\right)_P K \end{aligned} \tag{7.91}$$

where by definition

$$Z_P \equiv P^\dagger Z P \tag{7.92}$$

for some matrix $Z$. Noting that $P$ is a real matrix ($P^* = P$),

$$\sigma\mathbb{1} = P^2\left(K^*\right)_P K \tag{7.93}$$

On the left side we have a diagonal matrix. Since $\left(K^*\right)_P$ and $K$ are also diagonal, $P^2$ must be too. But $P^2$ is a permutation and the only permutation that has a diagonal form is the identity. The conclusion is that

$$P^2 = \mathbb{1} \tag{7.94}$$

$$\left(K^*\right)_P K = \sigma\mathbb{1} \tag{7.95}$$

The first of these equations is very interesting since it tell us that $P$ is a permutation that only contains 1-cycles and 2-cycles. For instance, if the irrep we are considering is three dimensional, then $P$ could be the matrix associated to permutations $e = (1)(2)(3)$, $(12)(3)$, $(13)(2)$, $(23)(1)$ but not $(123)$ nor $(132)$. That was to be expected since, as previously mentioned, the eigenvalues $\lambda_i$ of $X = \mathrm{diag}\,(\lambda_1, \lambda_2, ..., \lambda_n)$ are either real or come in conjugate pairs so that $X^*$ is obtainable from $X$ with the permutation of these pairs of conjugate eigenvalues. With this in mind, one can make further comments on the $K$ phase matrix based on equation 7.95 which says that $K = \sigma K_P$. Take for instance a 2-dimensional irrep. $P$ could be the identity, $\mathbb{1}$, or

$$P_{(12)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{7.96}$$

For $P = \mathbb{1}$ $K_P = K$ so that only $\sigma = +1$ is allowed and $K$ may be an arbitrary phase matrix ($K = \text{diag}\left(e^{i\alpha}, e^{i\beta}\right)$). On the other hand, if $P = P_{(12)}$, $\sigma$ is free to take the values $+1$ or $-1$ and $K$ must have the form $\text{diag}\left(e^{i\alpha}, \sigma e^{i\alpha}\right)$, i.e., only one phase is free. Let's go on to 3-dimensional irreps. Other than $P = \mathbb{1}$, we are left with permutation of the type $(xx)(x)$. Let's consider then $P = P_{(12)(3)}$:

$$P_{(12)(3)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.97}$$

For $K = \text{diag}\left(e^{i\alpha}, e^{i\beta}, e^{i\gamma}\right)$,

$$\begin{aligned} K_P &= P^\dagger K P \\ &= \text{diag}\left(e^{i\beta}, e^{i\alpha}, e^{i\gamma}\right) \end{aligned} \tag{7.98}$$

As for $\alpha$ and $\beta$, $K = \sigma K_P$ implies that $e^{i\beta} = \sigma e^{i\alpha}$, so that one phase is not free. But, when we look at $\gamma$ it is seen that while it is a free phase, it imposes $\sigma = +1$. The origin of this restriction is simple; it's the presence of a 1-cycle in permutation $(12)(3)$. The conclusion is that antisymmetric $\tau$'s ($\sigma = -1$) are only possible for irreps with even dimension.

Let's go back to the general case now. Assume that $P$ has $y$ 2-cycles and $z$ 1-cycles - without loss of generally we may consider it to be the permutation $(12)(34)...(2y-1\ 2y)(2y+1)(2y+2)...(2y+z)$. $K$ must be of the form $K = \text{diag}\left(e^{i\theta_{12}}, \sigma e^{i\theta_{12}}..., e^{i\theta_{2y-1\ 2y}}, \sigma e^{i\theta_{2y-1\ 2y}}, e^{i\theta_{2y+1}}, ..., e^{i\theta_{2y+z}}\right)$ where $\sigma = -1$ is only possible if $z = 0$ as seen previously, although we won't be making any assumptions on $z$. We wish now to separate the '$\sigma$ contribution' in $K$ from the $\theta$ phase angles. So, let's define $K \equiv \Sigma K'$ where $K' = \text{diag}\left(e^{i\theta_{12}}, e^{i\theta_{12}}..., e^{i\theta_{2y-1\ 2y}}, e^{i\theta_{2y-1\ 2y}}, e^{i\theta_{2y+1}}, ..., e^{i\theta_{2y+z}}\right)$ and $\Sigma$ is a diagonal matrix with 1s and $\sigma$'s in its diagonal. This separation ensures that

$$K' = K'_P \tag{7.99}$$

$$\Sigma = \sigma \Sigma_P \tag{7.100}$$

i.e., the symmetric/antisymmetric dependence of $\tau = PK = PK'\Sigma$ is isolated in $\Sigma$. As for $K'$, it is not the most general phase matrix since of each 2-cycle in the permutation $P$ the corresponding phases must be identical. This restriction is annoying to work with. One possible way out is to introduce a completely general phase matrix $K''$ i.e., with unrestricted phases, and notice that $K''_P K''$ is automatically of the form of $K'$.

In conclusion, in a basis where one of the irrep's matrices with distinct eigenvalues is in a diagonal form,

$$\tau = PK''K''_P\Sigma \tag{7.101}$$

where $P$ is a permutation matrix such as $P^2 = \mathbb{1}$, $K''$ is an arbitrary phase matrix and $\Sigma = \text{diag}\,(1, \sigma, 1, \sigma, ... , 1, 1, ...)$ where the pattern '$1, \sigma$' is due to 2-cycles in $P$ and the '1's to 1-cycles so that $\Sigma = \sigma\Sigma_P$ (again noting that $\sigma = -1$ implies that no 1-cycles exist).

The form of $\tau$ in 7.101 is just a means to prove that we can erase the phase factor $K''K''_P$ with an adequate choice of basis for the representation. Let's then check how $\tau$ varies with a change-of-basis transformation. If for all $i$

$$A_i \rightarrow A'_i = V A_i V^\dagger \tag{7.102}$$

and since we defining $\tau$ and $\tau'$ such that

$$A_i^* = \tau A_i \tau^\dagger \tag{7.103}$$

$$A'^*_i = \tau' A'_i \tau'^\dagger \tag{7.104}$$

then the old $\tau$ and the new one, $\tau'$, are related by the expression

$$\tau \quad \rightarrow \quad \tau' = V^* \tau V^\dagger \tag{7.105}$$

Note that this base transformation can't change a symmetric $\tau$ into an antisymmetric one, or vice-versa. Now, with $\tau$ of the form indicated in 7.101 and $V = K''$,

$$\begin{aligned} \tau \rightarrow \tau' &= K''^* \left( P K'' K_P'' \Sigma \right) K''^* \\ &= P \left( K''^* \right)_P K'' K_P'' K''^* \Sigma \\ &= P\Sigma \end{aligned} \tag{7.106}$$

Use was made of the fact that all matrices except $P$ commute between themselves since they are all diagonal. Let's check the symmetric/antisymmetric character of $\tau$ in this form explicitly:

$$\begin{aligned} \tau^T &= \Sigma^T P^T \\ &= \Sigma P \\ &= P \Sigma_P \\ &= \sigma P \Sigma \\ &= \sigma \tau \end{aligned} \tag{7.107}$$

Equation 7.106 is the main conclusion of this section: when $\tau$ is symmetric ($\sigma = +1$), $\Sigma$ is equal to the identity matrix and so there is a base where $\tau$ is equal to a permutation $P$ with $P^2 = \mathbb{1}$ .

Back to physics, we are potentially interested in 1, 2 and 3-dimensional irreps $A_i$. The possible $\tau$ are presented in table 7.4.

| Irrep dimension | Possible $\tau$ |
|---|---|
| 1 | $[1]$ |
| 2 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

Table 7.4: Qualitatively distinct forms allowed for the $\tau$ matrices in an appropriately chosen base (see text).

## 7.10 Can models with one Higgs doublet accommodate experimental data?

'Yes' is a cheater's answer to whether or not models based on discrete family symmetries, with one Higgs doublet, can accommodate the known fermion mass matrices. That's because, in the extreme case, we may set $L = R = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ where $\mathbf{1}$ is the trivial irrep, i.e., $\psi_L \rightarrow \psi_L$ and $\psi_R \rightarrow \psi_R$. Unsurprisingly, our previous analysis would lead to arbitrary $3 \times 3$ $M_D$ and $M_R$ matrices. This goes to show that we can't test the usefulness of discrete family symmetries by answering the naive question raised in the title of this section. Indeed, for these models to be successful they must 'explain' the know values of the mass matrices, i.e., accommodate the experimental data as well as restrict the parameter space of the theory. For quarks there are 6 masses plus 3 mixing angles plus 1 CP violating phase. The same happens for leptons if neutrinos are Dirac particles. If not then two additional Majorana phases must be taken into account. Restricting the parameter space means providing a mechanism that explains the values these

quantities. Can discrete family symmetries be a natural way to do so? The case I'll now try to make is that for single Higgs doublet models it is not possible to impose tri-bimaximal in the leptonic sector.

After extensively considering various scenarios for the groups action on $\psi_L$ and $\psi_R$ and for the representations $R$ and $L$ in previous sections, we should firstly settle on what's physically useful for us to consider. From the summary in table 7.1, synchronized action of representations $R$ and $L$ is the only potentially relevant situation. Additionally, nothing is gained from considering distinct groups from which to pick $R$ and $L$. It would be reasonable to go even further, imposing that the two representations share the same irreps. That's because if they don't, the Dirac mass matrices $M_D$ acquire null lines and columns which means that some masses are set to zero - that is something that can only be considered reasonable for neutrinos. Despite this observation, I shall assume no relation between $R$ and $L$.

Let's consider Dirac particles first (table 7.10). For quarks, there is no problem if we consider $V_{ckm} = \mathbb{1}$ good enough. Taking that to be the case and since up ($u$) and down ($d$) quarks have a clear mass hierarchy, only $L(Q_L) = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''}$ is viable[3]. As for $R(u_R)$ and $R(d_R)$, it is only necessary to ensure that in $H_D = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ none of the $\alpha$'s gets erased. For that, it suffices to choose $R(u_R) = R(d_R) = L(Q_L)$ and we get two diagonal $M_D$'s with three free eigenvalues each.

| $L$ representation | $H_D = M_D M_D^\dagger$ |
|---|---|
| $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2^* & \alpha_4 & \alpha_5 \\ \alpha_3^* & \alpha_5^* & \alpha_6 \end{bmatrix}$ |
| $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1'}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_2^* & \alpha_3 & 0 \\ 0 & 0 & \alpha_4 \end{bmatrix}$ |
| $\mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''}$ | $\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$ |
| $\mathbf{1} \oplus \mathbf{2}$ | $\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}$ |
| $\mathbf{3}$ | $\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{bmatrix}$ |

Table 7.5: Qualitatively distinct forms allowed for $H_D$ depending on the $L$ representation ($\psi_L \to L_i \psi_L$,$\psi_R \to R_i \psi_R$). The condition $H_D = H_D^\dagger$ was taken into account. The choice of the $R$ representation may bring some other restrictions to the $\alpha$'s though.

Crucially, any choice of group or indeed 1-dimensional irreps is the same. So, the simplest case would be to use the $Z_3$ group and its three 1-dimensional irreps. Explicitly, with

$$C \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{bmatrix}$$

invariance of the Lagrangian for the transformation

$$Q_L \to Q'_L = C^i Q_L \tag{7.108}$$
$$u_R \to u'_R = C^i u_R \tag{7.109}$$
$$d_R \to d'_R = C^i d_R \tag{7.110}$$

for $i = 0, 1, 2$ leads to 6 non-degenerate quark masses and $V_{ckm} = \mathbb{1}$.

[3]Since different right and left multiplets are to be considered, we need to specify these when indicating the various $R$ and $L$ representations.

As for leptons, if neutrinos are Dirac particles, the problem is that large mixing angles are needed. Tri-bimaximal mixing in particular requires certain values for these angles and the problem is that when $H_D$ is not diagonal ($L = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ or $L = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'$), there is no control over the non diagonal matrix entries - we can either kill them or let them take arbitrary values. So, no matter how the $L\,(L_L)$, $R\,(l_R)$, and $R\,(\nu_R)$ representations are chosen, HPS mixing can't be forced onto leptons in this way. One final observation: since $H_D$ is equal to $M_D M_D^\dagger$, if the entry $ii$ of $H_D$ is null, $(H_D)_{ii} = 0$, so must be the entire column $i$ and line $i$. That means that off diagonal entries $(H_D)_{ij}$ may take values different from 0 only if $(H_D)_{ii}$ and $(H_D)_{jj}$ are non zero themselves. For instance in table , with $L = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'$, it is not possible to have $\alpha_2 \neq 0$ while imposing $\alpha_1 = 0$ or $\alpha_3 = 0$.

What if neutrinos are Majorana particles? For $M_{eff}$, one of the mixing angles, say $\theta_{23}$, can be forced to $\pi/2$ by choosing $L = \mathbf{1} \oplus \mathbf{2}$ with $\tau\,(\mathbf{2}) = P_{(12)}$ or $L = \mathbf{3}$ with $\tau\,(\mathbf{3}) = P_{(23)}$:

$$L = \mathbf{1} \oplus \mathbf{2} \quad : \quad M_{eff} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \\ 0 & \alpha_2 & 0 \end{bmatrix} \tag{7.111}$$

$$L = \mathbf{3} \quad : \quad M_{eff} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 \\ 0 & \alpha_1 & 0 \end{bmatrix} \tag{7.112}$$

where in the first case it was assumed that irrep $\mathbf{1}$ is real ($\mathbf{1}^* = \mathbf{1}$). For charged leptons, both of these choices would lead to a diagonal $H_D$:

$$L = \mathbf{1} \oplus \mathbf{2} \quad : \quad H_D = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix} \tag{7.113}$$

$$L = \mathbf{3} \quad : \quad H_D = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{bmatrix} \tag{7.114}$$

In the limit of exact family symmetry, this double (triple) mass degeneracy on both light neutrinos and charged leptons means that the mixing angle $\theta_{12}$ ($\theta_{12}$,$\theta_{13}$ and $\theta_{23}$) of $V_{pmns}$ is undermined. Even if for some reason we could admit a small breaking of the symmetry we would not get tri-bimaximal mixing - only $V_{pmns}$'s $\theta_{23}$ would be significantly different from zero.

There is another possibility which is using an $L$ representation made up of three 1-dimensional irreps. I'll now use the subscript "R" and "I" to distinguish real irreps (that are self conjugate) from imaginary irrep (which are not self conjugate). Table 7.6 shows a list of all possible cases that may arise. Now, unless there is degeneracy in the irreps used, $H_D$ for the charged leptons will be diagonal so for $M_{eff}$ we would need to get $M_\nu^{hps}$ in order to have tri-bimaximal mixing. And that is not possible.

| $L$ representation | $M_{eff}$ | $L$ representation | $M_{eff}$ |
|---|---|---|---|
| $\mathbf{1_R} \oplus \mathbf{1'_R} \oplus \mathbf{1''_R}$ | $\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$ | $\mathbf{1_I} \oplus \mathbf{1_I^*} \oplus \mathbf{1_R}$ | $\begin{bmatrix} 0 & \alpha_1 & 0 \\ \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}$ |
| $\mathbf{1_R} \oplus \mathbf{1_R} \oplus \mathbf{1'_R}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & \alpha_4 \end{bmatrix}$ | $\mathbf{1_I} \oplus \mathbf{1'_I} \oplus \mathbf{1''_I}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| $\mathbf{1_R} \oplus \mathbf{1_R} \oplus \mathbf{1_R}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_5 & \alpha_6 \end{bmatrix}$ | $\mathbf{1_I} \oplus \mathbf{1_I^*} \oplus \mathbf{1'_I}$ | $\begin{bmatrix} 0 & \alpha_1 & 0 \\ \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| $\mathbf{1_I} \oplus \mathbf{1_R} \oplus \mathbf{1_R}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}$ | $\mathbf{1_I} \oplus \mathbf{1_I} \oplus \mathbf{1_I^*}$ | $\begin{bmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 \end{bmatrix}$ |
| $\mathbf{1_I} \oplus \mathbf{1'_I} \oplus \mathbf{1_R}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_1 \end{bmatrix}$ | | |

Table 7.6: Qualitatively distinct forms allowed for $M_{eff}$ depending on the $L$ representation ($\psi_L \to L_i\psi_L$,$\psi_R \to R_i\psi_R$). The condition $M_{eff} = M_{eff}^T$ was taken into account. Note that the primes in a complex irrep, such as in $\mathbf{1'_I}$, are meant to differentiate it from $\mathbf{1_I}$ **and** $\mathbf{1_I^*}$ so that $L = \mathbf{1_I} \oplus \mathbf{1_I^*} \oplus \mathbf{1_R}$ can't be seen as a special case of $L = \mathbf{1_I} \oplus \mathbf{1'_I} \oplus \mathbf{1_R}$.

# Chapter 8

# Models with more than one Higgs doublet

Dealing with just one invariant Higgs doublet is much simpler than considering many of them. The reason is simple: with one Higgs, calculating the invariant mass matrices is equivalent to finding trivial representations in Kronecker products of two irreps while with multiple Higgs we find ourselves calculating invariants in the Kronecker products of three irreps. In this latter case there is the additional problem, which I will not address, of finding the vevs of the Higgs fields.

In the previous chapter, the relative simplicity of single Higgs models motivated the study of many scenarios that varied in the choice of groups used and the way they were made to act on the fields. With the insight gained, I will now restrict the analysis to the most promising scenario - using a single group $G$ from which all irreps are picked, and making $G$ act simultaneously on all multiplets - $\psi_L$, $\psi_R$, and $\Phi$ (the multiplet of Higgs doublets), i.e., the Lagrangian must be invariant under the transformation

$$\psi_L \rightarrow \psi'_L = L_i \psi_L \tag{8.1}$$
$$\psi_R \rightarrow \psi'_R = R_i \psi_R \tag{8.2}$$
$$\Phi \rightarrow \Phi' = H_i \Phi \tag{8.3}$$

where $R$, $L$ and $H$ are all representations of group $G$. In contrast to $R$ and $L$, $H$'s dimension is arbitrary since a priori we are not setting restrictions on the number of Higgs doublets.

## 8.1 Paradigm of the analysis

For $n$ Higgs doublets, there will be several $3 \times 3$ Yukawa matrices $M_i$ [1]$(i = 1, ..., n)$ that contribute to the Dirac mass matrix $M_D$:

$$\mathscr{L}_Y = \sum_{i=1}^{n} \overline{\psi}_L M_i \psi_R \Phi_i + \text{h.c.} \tag{8.4}$$

Since we are going to avoid the problem of finding the Higgs vevs, all we need to consider is that there is a Higgs potential $V(\Phi)$ which is minimized for[2]

$$\langle \Phi \rangle = \begin{bmatrix} \langle \Phi_1 \rangle \\ \vdots \\ \langle \Phi_n \rangle \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} c_1^u \\ c_1^d \end{bmatrix} \\ \vdots \\ \begin{bmatrix} c_n^u \\ c_n^d \end{bmatrix} \end{bmatrix} \tag{8.5}$$

[1] I will often refer to the $M_i$'s as "mass matrices" although, strictly speaking, this is not true.

[2] The symmetry of the model connects different minimums of the Higgs potential. The practical consequences of this is discussed in appendix B (VI).

Not wanting to discriminate between $t_3 = +1/2$ and $t_3 = -1/2$ particles, I will henceforth use $c_i$ as a generic notation for the coefficients $c_i^u$ and $c_i^d$ so that the Dirac mass matrix is a linear combination of the $M_i$ Yukawa matrices, with coefficients $c_i$:

$$M_D = \sum_i c_i M_i \tag{8.6}$$

If we put the $M_i$ matrices and the $c_i$ coefficients in two vectors,

$$\mathbf{\Phi_0} \equiv \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \tag{8.7}$$

$$\mathbf{M} \equiv \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} \tag{8.8}$$

then

$$M_D = \mathbf{\Phi_0}^T \mathbf{M} \tag{8.9}$$

We need to work with $\mathbf{M}$, which is a $3n \times 3$ matrix. That involves some care. Take $Z$ to be some arbitrary $x \times y$ matrix. Then the $3x \times 3y$ matrix $\hat{Z}$ will be defined as

$$\hat{Z} = \begin{bmatrix} Z_{11}\mathbb{1}_{3\times3} & \cdots & Z_{1y}\mathbb{1}_{3\times3} \\ \vdots & \ddots & \vdots \\ Z_{x1}\mathbb{1}_{3\times3} & \cdots & Z_{xy}\mathbb{1}_{3\times3} \end{bmatrix} \tag{8.10}$$

where $\mathbb{1}_{3\times3}$ is the $3 \times 3$ identity matrix. It is also useful to take in a $x \times y$ matrix $Z$ and build the $3x \times 3y$ matrix $\tilde{Z}$

$$\tilde{Z} = \begin{bmatrix} Z & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Z \end{bmatrix} \tag{8.11}$$

Ignoring the Dirac and $SU(2)_L$ structure of the fields, which are of no importance to us, the Yukawa terms may be written as

$$\begin{aligned} \mathscr{L}_Y &= \psi_L^\dagger \left( \begin{bmatrix} \Phi_1\mathbb{1}_{3\times3} & \cdots & \Phi_n\mathbb{1}_{3\times3} \end{bmatrix} \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} \right) \psi_R + \text{h.c.} \\ &= \psi_L^\dagger \hat{\Phi}^T \mathbf{M} \psi_R + \text{h.c.} \end{aligned} \tag{8.12}$$

When the spinors undertake the transformations 8.1-8.3,

$$\begin{aligned} \mathscr{L}'_Y &= \psi_L^\dagger L_i^\dagger \left( \begin{bmatrix} (H_i)_{11}\mathbb{1}_{3\times3} & \cdots & (H_i)_{1n}\mathbb{1}_{3\times3} \\ \vdots & \ddots & \vdots \\ (H_i)_{n1}\mathbb{1}_{3\times3} & \cdots & (H_i)_{nn}\mathbb{1}_{3\times3} \end{bmatrix} \begin{bmatrix} \Phi_1\mathbb{1}_{3\times3} \\ \vdots \\ \Phi_n\mathbb{1}_{3\times3} \end{bmatrix} \right)^T \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} R_i \psi_R + \text{h.c.} \\ &= \psi_L^\dagger L_i^\dagger \hat{\Phi}^T \hat{H}_i^T \mathbf{M} R_i \psi_R + \text{h.c.} \\ &= \psi_L^\dagger \hat{\Phi}^T \hat{H}_i^T \tilde{L}_i^\dagger \mathbf{M} R_i \psi_R + \text{h.c.} \end{aligned} \tag{8.13}$$

Therefore, invariance of $\mathscr{L}_Y$ requires that

$$\mathbf{M} = \hat{H}_i^T \tilde{L}_i^\dagger \mathbf{M} R_i \tag{8.14}$$

To be perfectly clear, let's state explicitly the dimensions of these matrices:

- $\mathbf{M}$ is $3n \times 3$ resulting from stacking the $n$ Yukawa matrices $M_i$ in a vector;
- $\hat{H}_i$ and $\tilde{L}_i$ are commuting $3n \times 3n$ matrices derived in a natural way from $H_i$ and $L_i$;
- $R_i$ is the $3 \times 3$ matrix that appears in 8.2.

Given the $H$, $L$ and $R$ representations, we want $\mathbf{M}$ to calculate $M_D$. The problem with making statements on $\mathbf{M}$ without assuming any specific choice for the representations is that, unlike for 1 Higgs models, it is difficult to avoid doing the Kronecker product of $H$ and $L$ ,which is essentially the $\hat{H}_i^T \tilde{L}_i^\dagger$ factor in the last equation, and making its decomposition into irreps. That's something that amounts to evaluating Clebsch-Gordon coefficients. If that could be done in a general way, one would then be left with the simpler problem of having $\mathbf{M} = (\text{some irreps})\,\mathbf{M}\,(\text{some irreps}')$ which would be precisely a reduction to what we've seen for 1 Higgs models. Unfortunately finding the "some irreps" requires concrete knowledge of $H$ and $L$, which spoils our efforts to make general statements on $\mathbf{M}$.

Let's not forget $H_D$:

$$\begin{aligned} H_D &= M_D M_D^\dagger \\ &= \mathbf{\Phi_0}^T \mathbf{M} \mathbf{M}^\dagger \mathbf{\Phi_0^*} \end{aligned} \tag{8.15}$$

The $3n \times 3n$ matrix $\mathbf{M}\mathbf{M}^\dagger$ is subject to the condition

$$\mathbf{M}\mathbf{M}^\dagger = \left(\tilde{L}_i \hat{H}_i^*\right)^\dagger \mathbf{M}\mathbf{M}^\dagger \left(\tilde{L}_i \hat{H}_i^*\right) \tag{8.16}$$

Why not focus on $\mathbf{M}\mathbf{M}^\dagger$ instead of $\mathbf{M}$ much like in 1 Higgs models? That's because $\mathbf{M}\mathbf{M}^\dagger$ is potentially a much larger array of numbers than $\mathbf{M}$; $\mathbf{M}$ is a column vector of $3 \times 3$ matrices, meaning that $\mathbf{M}\mathbf{M}^\dagger$ is a matrix of matrices. And even by analyzing $\mathbf{M}\mathbf{M}^\dagger$ directly, we can't escape the problem of breaking the Kronecker product of $H$ and $L$ into irreps. The same can be said of $M_{eff}$ resulting from the seesaw mechanism. Since the Majorana mass matrix $M_R$ is associated with a term $\psi_R \psi_R$, it is immune to any consideration made about the Higgs fields so that we still get

$$M_R = R_i^T M_R R_i \tag{8.17}$$

and the observations of the last chapter apply here. As for a light mass matrix $M_{eff}$ resulting from the seesaw mechanism,

$$\begin{aligned} M_{eff} &= -M_D M_R^{-1} M_D^T \\ &= -\mathbf{\Phi_0}^T \mathbf{M} M_R^{-1} \mathbf{M}^T \mathbf{\Phi_0} \end{aligned} \tag{8.18}$$

The relevant constraint here is on $\mathbf{M} M_R^{-1} \mathbf{M}^T$:

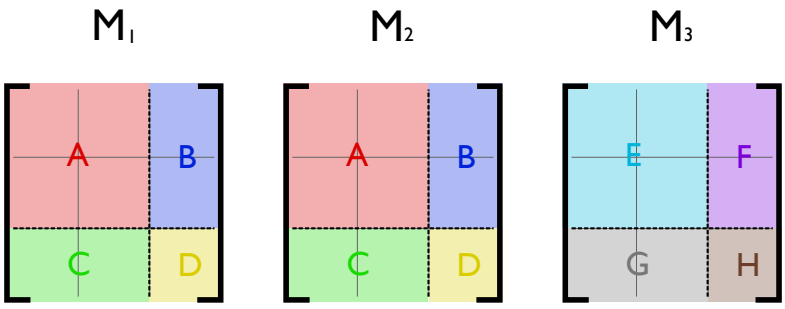
$$\mathbf{M} M_R^{-1} \mathbf{M}^T = \left(\tilde{L}_i^* \hat{H}_i\right)^T \mathbf{M} M_R^{-1} \mathbf{M}^T \tilde{L}_i^* \hat{H}_i \tag{8.19}$$

The same reason that discourages working directly with $\mathbf{M}\mathbf{M}^\dagger$ applies to $\mathbf{M} M_R^{-1} \mathbf{M}^T$ though.

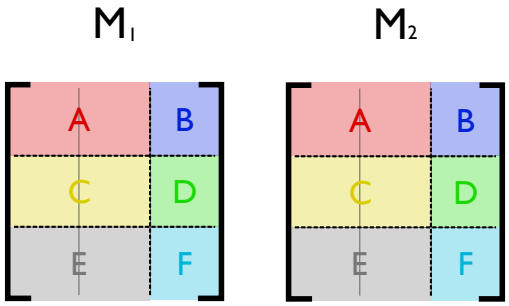
## 8.2 Properties of the $M_i$ matrices

Since the basis chosen for the $H$, $L$, and $R$ representations do not interfere with the physics of the model, we are free to use the basis that suits us best. And the natural choice once again is to have these representations in a reduced form so that the group irreps appear as blocks in $H$, $L$, and $R$'s diagonals. And just as for one Higgs models, where the representations break down into irreps provided a natural way to break the mass matrices $M$ in blocks $[M]_{\alpha\beta}$, for multiple Higgs something similar can be done. Now, for $n$ Higgs doublets there are $n$ mass matrices $M_i$ and the 'blocks' - which I shall call zones - can spread across several mass matrices. Figure 8.1 shows two examples.

One can best see the importance of this division of the mass matrices with an example. So let us consider the $S_3$ group again. For $L = H = R = \mathbf{1} \oplus \mathbf{2}$ the most general mass matrices $M_i$ are of the form




(a)




(b)

Figure 8.1: Two examples of division in zones of the $M_i$ matrices. In (a), $H = \mathbf{2} \oplus \mathbf{1}$, $L = \mathbf{2}' \oplus \mathbf{1}'$, and $R = \mathbf{2}'' \oplus \mathbf{1}''$. The connection between irreps and the various zones is as follows: zone A $= \mathbf{2} \otimes \mathbf{2}' \otimes \mathbf{2}''$, zone B $= \mathbf{2} \otimes \mathbf{2}' \otimes \mathbf{1}''$, zone C $= \mathbf{2} \otimes \mathbf{1}' \otimes \mathbf{2}''$, zone D $= \mathbf{2} \otimes \mathbf{1}' \otimes \mathbf{1}''$, zone E $= \mathbf{1} \otimes \mathbf{2}' \otimes \mathbf{2}''$, zone F $= \mathbf{1} \otimes \mathbf{2}' \otimes \mathbf{1}''$, zone G $= \mathbf{1} \otimes \mathbf{1}' \otimes \mathbf{2}''$, zone H $= \mathbf{1} \otimes \mathbf{1}' \otimes \mathbf{1}''$. In (b) it is used $H = \mathbf{2}$, $L = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$, and $R = \mathbf{2}' \oplus \mathbf{1}'''$ so that zone A $= \mathbf{2} \otimes \mathbf{1} \otimes \mathbf{2}'$, zone B $= \mathbf{2} \otimes \mathbf{1} \otimes \mathbf{1}'''$, zone C $= \mathbf{2} \otimes \mathbf{1}' \otimes \mathbf{2}'$, zone D $= \mathbf{2} \otimes \mathbf{1}' \otimes \mathbf{1}'''$, zone E $= \mathbf{2} \otimes \mathbf{1}'' \otimes \mathbf{2}'$, zone F $= \mathbf{2} \otimes \mathbf{1}'' \otimes \mathbf{1}'''$.

$$M_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & \lambda_3 \\ \lambda_4 & 0 & 0 \\ 0 & \lambda_5 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_5 \\ \lambda_4 & 0 & 0 \end{bmatrix} \tag{8.20}$$

for arbitrary $\lambda_i$'s. Notice how 'clean' and simple these mass matrices are when we use the representations in their reduced form. And more importantly for our present discussion is the fact that a given $\lambda_i$ is never present in two zones: $\lambda_1$ is in the $(H \otimes L \otimes R) = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ zone, $\lambda_2$ is in the $\mathbf{1} \otimes \mathbf{2} \otimes \mathbf{2}$ zone, $\lambda_3$ is in the $\mathbf{2} \otimes \mathbf{1} \otimes \mathbf{2}$ zone, $\lambda_4$ is in the $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{1}$ zone and finally $\lambda_5$ is in the $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ zone. This is easily understandable if we see things as follows. Each of these $\lambda_i$'s marks the presence of an invariant in the direct representation $H \otimes L \otimes R$ since each of them is the leading coefficient of some invariant. Through the rest of the chapter, this '$\lambda$ notation' will continue to be used. As an example, in 8.20 $\lambda_1$ indicates that $\Phi_1 \psi^*_{L\,1} \psi_{R\,1}$ is a singlet, while the placement of $\lambda_5$ indicates that $\Phi_2 \psi^*_{L\,3} \psi_{R\,2} + \Phi_3 \psi^*_{L\,2} \psi_{R\,3}$ is an invariant too. Imagine then a hypothetical $\lambda_6$ that would be in first row of $M_2$ and in the second row of $M_3$. That would mean that $\Phi_\alpha \psi^*_{L\,1} \psi_{R\,\beta} + \Phi_\gamma \psi^*_{L\,2} \psi_{R\,\delta}$ for some $\alpha, \beta, \gamma, \delta$ would be an invariant. The problem here is that, since $L = \mathbf{1} \oplus \mathbf{2}$, the symmetry never mixes $\psi_{L\,1}$ with $\psi_{L\,2}$ and $\psi_{L\,3}$ so that $\Phi_\alpha \psi^*_{L\,1} \psi_{R\,\beta}$ is changed to something proportional to $\Phi_{\alpha'} \psi^*_{L\,1} \psi_{R\,\beta'}$ for some $\alpha'$ and $\beta'$. Even if $\Phi_\alpha \psi^*_{L\,1} \psi_{R\,\beta} + \Phi_\gamma \psi^*_{L\,2} \psi_{R\,\delta}$ is an invariant, it is pointless to consider it a basic invariant.

In essence, the division in zones of the matrices $M_i$ define regions that do not mix under the transformation 8.1-8.3, so that the search for invariants can be made by looking individually at each of these zones separately. Of course, if $I_1$ and $I_2$ are two invariants each in its separate zone, for instance $I_1 + I_2$ is also invariant at it spans across two zones. As a concrete example, we can take the last three $M_i$ matrices and make the perfectly valid substitution $\lambda_3 \to \lambda'_3 = \lambda_3 + \lambda_5$:

$$M_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & \lambda_3 + \lambda_5 \\ \lambda_4 & 0 & 0 \\ 0 & \lambda_5 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & \lambda_3 + \lambda_5 & 0 \\ 0 & 0 & \lambda_5 \\ \lambda_4 & 0 & 0 \end{bmatrix} \tag{8.21}$$

Expressing things in this way puts $\lambda_5$ in two zones. So, what I'm stating here is not that we can't express the model's invariants in such a way that some of them span across different zones but rather that we can always choose the invariants so that every one of them is contained in a single zone (as in 8.20).

Before venturing any further, one question that may come to mind is how many $\lambda_i$'s can exist in each zone, i.e., how many invariants exist in the direct product representation $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ where $x, y$, and $z$ are the dimensions of the irreps. The exact number requires that we know more than just the dimensions $x, y$, and $z$. However, we can figure out a simple upper limit. This limit is somewhat intuitive and can be nicely formalized using character theory. As explained in the appendix V, the number of times $s_{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}}$ the trivial representation $\mathbf{1}$ is present in $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ is given by

$$s_{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}} = \sum_i \chi_i^{\mathbf{1}*} \chi_i^{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}} \tag{8.22}$$

where $\chi_i^R$ are the characters of some representation $R$ and the sum is taken over the conjugacy classes of the group (I've been calling it group $G$). Now, $\chi_i^{\mathbf{1}} = 1$ for any class and the character of the direct product representation $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ is just the product of the characters of $\mathbf{x}$,$\mathbf{y}$, and $\mathbf{z}$. So,

$$s_{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}} = \sum_i \chi_i^{\mathbf{x}} \chi_i^{\mathbf{y}} \chi_i^{\mathbf{z}} \tag{8.23}$$

This expression is very interesting because it can be read in four different way. The left side of this equation expresses:

1. The number of trivial $\mathbf{1}$ irreps inside $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$;
2. The number of $\mathbf{x}^*$ irreps inside $\mathbf{y} \otimes \mathbf{z}$;
3. The number of $\mathbf{y}^*$ irreps inside $\mathbf{x} \otimes \mathbf{z}$;
4. The number of $\mathbf{z}^*$ irreps inside $\mathbf{x} \otimes \mathbf{y}$.

While viewpoint 1 doesn't give us a clue for an upper limit on $s_{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}}$ (besides the obvious $xyz$), viewpoints 2, 3 and 4 are more fruitful in this regard. I'll use the second viewpoint as an example. Since the dimensions of the representations $\mathbf{x}^*$ and $\mathbf{y} \otimes \mathbf{z}$ are $x$ and $yz$ respectively, it is obvious that $\mathbf{x}^*$ cannot be contained in $\mathbf{y} \otimes \mathbf{z}$ more than $yz/x$ times. Indeed, we can refine this upper limit to the integer number equal or smaller than $yz/x$ - int $\left(\frac{yz}{x}\right)$. Extending this thought to viewpoints 3 and 4 we find the upper limit for $s_{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}}$ we sought:

$$s_{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}} \leq \text{int}\left[\frac{xyz}{\max(x,y,z)^2}\right] \tag{8.24}$$

For $n$ (the number of Higgs doublets) smaller or equal to 3, all possible cases cases that can arise in $H \otimes L \otimes R$ are contemplated in table 8.1. Looking back at figure 8.1 one finds from this consideration alone the maximum number of invariants in zone A is 2. For zones B, C, E, and H this number is 1 while zones D, F,and G can't contain any invariants at all. For our $S_3$ example in equation 8.20, these limits on the number of $\lambda_i$'s alone would only give space of an extra $\lambda_6$ in $\lambda_5$'s zone! Equally interesting is that $(M_1)_{12} = (M_1)_{13} = (M_1)_{21} = (M_1)_{31} = (M_2)_{11} = (M_3)_{11} = 0$ is also a consequence of these upper limits on the number of invariants in each zone.

| Representation | Maximum number of invariants | Representation | Maximum number of invariants |
|---|---|---|---|
| $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ | 1 | $\mathbf{1} \otimes \mathbf{3} \otimes \mathbf{3}$ | 1 |
| $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{2}$ | 0 | $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ | 2 |
| $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{3}$ | 0 | $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{3}$ | 1 |
| $\mathbf{1} \otimes \mathbf{2} \otimes \mathbf{2}$ | 1 | $\mathbf{2} \otimes \mathbf{3} \otimes \mathbf{3}$ | 2 |
| $\mathbf{1} \otimes \mathbf{2} \otimes \mathbf{3}$ | 0 | $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ | 3 |

Table 8.1: Upper limits on the number of invariants for some direct product representations.

Let us now establish some restrictions on the structure of these zones. From 8.40,

$$\mathbf{M}^\dagger\mathbf{M} \quad = \quad R_i^\dagger\mathbf{M}^\dagger\mathbf{M}R_i \tag{8.25}$$

which is a familiar type of restriction, this time on $\mathbf{M}^\dagger\mathbf{M} = \sum_i M_i^\dagger M_i$. In our old block notation,

$$\left[\mathbf{M}^\dagger\mathbf{M}\right]_{\alpha\beta} \quad = \quad \begin{cases} 0 & \text{if } [R_i]_{\alpha\alpha} \neq [R_i]_{\beta\beta} \\ c_{\alpha\beta}\mathbb{1} & \text{if } [R_i]_{\alpha\alpha} = [R_i]_{\beta\beta} \end{cases} \tag{8.26}$$

with $c_{\alpha\beta}$ being a constant. Take figure 8.1's example (a) where $L = H = R = \mathbf{2} \oplus \mathbf{1}$. For simplicity, I'll use the letters $A$ to $H$ when referring to the matrices that make ups the different zones. As for zones $A$ to $D$, since they are divided in two parts (one in $M_1$ and another in $M_2$), I'll add a subscript '1' and '2' to differentiate between these blocks so that

$$M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad M_3 = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \tag{8.27}$$

and $\mathbf{M}^\dagger\mathbf{M}$ is equal to

$$\begin{bmatrix} A_1^\dagger A_1 + C_1^\dagger C_1 + A_2^\dagger A_2 + C_2^\dagger C_2 + E^\dagger E + G^\dagger G & A_1^\dagger B_1 + C_1^\dagger D_1 + A_2^\dagger B_2 + C_2^\dagger D_2 + E^\dagger F + G^\dagger H \\ B_1^\dagger A_1 + D_1^\dagger C_1 + B_2^\dagger A_2 + D_2^\dagger C_2 + F^\dagger E + H^\dagger G & B_1^\dagger B_1 + D_1^\dagger D_1 + B_2^\dagger B_2 + D_2^\dagger D_2 + F^\dagger F + H^\dagger H \end{bmatrix} \tag{8.28}$$

This should be equal to

$$\begin{bmatrix} c_{11}\mathbb{1}_{2\times 2} & \mathbf{0} \\ \mathbf{0} & c_{22}\mathbb{1}_{1\times 1} \end{bmatrix} \tag{8.29}$$

The key here is to realize that these are not just four restriction. Since the $\lambda$s in each zone are arbitrary numbers and are independent from other $\lambda$s, we can effectively set each zone to zero and the above restriction should still hold. In particular, we can consider just one non-null zone at a time. For the two diagonal blocks, this means that

$$A_1^\dagger A_1 + A_2^\dagger A_2,\ C_1^\dagger C_1 + C_2^\dagger C_2,\ E^\dagger E,\ G^\dagger G \quad \propto \quad \mathbb{1}_{2\times 2} \tag{8.30}$$

$$B_1^\dagger B_1 + B_2^\dagger B_2,\ D_1^\dagger D_1 + D_2^\dagger D_2,\ F^\dagger F,\ H^\dagger H \quad \propto \quad \mathbb{1}_{1\times 1} \tag{8.31}$$

and, for the non diagonal blocks, leaving two non-null zones at a time,

$$A_1^\dagger B_1 + A_2^\dagger B_2,\ C_1^\dagger D_1 + C_2^\dagger D_2,\ E^\dagger F,\ G^\dagger H \quad = \quad \mathbf{0} \tag{8.32}$$

These diagonal conditions provide strong constraints on the structure of each zone and the non diagonal conditions go even further and relate the structure of different zones. But we have only used the $R$ representation. Surely we can extract from $H$ and $L$ some analogous conditions.

Having provided in matricial form some motivation on how to get constraints on the structure of each zone of $\mathbf{M}$, we will now venture into tensorial notation to get the full constraints provided by $L$, $H$ and $R$. To avoid unnecessary clutter, I will drop the $i$ index in the $L_i$, $H_i$, and $R_i$ matrices since it is irrelevant to keep track of the exact matrix of the representation that is being used. First, let us express the unitary nature of these matrices in terms of their components:

$$L_{ij}L_{kj}^* \quad = \quad \delta_{ik} \tag{8.33}$$

$$H_{ij}H_{kj}^* \quad = \quad \delta_{ik} \tag{8.34}$$

$$R_{ij}R_{kj}^* \quad = \quad \delta_{ik} \tag{8.35}$$

Einstein's summation convention is being used. Next, we shall define $m_{ijk} \equiv (M_j)_{ik}$ so that

$$\mathscr{L}_Y \quad = \quad m_{ijk}\psi^*_{L\,i}\Phi_j\psi_{R\,k} + \text{h.c.} \tag{8.36}$$

and the transformation 8.1-8.3 imposes the following condition on the $m_{ijk}$ coefficients:

$$m_{ijk} \quad = \quad m_{\alpha\beta\gamma}L^*_{\alpha i}H_{\beta j}R_{\gamma k} \tag{8.37}$$

Multiplying each side of the equation by its complex conjugate, contracting on the indexes $i$ and $j$ and using the unitarity nature of $L$, $H$, and $R$ one arrives at the condition

$$R_{ij}m^*_{\alpha\beta j}m_{\alpha\beta k} \quad = \quad m^*_{\alpha\beta i}m_{\alpha\beta j}R_{jk} \tag{8.38}$$

In matricial notation and introducing the matrix $C^R$ such that $\left(C^R\right)_{ij} = m^*_{\alpha\beta i}m_{\alpha\beta j}$ this can be written as

$$RC^R \quad = \quad C^R R \tag{8.39}$$

The application of Schur's lemmas here is to be taken together with what was stated some paragraphs above about the independence of different zones. We end up with

$$\sum_\alpha{}' \sum_\beta{}' m^*_{\alpha\beta j}m_{\alpha\beta k} \quad = \quad \begin{cases} c\delta_{jk} & \text{if } \psi_{j\,R} \text{ and } \psi_{k\,R} \text{ are associated to two equal irreps of } R \\ 0 & \text{if } \psi_{j\,R} \text{ and } \psi_{k\,R} \text{ are associated to two different irreps of } R \end{cases} \tag{8.40}$$

where $c$ is some constant. I've made the summation explicit, with a prime, to remind us that $\alpha$ and $\beta$ need not be summed over all allowed values. Indeed, they should only be summed over the values correspondent to one irrep. Consider this example: if $L = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ , $H = \mathbf{3} \oplus \mathbf{3}$, and $R = \mathbf{1} \oplus \mathbf{2}$, $\alpha$ is to be summed over the indexes $\{1\}$, $\{2\}$ or $\{3\}$ and $\beta$ over $\{1,2,3\}$ or $\{4,5,6\}$. As for the free indexes $(j,k)$, the choices $j \in \{1\}, k \in \{1\}$ and $j \in \{2,3\}, k \in \{2,3\}$ correspond to situations where $\psi_{R\,j}$ and $\psi_{R\,k}$ are associated to the same irrep of $R$ ($\mathbf{1}$ in the first case and $\mathbf{2}$ in the second). On the other hand, choices $j \in \{1\}, k \in \{2,3\}$ or $j \in \{2,3\}, k \in \{1\}$ correspond to situations where $\psi_{R\,j}$ and $\psi_{R\,k}$ are associated to different irreps of $R$. See figure 8.2 for a graphical view of this example.

If we multiply each side of the 8.37 by its complex conjugate and contract different pairs of indexes, we get similar constraints this time due to the $L$ and $H$ representations. So the full set of conditions on the $m_{ijk}$ coefficients are

$$R_{ij}m^*_{\alpha\beta j}m_{\alpha\beta k} \quad = \quad m^*_{\alpha\beta i}m_{\alpha\beta j}R_{jk} \tag{8.41}$$
$$H_{ij}m^*_{\alpha j\beta}m_{\alpha k\beta} \quad = \quad m^*_{\alpha i\beta}m_{\alpha j\beta}H_{jk} \tag{8.42}$$
$$L_{ij}m^*_{i\alpha\beta}m_{j\alpha\beta} \quad = \quad m^*_{i\alpha\beta}m_{j\alpha\beta}L_{jk} \tag{8.43}$$

resulting in

$$\sum_\alpha{}' \sum_\beta{}' m^*_{\alpha\beta j}m_{\alpha\beta k} \quad = \quad \begin{cases} c^R_{jk}\delta_{jk} & \text{if } \psi_{j\,R} \text{ and } \psi_{k\,R} \text{ are associated to two equal irreps of } R \\ 0 & \text{if } \psi_{j\,R} \text{ and } \psi_{k\,R} \text{ are associated to two different irreps of } R \end{cases} \tag{8.44}$$

$$\sum_\alpha{}' \sum_\beta{}' m^*_{\alpha j\beta}m_{\alpha k\beta} \quad = \quad \begin{cases} c^H_{jk}\delta_{jk} & \text{if } \Phi_j \text{ and } \Phi_k \text{ are associated to two equal irreps of } H \\ 0 & \text{if } \Phi_j \text{ and } \Phi_k \text{ are associated to two different irreps of } H \end{cases} \tag{8.45}$$

$$\sum_\alpha{}' \sum_\beta{}' m^*_{j\alpha\beta}m_{k\alpha\beta} \quad = \quad \begin{cases} c^L_{jk}\delta_{jk} & \text{if } \psi_{j\,L} \text{ and } \psi_{k\,L} \text{ are associated to two equal irreps of } L \\ 0 & \text{if } \psi_{j\,L} \text{ and } \psi_{k\,L} \text{ are associated to two different irreps of } L \end{cases} \tag{8.46}$$

It is important to note that Schur's lemmas make the constant $c^R_{jk}$ ($c^H_{jk}$ or $c^L_{jk}$) the same as $c^R_{j'k'}$ ($c^H_{j'k'}$ or $c^L_{j'k'}$) if $\psi_{j\,R}$ and $\psi_{j'\,R}$ ($\Phi_j$ and $\Phi_{j'}$; or $\psi_{j\,L}$ and $\psi_{j'\,L}$) as well as $\psi_{k\,R}$ and $\psi_{k'\,R}$ ($\Phi_k$ and $\Phi_{k'}$; or $\psi_{k\,L}$ and $\psi_{k'\,L}$) refer to the same irrep of $R$ ($H$ or $L$). Notice also that for different summation domains of $\alpha$ and $\beta$,

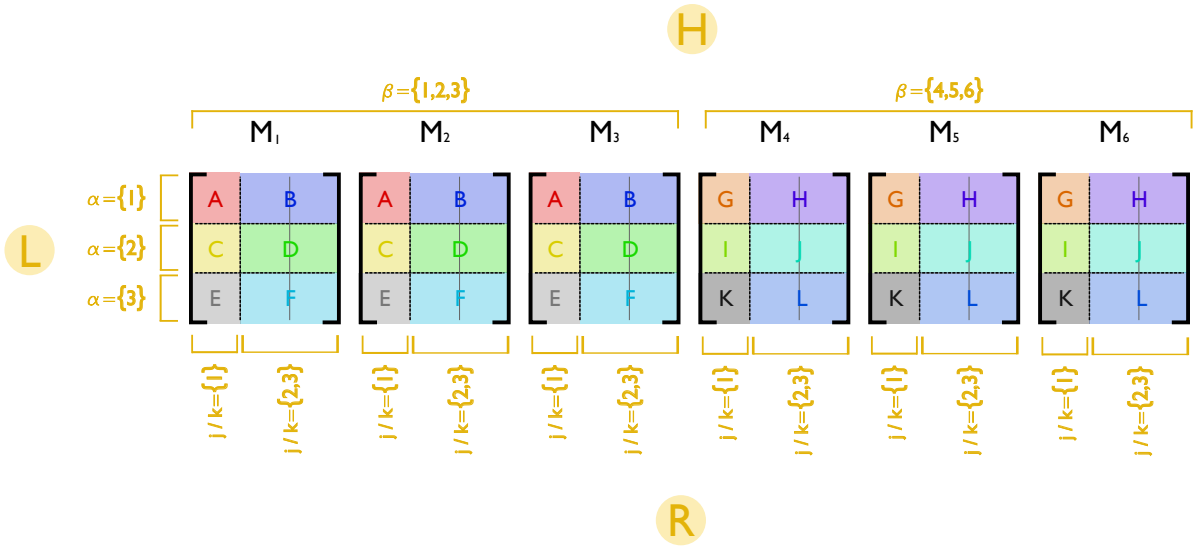



Figure 8.2: In this example $L = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ , $H = \mathbf{3} \oplus \mathbf{3}$, and $R = \mathbf{1} \oplus \mathbf{2}$. In equation 8.40 $\alpha$ is summed over the sets $\{1\}$, $\{2\}$ or $\{3\}$ and $\beta$ is summed over $\{1,2,3\}$ or $\{4,5,6\}$ resulting in a total of $3 \times 2 = 6$ possible summations. When $(j,k) = (1,1),(2,2),(2,3),(3,2)$ or $(3,3)$ the top condition on the right side of 8.40 is valid ('$\psi_{R\,j}$ and $\psi_{R\,k}$ are associated to the same irrep of $R$'). Otherwise the sum $\sum_\alpha{}' \sum_\beta{}' m^*_{\alpha\beta j} m_{\alpha\beta k}$ is null.

the corresponding constants $c^R_{jk}$ ($c^H_{jk}$ or $c^L_{jk}$) are not equal; different summation domains of $\alpha$ and $\beta$ are associated with independent $c^R_{jk}$ ($c^H_{jk}$ or $c^L_{jk}$) constants.

These constraints are easily seen as orthonormality relations between the entries of the $M_i$ matrices. The notation and conditions on the indexes used in these equations can be simplified by restricting our attention to a subset of these equations. Recalling example (a) in figure 8.1, we have written explicitly the restrictions coming from the $R$ representation in terms of matrix blocks (8.30-8.32). While restrictions 8.32 involve 2 matrix zones at a time, 8.30 and 8.31 are 'intra-zone' restrictions. Considering only these 'intra-zone' restrictions on some zone $z$ of $\mathbf{M}$, the last three equations collapse to

$$\sum_{\alpha,\beta}^{\text{over zone } z} m^*_{\alpha\beta i} m_{\alpha\beta j} = c^R_z \delta_{ij} \quad i,j \in \text{zone } z \tag{8.47}$$

$$\sum_{\alpha,\beta}^{\text{over zone } z} m^*_{\alpha i\beta} m_{\alpha j\beta} = c^H_z \delta_{ij} \quad i,j \in \text{zone } z \tag{8.48}$$

$$\sum_{\alpha,\beta}^{\text{over zone } z} m^*_{i\alpha\beta} m_{j\alpha\beta} = c^L_z \delta_{ij} \quad i,j \in \text{zone } z \tag{8.49}$$

So, for every zone $z$ of $\mathbf{M}$ there are these 3 constants $c^R_z$, $c^H_z$, and $c^L_z$. Are they independent? Clearly not. Contracting the indexes $i$ and $j$ in all three equations, one is summing on the left side $m^*_{\alpha\beta\gamma} m_{\alpha\beta\gamma}$ over all the entries of zone $z$. If zone $z$ comes from a direct product $\mathbf{d_H} \otimes \mathbf{d_L} \otimes \mathbf{d_R}$ ($H \otimes L \otimes R$) where $d_H, d_L$, and $d_R$ are the irreps dimensions, then

$$d_H c^H_z = d_L c^L_z = d_R c^R_z \equiv c_z \tag{8.50}$$

so

$$\sum_{\alpha,\beta}^{\text{over zone } z} m^*_{\alpha\beta i} m_{\alpha\beta j} = \frac{c_z}{d_R}\delta_{ij} \quad i,j \in \text{zone } z \tag{8.51}$$

$$\sum_{\alpha,\beta}^{\text{over zone } z} m^*_{\alpha i\beta} m_{\alpha j\beta} = \frac{c_z}{d_H}\delta_{ij} \quad i,j \in \text{zone } z \tag{8.52}$$

$$\sum_{\alpha,\beta}^{\text{over zone } z} m^*_{i\alpha\beta} m_{j\alpha\beta} = \frac{c_z}{d_L}\delta_{ij} \quad i,j \in \text{zone } z \tag{8.53}$$

These are **orthonormality relations** of a zone's entries[3]. The natural way to see it is shown in figure 8.3 where I present the $M_i$ matrices invariant for the choice $L^* = R = H = \mathbf{3_a}$ with $\mathbf{3_a}$ being one of the two 3-dimensional irreps of the $\Delta(27)$ group so that there is just one single zone that comprises 27 matrix entries in total[4].

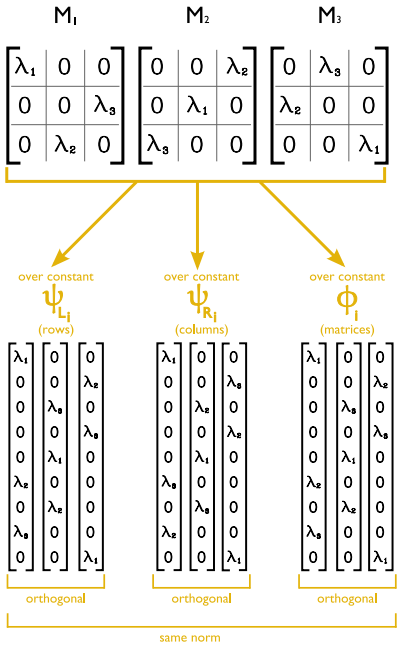



Figure 8.3: In the top are the three $M_i$ matrices that make up $\mathbf{M}$ for the choice of group $G = \Delta(27)$ and with $L^* = R = H = \mathbf{3_a}$. There are three different ways of stacking $\mathbf{M}$'s entries to make triplets of 9-dimensional vectors: for some $i = 1, 2, 3$ we may collect all entries associated with $\psi_{L\,i}$, $\psi_{R\,i}$ or $\Phi_i$. The resulting triplets of vectors are orthogonal and share the same norm between themselves. Actually the norm of all vectors is the same across triplets of vectors ($= |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2$) since the dimensions of the $L$, $H$ and $R$ representations are the same.

While placing strong restrictions on the structure of each zone of $\mathbf{M}$, these relations by themselves do not explain why we often get very 'clean' mass matrices. By this I mean two things:

1. The $\lambda$'s appear in the entries of $M_j$ multiplied by a coefficient with absolute value one;
2. The invariants appear to be 'untangled', i.e., in a given entry of the $M_j$'s we either find 0 or something proportional to some $\lambda$ but never a linear combination of two or more $\lambda$'s.

This is curious indeed. I shall first give explicit examples. For instance, the mass matrices

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 2\lambda_1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} -2\lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} \tag{8.54}$$

[3] Clearly, these relations do not impose unit norm on the vectors built from the $m_{ijk}$'s. Nevertheless, I'll use the expression 'orthonormality relations' since they do impose condition on the norms of these vectors.

[4] This is a peculiar example because there are 3 invariants inside this zone, the maximum allowable for $3 \times 3 \times 3$ zone.

would be allowed under the orthonormality relations (assuming just two families of fermions) but they go against observation one. As for the second observation,

$$M_1 = \begin{bmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} \tag{8.55}$$

would be forbidden but not

$$M_1 = \begin{bmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{bmatrix} \tag{8.56}$$

and

$$M_1 = \begin{bmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_1 - \lambda_2 \end{bmatrix} \tag{8.57}$$

In the last two cases, we can redefine the $\lambda$'s in such a way that the invariants represented by $\lambda_1$ and $\lambda_2$ get 'separated'. All that it takes is the substitution $(\lambda_1, \lambda_2) \to \left(\frac{\lambda_1+\lambda_2}{2}, \frac{\lambda_1-\lambda_2}{2}\right)$ which leads to

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_1 \end{bmatrix} \tag{8.58}$$

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{8.59}$$

respectively. This transformation have clarified the invariant's structure and, in the first equation in particular, it clearly shows that the two initial $\lambda$'s actually stand for one invariant only.

This freedom to redefine the $\lambda$'s means that we should be careful formalizing the above observations. We wish to show that with a suitable definition of the $\lambda$'s we *can* put $\mathbf{M}$ in such a form that all its entries $m_{ijk}$ have the form

$$m_{ijk} \quad = \quad \begin{cases} 0 & \\ k_{ijk}\lambda_l\,, & |k_{ijk}| = 1 \end{cases} \qquad \forall_{i,j,k} \tag{8.60}$$

for some $\lambda_l$s. While I am unable to cast it into a proof for all irreps, this statement is valid for many of them. It is true at least for the 92 groups with less than 31 elements and their 1147 irreducible representations. Mathematically then, when is this true? The statement is true whenever the representations $L$, $H$ and $R$, and consequently $H \otimes L \otimes R$, are such that every one of its matrices have only one non-null entry per row and column. As an example, $\Delta(27)$'s $\mathbf{3_a}$ representation can be generated with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e(3) & 0 \\ 0 & 0 & e(3)^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{8.61}$$

Take $X$ to be a representation of a group $G$ possessing this characteristic and $\psi$ being a multiplet of fields that transforms according to it. All of $X$'s invariants are found by computing the (orthogonal) projection matrix $P = \sum_{g_i \in G} X(g_i)$ and applying this operator to the various components of $\psi$ ($\psi_i$). This is obviously an invariant, for all $i$. And if we go over all $\psi_i$'s there is no doubt that we will find all invariants that can be built from the components of $\psi$ (some of them may appear more than once, while for some $i$s $P\psi_i$ may be null). For simplicity and without loss of generality, let's assume that a term in $\psi_1$ is in the invariant $I \propto P\psi_1$. This invariant is normalized so that $\psi_1$ appears in it with a coefficient 1. Since all $X_i \equiv X(g_i)$ contain one non-null component for each row and column we know that applying any $X_i$ to some $\psi_j$ results in $\alpha_{jk}\psi_k$ (no sum here). From $X_i$'s unitarity, the $\alpha_{jk}$'s have absolute value equal to one. So, if $I$ is an invariant by applying $X_i$ with some $i$ , $\psi_1 \to \alpha_{1k}\psi_k$ and since no other component of $\psi$ gets sent to $\psi_k$ in this transformation (it's just $\psi_1$) the term $\alpha_{1k}\psi_k$ must show up in $I$. And this term gets sent to $\alpha_{1k}\alpha_{kl}\psi_l$ which must also appear in $I$. And so on, until we get back to $\psi_1$. If we do this for all $X_i$ matrices we go over all the terms in $I$. So, if we write

$$I \quad = \quad \sum_j \beta_j \psi_j \tag{8.62}$$

where $\beta_1 = 1$ by our normalization then

1. All the participating $\psi_j$ (those with $\beta_j \neq 0$) can't be part in another invariant $I'$ since, as we have seen, the presence of one $\psi_j$ 'drags' into the invariant all the $\psi_k$s to which it can be transformed into. In fact, the sum of these terms is proportional to $I$ itself;

2. For all $j$, $|\beta_j| = 1$ due to the representation's unitarity.

If $X$ is the direct product of 3 irreps of $L$, $H$ and $R$, these two observations prove what we were looking after: with proper normalizations, the coefficient of an invariant are all unitary (observation two) and the various invariants can be chosen in such a way that they do not share terms $\Phi_i \psi^*_{L\,j} \psi_{R\,k}$ (observation one). I'll call these the **chain relations** of the $m_{ijk}$s.

One interesting result comes from the combination of the orthonormality relations and the chain relations. Consider zone $z$ that comes from the irrep product $\mathbf{d_H} \otimes \mathbf{d_L} \otimes \mathbf{d_R}$ ($H \otimes L \otimes R$) where $d_H, d_L$, and $d_R$ are the irreps' dimensions. We just need to consider one invariant, i.e. only one particular $\lambda_i$. From the chain relations the square of the norm of any vector that we make up the $m_{ijk}$ entries is $|\lambda_i|^2$ times the number of occurrences of $\lambda_i$ in the vector. That is "norm$^2$ $\propto$ number of $\lambda_i$'s". This is very curious since the orthonormality relations impose conditions between the norms of the vectors built from the $m_{ijk}$s, like the 9 vectors in figure 8.3. What these conditions state is that the square norm of the vectors built by picking all the $m_{ijk}$ associated with one component of irrep $\mathbf{d_H}$ ($h$-vectors),$\mathbf{d_L}$ ($l$-vectors), or $\mathbf{d_R}$($r$-vectors) are all related: $|\text{h-vectors}|^2 d_H = |\text{l-vectors}|^2 d_L = |\text{r-vectors}|^2 d_R$ . This means that the total number of $\lambda_i' s$ that show up in zone $z$ (and in all of $\mathbf{M}$) is a multiple of $\text{lcm}\,(d_H, d_L, d_R)$ (lcm = least common multiple). Otherwise these relations between norms can't be satisfied (figure 8.4).

Again, the chain relations rest on the assumption that the various irreps can be put in a basis such that their matrices have one non-null entry per row and column. This is possible for most for the irreps of groups with less than 31 elements[5] and there is no reason, that I know of, to believe that for higher order groups there is a dramatic change in this regard. So I'll be assuming that the irreps used do indeed have this property and are in this special base.

## 8.3 Linear combinations of $M_i$s

Now that we know some properties of the $M_i$ matrices, we wish the see what happens when we take linear combinations of these matrices since the Dirac mass matrix $M_D$ is given by

$$M_D \quad = \quad {\mathbf{\Phi_0}}^T \mathbf{M} = \sum_i c_i M_i \tag{8.63}$$

where nothing is known about the $c_i$'s. Some information is lost in the process of going from $\mathbf{M}$ to $M_D$. There are two particularly important situations. On possibility is that the $c_i$s take some special form as a consequence of the symmetry of the Higgs potential. Say, it could be that $c_1 = c_2 = \cdots = c_n$ or $c_1 = 1$ and $c_2 = \cdots = c_n = 0$. This could be a problem since $c_i$s with special forms could conceivably erase non-null entries of the $M_i$s from $M_D$. That is what happens for the following case[6]:

$$M_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & e\,(3)\,\lambda_1 & 0 \\ 0 & 0 & e\,(3)^2\,\lambda_1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} -e\,(3)\,\lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -e\,(3)^2\,\lambda_1 \end{bmatrix} \tag{8.64}$$

[5]There are 92 district groups with less than 31 elements. These contain in total 1147 irreps: 1000 are 1-dimensional, 134 are 2-dimensional, 12 are 3-dimensional, and 1 is 4-dimensional. The matrices of the 1-dimensional irreps obviously have one non-null entry per row and column since there is just one column and one row! So there is just 147 irreps in this universe that could fail to have this property. Of these, only 3 2-dimensional irreps seem not to have a basis where the representations matrices all have just one non-null entry per row and column.

[6]The group used is (group order, group index) = (24, 12). The indexing of groups as well as irreps is done in accordance to the software "Groups, Algorithms, Programming" (GAP). $H = \text{Irrep}\,(24, 12, 3)$, $L = \text{Irrep}\,(24, 12, 3)$, $R = \text{Irrep}\,(24, 12, 3)$ [format: (group order, group index, irrep index)].

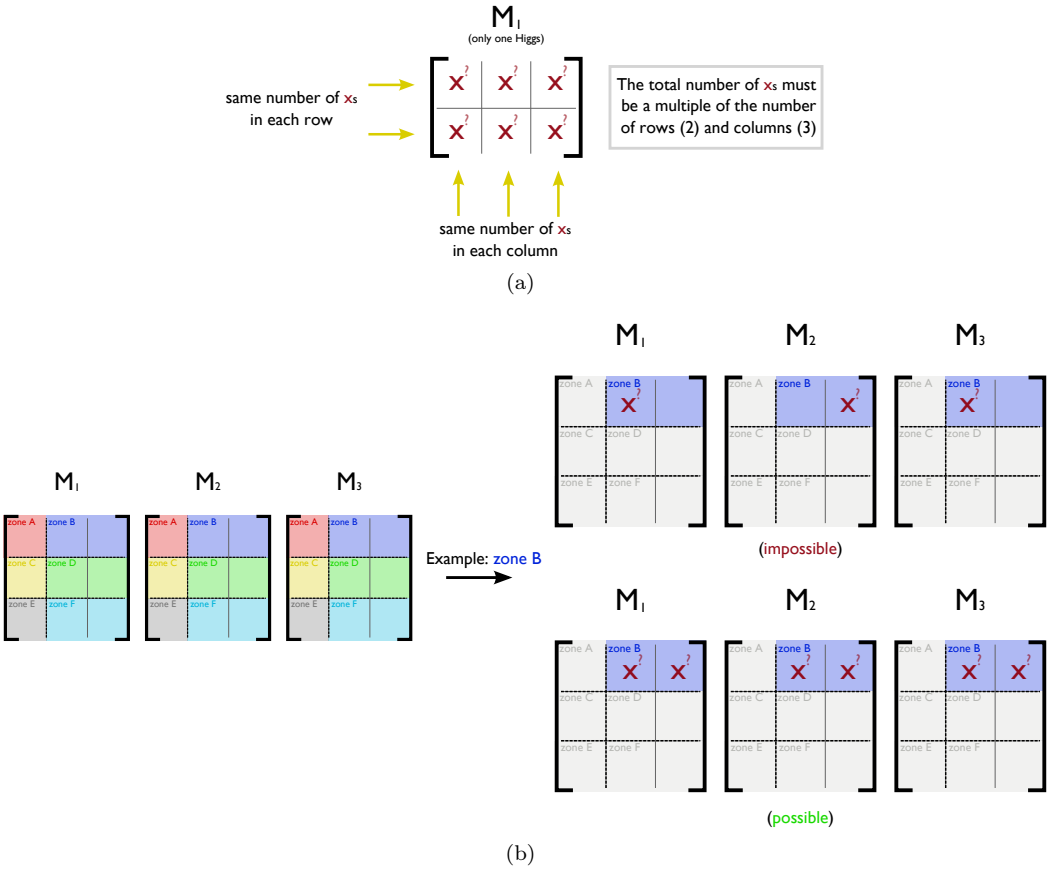



Figure 8.4: In (a) it is assumed that there is only one Higgs doublet and two left-handed particles. In this simplified example, one assumes that the whole matrix $M_1$ is a zone. The question is: for a particular invariant, how many of the associated $\lambda_i$s would show up in $M_1$? The chain relations state that, whenever a $\lambda_i$ shows up, its coefficient has absolute value 1. So, to calculate norms of row or columns vectors, we just need to consider where are the $\lambda_i$s (an 'x' is used for that). So, how many 'x's are there? According to the orthonormality relations, the number of 'x's in each row must be equal. The same is true for columns. So in this example, the number of 'x' must be a multiple of 2 and 3 at the same time, meaning that it must be a multiple of 6. In (b), a more practical examples is given. Focusing on zone B, one sees that the number of $\lambda_i$s (marked with 'x's) would have to be a multiple of the number of rows, columns, and matrices that zone B spans across. These are just two examples of the consequence of combining the chain and orthonormality relations since the dimensions of the zones considered in both (a) and (b) actually do not allow invariants (see table 8.1).

The values $c_1 = c_2 = 1$ would then conspire to erase the presence of $\lambda_1$ from $(M_D)_{33}$:

$$\begin{aligned} M_D &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & e(3)\lambda_1 & 0 \\ 0 & 0 & e(3)^2\lambda_1 \end{bmatrix} + \begin{bmatrix} -e(3)\lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -e(3)^2\lambda_1 \end{bmatrix} \\ &= [1-e(3)] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{8.65}$$

In $M_D$ the information on the dimension of the irreps of $H$ can be difficult to retrieve. But for the $L$ and $R$ representations, we could have been tempted to say that each consisted of the direct sum of a 2-dimensional irrep and a 1-dimensional irreps from the fact that $\lambda_1$ is on just two rows and two columns. That is nonetheless false and we would have detected that from $M_1$ and $M_2$ if we knew them.

Another possibility is that some $c_i$s are free parameters, i.e., they are not constrained by the discrete symmetry. As an example, using the above $M_1$ and $M_2$, arbitrary $c_1$ and $c_2$ lead to

$$M_D = \lambda_1 \begin{bmatrix} c_1 + e(3)c_2 & 0 & 0 \\ 0 & e(3)c_1 - c_2 & 0 \\ 0 & 0 & e(3)^2 c_1 - e(3)^2 c_2 \end{bmatrix} \tag{8.66}$$

which for all purposes is of the form

$$M_D = \begin{bmatrix} c_1' & 0 & 0 \\ 0 & c_2' & 0 \\ 0 & 0 & -i\sqrt{3}c_1' + c_2' \end{bmatrix} \tag{8.67}$$

Again, this could be misleading since it does not allow us to see if two different free parameters ($c_1'$ and $c_2'$) come from different invariants of if they are simply due to the $c_i$ coefficients. In this example, the first of these interpretations would wrongly lead us to think that the above $M_D$ matrix is impossible to obtain.

Having this in mind, we next analyze the possibility of obtaining tri-bimaximal mixing.

## 8.4 Is tri-bimaximal mixing possible?

Up to this point, we analyzed the Yukawa interactions of models with multiple Higgs doublets. It was crucial not no enter in the description of a particular model or indeed to make assumptions other than general ones. We have been able to get away with this so far. In this section though, the broadness and generality of the models we have been describing finally cause real problems, turning it impossible to say if tri-bimaximal mixing is obtainable or not from these symmetries[7]. I can give three reasons for this:

1. Not knowing the number of Higgs doublets and, even worse, knowing nothing about the $c_i$ coefficients makes it very difficult to eliminate candidate models;

2. I have pointed out that the representation's basis is irrelevant for the physics. But if we want to get tri-bimaximal mixing, we have to ask ourselves what are the mass matrices we seek and, for that, it is necessary to choose a particular weak basis. But noting assures us that this weak basis matches the basis that completely reduces the $L$ and $R$ representations;

3. Even if we agree on the weak basis that matches the 'irrep basis' implicit in $L$ and $R$, we can get tri-bimaximal mixing with various assumptions on the fermions masses. For instance, we may seek a 3 parameter neutrino mass matrix that leads to 3 independent masses, or we may look for a matrix with less parameters/independent masses. This latter case would be of greater interest since, if such a mass matrix could be obtained from a discrete symmetry, not only could we claim success in determining the fermions' mixing matrix but also some relations between the particles' masses.

[7] One should keep in mind that models with discrete family symmetries, in more complex settings and with ad hoc assumptions, have been show to lead to HPS.

I will elaborate on the second observation. If neutrinos are Dirac particles, in the $SU(2)_L$ gauge basis

$$M_D(l) = U_1 D_l U_2 \tag{8.68}$$
$$M_D(\nu) = U_1 V_{hps} D_\nu U_3 \tag{8.69}$$

where $U_1$, $U_2$, and $U_3$ are arbitrary unitary matrices and $D_l$, $D_\nu$ are diagonal matrices with the charged leptons and the neutrinos masses, respectively. Observation 2 relates to the arbitrariness in $U_1$, $U_2$, and $U_3$ and observation 3 to the parameters in $D_l$ and $D_\nu$. On the other hand, if neutrinos are Majorana particles then, in the flavour basis,

$$\left[M_D(\nu) M_R^{-1} M_D^T(\nu)\right]^* = (KV_{hps})^* D_\nu (KV_{hps})^\dagger \tag{8.70}$$

where $K$ is the unitary diagonal matrix carrying the two Majorana phases. I'll assume that $M_R^{-1}$ is factorizable into the form

$$M_R^{-1} \equiv \beta^\dagger D_R^{-1} \beta^* \tag{8.71}$$

where $\beta$ is unitary and $D_R$ is diagonal. Then,

$$\left[M_D(\nu)\beta^\dagger D_R^{-1/2}\right]\left[M_D(\nu)\beta^\dagger D_R^{-1/2}\right]^T = \left(KV_{hps}D_\nu^{1/2}\right)\left(KV_{hps}D_\nu^{1/2}\right)^T \tag{8.72}$$

The exponent $1/2$ in the diagonal matrices is used to indicate the matrix that, when squared, gives the original one, i.e., $\left(D_R^{-1/2}\right)^2 \equiv D_R^{-1}$ and $\left(D_\nu^{1/2}\right)^2 \equiv D_\nu$. From the last equation,

$$\begin{aligned} M_D(\nu)\beta^\dagger D_R^{-1/2} &= KV_{hps}D_\nu^{1/2}O \\ &\Leftrightarrow \\ M_D(\nu) &= KV_{hps}D_\nu^{1/2}OD_R^{1/2}\beta \end{aligned} \tag{8.73}$$

where $O$ is an arbitrary complex orthogonal matrix ($OO^T = \mathbb{1}$). So, in the $SU(2)_L$ gauge basis

$$M_D(l) = U_1 D_l U_2 \tag{8.74}$$
$$M_D(\nu) = U_1 K V_{hps} D_\nu^{1/2} O D_R^{1/2} \beta \tag{8.75}$$

In this case, the second difficulty noted above is in $U_1$, $U_2$, and $O$. Before moving on, I'll just comment on the forms of $\beta$ and $D_R^{1/2}$. For non-pathological choices of the $R$ representation, $M_R^{-1}$ can take the following forms (not considering possible permutations of lines and columns):

$$M_R^{-1} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \Rightarrow D_R^{1/2} = \begin{bmatrix} r_1^{1/2} & 0 & 0 \\ 0 & r_2^{1/2} & 0 \\ 0 & 0 & r_3^{1/2} \end{bmatrix}; \beta = \mathbb{1} \tag{8.76}$$

$$M_R^{-1} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & 0 & r_2 \\ 0 & r_2 & 0 \end{bmatrix} \Rightarrow D_R^{1/2} = \begin{bmatrix} r_1^{1/2} & 0 & 0 \\ 0 & r_2^{1/2} & 0 \\ 0 & 0 & r_2^{1/2} \end{bmatrix}; \beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{8.77}$$

The $r_i$s may or may not be distinct.

### 8.4.1 An example of the usefulness of the properties of the Yukawa interactions

No really meaningful analysis can be made without addressing the problems I have just mentioned. However, in order to exemplify the kind of analysis that it is possible to make when the form of the target mass matrices in the irreps' basis is known, I'll assume that

$$M_D(l) = D_l = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \tag{8.78}$$

$$M_D(\nu) = V_{hps} D_\nu V_{hps}^T = \begin{bmatrix} a & b & b \\ b & a+c & b-c \\ b & b-c & a+c \end{bmatrix} \tag{8.79}$$

A quick look at $M_D(l)$ would suggest that $L$ is $\mathbf{1} \otimes \mathbf{1'} \otimes \mathbf{1''}$. But that is not necessarily so. Since $x$, $y$, and $z$ are arbitrary parameters, $\mathrm{diag}(x, y, z)$ is equivalent to $\mathrm{diag}(x+y, x-y, z)$, which suggests $L = \mathbf{2} \otimes \mathbf{1}$, or $\mathrm{diag}(-x+y+z, x-y+z, x+y-z)$ which suggests $L = \mathbf{3}$. So, even a diagonal matrix can be treacherous. $M_D(\nu)$ gives us more information. In it, the $b$ parameter appears in all 3 rows and columns, giving away that $L = \mathbf{3}$ and $R_\nu = \mathbf{3'}$[8].

We know that $L$ is a single 3-dimensional irrep then. Going back to the charged leptons this suggests viewing $M_D(l)$ as $\mathrm{diag}(-x+y+z, x-y+z, x+y-z)$ which would imply $R_l = \mathbf{3''}$.

One is left figuring out the Higgs representation. Looking at $H \otimes L \otimes R_l = H \otimes \mathbf{3} \otimes \mathbf{3''}$, a 1-dimensional irrep in $H$ would give at most 1 invariant. A 2-dimensional irrep with the help of Higgs vevs $c_1$ and $c_2$ could appear at best as two independent parameters in $M_D(l)$. We have three parameters in $M_D(l)$ though ($x$, $y$, and $z$) . So, at least 3 Higgs are needed.

Let's summarize things. So far, one the dimensions of the representations we have seen the following:

- Unavoidable facts: $L = \mathbf{3}$; $R_\nu = \mathbf{3'}$;
- Strong possibilities: $R_l = \mathbf{3''}$; $H$ has 3 or more dimensions.

I won't go through all possibilities; I will just elaborate on this particular one.

Notice that in $M_D(\nu)$, the $c$ parameter spans over only two columns and two rows. It is impossible for the $M_i$s behind $M_D(\nu)$ not to have $c$s in the first row and column. This requires what I have called a conspiracy from the $c_i$ coefficient (the Higgs vevs) to make these $c$s disappear from $M_D(\nu)$. Let us focus then on the irrep of $H$ from which the $c$ parameter originates. A 1-dimensional irrep in $H$ won't do, since we would get just one Yukawa matrix $M_1$ from which we couldn't make these $c$s disappear. Not so obvious is the fact that a 2-dimensional irrep won't do the trick either. This is due to the orthonormality relations. I know of no such problem for a 3-dimensional irrep, but these are disfavored by the fact that we would expect $\mathrm{lcm}(3,3,3) = 3$ $c$s in total in the irrep product $\mathbf{3'''} \otimes \mathbf{3} \otimes \mathbf{3'}$ ($H \otimes L \otimes R$). We need to explain at least the 4 visible $c$s in $M_D(\nu)$ though. $H = \mathbf{4}$ would point to $\mathrm{lcm}(4,3,3) = 12$ $c$s (at least), 3 in each of the four resulting $M_i$s. Now, this would require some massive conspiracy from the coefficients $c_1$,$c_2$, $c_3$ and $c_4$ to hide in $M_D(\nu)$ 8 of these $c$s. Unfortunately, going up to 9-dimensional irreps (no invariant is allowed in $H \otimes \mathbf{3} \otimes \mathbf{3}$ for bigger irreps) does not produce better candidates. One way out would be to consider free Higgs expectation values $c_i$. For instance, that could make one single invariant of the $M_i$ matrices originate both $c$ and $b$ in the way shown in the last chapter.

When discussing and successfully finding a possible origin for the $c$ parameter, we can always deal with the other parameters ($a$ and $b$) by saying that $H$ is the direct sum of more than one irrep so that multiple zones get created, although this is not a minimal solution. In this way, it is easy to separately find an origin for the different parameters. The cost of such a procedure is loss of simplicity though. So, regardless of $c$, if $a$ and $b$ come from the same irrep of $H$ (i.e. same zone), why are there 3 $a$s and 6 $b$s? Only a 1-dimensional or a 3-dimensional irrep in $H$ would allow 3 $a$s in $M_D(\nu)$ without a special alignment of the Higgs vevs. But in both cases, $\mathrm{lcm}(1,3,3) = \mathrm{lcm}(3,3,3) = 3$, making it somewhat strange that $b$ gets away with six copies.

Since we do not have a limit on the number of Higgs doublets and we are not making any assumption about their vevs, even in this particular basis it is not clear if tri-bimaximal mixing is possible. It is certainly not trivial and the non-trivial aspects of it are the ones outlined here. A search for particular groups and irreducible representations would have to start by providing specific solutions to these issues.

[8] This is the representation of the right-handed neutrinos, as opposed to $R_l$, the representation for the right-handed charged leptons.

# Chapter 9

# Conclusion

The Standard Model of particle physics has had great success in explaining a tremendous amount of experimental data. Even the recent discovery of neutrino mass can easily be incorporated into the SM in various ways. In spite of this, it is a model with many parameters whose values must be inputted since no prediction is made about them. Notably, fermion masses and mixing parameters are unconstrained from a theoretical point of view. This thesis elaborates on the use of discrete family symmetries to restrict the values of these parameters.

The different ways of building models based on flavour symmetries are virtually infinite. A comprehensive study of these models seems all but impossible. In this thesis the focus is on a subset of this universe; I only consider extensions of the SM with one or more Higgs doublets and with the inclusion of either pure Dirac or type-I seesaw neutrino masses. No ad hoc assumptions are made. Some reasons can be given for limiting the models considered but probably the best one is precisely the vastness of the original set of models with discrete family symmetries. Arguably, the initial 'model space' is so big that it even surpasses in size the parameter space of the SM and its minimal extensions, meaning that it is likely that one can easily find far-from-minimal, hand-picked models that can explain the values of these parameters. In fairness, just by considering multiple Higgs doublets one is already deviating from the SM's prediction $\rho = m_W^2/m_Z^2 \cos^2\theta_w = 1$ which is in good agreement with experimental data.

For models with one invariant Higgs doublet, a comprehensive study showed that leptonic tri-bimaximal mixing is unobtainable without further assumptions. It became clear that little or no benefit at all comes from considering acting on the fields in a representation other than a fully reduced one. In such a basis, all one needs to do is compare irreducible representations to other irreducible representations and their conjugates, in order to obtain the invariant mass matrices. As a matter of fact, in all the different ways we have made groups act on the fields, the only characteristics of the irreps that affected the mass matrices were their dimensions and the way they relate to their conjugates representation. For instance, we could have hoped after unsuccessfully trying millions of groups, that one could still find some new group that would lead to mass matrices with some new special structure. That is false though; for up to three dimensional irreps, all the possible variability is exhausted in a very limited set of small groups.

As for models with more than one Higgs doublet, no conclusive result was obtained on the possibility of having tri-bimaximal mixing for leptons. In spite of this, it was possible to arrive at a set of strong constraints for the Yukawa interactions that are not specific to a particular group or representation. Without assuming anything on the group representations used and even not knowing the Higgs' vevs, it is still possible to, given a target mass matrix, make some statements on the possibility of achieving such a matrix with these models and to deduce some properties of the irreps needed for that. Thus, knowledge of the target mass matrices in the basis where the group representations are fully reduced is important. For this reason, progress in the generic analysis of the these models requires the development of methods to achieve this. Equally important is to have some knowledge of the vacuum expectation values of the Higgs fields of these models.

In this work, for models with more than one Higgs doublet, it was not possible either to obtain leptonic tri-bimaximal mixing or to exclude the possibility of that being possible. Nonetheless, the methods shown should prove useful in the discussion of this, or any other type of mixing, in the class of models considered.

# Part V

# Appendix A - Basic notions in group theory

# Chapter 10

# General

## 10.1 Group

A group $(G, \cdot)$ is a set $G$ with a binary operation $\cdot$ on $G$ that satisfies the following conditions:

1. $a \cdot b \in G$ for all $a, b \in G$ (Closure);
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$ (Associativity);
3. There is an element $e \in G$ that satisfies $a \cdot e = e \cdot a = a$ for all $a \in G$ (Existence of the identity element);
4. For all $a \in G$, there is an element $b \in G$ such that $a \cdot b = b \cdot a = e$ (Existence of the inverse element; $b \equiv a^{-1}$).

See also [5, 6].

## 10.2 Order of a group

The order of a finite group $G$, $|G|$, is the number of elements in $G$.

## 10.3 Order of an element

The order of an element $g$ of a finite Group $G$, $|g|$, is the smallest positive integer $n$ such that $g^n = e$.

1. For all $g \in G$, $|g|$ divides $|G|$.

## 10.4 Rearrangement lemma

For $a, b, c \in G$, $ab = ac \Rightarrow b = c$. Therefore, the application of an element $a \in G$ to the set of all elements of $G$ produces a permutation of them.

## 10.5 Subgroup

A subset $H$ of a group $G$ is a subgroup of $G$ if it forms a group under the binary operation of $G$.

1. For a finite group, $|H|$ divides $|G|$ (Lagrange Theorem).

## 10.6 Conjugate elements

An element $b \in G$ is conjugate to $a \in G$, $a \sim b$, if there is an element $p \in G$ such that $b = pap^{-1}$.

1. Conjugation is an equivalence relation;
2. If $a \sim b$, then $|a| = |b|$.

## 10.7 (Conjugacy) class

For any $g \in G$, the conjugacy class of $g$ is the set $[g] = \left\{hgh^{-1} : h \in G\right\}$. In other words, it is the set of all elements in $G$ conjugate to $g$.

1. Two conjugacy classes are either equal or disjoint;
2. For all $g \in G$, the number of elements of $[g]$ divides $|G|$.

## 10.8 Invariant subgroup

A subgroup $H$ of $G$ is invariant/normal if it is invariant under conjugation (for all $h \in H, g \in G$ the element $ghg^{-1}$ is in $H$).

## 10.9 Cosets

For a subgroup $H$ of $G$, $pH = \{ph : h \in H\}$ is a left coset of $H$ and $Hp = Hp = \{hp : h \in H\}$ is a right coset of $H$ ($p \in G$).

## 10.10 Factor group

If $H$ is an invariant subgroup of $G$, the set of cosets of $H$ under the binary operation $(pH) \cdot (qH) = (pq) H$ forms the factor group $G/H$ ($p, q \in G$).

1. $|G/H| = \frac{|G|}{|H|}$.

## 10.11 Direct product group

The direct product of two groups $(G, \cdot)$ and $(H, \star)$, $G \times H$, is the set $\{(g, h) : g \in G, h \in H\}$ with the binary operation $\times$ such that $(g, h) \times (g', h') = (g \cdot g', h \star h')$.

1. $|G \times H| = |G| \, |H|$;
2. $G$ and $H$ are invariant subgroups of $G \times H$.

## 10.12 Classification of groups

### 10.12.1 Abelian group

An abelian group $G$ is one for which the group multiplication is commutative ($a \cdot b = b \cdot a$ for all $a, b \in G$).

### 10.12.2 Finite group

A group $G$ is finite if $|G|$ is finite.

### 10.12.3 Isomorphic groups

Two groups $(G, \cdot)$ and $(G', \star)$ are isomorphic, $(G, \cdot) \cong (G', \star)$, if there is a bijection $f : G \to G'$ such that $f(a \cdot b) = f(a) \star f(b)$ for all $a, b \in G$.

### 10.12.4 Simple and semi-simple groups

A group that does not contain any non-trivial subgroups is simple. A group that does not contain any abelian invariant subgroup is semi-simple.

### 10.12.5 Lie Group

A group $G$ is a Lie group if it forms a finite dimensional real-analytic manifold, with the group multiplication and inversion operations being analytical functions.

# Chapter 11

# Group Representations

## 11.1 Representation of a group

A representation $\rho$ of group $G$ on the vector space $V$ is a function $\rho : G \to GL(V)$ with $\rho(g_1 g_1) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$.

1. $GL(V)$ is the general linear group on $V$. If $V = \mathbb{C}^n$, the general linear group is identifiable with the group of $n \times n$ complex invertible matrices;
2. Often, $\rho(G)$ or $V$ itself is called the representation.
3. The dimension of the representation is given by the dimension of the vector space $V$ ($n$ for $V = \mathbb{C}^n$);

## 11.2 Faithful representation

If the representation $\rho : G \to GL(V)$ is injective, it is said to be faithful. That means that $g_1 \neq g_2 \Rightarrow \rho(g_1) \neq \rho(g_2)$ for all $g_1, g_2 \in G$. Otherwise the representation is degenerate.

## 11.3 Equivalent representations

For $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$, $\rho(G)$ and $\rho'(G)$ are equivalent representations ($\rho(G) \sim \rho'(G)$) if there is an isomorphism $S : V \to V'$ such that $S \circ \rho(g) \circ S^{-1} = \rho'(g)$.

1. In the special case $V' = V = \mathbb{C}^n$: $S$, $\rho(g)$ and $\rho'(g)$ are $n \times n$ complex invertible matrices.

## 11.4 Invariant subspace

A subspace $V_1$ of $V$ is invariant with respect to the representation $\rho : G \to GL(V)$ if $[\rho(G)](V_1) \subset V_1$. $V_1$ is minimal if additionally it does not contain any nontrivial invariant subspace with respect to $\rho$.

## 11.5 Irreducible representation

An irreducible representation $\rho$ of a group $G$ is a group representation that has no nontrivial invariant subspaces. Otherwise it is reducible. In this latter case, if the orthogonal complement of the invariant subspace is also an invariant subspace, then $\rho$ is fully reducible.

1. The number $n$ of inequivalent irreducible representations of $G$ is equal to the number of its classes;
2. If $d_i$ is the dimension of the $i$th irreducible representation of $G$,
   (a) $d_i$ divides $|G|$ for all $i$;
   (b) $\sum_i d_i^2 = |G|$.

## 11.6 Unitary representation

If $V$ is an inner product space, the representation $\rho : G \to GL(V)$ is unitary if $\rho(g)$ is a unitary operator, for all $g \in G$.

1. If $V$ is an inner product space and $G$ is a finite group then every representation of $G$ in $V$ is equivalent to a unitary representation;

2. A reducible unitary representation is fully reducible.

## 11.7 Direct sum representation

For two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ their direct sum representation, $\rho \oplus \rho' : G \to GL(V \oplus V')$, is such that $[\rho \oplus \rho'(g)](v, v') = \{[\rho(g)] v\} \oplus \{[\rho'(g)] v'\}$ for all $g \in G, v \in V, v' \in V'$.

1. In the special case $V = \mathbb{C}^n$ and $V' = \mathbb{C}^m$, $\rho \oplus \rho'(g)$ is the $(n+m) \times (n+m)$ matrix that results form the direct sum of the $n \times n$ matrix $\rho(g)$ with the $m \times m$ matrix $\rho'(g)$ .

## 11.8 Direct product representation

For two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ their direct product representation, $\rho \otimes \rho' : G \to GL(V \otimes V')$, is such that $[\rho \otimes \rho'(g)](v \otimes v') = \{[\rho(g)] v\} \otimes \{[\rho'(g)] v'\}$ for all $g \in G, v \in V, v' \in V'$.

1. In the special case $V = \mathbb{C}^n$ and $V' = \mathbb{C}^m$, $\rho \otimes \rho'(g)$ is the $(n \times m) \times (n \times m)$ matrix that results from the Kronecker product of the $n \times n$ matrix $\rho(g)$ with the $m \times m$ matrix $\rho'(g)$ .

## 11.9 Schur's lemma 1

Let $A$ be an operator on a vector space $V$ and $\rho(G)$ an irreducible representation of group $G$ in $V$. If $A\rho(g) = \rho(g) A$ for all $g \in G$ then $A$ is a multiple of the identity operator $E$.

1. An important consequence of this lemma is that all irreducible representations of an abelian group are one dimensional.

## 11.10 Schur's lemma 2

Let $A$ be a linear transformation from vector spaces $V'$ to $V$. For two irreducible representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$, if $A\rho'(g) = \rho(g) A$ for all $g \in G$ then either $A = 0$ or $V$ is isomorphic to $V'$ and $\rho(G) \sim \rho'(G)$.

## 11.11 Characters of a representation

The characters of the representation $\rho : G \to GL(V)$ are given by $\chi_\rho(g) = \mathrm{tr}[\rho(g)]$.

1. All elements of a conjugacy class have the same character: characters are class functions;

2. Equivalent representations have the same characters;

3. By irreducible characters it is meant the characters of an irreducible representation;

4. $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$ and $\chi_{\rho \otimes \rho'} = \chi_\rho \chi_{\rho'}$.

## 11.12 Orthonormality and completeness relations of irreducible characters

Let $G$ be a finite group and $n_i$ the number of elements in class-$i$. Since characters are class functions, $\chi_i^\mu$ shall stand for the class-$i$'s character in the irreducible representation $\mu$. Then the following relations hold:

$$\frac{1}{|G|}\sum_i n_i \left(\chi_i^\mu\right)^* \chi_i^\nu = \delta^{\mu\nu} \tag{11.1}$$

$$\frac{n_i}{|G|}\sum_\mu \left(\chi_i^\mu\right)^* \chi_j^\mu = \delta^{ij} \tag{11.2}$$

## 11.13 Character table

The entries of the character table contain the irreducible characters of the classes of a group. The character table is a square table since the number of classes and the number of irreducible representations is the same. If the characters are properly normalized ($\tilde{\chi}_i^\mu \equiv \sqrt{\frac{n_i}{|G|}}\chi_i^\mu$), the resulting normalized character table is a unitary matrix.

## 11.14 Reduction of a representation

The decomposition of a representation $R$ of group $G$ on $V$ can be obtained with the knowledge of its class characters $\chi_i^R$ and those of the irreducible representations of the group, $\chi_i^\mu$. The number of times irrep $\mu$ occurs in $R$ is given by

$$a_\mu = \frac{1}{|G|}\sum_i n_i \left(\chi_i^\mu\right)^* \chi_i^R \tag{11.3}$$

($n_i$ is the number of elements in class-$i$). The following is another useful result if we restrict ourselves to $V = \mathbb{C}^m$. $\rho_\mu(g)$ and $\rho_R(g)$ for $g \in G$ are $n_\mu \times n_\mu$ and $n_R \times n_R$ matrices, respectively. Defining the operators

$$P_{\mu i}^j = \frac{n_\mu}{|G|}\sum_g \left[\rho_\mu^{-1}(g)\right]_{ji} \rho_R(g) \tag{11.4}$$

, for a given $\mu, j$, and $v \in V$ fixed $\left\{w^i \equiv P_{\mu i}^j v : i = 1, \cdots, n_\mu\right\}$ forms a set of orthogonal vectors that transform under $\rho_R(g)$ according to the $\mu$ irreducible representation (if they are not all null). This means that

$$\rho_R(g)\, w^i = \sum_{i=1}^{n_\mu} \left[\rho_\mu(g)\right]_{ki} w^k \tag{11.5}$$

Appropriate use of this result allows for the decomposition of $V$ in invariant subspaces with respect to the representation $R$, as well as fully reduce $R$ if the group $G$ is finite.

# Part VI

# Appendix B - The impact of choosing different Higgs vacua connected by the discrete symmetry

What is the influence of choosing different minimums of the Higgs potential $V$ that are connected by the discrete symmetry? If $\Phi = \Phi_0$ is a solution of

$$V(\Phi) = 0 \tag{11.6}$$

then, since $V(\Phi) = V(H_i\Phi)$ for any $i$,

$$\Phi_0^i \equiv H_i\Phi_0 \tag{11.7}$$

is also a minimum of the potential. Notice that $\Phi_0$ does not need to be equal to $\Phi_0^i$; that would be symmetry breaking. The difference is very important. Consider for instance that there are three Higgs doublets $\Phi_j$ and that the potential $V$ is invariant for permutations of them. One possible vacuum solution is [36]

$$\Phi_0 \equiv \begin{bmatrix} \langle\Phi_1\rangle \\ \langle\Phi_2\rangle \\ \langle\Phi_3\rangle \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} \\ \begin{bmatrix} 0 \\ x \end{bmatrix} \\ \begin{bmatrix} 0 \\ y \end{bmatrix} \end{bmatrix} \tag{11.8}$$

If we permute these three vacuum values of the $\Phi_j$'s to obtain the different $\Phi_0^i$'s, clearly they need not be equal to $\Phi_0$ since this vacuum is not $S_3$ symmetric. But we readily recognize that all the $\Phi_0^i$'s are also vacuum solutions since the Higgs potential is $S_3$ symmetric.

So, is it legitimate to use the vacuum $\Phi_0$ and not some other $\Phi_0^i$? That is irrelevant since the masses and mixing matrices remain the same due to the symmetry of the Yukawa interactions. To see it explicitly, let's define

$$\mathbf{\Phi_0^i} \equiv \hat{H}_i\mathbf{\Phi_0} \tag{11.9}$$

$$H_D^i \equiv \mathbf{\Phi_0^i}^T\mathbf{M}\mathbf{M}^\dagger\mathbf{\Phi_0^{i*}} \tag{11.10}$$

Then

$$\begin{aligned} H_D^i &= \mathbf{\Phi_0}^T\hat{H}_i^T\mathbf{M}\mathbf{M}^\dagger\hat{H}_i^*\mathbf{\Phi_0^{i*}} \\ &= \mathbf{\Phi_0}^T\tilde{L}_i\mathbf{M}\mathbf{M}^\dagger\tilde{L}_i^\dagger\mathbf{\Phi_0^{i*}} \\ &= L_iH_DL_i^\dagger \end{aligned} \tag{11.11}$$

where use was made of relation 8.16. Similarly for $M_{eff}$, if we define $M_{eff}^i \equiv -\mathbf{\Phi_0^i}^T\mathbf{M}M_R^{-1}\mathbf{M}^T\mathbf{\Phi_0^i}$, we arrive at the relation

$$M_{eff}^i = L_iM_{eff}L_i^T \tag{11.12}$$

Clearly, for Majorana or Dirac particles and whatever $L_i$ is, the physical content of the model stays the same if we pick different minimums of the Higgs potential that are connected by the discrete symmetry.

# Bibliography


[1] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*. Dover Publications, 2005.

[2] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*. Pearson Higher Education, 3 ed., 2002.

[3] T.-P. Cheng and L.-F. Li, *Gauge theory of elementary particle physics*. Oxford University Press, 1988.

[4] D. Griffiths, *Introduction to Elementary Particles*. Wiley, 1987.

[5] W.-K. Tung, *Group Theory in Physics*. World Scientific Publishing Company, 1985.

[6] A. Messiah, *Quantum Mechanics*. Dover Publications, 1999.

[7] W.-M. et al. (Particle Data Group) *J. Phys. G 33, 1 (2006) and 2007 partial update for the 2008 edition* .

[8] A. Hocker and Z. Ligeti, "CP violation and the CKM matrix," *Ann. Rev. Nucl. Part. Sci.* **56** (2006) 501–567, arXiv:hep-ph/0605217.

[9] L. Wolfenstein, "Parametrization of the kobayashi-maskawa matrix,"*Phys. Rev. Lett.* **51** (Nov, 1983) 1945–1947.

[10] **CKMfitter Group** Collaboration, J. Charles *et al.*, "CP violation and the CKM matrix: Assessing the impact of the asymmetric B factories," *Eur. Phys. J.* **C41** (2005) 1–131, arXiv:hep-ph/0406184.

[11] H. Nunokawa, S. J. Parke, and J. W. F. Valle, "CP Violation and Neutrino Oscillations," *Prog. Part. Nucl. Phys.* **60** (2008) 338–402, arXiv:0710.0554 [hep-ph].

[12] G. C. Branco *et al.*, "Minimal scenarios for leptogenesis and CP violation," *Phys. Rev.* **D67** (2003) 073025, arXiv:hep-ph/0211001.

[13] H. J. Lipkin, "Quantum theory of neutrino oscillations for pedestrians: Simple answers to confusing questions," *Phys. Lett.* **B642** (2006) 366–371, arXiv:hep-ph/0505141.

[14] B. Kayser, "Neutrino Mass, Mixing, and Flavor Change," arXiv:0804.1497 [hep-ph].

[15] A. Strumia and F. Vissani, "Neutrino masses and mixings and.," arXiv:hep-ph/0606054.

[16] J. I. Silva-Marcos, "The problem of large leptonic mixing," *JHEP* **07** (2003) 012, arXiv:hep-ph/0204051.

[17] J. I. Silva-Marcos, "Symmetries, large leptonic mixing and a fourth generation," *JHEP* **12** (2002) 036, arXiv:hep-ph/0204217.

[18] G. C. Branco and J. I. Silva-Marcos, "The symmetry behind extended flavour democracy and large leptonic mixing," *Phys. Lett.* **B526** (2002) 104–110, arXiv:hep-ph/0106125.

[19] G. Altarelli, "Lectures on Models of Neutrino Masses and Mixings," arXiv:0711.0161 [hep-ph].

[20] W. G. Scott, "Status of trimaximal neutrino mixing," arXiv:hep-ph/0010335.

[21] P. F. Harrison, D. H. Perkins, and W. G. Scott, "Tri-bimaximal mixing and the neutrino oscillation data," *Phys. Lett.* **B530** (2002) 167, arXiv:hep-ph/0202074.

[22] E. Ma, "A(4) Symmetry and Neutrinos," arXiv:0710.3851 [hep-ph].

[23] E. Ma and U. Sarkar, "Neutrino masses and leptogenesis with heavy Higgs triplets," *Phys. Rev. Lett.* **80** (1998) 5716–5719, arXiv:hep-ph/9802445.

[24] E. Ma, "A(4) symmetry and neutrinos with very different masses," *Phys. Rev.* **D70** (2004) 031901, arXiv:hep-ph/0404199.

[25] E. Ma and G. Rajasekaran, "Softly broken A(4) symmetry for nearly degenerate neutrino masses," *Phys. Rev.* **D64** (2001) 113012, arXiv:hep-ph/0106291.

[26] E. Ma, "Plato's fire and the neutrino mass matrix," *Mod. Phys. Lett.* **A17** (2002) 2361–2370, arXiv:hep-ph/0211393.

[27] K. S. Babu, E. Ma, and J. W. F. Valle, "Underlying A(4) symmetry for the neutrino mass matrix and the quark mixing matrix," *Phys. Lett.* **B552** (2003) 207–213, arXiv:hep-ph/0206292.

[28] M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle, and A. Villanova del Moral, "Phenomenological tests of supersymmetric A(4) family symmetry model of neutrino mass," *Phys. Rev.* **D69** (2004) 093006, arXiv:hep-ph/0312265.

[29] G. Altarelli and F. Feruglio, "Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions," *Nucl. Phys.* **B720** (2005) 64–88, arXiv:hep-ph/0504165.

[30] K. S. Babu and X.-G. He, "Model of geometric neutrino mixing," arXiv:hep-ph/0507217.

[31] I. de Medeiros Varzielas, S. F. King, and G. G. Ross, "Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry," *Phys. Lett.* **B648** (2007) 201–206, arXiv:hep-ph/0607045.

[32] I. de Medeiros Varzielas and G. G. Ross, "SU(3) family symmetry and neutrino bi-tri-maximal mixing," *Nucl. Phys.* **B733** (2006) 31–47, arXiv:hep-ph/0507176.

[33] W. Grimus and L. Lavoura, "A model realizing the Harrison-Perkins-Scott lepton mixing matrix," *JHEP* **01** (2006) 018, arXiv:hep-ph/0509239.

[34] E. Ma, "Non-Abelian Discrete Flavor Symmetries," arXiv:0705.0327 [hep-ph].

[35] C. Luhn, S. Nasri, and P. Ramond, "The flavor group Delta(3n**2)," *J. Math. Phys.* **48** (2007) 073501, arXiv:hep-th/0701188.

[36] E. Derman, "Flavor unification, tau decay and b decay within the six quark six lepton Weinberg-Salam Model," *Phys. Rev.* **D19** (1979) 317.