Detection of Outer Sound Sources Through Measurements of Amplitude on a Body Surface

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Abstract

The problem of detecting radiating sources without using direct observation is an old problem in the applied sciences. In this work, a detailed study is made about the exterior two dimensional problem of determination of sound sources by reading boundary data on a measuring object. We solve the two dimensional exterior Helmholtz problem using the Method of Fundamental Solutions, and we determine the position and intensity of unknown sources using the Reciprocity Functional coupled with the Levenberg-Marquardt algorithm; alternative efficient methods are presented for the simplified case of one sound source. Accurate results are obtained for at most three sources, with convergence problems for more sources.
Key Words: Acoustic sources, Source detection, Inverse problems, Method of fundamental solutions.

1 Introduction

In this work we consider the problem of detecting acoustic point-sources by boundary measurements on some body surface. The number and location of these point-sources is unknown, and only the wavenumber is assumed to be known. Therefore the unknown outer sources generate a time-harmonic incident wave of the form $U(x,t) = e^{-i\omega t}u^{inc}(x)$, where ω is the frequency and $k = \omega/C$ is the wavenumber (being C the propagation speed of the wave),

$$u^{inc}(x) = \sum_{j=1}^{n} c_i \Phi(x - s_i)$$

here Φ is the fundamental solution of the Helmholtz equation, thus verifying$(\Delta + k^2)u^{inc} = 0$, outside the source points. However, the only possible measurements are of the total wave $u = u^{inc} + u^{sc}$.

Depending on the measurement body, different boundary conditions are then to be considered – Dirichlet (sound soft), Neumann (sound hard) or Robin (impedance) problems. In this work we will focus on the Dirichlet problem, and therefore the total wave u vanishes on the sound soft boundary.

In this context, possible applications consist in determining the location of unknown sources that generate an anomalous noise associated with mechanical defects, using measurements on a surface of a body.

In this work, we begin by discussing the direct problem, solved numerically using the Method of Fundamental Solutions (MFS), already used in other contexts (as in [7],[6]). Then, we discuss the inverse problem, and we deduce a numerical method based on a Reciprocity Functional, and we finalize the work with some numerical simulations and results.

2 Direct Problem

Suppose we have n sound sources, with known positions $s_j \in \mathbb{R}^2$, $j = 1, \ldots, n$, and magnitudes c_j, $j = 1, \ldots, n$. Let Ω be a simply-connected bounded non-empty open set, and Γ its boundary; Ω represents the body of measurement, that we will assume to be sound-soft.

Let us define $u := u^{sc} + u^{inc}$ as the total field, with u^{inc} being the incident field (generated by the sound sources, plus some background field), and u^{sc} being the scattered field. We therefore have $u^{inc}(x) = \sum_{j=1}^{n} c_j \Phi_{s_j}(x) + u^{bg}(x), \quad x \in \mathbb{R}^2$, in which

$$\Phi(x) = \frac{i}{4} H_0^{(1)}(k|x|), \quad \Phi_y(x) = \Phi(y - x) \tag{1}$$

Φ is the fundamental solution of the Helmholtz equation, i.e., $\Delta\Phi + k^2\Phi = -\delta$, with δ being Dirac's delta; $H_0^{(1)}$ is the Hankel function of first kind, defined by $H_0^{(1)}(x) = J_0^{(1)}(x) + iY_0^{(1)}(x)$, where $J_0^{(1)}(x)$ and $Y_0^{(1)}(x)$ are respectively Bessel Functions of the First and Second Kind. The known background field is denoted by u^{bg}. Supposing every considered field is radiating, and that the body is sound-soft, the scattered field is u^{sc} verifying by the following equations:

$$\Delta u^{sc} + k^2 u^{sc} = 0, \qquad \mathbb{R}^2 \backslash \bar{\Omega} \tag{2}$$

$$u^{sc} = -\sum_{j=1}^{n} c_j \Phi_{s_j} - u^{bg}, \qquad \Gamma \equiv \partial\Omega \tag{3}$$

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^{sc}(x)}{\partial r} - iku^{sc}(x)\right) = 0, \tag{4}$$

And the total field verifies:

$$\Delta u + k^2 u = -\sum_{j=1}^{n} c_j \delta_{s_j}, \qquad \mathbb{R}^2 \backslash \bar{\Omega} \tag{5}$$

$$u = 0, \qquad \Gamma \equiv \partial\Omega \tag{6}$$

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u(x)}{\partial r} - iku(x)\right) = 0 \tag{7}$$

Solving these equations corresponds to solving an well-posed (see [2], [4]) exterior PDE problem with Dirichlet boundary conditions. The Method of Fundamental Solutions, used to solve this equation, is derived from the theory of boundary integral equations (see [3], [5]), and not from regular PDE theory.

2.1 Numerical Solution

We will use the Method of Fundamental Solutions to approximate the solution (as in [1], [5], [6]). This method, like other boundary methods is adapted to solve exterior problems, due to not requiring meshing.

2.1.1 Density

The following density result justifies the approximation by fundamental solution (point-sources):

Theorem 1 *Let Ω be an open with compact closure, Γ its boundary, and consider $\gamma = \partial\omega$, where ω is an open set with C^1 boundary, $\bar{\omega} \in \Omega$. Furthermore, assume that k^2 is not an eigenfrequency of the Laplace-Dirichlet operator for the domain ω. The space generated by $\{\Phi_y, y \in \gamma\}$ is dense in $L^2(\Gamma)$.*

In general, this result tells us that we can approximate $L^2(\Gamma)$ functions if we have "enough" functions of the form Φ_y, with y belonging to the boundary γ of some open set ω. Nevertheless, the Theorem does not give us any convergence properties for this form of approximation, therefore we must confirm its quality experimentally.

2.1.2 Method of Fundamental Solutions

We consider k such that k^2 is not an eigenvalue of the Laplace-Dirichlet operator for the chosen ω, and we assume Ω to be B_1 (unit circle). We want to approximate u by $\tilde{u}$ such that it solves equations (1) and (3), and such that it minimizes $\|\tilde{u} - u^{sc}\|_{2,\Gamma} = \sqrt{\int_\Gamma |\tilde{u} - u^{sc}|^2 \mathrm{ds}}$.

Define $\gamma = r\Gamma, r \in (0,1)$. According to Theorem 1, it is possible to approximate our exact solution u^{sc} by $\tilde{u} = \sum_{l=1}^{m} d_l \Phi_{y_l}$, with $y_l \in \gamma$, $l = 1, \ldots, m$, fitting the complex valued d_l. We recall that each Φ_{y_l} solves (1) and (3).

In order to calculate error, we must approximate $\|\tilde{u}-u^{sc}\|_{2,\Gamma}^2 = \int_\Gamma |\tilde{u} - u^{sc}|^2 \mathrm{ds}$. We do so by picking N evenly distributed points $z_j = 1, \ldots, N$ over Γ, and by approximating the boundary by straight lines. Likewise, we may use a first-order rule to approximate numerically the L^2 norm.

We do not have access to a measure of relative error, so we measure error using *scaled error*:

$$e_{sc} := \frac{\|\tilde{u} - u^{sc}\|_{2,\Gamma}}{\|\partial_n \tilde{u}\|_{2,\Gamma}} \tag{8}$$

Assume $m < N$ (fitting m functions in N points); we will consider $N = 2m$. Fitting $\tilde{u}$ to u is equivalent to solving the overdetermined linear system $\tilde{u}(z_j) = -u^{inc}(z_j)$, $j = 1, \ldots, N$, which can be represented by the following linear system:

$$[A_{ij}][d_j] = -[u^{inc}(z_i)] \tag{9}$$

with A_{ij} given by $\Phi_{y_j}(z_i)$, $i = 1, \ldots, N, j = 1, \ldots, m$. This system does not in general have a solution; we can, however, solve it using a least squares approach, which involves an ill-conditioned system; regularization procedures are to be considered, and in particular, we use Tikhonov Regularization, which consists in solving the linear system given by:

$$(\alpha I + [A_{ij}]^T [A_{ij}]).[d_j] = -[A_{ij}]^T.[u^{inc}(z_i)] \tag{10}$$

with α determined such that there is a balance between good conditioning (for larger α) and ill-conditioning, also due to the least squares solution (for small α). In this work, we chose to increase α, *ad hoc*, until the condition number of $\alpha I + A^* A$ is lesser than 10^{15} (for machine precision of 10^{-16}), in which case it is considered to be well conditioned.

2.2 Numerical Results

We tested the MFS, considering $\Omega = B_1$, $\gamma = r\Gamma$, $r = 0.7$.

In the context of the Inverse Problem, we need to solve the Direct Problem at most once, and so the most important goal is to minimize the scaled error below a fixed threshold, at the expense of as many fundamental solutions as needed to obtain those results. The following results were obtained:

- For a varying number of fundamental solutions in the MFS, results improved exponentially with a growing number of fundamental solutions, but the results were limited by ill-conditioning, even for improved results using Tikhonov's Regularization;

m	10	20	40	80	160
e^1_{sc} MFS	0.0039	7.44e-005	4.68e-008	4.80e-009	2.63e-007
e^1_{sc} MFS-R	-	-	-	1.01e-009	6.59e-010
e^2_{sc} MFS	0.0053	9.86e-005	5.90e-008	4.13e-009	2.05e-007
e^2_{sc} MFS-R	-	-	-	7.4863e-010	8.15e-010
e^3_{sc} MFS	0.0029	7.16e-005	7.74e-007	6.21e-009	1.22e-007
e^3_{sc} MFS-R	-	-	-	5.01e-009	7.99e-009

Table 1: Table of scaled error for the MFS using Tikhonov regularization (MFS-R) or without (MFS). The number of sources m changes from 10 up to 160.

- The performance was independent from number, location, and intensity of the sound sources;

- For low k, the system is more ill-conditioned, which is solved by adequate conditioning;

- Results for existent background fields (which are, themselves, generated by fundamental solutions inside Ω) are poorer than previous results; it is possibly due to the similar nature between the approximated solution (composed by fundamental solutions centered inside Ω) and the background field (which is generated the same way).

m	10	20	40	80	160
e^1_{sc}	0.0310	0.0151	0.0095	0.0049	0.0036
e^2_{sc}	0.0383	0.0201	0.0089	0.0082	1.17e-009

Table 2: Table of scaled error for MFS, for a background field, depending on m.

3 Inverse Problem

In the case of the Inverse Problem, we do not have full knowledge of the system of equations - the number, position, and intensity of the sound sources are unknown - and our aim is to recover this information using boundary data.

The Inverse Problem raises different and sometimes more complicated issues than the Direct Problem: we must determine if the given boundary data is sufficient for determination, we may have boundary data not related to the sound sources, we may have noise, and we face a non-linear optimization problem.

3.1 Inverse Problem

The Inverse Problem we address consists in finding c_j, s_j and n verifying (5)–(7).

The incident field is generated by the sound sources, and also by a background field, u^{bg}, which is a solution to

$$\Delta u^{bg} + k^2 u^{bg} = 0, \qquad \mathbb{R}^2 \backslash \bar{\Omega} \tag{11}$$

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^{bg}(x)}{\partial r} - iku^{bg}(x)\right) = 0, \tag{12}$$

which is not enough to determine it uniquely, and therefore it is unknown. As additional data, we know $g \equiv \partial_n u$ along $\Gamma^* \subset \Gamma$ (we assume Γ^* is an open set in the Γ topology).

Our aim is to determine an unknown number of n sources, which ammounts to $4n$ unknown real variables.

The first question we pose is if the same boundary data can be generated by two different sets of sources.

Theorem 2 *Let u_1 be a solution of*

$$\Delta u_1 + k^2 u_1 = -\sum_{j=1}^{n_1} c_j \delta_{s_j}, \qquad \mathbb{R}^2 \backslash \bar{\Omega} \tag{13}$$

$$u_1 = 0, \qquad \Gamma \equiv \partial\Omega \tag{14}$$

$$\partial_n u_1 = g, \qquad \Gamma^* \tag{15}$$

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u_1(x)}{\partial r} - iku_1(x)\right) = 0 \tag{16}$$

and u_2 be a solution of

$$\Delta u_2 + k^2 u_2 = -\sum_{j=1}^{n_2} \tilde{c}_j \delta_{\tilde{s}_j}, \qquad \mathbb{R}^2 \backslash \bar{\Omega} \tag{17}$$

$$u_2 = 0, \qquad \Gamma \equiv \partial\Omega \tag{18}$$

$$\partial_n u_2 = g, \qquad \Gamma^* \tag{19}$$

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u_2(x)}{\partial r} - iku_2(x)\right) = 0 \tag{20}$$

Then $n_1 = n_2$, $c_j = \tilde{c}_j$, $s_j = \tilde{s}_j$,$j = 1, \ldots, n_1$, and $u_1 \equiv u_2$.

Uniqueness is as far as we can go in regards to well-posedness. The solution does not depend continuously on the data, and existence of solution must be assumed.

3.2 Reciprocity Functional

In this work, we solve the Inverse Problem using the Reciprocity Functional. We assume in this subsection that $\Gamma^* = \Gamma$, and we do the same for the remainder of the work. We define the Functional of Reciprocity:

$$A(v) := \int_\Gamma v\partial_n u - u\partial_n v \text{ ds} \tag{21}$$

for which the domain is defined by the *test functions*' space V, composed by radiating solutions to the homogeneous Helmholtz equation. In order to use Green's Theorem, we define also

$$A_R(v) := \int_{\Gamma_R - \Gamma} u\partial_n v - v\partial_n u \text{ ds} \tag{22}$$

where $\Gamma_R = \{x \in \mathbb{R}^2 : |x| = R\}$, R large enough such that $s_k \in B_R \backslash \bar{\Omega}$ for all k, and $v \in V$. We have that $\partial_n u = \partial_r u$ over Γ_R. Using the Green's Theorem, as well as the fact that both v and u are radiating, and taking R to the infinity, we get $A_R \to A$, and:

$$A(v) = \int_\Gamma v\partial_n u \text{ ds} = \sum_{j=1}^{n} c_j v(s_j) \tag{23}$$

$$\Leftrightarrow \int_\Gamma v\partial_n u \text{ ds} - \sum_{j=1}^{n} c_j v(s_j) = 0, \forall v \in V \tag{24}$$

We can prove a stronger result:

Theorem 3 *Let u be such that*

$$u = 0, \qquad \Gamma \equiv \partial\Omega \tag{25}$$

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u(x)}{\partial r} - iku(x)\right) = 0. \tag{26}$$

And let k be such that k^2 is not an eigenfrequency for the Laplace-Dirichlet operator for ω. Then u is a solution to

$$\Delta u + k^2 u = -\sum_{j=1}^{n} c_j \delta_{s_j}, \qquad \mathbb{R}^2 \backslash \bar{\Omega} \tag{27}$$

iff

$$\int_\Gamma v\partial_n u \; ds = \sum_{j=1}^{n} c_j v(s_j), \qquad \forall v \in V \tag{28}$$

Based on the above result, we can define the main method of this work, the *Reciprocity Method*. We minimize the error in (28) for a growing number of hypotethical sources, for several v, instead of minimizing $\|\partial_n u - \partial_n \tilde{u}\|_{\Gamma,2}$. We start by minimizing (28) for one source, and then we add sources until we find an approximate zero. More precisely, for each step in which we are searching for m sources (which may be different from the real number n of sources), we aim to minimize

$$f_m(\bar{c}, \bar{s}) := \sum_{k=1}^{M} r_k(\bar{c}, \bar{s})^2 \tag{29}$$

where $\bar{c} = (c_1^1, c_1^2, \ldots, c_m^1, c_m^2)$, with c_j^1 and c_j^2 being respectively the real and imaginary parts of c_j, and $\bar{s} = (s_1^1, s_1^2, \ldots, s_m^1, s_m^2)$, with s_j^1 and s_j^2 being the first and second coordinates of s_j, and

$$r_k(\bar{c}, \bar{s}) = |\int_\Gamma \partial_n u v_k \text{ ds} - \sum_{j=1}^{m} c_j v_k(s_j)| \tag{30}$$

With each $v_k \in V$, $k = 1, \ldots, M$. This minimization method is called the *Reciprocity Method*. The search for a growing number of sources, using the Reciprocity Method, until an optimal solution is found, is called the *Reciprocity Algorithm*.
The Reciprocity Algorithm provides us with two features. We do not need explicit values for $\partial_n \tilde{u}$, and therefore we do not need to solve the direct problem. We also do not need to separate the field generated by the sound sources and by background sources; the value of $\int_\Gamma v \partial_n u$ ds is invariant by those background fields.

If we suppose we have a single source with position s and constant c, simplifications are available. It is possible to estimate only the position and calculate the constant afterwards. We have access to the following identity for v, w in V:

$$v(s) \int_\Gamma w \partial_n u \text{ ds} = w(s) \int_\Gamma v \partial_n u \text{ ds} \tag{31}$$

And therefore, in theory, we can find our position s using only two test functions, v and w, by minimizing

$$f(s) := \left| v(s) \int_\Gamma w \partial_n u \text{ ds} - w(s) \int_\Gamma v \partial_n u \text{ ds} \right| \tag{32}$$

and calculating c afterwards. For more test functions, we have:

$$f_M(s^1, s^2) := \sum_{k=1}^{M} r_k(s^1, s^2)^2 \tag{33}$$

with

$$r_k(s) = \left| v_{k,1}(s) \int_\Gamma v_{k,2} \partial_n u \text{ ds} - v_{k,2}(s) \int_\Gamma v_{k,1} \partial_n u \text{ ds} \right| \tag{34}$$

where $v_{k,1}$, $v_{k,2}$ are functions from V.

4 Simulations

4.1 Minimization

The function to be minimized in the case of the Single Source, is given by (33), while the function to be minimized in the general case is given by (29). Furthermore, functions in the function space V, which, we recall, is the space of radiating solutions to the homogeneous Helmholtz equation can be represented by sums of functions of the form Φ_z, with $z \in \gamma' = r'\Gamma$ (due to the density result in subsubsection 1).

In order to minimize any of these functions we use either the Levenberg-Marquardt (LM) method (see [8], [9]), or a Modified Levenberg-Marquardt method (MLM, for which the difference in relation to LM is that λ is never updated to increase), which behaves more like Gauss-Newton. For any of these methods, a finite difference approximation to the Hessian is made.

4.2 Initial Data

In order to have boundary data, we have to solve the Direct Problem, for selected sources. For solving it we use the Method of Fundamental Solutions presented in Chapter 1, using a curve $\gamma = r\Gamma$, $r \in (0,1)$. Our aim is not to test once again the MFS, but to solve the Inverse Problem, so any initial data will be generated with as much precision as possible, regardless of speed or computational work.

Noise is introduced in the data, point-wise:

$$\partial_n u^\delta(x_k) = \partial_n u(x_k)(1 + \epsilon R_k), \quad k = 1, \ldots, N \tag{35}$$

(with R_k being a random complex number with absolute value between 0 and 1), which produces a perturbed solution with relative error of order ϵ.

We note that, although we have no (known) regularization process for smoothing or minimizing the noise's effect, the nature of our function to be minimized, which is given by integrals, may introduce some natural regularization for the noise, because introduced error in $\partial_n u$ is averaged by the integrals in r_k, reducing the effects of pontual noise.

4.3 One Sound Source

Tests for one sound source were performed, using the reciprocity method. Relative error for the sources is calculated using the norm

$$e_{rel} = \frac{\|\mathbf{c} - \tilde{\mathbf{c}}\|_{\ell^2} + \|\mathbf{s} - \tilde{\mathbf{s}}\|_{\ell^2}}{\|\mathbf{c}\|_{\ell^2} + \|\mathbf{s}\|_{\ell^2}}. \tag{36}$$

Furthermore, we fix $\Omega = B_1$, the sources' intensities at magnitude 1, and null background field. Results were the following:

- The simplification available for a single source makes the convergence much faster, but diverges if the initial iterate is not in a good initial direction;

c_0	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)
s_0	(1,2)	(2,1)	(2,0)	(0,-2)	(0,2)
e_{rel} simplified	2.8438e-006	1.3560e-005	*	*	*
Iterations	7	7	*	*	*
e_{rel} LM	0.0060	2.61e-006	3.32e-004	0.0066	0.0061
Iterations	464	60	261	1862	331
e_{rel} MLM	3.45e-004	1.96e-005	2.68e-005	3.82e-005	2.15e-005
Iterations	28	11	10	49	13

Table 3: Comparison of performance between simplified LM for single source, LM, and Modified LM, for $s = (3, 4)$. An asterisk is given for non-convergence.

- LM showed to be a slower but almost always converging method, and MLM was a much faster method which does not have reliable convergence; LM was used for single sources.
- The method converges slower for sources farther from the measurement body; for sources with radiuses $r > 10$, iteration limit is probably achieved, posing a problem for the method;
- Noise affects proportionately the error. However, given sufficient relaxation of the stopping criteria, convergence does not suffer with noise;
- The number of test functions used does not affect significantly the results;

4.4 Multiple sources

4.4.1 Initial estimate vs. no initial estimate

For multiple sources, there are added problems of convergence speed, for the LM method. We perform some tests, in which we consider no background field, introduced noise equal to $\delta = 10^{-4}$, $k = 1$, and $M = 50$, and random sources. Results for initial estimates and no estimates are presented in the table below. For the case of no initial estimate, the initial sources are fixed, with evenly spaced starting points on the circumference of radius 3, and initial constants equal to 1. For the case of the initial estimate, the starting points are assumed to verify $|\tilde{s}_k - s_k| \leq 2$, for each k. The method is reliable for up to three sources,

Number of sources	No estimate	Estimate
1	$95,2\%$	100%
2	$65,8\%$	$90,5\%$
3	$54,9\%$	$73,7\%$
4	$3,1\%$	$52,2\%$

Table 4: Table with percentages of sucess (relative error lesser than 10^{-1}, for random sources.

provided an initial guess is given; otherwise, convergence is obtained for three or less sources.

4.5 Conclusions

The MFS is an effective method for approximating the exact solution on Γ, but the approximation cannot get indefinetely better, due to ill-conditioning. For fields generated by the outside sources, MFS behaves independently of the given sources. However, for the considered background fields, results might be poorer.

The Reciprocity Algorithm is reliable for finding three or less sources, provided an initial estimate is given. Without initial guesses, convergence for four or more sources is nearly impossible, and for three or more sources is unreliable. Therefore, the Reciprocity Algorithm, as presented, is not suited for finding more than three sources.

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