Summary

In these pages we intend to give a picture of the work developed in the thesis 'A Combinatorial Approach to the HOMFLY n-Specializations'. We will present the principal ideas and results of the original text leaving aside the technical details, which enables us to slightly reorganize the text in order to make it more flowing.

1 Abstract

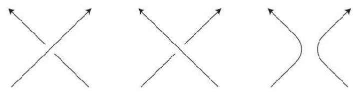
Kauffman discovered a combinatorial definition of the Jones polynomial which allows one to compute this knot invariant in an efficient way. Khovanov's categorification of the Jones polynomial gave birth to an invariant which is stronger than the Jones polynomial and also led to new topological insights into this polynomial. Both these important discoveries boosted mathematicians in their search for combinatorial definitions and categorifications of other polynomial invariants. One important example of such a polynomial invariant is the HOMFLY polynomial and it n-specializations.

In this work we give a definition of the HOMFLY n-specializations by purely combinatorial methods which allows for the systematical computation of these invariants. The approach taken is analogous to Kauffman's use of his bracket for defining the Jones polynomial, and requires the evaluation of a bracket on a class of 3-valent graphs with two types of edges. Our definition follows an alternative route to that taken by H. Murakami, T. Ohtsuki and S. Yamada - MOY (who give an explicit but impractical formula for the bracket), by defining the bracket on graphs by means of a list of properties. We show that these properties can be used to construct an algorithm for evaluating any graph, and then uniqueness follows from the MOY formula, which satisfies these properties, as we show in a slightly different way to the MOY paper. Our results may be useful for a better understanding of the categorification of the HOMFLY n-specializations due to Khovanov and Rozansky since their categorification was based precisely on the MOY relations.

2 The Kauffman Bracket

In the first chapter we start by introducing some fundamental notions in knot theory. We introduce the notion of knot, link and link diagram together with examples and a discussion of knot equivalence; we introduced the definition of knot invariant, braid, the Reidemeister theorem and the Alexander theorem for the closure of a braid. We close the section on preliminaries with a discussion of the notion of skein-relation by means of the Conway theorem for the Alexander polynomial. We follow the preliminaries with a detailed treatment of the work of Kauffman on the Jones polynomial, as summarised below.

Let $\mathbb{L}$ be the set of all oriented link diagrams and let L_{-}, L_{+} and L_0 be as in the following figure.



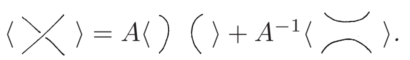
$L+$ $\quad$ $L-$ $\quad$ L_0

Definition 2.1 *The Jones polynomial is a map $V : \mathbb{L} \to \mathbb{Z}[q, q^{-1}]$, defined by the following rules:*

1. *If $L \sim L^{'}$ then $V(L) = V(L^{'})$.*
2. *$V(O) = 1$, where O is the trivial knot.*
3. *$q^2 V(L_{+}) - q^{-2} V(L_{-}) = (q - q^{-1}) V(L_0)$*

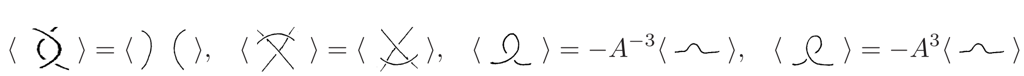
All diagrams are considered up to planar isotopy and only the part of the diagram which is represented changes. Outside this neighbourhood the three diagram are equal. If we substitute 2 by the rule $V(O) = q+q^{-1}$, we would get what is called the unnormalized Jones polynomial.

Definition 2.2 *Let L be an unoriented link diagram. The Kauffman bracket, $\langle \quad \rangle$, is defined by the following rules:*

1. *$\langle O \rangle = 1$*
2. *$\langle L \cup O \rangle = (-A^2 - A^{-2}) \langle L \rangle$,*
3. 

Remark 2.3 *To compute the bracket: first apply rule 3 at each crossing to obtain 2^n diagrams, where n is the number of crossings. Each of these diagrams are made of a finite union of disjoint 'circles'. Then apply rule 1 and 2 until you find the final value. In fact, there is an explicit formula for the bracket and so there is no ambiguity in the final value.*

The Kauffman bracket satisfies the following conditions:



which means that it is a 'generalized' invariant, i.e. invariant under moves 2, 3 and invariant up to a factor or its inverse under move 1. To make it a link invariant there is a trick of Kauffman.

Definition 2.4 *We define the **X-polynomial** for $L \in \mathbb{L}$ to be*

$$X(L) = (-A^3)^{-\omega(L)} \langle |L| \rangle$$

where ω is the writhe and $|.|$ ignores the orientation.

As $\langle L\rangle$ and $\omega(L)$ are unaffected by moves 2 and 3, therefore $X(L)$ is also unaffected by these moves. It can be shown that $X(L)$ is also invariant under move 1, thus it is a link invariant. Furthermore, by direct substitution it can be shown that $X(L)$ satisfies rule 3 in definition 2.1. An argument by induction proves the uniqueness of a function satisfying the three rules of this definition. We thus conclude that $X(L)$ is the Jones polynomial. By remark 2.3 there is no ambiguity in the computation of the bracket, and then we can use rule 3 in def. 2.1 to compute the polynomial for each diagram.

3 A Bracket for the HOMFLY n-Specializations

Definition 3.1 *Let $n \in \mathbb{N}$. The **HOMFLY n-specialization** is a map $J_n : \mathbb{L} \to \mathbb{Z}[q, q^{-1}]$, defined by the following rules:*

1. *If $L \sim L^{'}$ then $J_n(L) = J_n(L^{'})$.*
2. *$J_n(O) = [n]$, where O is the trivial knot.*
3. *$q^n J_n(L_+) - q^{-n} J_n(L_-) = (q - q^{-1}) J_n(L_0)$*

In rule 2 we used the notation $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1}$.

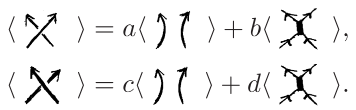
Definition 3.2 *Let Γ be an oriented trivalent planar graph. We say that Γ is classic if:*

1. *At each vertex two edges are "in" and one edge is "out", or two are "out" and one is "in".*
2. *An edge that at one vertex is the single "outgoing" edge must be the single "ingoing" edge at its other vertex and vice-versa. We call it a thick edge and we represent it as in figure 1.*

Figure 1: A thick edge. It is supposed to be oriented in the obvious way.

The graph with no vertices and one connected component (we will denote it by O), and the empty graph are also classic graphs. We will call O the trivial graph.

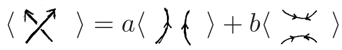
In order to follow an approach similar to that of Kauffman we define the n**-bracket** by the following two rules:



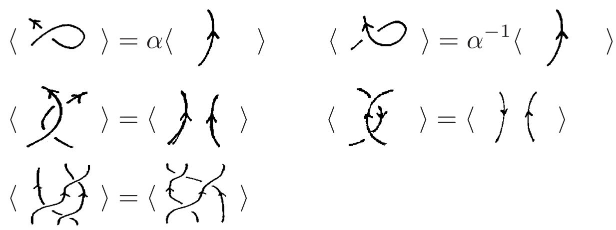
(1)

(2)

Here we introduce a bracket for classic graphs, the **Γ-bracket**, because we want to consider oriented diagrams. Without the graphs the first rule would look like



and there is an incompatibility with the orientations in the last term. Now we want the n-bracket to become a generalized invariant, that is:



(3)

(4)

(5)

Remark 3.3 *Since the other oriented versions of Reidemeister move 3 can be obtained from 4 and 5, we only have to consider the case of move 3 where all the crossings are positive, i.e. 5.*

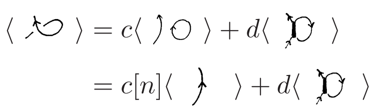
It is clear that for the relations (3), (4) and (5) to hold we have to find appropriate values for a, b, c, d and impose conditions on the Γ-bracket. A natural rule for the Γ-bracket is

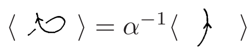
$$\langle G \cup O \rangle = [n]\langle G \rangle, \tag{6}$$

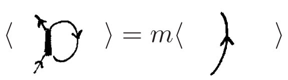
where ∪ denotes the juxtaposition of two graphs. If we set $\langle \emptyset \rangle = 1$ the previous rule gives

$$\langle O \rangle = [n] \tag{7}$$

As an example, we compute move 1:



hence we have  if and only if



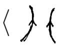
(8)

where $m \in \mathbb{Z}[q, q^{-1}]$ and we have

$$\alpha^{-1} = c[n] + dm. \tag{9}$$

From the other type of orientation we conclude that

$$\alpha = a[n] + bm. \tag{10}$$

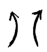
By computing the bracket for move 2 we can conclude that $\langle$  $\rangle = \langle$  $\rangle$ if $ac = 1$ and

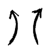
$\langle$  $\rangle = -(ad + bc)(db)^{-1}\langle$  $\rangle$. (11)

Now we recall that our objective is to use this bracket to get an invariant which satisfies the skein-relation of the HOMFLY n-specialization. Suppose that the n-bracket satisfies equations (3), (4) and (5), and using Kauffman's trick define

$$I(L) = \alpha^{-\omega(L)}\langle L \rangle \tag{12}$$

which easily can be checked to be an invariant. At this point we show that the n-bracket satisfies the desired skein-relation if $a = c = 1$, $d = -q$, $b = -q^{-1}$ and $m = [n-1]$, which then allows us to rewrite it as

$\langle$  $\rangle = \langle$  $\rangle - q^{-1}\langle$  $\rangle$ (13)

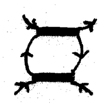
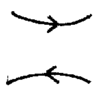
$\langle$  $\rangle = \langle$  $\rangle - q\langle$  $\rangle$ (14)

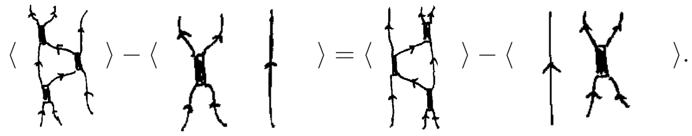
and we use (13) and (14) to compute the second version of Reidemeister move 2 (4) and Reidemeister move 3 (5) . In the end we have the following list of properties that we need the Γ-bracket to satisfy:

$$\langle G \cup O \rangle = [n]\langle G \rangle, \tag{15}$$

$\langle$  $\rangle = [n-1]\langle$  $\rangle$, (16)

$\langle$  $\rangle = (q + q^{-1})\langle$  $\rangle = [2]\langle$  $\rangle$, (17)

$\langle$  $\rangle = \langle$  $\rangle + [n-2]\langle$  $\rangle$, (18)



(19)

4 The Γ-bracket and the MOY Formula

In the final chapter 3 we analyse the MOY formula. This explicit formula satisfies all the properties of the Γ-bracket and allows us to conclude that when computing the bracket for a graph we can simplify the graph by using the rules in any order. Also, these rules are enough to give a value for any graph, which means that we can define the bracket by the rules. Thus the n-bracket is well defined and we are able to compute J_n from the skein-relation.

Here we proceed with the definition of the MOY formula and then we summarize the most important aspects.

Let $n \in \mathbb{N}$ and $n \geq 2$ and put $X = \{-(n-1), -(n-1)+2, ..., n-1\}$.

Remark 4.1 *X is the set of exponents appearing in the sum formula for $[n]$.*

A **state**, σ, of a classic graph is an assignment of an element $e \in X$ to each normal edge such that if two normal edges end in the same thick edge they must have different elements assigned to them and, moreover, if two elements go into a thick edge the same two elements must also go out of the same edge at the end. Figure 2 illustrates a state.

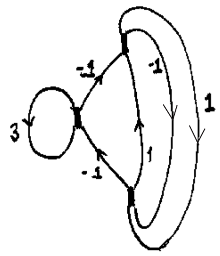
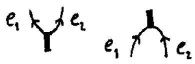
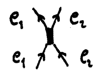
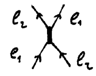
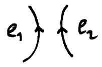


Figure 2: Example of a state for $n = 4$

Given a state, σ, we define the **weight** of a vertex v to be

$$w(v, \sigma) = q^{\delta},$$

where $\delta = 1/2$ if $e_1 < e_2$ or $\delta = -1/2$ if $e_1 > e_2$ and the outgoing/incoming edges are labelled as follows  .

After fixing a state, at each thick edge we must have  or  . Now replace every thick edge by  or  , respectively. This gives rise to a finite set of simple closed curves (which may intersect each other) such that each curve is associated with an element $e(C) \in X$ (see fig. 3). Then we define the **rotation number**, $rot(\sigma)$ to be

$$\sum_C e(C) rot(C)$$

where the sum is over all the curves and $rot(C)$ is -1 or 1 if C is oriented clockwise or counter-clockwise, respectively.

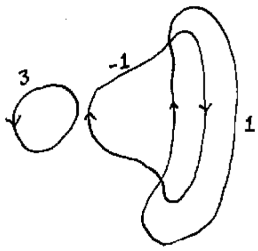


Figure 3: Curves corresponding to the state in fig. 2

As an example, the state represented in figure 2 with curves in fig. 3 has rotation number $(3)(1) + (-1)(-1) + (1)(-1) = 3$. Finally, we define the Γ-bracket to be

$$\langle G \rangle_\Gamma = \sum_\sigma \left\{ \prod_v w(v, \sigma) \right\} q^{rot(\sigma)}, \tag{20}$$

where σ runs over all states of G and v runs over all vertices of G. We also put $\langle \emptyset \rangle_\Gamma = 1$. It is clear that $\langle \quad \rangle_\Gamma$ is invariant under planar isotopy.

The next theorem extablishes that we can define the Γ-bracket by the properties (15)-(19).

Theorem 4.2 *The Γ-bracket satisfies the rules (15), (16), (17), (18), (19) and is explicitly given by (20). These rules can be applied without ambiguity in order to compute the Γ-bracket instead of using the formula.*

Proof: Here we prove that the Γ-bracket satisfies (15) and little coments are made on the proofs of the other relations; we will denote the Γ-bracket simply by $\langle \quad \rangle$.

The following reasoning is independent of the orientation of the separated component, because of the symmetry in the elements of X. Observe that a state, σ, of $G \cup O$ is given by a state of G, σ_G, and an element $k \in X$ associated to O; the vertices of $G \cup O$ are those of G and $rot(\sigma) = rot(\sigma_G) + rot(O)$. Then we have

$$\begin{aligned}
\langle G \cup O \rangle &= \sum_{\sigma} \left\{ \prod_{v} w(v,\sigma) \right\} q^{rot(\sigma)} = \sum_{(\sigma_G,k)} \left\{ \prod_{v_G} w(v_G,\sigma_G) \right\} q^{rot(\sigma_G)+rot(O)} \\
&= \sum_{k\in X} \left(\sum_{\sigma_G} \left\{ \prod_{v_G} w(v_G,\sigma_G) \right\} q^{rot(\sigma_G)} \right) q^k \\
&= \sum_{k\in X} \langle G \rangle q^k = [n]\langle G \rangle.
\end{aligned}$$

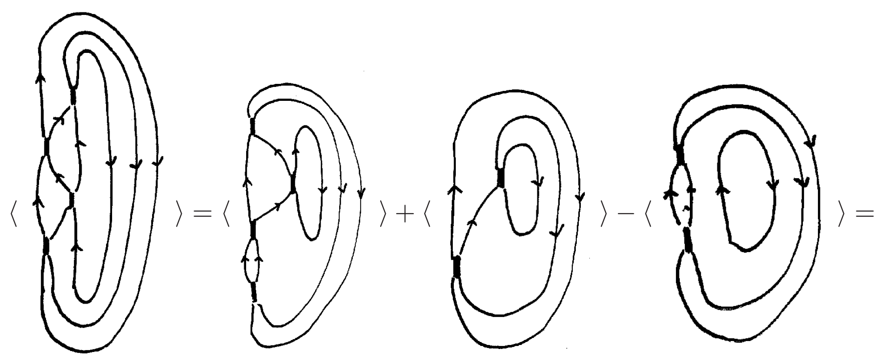
In (17), independently of the state the two extra vertices in the left hand side contribute with $q + q^{-1} = [2]$; in (16) the left hand side contributes with $q^{sign(\beta-\alpha)-\beta}$, where α is the labelling of the bottom and upper strands and β is the labelling of the extra strand; (18) has a few cases to consider but follows similarly to the previous case. For (19), first we show that we only need to analyse what happens with the part of the graphs which is shown and then we must perform an exhaustive case-by-case inspection of which labellings are possible and verify that the relation holds in each case.

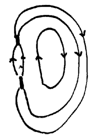
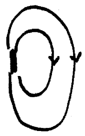
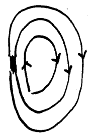
■

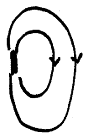
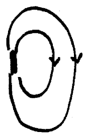
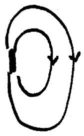
We end with a few observations about how the relations allow to simplify and compute the value of the Γ-bracket for each graph. In fact, we only need to consider graphs coming from braids, because there is a theorem of Alexander which states that each link is equivalent to the closure of a braid.

First remember that (15) implies $\langle O \rangle = [n]$, which means that we have a value for the trivial graph. Given a classic graph, if we are able to turn it into a linear combination of trivial graphs then we are able to get a value for the initial graph; we just have to take the sum of the coefficients times $[n]$, as in the next example.

Example 4.3



$= [2][n-1]\langle$  $\rangle + [n-1]\langle$  $\rangle - [2]\langle$  $\rangle$

$= [2]^2[n-1]\langle$  $\rangle + [n-1]\langle$  $\rangle - [2][n]\langle$  $\rangle$

$$= [n-1]([2]^2[n-1]\langle O\rangle + [n-1]\langle O\rangle - [2][n]\langle O\rangle)$$

$$= [n][n-1]^2[2]^2 + [n-1]^2[n] - [n]^2[n-1][2]$$

For this process to work in every case we state that for a graph coming from a braid the possibilities for a face with two edges are  and  ; moreover, for a face with four edges the possibilities are  and  . Thus we can always apply (16) or (17) or (19) in any graph because of the follwoing lemma.

Lemma 4.4 *Every classic graph ($\neq \emptyset, O$) has a face with two or four edges.*

Although it is obvious from the expressions of (16), (17) (and also (18)) that these rules simplify (i.e. diminuish the number of thick edges) a graph, we need to be more careful with (19). In the thesis we describe an algorithm which simplifies graphs where only (19) can be performed. Here we will summarize the main ideas of this algorithm.

The algorithm consists in applying (19) repeatedly in a way which will lead us to a configuration that can be simplified using (16). This is achieved by successively reducing the number of thick edges between the last two strands (i.e. furthest to the right), because when this number is one we are able to perform (16) (in example 4.3, we only needed to apply (19) once before we were able to simplify the graph). The main part of the algorithm consists of looking into the band between the two uppermost thick edges in the last two strands and search, in a special way, for a configuration where we can perform (19); we repeat this with the resulting graph until we reach a graph where we can apply (19) with the two thick edges that determined the band. Doing this we reduce by one the number of thick edges between the last two strands.