C^* crossed products. Invertibility in an algebra of functional operators.

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Contents

1 Introduction

The objective of this work is to give an introduction to theory of representations of C^* crossed products (by groups and semigroups), and to apply this theory to establish the invertibility theory of a C^*-algebra of functional operators generated by all multiplication operators by piecewise slowly oscillating functions in $\mathbb{T}$ and by a unitary representation of an amenable discrete group of diffeomorphisms of $\mathbb{T}$ with the same set of fixed points.

2 C^* crossed products

2.1 C^* dynamical systems

When $\mathcal{A}$ is a C^*-algebra we denote by $Aut(\mathcal{A})$ its *-automorphisms group, endowed with the pointwise convergence topology.

2.1. Definition - *A C^** **dynamical system** *is a triple $(\mathcal{A}, G, \alpha)$ consisting of a C^*-algebra $\mathcal{A}$, a locally compact group G and a continuous homomorphism $\alpha : G \to Aut(\mathcal{A})$.*

2.2. Theorem - *Let X be a locally compact Hausdorff space and G a locally compact group. If G acts continuously on X, then $(C_0(X), G, \alpha)$ is a C^* dynamical system, where α is defined by*

$$\alpha_s(f)\,(x) := f(s^{-1} \cdot x)\,, \qquad s \in G,\, x \in X.$$

Conversely, every C^* dynamical system in $C_0(X)$ is induced by a continuous action of G in X.

2.3. Theorem - *Let H be a Hilbert space. If $(B(H), G, \alpha)$ is a C^* dynamical system in $B(H)$, then there is a unitary representation $U : G \to B(H)$ such that*

$$\alpha_s(T) = U_s T U_s^*\,, \qquad s \in G\,, T \in B(H)$$

Given a C^* dynamical system $(\mathcal{A}, G, \alpha)$ we are interested in representations that preserve the structure of the dynamical system.

2.4. Definition - *A* **covariant representation** *of a C^* dynamical system $(\mathcal{A}, G, \alpha)$ is a pair (π, U), where $\pi : \mathcal{A} \to B(H)$ and $U : G \to B(H)$ are, respectively, a representation of $\mathcal{A}$ and a unitary representation of G in a Hilbert space H, satisfying*

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*\,, \qquad s \in G\,, a \in \mathcal{A}$$

2.2 C^* crossed product

The study of C^* dynamical systems leads naturally to the consideration of a new C^*-algebra, the C^* crossed product, that encodes information from the dynamical system.

2.5. Theorem - *Let $(\mathcal{A}, G, \alpha)$ be C^* dynamical system. The following product, involution and norm are well defined in $C_c(G, \mathcal{A})$ and make it a normed *-algebra:*

- $(f * g)(s) := \int_G f(t)\alpha_t(g(t^{-1}s))\, d\mu(t)\,, \qquad s \in G$
- $f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\,, \qquad s \in G$
- $\|f\|_1 := \int_G ||f(t)||\, d\mu(t).$

2.6. Definition - *The C^** **crossed product** *of $(\mathcal{A}, G, \alpha)$ is the enveloping C^*-algebra of $C_c(G, \mathcal{A})$, and is denoted by $\mathcal{A} \rtimes_\alpha G$.*

The main tool for constructing representations of $C_c(G, \mathcal{A})$ is the following:

2.7. Definition - *Let (π, U) be a covariant representation of $(\mathcal{A}, G, \alpha)$ in a Hilbert space H. The mapping $\pi \rtimes U\,(f) : C_c \to B(H)$ defined by*

$$\pi \rtimes U\,(f) := \int_G \pi(f(s))U_s\, d\mu(s)\,, \qquad f \in C_c(G, \mathcal{A})$$

is called the **integral form** *of (π, U).*

2.8. Theorem - *Let (π, U) be a covariant representation of $(\mathcal{A}, G, \alpha)$. The integral form $\pi \rtimes U$ defines a continuous representation of $C_c(G, \mathcal{A})$. Moreover, $\pi \rtimes U$ is non-degenerate (irreducible) if and only if (π, U) is non-degenerate (irreducible).*

The converse of Theorem[2.8] also holds:

2.9. Theorem - *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system. Every non-degenerate continuous representation of $C_c(G, \mathcal{A})$ is the integral form of a non-degenerate covariant representation.*

2.10. Corollary - *There is a bijective correspondence between non-degenerate covariant representations of* $(\mathcal{A}, G, \alpha)$ *and non-degenerate representations of* $\mathcal{A} \rtimes_\alpha G$*. This correspondence preserves irreducibility.*

The existence of "plenty" of representations of a C^*-algebra allows us to construct many covariant representations of a dynamical system.

2.11. Definition - *Let* $(\mathcal{A}, G, \alpha)$ *be a* C^* *dynamical system and* π *a representation of* $\mathcal{A}$ *in the Hilbert space* H*. We define a representation* π_α *of* $\mathcal{A}$ *and a unitary representation* λ *of* G *in the Hilbert space* $L^2(G, H)$ *by*

$$[\pi_\alpha(a)\xi](t) := \pi(\alpha_t^{-1}(a))\xi(t)\,, \qquad \xi \in L^2(G,H)\,, t \in G$$
$$[\lambda_s \xi](t) := \xi(s^{-1}t)\,, \qquad \xi \in L^2(G,H)\,, s, t \in G$$

The pair (π_α, λ) *is called a* **regular representation** *of* $(\mathcal{A}, G, \alpha)$*.*

2.12. Theorem - *Let* $(\mathcal{A}, G, \alpha)$ *be a* C^* *dynamical system,* π *a representation of* $\mathcal{A}$ *and* (π_α, λ) *the associated regular representation. We have that* (π_α, λ) *is a covariant representation, which is non-degenerate if* π *is so. Moreover, if* π *is faithful, then* $\pi_\alpha \rtimes \lambda$ *is also faithful.*

2.13. Corollary - $\mathcal{A} \rtimes_\alpha G$ *is a completion of* $C_c(G, \mathcal{A})$ *in the norm defined by* $\|f\|_u := \sup_\pi \|\pi(f)\|$*, where the supreme is taken on the class of continuous representations of* $C_c(G, \mathcal{A})$*.*

2.3 Reduced C^* crossed product

In general, C^* crossed products may be very difficult to handle, since the norm $\|\cdot\|_u$ requires the knowledge of all covariant representations of the dynamical system. Therefore, it is often useful to restrict to regular representations.

2.14. Definition - *We define the* **reduced norm** $\|\cdot\|_r$ *in* $C_c(G, \mathcal{A})$ *by*

$$\|f\|_r := \sup_{(\pi_\alpha, \lambda)} \|\pi_\alpha \rtimes \lambda(f)\|$$

where the supreme is taken over the regular representations of $(\mathcal{A}, G, \alpha)$*.*

2.15. Definition - *The* **reduced** C^* **crossed product** *of* $(\mathcal{A}, G, \alpha)$ *is the completion of* $C_c(G, \mathcal{A})$ *in the norm* $\|\cdot\|_r$*, and is denoted by* $\mathcal{A} \rtimes_\alpha^r G$*.*

2.16. Theorem - $\mathcal{A} \rtimes_\alpha^r G$ *is always a quotient of* $\mathcal{A} \rtimes_\alpha G$*. Moreover, the following inequality holds on* $C_c(G, \mathcal{A})$ *:* $\|\cdot\|_r \leq \|\cdot\|_u \leq \|\cdot\|_1$*.*

2.17. Theorem - *If* G *is amenable then the norms* $\|\cdot\|_u$ *and* $\|\cdot\|_r$ *coincide. Hence, there is a *-isomorphism* $\mathcal{A} \rtimes_\alpha^r G \cong \mathcal{A} \rtimes_\alpha G$*.*

2.4 C^* crossed products by discrete groups

We now consider C^* dynamical systems $(\mathcal{A}, G, \alpha)$ with G a discrete group.

2.18. Proposition - *Let* $(\mathcal{A}, G, \alpha)$ *be a dynamical system with* G *discrete. The following inequalities hold in* $C_c(G, \mathcal{A})$ *:* $\|\cdot\|_\infty \leq \|\cdot\|_r \leq \|\cdot\|_u \leq \|\cdot\|_1$*.*

The evaluation mapping $E_s : C_c(G, \mathcal{A}) \to \mathcal{A}$ in $s \in G$, given by $E_s(f) := f(s)$, is continuous in the norms $\|\cdot\|_u$ and $\|\cdot\|_r$. Hence, it extends by continuity to mappings $E_s : \mathcal{A} \rtimes_\alpha G \to \mathcal{A}$ and $E_s : \mathcal{A} \rtimes_\alpha^r G \to \mathcal{A}$.

2.19. Theorem - *Let* $f \in \mathcal{A} \rtimes_\alpha^r G$*. If* $E_s(f) = 0$ *for all* $s \in G$*, then* $f = 0$*.*

2.5 Non-classical C^* crossed products

Recently, there has been interest in generalizing the notion of C^* crossed product to the case of dynamical systems associated with semigroups. Although there are several approaches to this problem, under different hypothesis, presently there is still no general theory of crossed products by these type of dynamical systems.

Let $\mathcal{A}$ be a C^*-algebra with unit and Γ an abelian discrete totally ordered group. Let $\Gamma^+ := \{s \in \Gamma : 0 \leq s\}$ be the semigroup of positive elements of Γ.

2.20. Definition - *If we fix an homomorphism* $\alpha : \Gamma^+ \to End(\mathcal{A})$*, the triple* $(\mathcal{A}, \Gamma^+, \alpha)$ *is called a* C^* **dynamical system***.*

2.21. Definition - *$(\mathcal{A}, \Gamma^+, \alpha)$ is said to be* **finely representable** *if there is a mapping $L : \Gamma^+ \to End(A)$ such that*

- $L_{s+t} = L_s \circ L_t\,,$ $\qquad\forall s, t \in \Gamma^+,$
- $L_s(\alpha_s(a)b) = aL_s(b)\,,$ $\qquad\forall s \in \Gamma^+\,, a, b \in \mathcal{A},$
- $\alpha_s(L_s(a)) = \alpha_s(1)a\alpha_s(1)\,,$ $\qquad\forall s \in \Gamma^+\,, a \in \mathcal{A}.$

Let $C_c(\Gamma, \alpha, \mathcal{A})$ be the space of functions $a : \Gamma \to \mathcal{A}$ such that

$$\begin{cases} a_s \in \mathcal{A}\alpha_s(1), & \text{if } \; s \geq 0 \\ a_s \in \alpha_{-s}(1)\mathcal{A}, & \text{if } \; s \leq 0 \end{cases}$$

2.22. Theorem - *The following product, involution and norm are well defined in $C_c(\Gamma, \alpha, \mathcal{A})$ and turn it into a normed *-algebra:*

$$(a * b)_s := \begin{cases} \displaystyle\sum_{\substack{s=t-r \\ 0<t,r}} a_t\alpha_s(b_{-r}) + \sum_{\substack{s=r-t \\ 0<t,r}} L_t(a_{-t}b_r) + \sum_{\substack{s=t+r \\ 0\leq t,r}} a_t\alpha_t(b_r), & 0 \leq s \\ \displaystyle\sum_{\substack{s=t-r \\ 0<t,r}} \alpha_{-s}(a_t)b_{-r} + \sum_{\substack{s=r-t \\ 0<t,r}} L_r(a_{-t}b_r) + \sum_{\substack{s=-t-r \\ 0\leq t,r}} \alpha_r(a_{-t})b_{-r}, & 0 > s \end{cases}$$

$$(a^*)_s := (a_{-s})^*$$

$$\|a\|_1 := \sum_{s\in\Gamma} \|a_s\|$$

2.23. Definition - *The* **crossed product** *$\mathcal{A} \rtimes_\alpha \Gamma^+$ of $(\mathcal{A}, \Gamma^+, \alpha)$ is the enveloping C^*-algebra of $C_c(\Gamma, \alpha, \mathcal{A})$.*

2.24. Definition - *A* **covariant representation** *of a finely representable system $(\mathcal{A}, G, \alpha)$ is a pair (π, U), where $\pi : \mathcal{A} \to B(H)$ is a representation of $\mathcal{A}$ and $U : \Gamma^+ \to B(H)$ is an homomorphism, satisfying*

$$U_s\pi(a)U_s^* = \pi(\alpha_s(a))\,, \qquad U_s^*\pi(a)U_s = \pi(L_s(a))$$

2.25. Theorem - *There is a bijective correspondence between non-degenerate covariant representations of $(\mathcal{A}, \Gamma^+, \alpha)$ and non-degenerate representations of $\mathcal{A} \rtimes_\alpha \Gamma^+$.*

3 Invertibility in a C^*-algebra of functional operators

3.1 Isomorphism with the crossed product and the local trajectory method

Let $\mathcal{A}$ be C^*-subalgebra of $B(H)$ and $\mathcal{Z}$ a central C^*-subalgebra of $\mathcal{A}$, both possessing the identity operator. Let $U : G \to B(H)$ be a unitary representation of the discrete group G. Assume that:

(A1). *$\alpha_s(a) := U_s a U_s^*$ are *-automorphisms of $\mathcal{A}$ and $\mathcal{Z}$, for all $s \in G$.*

(A2). *G is an amenable group.*

(A3). *There exists a set $M_0 \subset M$ such that, for every finite set $G_0 \subset G$ and every open set $W \subset P_{\mathcal{A}}$ there is $\nu \in W$ such that $m_\nu := \operatorname{Ker} \nu|_{\mathcal{Z}}$ belongs to the G-orbit of M_0 and $\beta_s(m_\nu) \neq m_\nu$, for all $s \in G_0 \setminus \{e\}$.*

3.1. Theorem - *Under conditions* **(A1)**, **(A2)** *and* **(A3)** *there is a *-isomorphism* $\operatorname{alg}\{\mathcal{A}, U_G\} \cong \mathcal{A} \rtimes_\alpha G$.

Let Ω denote the set of G-orbits of the points of M_0.

For each $\omega \in \Omega$ fix a point m_ω in ω, and let $\pi_\omega : \operatorname{alg}\{\mathcal{A}, U_G\} \to B(L^2(G, H_m))$ be the representation defined by

$$[\pi_\omega(a)\xi](t) := \widetilde{\pi}_\omega\big(\alpha_t^{-1}(a)\big)\xi(t)\,, \qquad [\pi_\omega(U_s)\xi](t) := \xi(s^{-1}t)$$

where $\widetilde{\pi}_\omega : \mathcal{A} \to B(H_m)$ is a non-degenerate representation of $\mathcal{A}$ whose kernel is the closed ideal generated by m_ω.

3.2. Theorem - *Under conditions* **(A1)**, **(A2)** *and* **(A3)** *an element $b \in \operatorname{alg}\{\mathcal{A}, U_G\}$ is invertible if and only if $\pi_\omega(b)$ is invertible for every $\omega \in \Omega$ and*

$$\sup\{\|\pi_\omega(b)^{-1}\| : \omega \in \Omega\} < \infty$$

3.2 Invertibility in the algebra of functional operators

3.3. Definitions - *A function $f : \mathbb{T} \to \mathbb{C}$ is said to be* **slowly oscillating** *at $\lambda \in \mathbb{T}$ if*

$$\lim_{\epsilon \to 0} \operatorname{osc}(f, \{z \in \mathbb{T} : \tfrac{1}{2}\epsilon \leq |z - \lambda| \leq \epsilon\}) = 0$$

Define $SO(\mathbb{T})$ *as the* C^**-subalgebra of* $L^\infty(\mathbb{T})$ *of slowly oscillating functions.*

Define $PC(\mathbb{T})$ *as the* C^**-subalgebra of* $L^\infty(\mathbb{T})$ *of functions which have one-sided limits at each point of* $\mathbb{T}$*.*

Define $PSO(\mathbb{T}) := \text{alg}\{SO(\mathbb{T}), PC(\mathbb{T})\}$.

The algebra of functional operators whose invertibility theory we are going to study is

$$\mathcal{B} := \text{alg}\{PSO(\mathbb{T})\,, U_G\}$$

where $U : G \to B(L^2(\mathbb{T}))$ is the unitary representation of G given by

$$[U_s\xi]\,(t) := |(s^{-1})'(t)|^{1/2}\,\xi(s^{-1}(t))$$

and G is an amenable discrete group of orientation-preserving diffeomorphisms of $\mathbb{T}$, such that all diffeomorphisms $s \in G \setminus \{e\}$ have the same set of fixed points.

Let Λ denote the set of fixed points of the diffeomorphisms $s \in G \setminus \{e\}$. Consider the partition of $\mathbb{T}$ in the sets Λ°, $\mathbb{T}^*_{arc}$ and Λ_*, defined by

- Λ° is the interior of Λ.
- $\mathbb{T}^*_{arc} := \mathbb{T} \setminus \overline{\Lambda^\circ}$.
- $\Lambda_* := \mathbb{T} \setminus \left(\Lambda^\circ \cup \mathbb{T}^*_{arc}\right)$.

Restricting to Λ°, $\mathbb{T}^*_{arc}$ and Λ_* we obtain the following C^*-subalgebras of $B(L^2(\Lambda^\circ))$, $B(L^2(\mathbb{T}^*_{arc}))$ and $B(L^2(\Lambda_*))$, respectively,

- $\mathcal{B}^\circ := \chi^\circ\mathcal{B}$
- $\mathcal{B}_{arc} := \chi_{arc}\mathcal{B}$.
- $\mathcal{B}_* := \chi_*\mathcal{B}$.

3.4. Theorem - *An operator* $A \in \mathcal{B}$ *is invertible if and only if* $\chi^\circ A$, $\chi_{arc}A$ *and* $\chi_* A$ *are invertible in* $\mathcal{B}^\circ$, $\mathcal{B}_{arc}$ *and* $\mathcal{B}_*$ *respectively.*

Let $\mathcal{A}$ be a C^*-subalgebra of $L^\infty(\mathbb{T})$ that contains $C(\mathbb{T})$ and $\mathcal{M}$ its maximal ideal space. If Δ is a subset of $\mathbb{T}$ we define $\mathcal{M}_\Delta$ as the set of fibers in $\mathcal{M}$ over the points $t \in \Delta$.

3.5. Theorem - *If Δ is open in $\mathbb{T}$ then $\chi_\Delta \mathcal{A} \cong C(\overline{\mathcal{M}_\Delta})$.*

3.6. Theorem - *Suppose also that $\mathcal{A}$ has a dense subspace of functions which are continuous in $\mathbb{T}$ except at most in a finite number of points.*

If $\Delta \subset \mathbb{T}$ is closed with empty interior then $\mathcal{M}_\Delta$ has empty interior.

3.2.1 Invertibility in $\mathcal{B}_{arc}$

The C^*-algebra $\mathcal{B}_{arc}$ satisfies conditions **(A1)**, **(A2)** and **(A3)**. Hence, we can apply the local trajectory method.

Let $\mathcal{O}_{arc}$ be a subset of $\mathbb{T} \setminus \Lambda$ containing exactly one point of each G-orbit. For each $\psi \in \mathcal{M}_{\mathcal{O}_{arc}}$ let $\pi_\psi : \mathcal{B}_{arc} \to B(L^2(G))$ be the representation

$$[\pi_\psi(\chi_{arc} a)\, \xi](t) := (\widehat{a \circ t})(\psi)\, \xi(t)$$
$$[\pi_\psi(\chi_{arc} U_s)\, \xi](t) := \xi(s^{-1} t)$$

3.7. Theorem - *An operator $A \in \mathcal{B}_{arc}$ is invertible if and only if for every $\psi \in \mathcal{M}_{\mathcal{O}_{arc}}$ the operator $\pi_\psi(A)$ is invertible in $L^2(G)$ and*

$$\sup_{\psi \in \mathcal{M}_{\mathcal{O}_{arc}}} \|\pi_\psi(A)^{-1}\| < \infty$$

3.2.2 Invertibility in $\mathcal{B}^\circ$

Since Λ° contains only fixed points of G, the operators $\chi^\circ U_s$, with $s \in G$, are all the identity operator in $L^2(\Lambda^\circ)$. Hence, $\mathcal{B}^\circ = \chi^\circ PSO(\mathbb{T}) \cong C(\overline{\mathcal{M}_{\Lambda^\circ}})$ is a commutative C^*-algebra whose maximal ideal space is $\overline{\mathcal{M}_{\Lambda^\circ}}$.

3.8. Theorem - *An operator $A \in \mathcal{B}^\circ$ is invertible if and only if $\widehat{A}(\psi) \neq 0$ for every $\psi \in \overline{\mathcal{M}_{\Lambda^\circ}}$.*

3.2.3 Invertibility in $\mathcal{B}_*$

Since Λ_* contains only fixed points of G, the operators $\chi_* U_s$, with $s \in G$, are all the identity operator in $L^2(\Lambda_*)$. Hence, $\mathcal{B}_* = \chi_* PSO(\mathbb{T})$ is a commutative C^*-algebra.

3.9. Proposition - *The maximal ideal space of $\mathcal{B}_*$ is canonically identified with a subset of $\mathcal{M}_{\Lambda_*}$.*

Since Λ_* has empty interior it follows that $\mathcal{M}_{\Lambda_*}$ has empty interior too. Hence, $\mathcal{M}_{\Lambda_*} \subset \overline{\mathcal{M}_{\mathbb{T}^*_{arc}}} \cup \overline{\mathcal{M}_{\Lambda^\circ}}$. Applying the invertibility criteria for the algebras $\mathcal{B}_{arc}$ and $\mathcal{B}^\circ$ one can indeed prove the following:

3.10. Theorem - *Let $A \in \mathcal{B}$. If $\chi_{arc}A$ and $\chi^\circ A$ are invertible in $\mathcal{B}_{arc}$ and $\mathcal{B}^\circ$, respectively, then $\chi_* A$ is invertible in $\mathcal{B}_*$.*

3.2.4 Invertibility in $\mathcal{B}$

Combining the invertibility theories of the algebras $\mathcal{B}^\circ$, $\mathcal{B}_{arc}$ and $\mathcal{B}_*$ we obtain the following invertibility criterion for the algebra $\mathcal{B}$:

3.11. Theorem - *An operator $A \in \mathcal{B}$ is invertible if and only if the following conditions are satisfied*

(i) $\widehat{\chi^\circ A}\,(\psi) \neq 0$ *for all* $\psi \in \overline{\mathcal{M}_{\Lambda^\circ}}$,

(ii) $\pi_\psi(\chi_{arc}A)$ *is invertible in* $L^2(G)$ *for all* $\psi \in \mathcal{M}_{\mathcal{O}_{arc}}$ *and*

$$\sup_{\psi \in \mathcal{M}_{\mathcal{O}_{arc}}} \|\pi_\psi(\chi_{arc}A)^{-1}\| < \infty$$

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