Modelling of Wave Energy Conversion

by

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1. Introduction to wave energy conversion

1.1. Introduction

1.1.1. The early years

The energy from ocean waves is the most conspicuous form of ocean energy, possibly because of the, often spectacular, wave destructive effects. The waves are produced by wind action and are therefore an indirect form of solar energy.

The possibility of converting wave energy into usable energy has inspired numerous inventors: more than one thousand patents had been registered by 1980 [1.1] and the number has increased markedly since then. The earliest such patent was filed in France in 1799 by a father and a son named Girard [1.2].

Several reviews on wave energy conversion have been published in book form, as conference and journal papers, and as reports. One should mention first the pioneering book by McCormick [1.1] published in 1981 (reprinted in 2007), and also the books by Shaw [1.3], Charlier and Justus [1.4] (their long chapter on wave energy was probably completed by 1986), Ross [1.2] (written from a non-technical point of view by a freelance journalist), Brooke [1.5] and Cruz [1.6]. A report prepared in 1999 for the UK Department of Energy [1.7] and the final report [1.8] from the European Thematic Network on Wave Energy (a project sponsored by the European Commission) provide abundant information on the state-of-the-art at the time. Shorter reviews can be found in [1.9-1.15]. An overview about the current status of wave energy conversion can be found in [1.16]. This chapter is to a large extent an updated and more extensively illustrated version of some parts of [1.14].

Fig. 1.1. Commander Yoshio Masuda (right) with Dr A.W. Lewis, in 2001 (courtesy of A.W. Lewis, University College Cork).

Yoshio Masuda (1925-2009) (Fig. 1.1), a former Japanese navy officer, may be regarded as the father of modern wave energy technology, with studies in Japan since the 1940s. He developed a navigation buoy powered by wave energy, equipped with an
air turbine (Fig. 1.2), which was in fact what was later named as a (floating) oscillating water column (OWC). These buoys were commercialized in Japan since 1965 (and later in USA) [1.17,1.18]. Later, in Japan, Masuda promoted the construction, in 1976, of a much larger device: a barge (80m×12m), named Kaimei (Fig. 1.3), used as a floating testing platform housing several OWCs equipped with different types of air turbines [1.19]. Probably because this was done at an early stage when the science and the technology of wave energy conversion were in their infancy, the power output levels achieved in the Kaimei testing program were not a great success.

Fig. 1.2. Outline of Japanese navigation buoy equipped with air turbine (based on [1.18]).

Fig. 1.3. Japanese wave energy converter Kaimei.

The oil crisis of 1973 induced a major change in the renewable energies scenario and raised the interest in large-scale energy production from the waves. A paper published in 1974 in the prestigious journal Nature by Stephen Salter [1.20], of the University of Edinburgh, became a landmark and brought wave energy to the attention of the international scientific community. The British Government started in 1975 an important research and development program in wave energy (Fig. 1.4) [1.21], followed shortly afterwards by the Norwegian Government. The first conferences devoted to wave energy took place in England (Canterbury, 1976, and Heathrow, 1978). This was followed in 1979 by two more genuinely international conferences: Power from Sea Waves (Edinburgh, June) and the First Symposium on Wave Energy Utilization (Gothenburg, October-November). The Second International Symposium on Wave
Energy Utilization (Trondheim, Norway, 1982) coincided with a marked decline in Government funding of the British wave energy program.

Fig. 1.4. Some of the devices whose development was funded by the British wave energy program 1975-82. Clockwise, from top left: the Cockrell raft, the Salter duck, the Bristol cylinder and the NEL oscillating water column [1,21].

Fig. 1.5. Shoreline prototypes installed in 1985 in Toftestallen, Norway: 500 kW

In Norway the activity went on to the construction, in 1985, of two full-sized (350 and 500 kW rated power) shoreline prototypes at Toftestallen, near Bergen (Fig. 1.5). In
the following years, until the early 1990s, the activity in Europe remained mainly at the academic level, the most visible achievement being a small (75 kW) OWC shoreline prototype deployed at the island of Islay, Scotland (commissioned in 1991) (Fig. 1.6) [1.22]. In 1990, two OWC prototypes were constructed in Asia: a 60 kW converter integrated into a breakwater at the port of Sakata, Japan, (Fig. 1.7) [1.23] and a bottom-standing 125 kW plant at Trivandrum, India, (Fig. 1.8) [1.24].

![Fig. 1.6. 75 kW oscillating water column prototype installed in 1991 on the island of Islay, Scotland, UK.](image1)

![Fig. 1.7. OWC plant integrated into a breakwater at Sakata harbour, Japan, 1990. Rated power 60 kW.](image2)

The wave energy absorption is a hydrodynamic process of considerable theoretical difficulty, in which relatively complex diffraction and radiation wave phenomena take place. This explains why a large part of the work on wave energy published in the second half of the 1970s and early 1980s was on theoretical hydrodynamics, in which several distinguished applied mathematicians took leading roles, with special relevance to Johannes Falnes, in Norway, and David V. Evans, in UK. For a review of early work, see [1.25,1.26].

In the development and design of a wave energy converter, the energy absorption may be studied theoretically/numerically, or by testing a physical model in a wave basin or wave flume. The techniques to be applied are not very different from those in the hydrodynamics of ships in a wavy sea. Numerical modelling is to be applied in the first
stages of the plant design. This is in most cases based on linear water wave theory, the main limitations of which lie in its being unable to account for losses in water due to real (viscous) fluid effects (large eddy, turbulence) and not being capable to model accurately large amplitude water oscillations (nonlinear waves). Such effects are known to be important (they also occur in naval engineering and in off-shore structures, where more or less empirical corrections are currently applied). For these reasons, model tests (scales 1:80 to 1:10) are carried out in wave basin when the final geometry of the plant is already well established. Stephen Salter is widely regarded as the pioneer in model testing of wave energy converters. In 1974 he started the experimental development of the “duck” concept in a narrow wave flume at the University of Edinburgh [1,27] (Fig. 1.9).

Salter’s experimental facilities were greatly improved with the construction, in 1977, of the 10 m × 27.5 m × 1.2 m “wide tank” equipped with 89 independently driven paddles, that made Edinburgh the leading centre for the experimental development in wave energy conversion (for detailed information, including early photographs, see [1,27]). Later, as the development of wave energy converter concepts progressed towards the prototype construction stage, the need of larger-scale testing required the use of very large laboratory facilities. This was the case, in Europe, of the large wave tanks in Trondheim (Norway), Wageningen (Netherlands) and Nantes (France).
The utilization of wave energy involves a chain of energy conversion processes, each of which is characterized by its efficiency as well as the constraints it introduces, and has to be controlled. Particularly relevant is the hydrodynamic process of wave energy absorption. The early theoretical studies on oscillating-body and OWC converters revealed that, if the device is to be an efficient absorber, its own frequency of oscillation should match the frequency of the incoming waves, i.e. it should operate at near-resonance conditions. The ignorance of this rule underlies many failures by inventors who regarded such systems as quasi-static (i.e. simply follow the wave surface motion) rather than dynamic. In practice, the frequency-matching meets with serious difficulties: (i) in most cases, except if the body or the OWC is quite large (this meaning possibly sizes substantially larger than ten metres), its own frequency of oscillation is too high as compared with typical ocean-wave frequencies; (ii) real waves are not single-frequency. Acting on the power take-off system (PTO) to achieve resonance has been named phase-control. Several phase-control strategies have been proposed, including for devices in real irregular waves (for a review, see Falnes, [1.28]). A control method that avoids the energy flow reversal was proposed by Budal and Falnes [1.29] (see also [1.30]) and consists in latching the device in a fixed position during certain intervals of the wave cycle so as to achieve approximate optimal phase control. Apart from the pioneers Falnes and Budal, phase control (including latching) was the object of theoretical studies from other researchers, namely Naito and Nakamura [1.31], who established the relation between causality and optimum control of wave energy converters, Nancy Nichols and her co-workers [1.32,1.33], who applied the maximum principle of Pontryagin to numerically solve the problem, and Korde [1.34] who studied the phase control of converters with several degrees of freedom. Optimal phase control in real random waves and its practical implementation in wave energy converters remain an open problem.

1.1.2. From the 1990s up to now

The situation in Europe was dramatically changed by the decision made in 1991 by the European Commission of including wave energy in their R&D program on renewable energies. The first projects started in 1992. Since then, more than thirty projects on wave energy were funded by the European Commission involving a large number of teams active in Europe. A few of these projects took the form of coordination activities, namely one in 2000-2003 with 18 partners and, more recently (2004-2007), the Coordination Action in Ocean Energy, with forty partners. Also sponsored (and in some cases partly funded) by the European Commission were a series of European Wave Energy Conferences (the more recent ones including also Tidal Energy): Edinburgh, UK (1993), Lisbon, Portugal (1995), Patras, Greece (1998), Aalborg, Denmark (2000), Cork, Ireland (2003), Glasgow, UK (2005), Porto, Portugal (2007), Uppsala, Sweden (2009), Southampton, UK (2011), Aalborg, Denmark (2013). The equally biennial International Conference on Ocean Energy, in which commercial, economic and environmental issues were object of special attention, took place in Bremerhaven, Germany (2006), Brest, France (2008), Bilbao, Spain (2010) and Dublin, Ireland (2012). Sessions on ocean energy (with a major or dominant contribution of papers on wave energy) are becoming increasingly frequent in annual conferences on ocean engineering (namely the OMAE and ISOPE conferences) and on energy (the case of the World Renewable Energy Congresses).

In 2001, the International Energy Agency established an Implementing Agreement on Ocean Energy Systems (IEA-OES, presently with 19 countries as contracting
parties) whose mission is to facilitate and co-ordinate ocean energy research, development and demonstration through international co-operation and information exchange. Surveys of ongoing activities in wave energy worldwide can be found in the IEA-OES annual reports.

In the last few years, growing interest in wave energy is taking place in northern America (USA and Canada), involving the national and regional administrations, research institutions and companies, and giving rise to frequent meetings and conferences on ocean energy [1.35,1.36].

The main disadvantage of wave power, as with the wind from which it originates, is its (largely random) variability in several time-scales: from wave to wave, with sea state, and from month to month (although patterns of seasonal variation can be recognized). The assessment of the wave energy resource is a basic prerequisite for the strategic planning of its utilization and for the design of wave energy devices. The characterization of the wave climate had been done before for other purposes, namely navigation, and harbour, coastal and offshore engineering (where wave energy is regarded as a nuisance), for which, however, the required information does not coincide with what is needed in wave energy utilization planning and design. The studies aiming at the characterization of the wave energy resource, having in view its utilization, started naturally in those countries where the wave energy technology was developed first. In Europe, this was notably the case of the United Kingdom [1.37,1.38]. When the European Commission decided, in 1991, to start a series of two-year (1992-93) Preliminary Actions in Wave Energy R&D, a project was included to review the background on wave theory required for the exploitation of the resource and to produce recommendations for its characterization [1.39]. The WERATLAS, a European Wave Energy Atlas, also funded by the European Commission, was the follow-up of those recommendations [1.40]. The WERATLAS remains the basic tool for wave energy planning in Europe. Reviews on wave energy resource characterization can be found in [1.41,1.42].

1.1.3. The technologies

Unlike large wind turbines, there is a wide variety of wave energy technologies, resulting from the different ways in which energy can be absorbed from the waves, and also depending on the water depth and on the location (shoreline, near-shore, offshore). Recent reviews identified about one hundred projects at various stages of development. The number does not seem to be decreasing: new concepts and technologies replace or outnumber those that are being abandoned.

In general, the development, from concept to commercial stage, has been found to be a difficult, slow and expensive process. Although substantial progress has been achieved in the theoretical and numerical modelling of wave energy converters and of their energy conversion chain, model testing in wave basin — a time-consuming and considerably expensive task — is still essential. The final stage is testing under real sea conditions. In almost every system, optimal wave energy absorption involves some kind of resonance, which implies that the geometry and size of the structure are linked to wavelength. For these reasons, if pilot plants are to be tested in the open ocean, they must be large structures. For the same reasons, it is difficult, in the wave energy technology, to follow what was done in the wind turbine industry (namely in Denmark): relatively small machines where developed first, and were subsequently scaled up to larger sizes and powers as the market developed. The high costs of constructing, deploying, maintaining and testing large prototypes under sometimes very harsh
environmental conditions, has hindered the development of wave energy systems; in most cases such operations were possible only with substantial financial support from governments (or, in the European case, from the European Commission).

Several methods have been proposed to classify wave energy systems, according to location, to working principle and to size ("point absorbers" versus "large" absorbers). The classification in Fig. 1.10 is based mostly on working principle. The examples shown are not intended to form an exhaustive list and were chosen among the projects that reached the prototype stage or at least were object of extensive development effort. The device categories shown in Fig. 1.10 are addressed hereafter.

![Diagram of wave energy device categories](image)

**Fig. 1.10. Various wave energy technologies.**

### 1.2. The oscillating water column (OWC)

#### 1.2.1. Fixed-structure OWC

Based on various energy-extracting methods, a wide variety of systems has been proposed but only a few full-sized prototypes have been built and deployed in open coastal waters. Most of these are or were located on the shoreline or near shore, and are sometimes named first generation devices. In general these devices stand on the sea bottom or are fixed to a rocky cliff. Shoreline devices have the advantage of easier installation and maintenance, and do not require deep-water moorings and long underwater electrical cables. The less energetic wave climate at the shoreline can be partly compensated by natural wave energy concentration due to refraction and/or diffraction (if the device is suitably located for that purpose). The typical first generation device is the oscillating water column. Another example is the overtopping device Tapchan (Tapered Channel Wave Power Device), a prototype of which was built on the Norwegian coast in 1985 and operated for several years (see section 4 and Fig. 1.5).
The oscillating water column (OWC) device comprises a partly submerged concrete or steel structure, open below the water surface, inside which air is trapped above the water free surface (Fig. 1.11). The oscillating motion of the internal free surface produced by the incident waves makes the air to flow through a turbine that drives an electrical generator. The axial-flow Wells turbine was invented in 1976 by Alan A. Wells (1924-2005) (at that time professor at Queen’s University, Belfast, UK) for OWC applications [1.43] and has the advantage of not requiring rectifying valves (it is self-rectifying). It has been used in most prototypes (Fig. 1.12). The most popular alternative to the Wells turbine seems to be the self-rectifying impulse turbine, patented by I.A. Babinsten in 1975 [1.44]. Its rotor is basically identical to the rotor of a conventional single-stage steam turbine of axial-flow impulse type (the classical de Laval steam turbine patented in 1889). Different versions of both self-rectifying turbines have been developed and constructed [1.14,1.18,1.18a].

Fig. 1.11. Cross-sectional view of a bottom- standing OWC (Pico plant).

Fig. 1.12. Wells turbine with double row of guide vanes (400 kW Pico plant, Azores, Portugal, 1999).

The OWC was one of the technologies whose development was funded by the British wave energy program in the second half of the 1970s: the so-called NEL (National Engineering Laboratory) oscillating water column device was made up of a series of bottom-standing OWC chambers arranged as a terminator (its longest dimension parallel to the wave crests during normal operation) (Fig. 1.4) [1.45]. Full sized OWC prototypes were built in Norway (in Toftestallen, near Bergen, 1985, Fig. 1.4 [1.46]), Japan (Sakata, 1990, Fig. 1.7 [1.23]), India (Vizhinjam,
Trivandrum, Kerala state, 1990, Fig. 1.8 [1.24]), Portugal (Pico, Azores, 1999, Fig. 1.13 [1.47]), UK (the LIMPET plant in Islay island, Scotland, 2000, Fig. 1.14 [1.48]). The largest

Fig. 1.13. Back view of the 400 kW OWC plant on the island of Pico, Azores, Portugal, 1999.

Fig. 1.14. LIMPET OWC plant, rated 500 kW, installed in 2000 on the island of Islay, Scotland, UK.

Fig. 1.15. 100 kW shoreline OWC built in 2001 in Guangdong Province, China [1.50].
of all, a nearshore bottom-standing plant (named OSPREY) was destroyed by the sea (in 1995) shortly after having been towed and sunk into place near the Scottish coast. In all these cases, the structure is fixed (bottom-standing or built on rocky sloping wall) and the main piece of equipment is the Wells air turbine driving an electrical generator. Except for the OSPREY, the structure was made of concrete. The cross-sectional area of these OWCs (at mid water-free-surface level) lies in the range 80-250 m\(^2\). Their installed power capacity is (or was) in the range 60-500 kW (2 MW for OSPREY). Less powerful shoreline OWC prototypes (also equipped with Wells turbines) were built in Islay, UK, in 1991, Fig. 1.6 (75 kW, [1.49]), and in Guangdong, China, in 2001, Fig. 1.15 (100 kW, [1.50]).

It has been found theoretically [1.51] and experimentally since the early 1980s that the wave energy absorption process can be enhanced by extending the chamber structure by protruding (natural or man-made) walls in the direction of the waves, forming a harbour or a collector. This concept has been put into practice in most OWC prototypes. The Australian company Energetech developed a technology using a large parabolic-shaped collector (shaped like a Tapchan collector) for this purpose; a nearshore prototype, Oceanlinx Mk 1, was tested at Port Kembla, Australia, in 2005 [1.52], (Fig. 1.16).

![Fig. 1.16. Oceanlinx Mk1, a 500 kW nearshore bottom-standing steel-made OWC converter installed at Port Kembla, Australia, in 2005.](image)

The design and construction of the structure (apart from the air turbine) are the most critical issues in OWC technology, and the most influential on the economics of energy produced from the waves. In the present situation, the civil construction dominates the cost of the OWC plant. The integration of the plant structure into a breakwater has several advantages: the constructional costs are shared, and the access for construction, operation and maintenance of the wave energy plant become much easier. This has been done successfully for the first time in the harbour of Sakata, Japan, in 1990 (Fig. 1.7, [1.23]), where one of the caissons making up the breakwater had a special shape to accommodate the OWC and the mechanical and electrical equipment. The option of the “breakwater OWC” was adopted in the breakwater constructed at the port of Mutriku, in northern Spain (2008-10), with 16 chambers and 16 Wells turbines rated 18.5 kW each (Figs 1.17,1.17a, [1.53]). A different geometry for an OWC embedded into a breakwater was proposed by Boccotti [1.54], approaching a quasi-two-dimensional terminator configuration, with an OWC that is long in the wave crest direction but narrow (small
aperture) in the fore-aft direction. The OWC cross-section is J-shaped, with its outer opening facing upwards. A field experiment was carried out about 2005 off the eastern coast of the straits of Messina, in southern Italy [1.55].

Fig. 1.17. Multi-chamber OWC plant integrated into a breakwater, Mutriku harbour, Basque Country, Spain, 2008-10. Eighteen chambers and 18 Wells turbines (rated 18.5 kW each).

Fig. 1.17a. One of the four machine rooms of the Mutriku plant, showing four turbine-generators sets.

1.2.2. Floating-structure OWC

As mentioned above, the first OWC converters deployed in the sea were floating devices developed in Japan in the 1960s and 1970s under the leadership of Yoshio Masuda: the wave-powered navigation buoys and the large Kaimei barge. The Kaimei had thirteen open-bottom chambers built into the hull, each having a water plane area of 42 to 50 m² (Fig. 1.3). It was deployed off the western coast of Japan in 1978-80 and again in 1985-86. Several air turbines were tested, both one-directional (which required the use of non-return rectifying valves) and self-rectifying turbines.
Masuda realized that the wave-to-pneumatic energy conversion of Kaimei was quite unsatisfactory and conceived a different geometry for a floating OWC: the Backward Bent Duct Buoy (BBDB). In the BBDB, the OWC duct is bent backward from the incident wave direction (Fig. 18) (which was found to be advantageous in comparison with the frontward facing duct version) [1.56]. In this way, the length of the water column could be made sufficiently large for resonance to be achieved, while keeping the draught of the floating structure within acceptable limits. The BBDB converter was studied (including model testing) in several countries (Japan, China, Denmark, Korea, Ireland) and was used to power about one thousand navigation buoys in Japan and China [1.50,1.57,1.58]. In the last few years, efforts have been underway in Ireland to develop a large BBDB converter for deployment in the open ocean. A 1:4th-scale 12 m long model equipped with a horizontal-axis Wells turbine (and later an impulse turbine) has been tested in the sheltered sea waters of Galway Bay (western Ireland) since the end of 2006 [1.59], Fig. 1.19.

The Mighty Whale, another floating OWC converter, was developed by the Japan Marine Science and Technology Center. After theoretical investigations and wave tank testing, a full-sized prototype was designed and constructed. The device consists of a floating structure (length 50 m, breadth 30 m, draught 12 m, displacement 4400 t) which has three air chambers located at the front, side by side, and buoyancy tanks [1.60]. Each air chamber is connected to a Wells air turbine that drives an electric generator.
The total rated power is 110 kW. The device was deployed near the mouth of Gokasho Bay, in Mie Prefecture, Japan, in 1998 (Fig. 1.20) and tested for several years.

Fig. 1.20. Mighty Whale, a three-chamber floating OWC equipped with Wells turbines, deployed in 1998 in Gokasho Bay, Japan. Rated power 110 kW.

The Spar Buoy is possibly the simplest concept for a floating OWC. It is an axisymmetric device (and so insensitive to wave direction) consisting basically of a (relatively long) submerged vertical tail tube open at both ends, fixed to a floater that moves essentially in heave. The length of the tube determines the resonance frequency of the inner water column. The air flow displaced by the motion of the OWC relative to the buoy drives an air turbine. Several types of wave-powered navigation buoys have been based on this concept, which has also been considered for larger-scale energy production. The spar buoy is possibly the first wave energy converter type to be object of a detailed theoretical study [1.61, 1.62]. A version of the Spar-buoy is being developed at Instituto Superior Técnico, Lisbon: a 1:10th-scale model was tested in 2012 at NAREC, Northern England, Fig. 1.20a. The Sloped Buoy has some similarities with the Spar Buoy and consists of a buoy with three sloped immersed tail tubes such that the buoy-tube set is made to oscillate at an angle intermediate between the heave and surge directions.

Fig. 1.20a. Model (1:10th-scale) of Spar-buoy tested in 2012 at NAREC, UK.

A report prepared for the British Department of Trade and Industry (DTI) compared three types of floating OWCs for electricity generation in an Atlantic environment: BBDB, Sloped Buoy and Spar Buoy [1.63].
The Australian company Oceanlinx deployed, in 2010, off Port Kembla, Australia, a one-third scale grid-connected model of their most recent OWC device, the Mk3, which (like the Kaimei three decades earlier) is a floating platform with several OWC chambers (in this case eight chambers) each with an air turbine. During the tests only two turbines (of different types) were installed (Fig. 1.21).

Fig. 1.21. One-third-scale Oceanlinx Mk3 multi-chamber floating OWC device. The tests took place off Port Kembla, Australia, in 2010, with two grid-connected turbine-generator sets of different types.

The structures of the floating OWC prototypes briefly described above are slack-moored to the sea bed and so are largely free to oscillate (which may enhance the wave energy absorption if the device is properly designed for that).

1.3. Oscillating body systems

Offshore devices (sometimes classified as third generation devices) are basically oscillating bodies, either floating or (more rarely) fully submerged. They exploit the more powerful wave regimes available in deep water (typically more than 40m water depth). Oscillating bodies produce energy by reacting against the sea bottom (or a fixed structure like a breakwater) or against another oscillating body. Offshore wave energy converters are in general more complex compared with first generation systems. This, together with additional problems associated with mooring, access for maintenance and the need of long underwater electrical cables, has hindered their development, and only recently some systems have reached, or come close to, the full-scale demonstration stage.

1.3.1. Single-body heaving buoys

The simplest oscillating-body device is the heaving buoy reacting against a fixed frame of reference (the sea bottom or a bottom-fixed structure). In most cases, such systems are conceived as point absorbers (i.e. their horizontal dimensions are much smaller than the wavelength).

An early attempt was a device named G-1T, consisting of a wedge-shaped buoy of rectangular planform (1.8 m × 1.21 m at water line level and 1.2 m water draft) whose vertical motion was guided by a steel structure fixed to a breakwater. The used PTO was an early example of the hydraulic ram in a circuit including a hydraulic motor and a gas accumulator. The tests, performed in Tokyo Bay in 1980, are reported in [1.64].

Another early example was the Norwegian buoy, consisting of a spherical floater which could perform heaving oscillations relative to a strut connected to an anchor on the sea bed through a universal joint [1.65]. The buoy could be phase-controlled by
latching and was equipped with an air turbine. A model (buoy diameter = 1 m), in which the air turbine was simulated by an orifice, was tested (including latching control) in the Trondheim Fjord in 1983 (Fig. 1.22).

Fig. 1.22. Norwegian heaving buoy in Trondheim Fjord, 1983 (courtesy of J. Falnes).

An alternative design is a buoy connected to a bottom-fixed structure by a cable which is kept tight by a spring or similar device. The relative motion between the wave-activated float on the sea surface and the seabed structure activates a PTO system. In the device that was tested in Denmark in the early 1990s, the PTO (housed in a bottom-fixed structure) consisted in a piston pump supplying high-pressure water to a hydraulic turbine [1.66].

A version of the taut-moored buoy concept is being developed at Uppsala University, Sweden, and uses a linear electrical generator (rather than a piston pump) placed on the ocean floor [1.67]. A line from the top of the generator is connected to a buoy located at the ocean surface, acting as power takeoff. Springs attached to the translator of the generator store energy during half a wave cycle and simultaneously act as a restoring force in the wave troughs (Fig. 1.23). Sea tests off the western coast of Sweden of a 3 m diameter cylindrical buoy are reported in [1.67].

Another system with a heaving buoy driving a linear electrical generator was developed at Oregon State University, USA [1.68]. It consists of a deep-draught spar and an annular saucer-shaped buoy (Fig. 1.24). The spar is taut-moored to the sea bed by a cable. The buoy is free to heave relative to the spar, but is constrained in all other degrees of freedom by a linear bearing system. The forces imposed on the spar by the relative velocity of the two bodies is converted into electricity by a permanent magnet linear generator. The spar is designed to provide sufficient buoyancy to resist the generator force in the down direction. A 10kW prototype L-10 (buoy outer radius 3.5 m, spar length 6.7 m) was deployed off Newport, Oregon, in September 2008, and tested [1.68].
Fig. 1.23. Swedish heaving buoy with linear electrical generator (courtesy of Uppsala University).

Fig. 1.24. L-10 wave energy converter with linear electrical generator, developed at Oregon State University.

1.3.2. Two-body heaving systems

The concept of a single floating body reacting against the sea floor may raise difficulties due to the distance between the free surface and the bottom and/or to tidal oscillations in sea level. Multi-body systems may be used instead, in which the energy
is converted from the relative motion between two bodies oscillating differently. The hydrodynamics of two-body systems was theoretically analysed in detail by Falnes [1.69]. Multi-body wave energy converters raise special control problems [1.34,1.70,1.71].

The Bipartite Point Absorber concept [1.72] is an early example (1985) of a two-point heaving system. It consists of two floaters, the outer one (with very low resonance frequency) being a structure that acts as the reference and the inner one acting as the resonating absorber. This device incorporates a concept that was later to be adopted in the Wavebob (see below): the mass of the inner body is increased (without significantly affecting the diffraction and radiation damping forces) by rigidly connecting it to a fully submerged body located sufficiently far underneath.

Fig. 1.25. Schematic representation of the IPS buoy.

One of the most interesting two-body point absorbers for wave energy conversion is the IPS buoy, invented by Sven A. Noren in 1978 [1.73] and initially developed in Sweden by the company Interproject Service (IPS). This consists of a buoy rigidly connected to a fully submerged vertical tube (the so-called acceleration tube) open at both ends (Fig. 1.25). The tube contains a piston whose motion relative to the floater-tube system (motion originated by wave action on the floater and by the inertia of the water enclosed in the tube) drives a power take-off (PTO) mechanism. The same inventor later (1981) introduced an improvement that significantly contributes to solve the problem of the end-stops: the central part of the tube, along which the piston slides, bells out at either end to limit the stroke of the piston [1.74]. A half-scale prototype of the IPS buoy was tested in sea trials in Sweden, in the early 1980s [1.75]. The AquaBuOY is a wave energy converter, developed in the 2000s, that combines the IPS buoy concept with a pair of hose pumps to produce a flow of water at high pressure that drives a Pelton turbine [1.79]. A prototype of the AquaBuOY was deployed and tested in 2007 in the Pacific Ocean off the coast of Oregon. A variant of the initial IPS buoy concept, due to Stephen Salter, is the sloped IPS buoy: the natural frequency of the
converter may be reduced, and in this way the capture width enlarged, if the buoy-tube set is made to oscillate at an angle intermediate between the heave and the surge directions. The sloped IPS buoy has been studied since the mid-1990s at the University of Edinburgh, by model testing and numerical modelling [1.77-1.79].

The Wavebob, under development in Ireland, is another two-body heaving device [1.80]. It consists of two co-axial axisymmetric buoys, whose relative axial motions are converted into electric energy through a high-pressure-oil system (Fig. 1.26). The inner buoy (body 2 in Fig. 1.26) is rigidly connected to coaxial submerged body located underneath, whose function is to increase the inertia (without reduction in the excitation and radiation hydrodynamic forces) and allow the tuning to the average wave frequency. A large (1:4th scale) model has been tested in the sheltered waters of Galway Bay (Ireland) in the last few years.

The American company Ocean Power Technologies developed another axisymmetric two-body heaving wave energy converter named PowerBuoy. A disc-shaped floater reacts against a submerged cylindrical body, terminated at its bottom end by a large horizontal damper plate whose function is to increase the inertia through the added mass of the surrounding water. The relative heaving motion between the two bodies is converted into electrical energy by means of a hydraulic PTO. A 40 kW prototype without grid connection was deployed off the coast of Santoña, in northern Spain, in September 2008 (Fig. 1.27).

1.3.3. Fully submerged heaving systems

The Archimedes Wave Swing (AWS), a fully submerged heaving device, was basically developed in Holland, and consists of an oscillating upper part (the floater) and a bottom-fixed lower part (the basement) (Fig. 1.28) [1.81]. The floater is pushed
down under a wave crest and moves up under a wave trough. This motion is resisted by a linear electrical generator, with the interior air pressure acting as a spring. The AWS device went for several years through a programme of theoretical and physical modelling. A prototype was built, rated 2 MW (maximum instantaneous power). After unsuccessful trials in 2001 and 2002 to sink it into position off the northern coast of Portugal, it was finally deployed and tested in the second half of 2004 (Fig. 1.29) [1.82]. The AWS was the first converter to use a linear electrical generator.

Fig. 1.27. The PowerBuoy prototype deployed off Santoño, Spain, in 2008 (courtesy of Ocean Power Technologies).

Fig. 1.28. Schematic representation of the Archimedes Wave Swing (AWS).

Another fully submerged, nominally heaving, body is the CETO, developed in Australia. The vertical motion of the tight-moored buoy drives a high-pressure water piston-pump. The rest of the PTO is located onshore. A 7-meter-diameter 80 kW device was tested in 2011 (Fig. 1.29a).
Although not exactly a heaving body, reference should be made to the so-called Bristol cylinder, a concept invented in the late 1970s by David V. Evans, a mathematician from the University of Bristol, UK. Based on linear water wave theory, Evans showed that, in two dimensions, a fully submerged horizontal circular cylinder whose axis is parallel to the crests of the incoming regular waves is capable of completely absorbing the incident wave power, provided that the cylinder centre is made to move in a circle of small radius; this was later confirmed approximately by testing a model in wave tank [1.83]. The concept was to be realized by including dampers and springs in the tight-mooring system of the buoyant submerged cylinder (Fig. 1.4). The Bristol cylinder was one of the devices whose development was funded by the British wave energy program 1975-82 [1.21].
1.3.4. Pitching devices

The oscillating-body wave energy converters briefly described above are nominally heaving systems, i.e. the energy conversion is associated with a relative translational motion. (It should be noted that, in some of them, the mooring system allows other oscillation modes, namely surge and pitch). There are other oscillating-body systems in which the energy conversion is based on relative rotation (mostly pitch) rather than translation. This is remarkably the case of the nodding Duck (created by Stephen H. Salter, from the University of Edinburgh) probably the best known offshore device among those that appeared in the 1970s and early 1980s [1.20], and of which several versions were developed in the following years [1.84]. Basically it is a cam-like floater oscillating in pitch. The first versions consisted of a string of Ducks mounted on a long spine aligned with the wave crest direction (Fig. 1.4), with a hydraulic-electric PTO system. Salter later proposed the solo duck, in which the frame of reference against which the nodding duck reacts is provided by a gyroscope (Fig. 1.30). Although the Duck concept was object of extensive R&D efforts for many years, including model testing at several scales [1.2], it never reached the stage of full-scale prototype in real seas.

Among the wide variety of devices proposed in the 1970s and 1980s that did not succeed in reaching full-size testing stage (see [1.2]), reference should be made to the Raft invented by Sir Christopher Cockerell (who was also the inventor of the Hovercraft) (Fig. 1.4). This was actually a series of rafts or pontoons linked by hinges, that followed the wave contour, with a PTO system (possibly hydraulic) located at each hinge [1.2.1.21]. The Cockerell Raft may be regarded as the predecessor of a more successful device, the Pelamis, and also of the McCabe Wave Pump (see below).

Fig. 1.30. The Duck version of 1979 equipped with gyroscopes (courtesy of University of Edinburgh).

The Pelamis, developed in UK, is a snake-like slack-moored articulated structure composed of four cylindrical sections linked by hinged joints, and aligned with the wave direction. The wave-induced motion of these joints is resisted by hydraulic rams, which pump high-pressure oil through hydraulic motors driving three electrical generators. Gas accumulators provide some energy storage. As other devices that reached full size, the Pelamis was the object of a detailed development program over
several years, that included theoretical/numerical modelling and physical model testing at several scales [1.85,1.86]. Sea trials of a full-sized prototype (120 m long, 3.5 m diameter, 750 kW rated power) took place in 2004 in Scotland. A set of three Pelamis devices was deployed off the Portuguese northern coast in the second half of 2008 (Fig. 1.31), making it the first grid-connected wave farm worldwide.

Fig. 1.31. The three-unit 3 x 750 kW Pelamis wave farm in calm sea off northern Portugal, 2008 (courtesy of R. Barros).

The McCabe Wave Pump has conceptual similarities to the Cockerell Raft and the Pelamis: it consists of three rectangular steel pontoons hinged together, with the heaving motion of the central pontoon damped by a submerged horizontal plate [1.87] (Fig. 1.32). Two sets of hydraulic rams and a hydraulic PTO convert the relative rotational motions of the pontoon into useful energy. A 40 m long prototype was deployed in 1996 off the coast of Kilbaha, County Clare, Ireland.

Fig. 1.32. Side and plan views of the McCabe Wave Pump.

The SeaRay, developed in USA by Columbia Power and by Oregon State University, is largely similar to the McCabe Wave Pump. It consists of two bodies that can oscillate in pitch with respect to a centrally-positioned long cylindrical body whose inertia is enlarged by a submerged plate. The relative motion is converted directly by an electrical generator. Model testing took place in 2011 off Seattle (Fig. 1.32a).
Two-body systems have been conceived in which only one body is in contact with the water: the other body is located above the water or is totally enclosed inside the wetted one (see [1.88] for an early example). The theoretical modelling and control of such devices (especially heaving ones and including also three-body systems) has been analysed by Korde [1.34,1.89].

A typical device based on the totally enclosed hull concept is the Frog, of which several offshore point-absorber versions have been developed at Lancaster University, UK. The PS Frog Mk 5 consists of a large buoyant paddle with an integral ballasted handle hanging below it [1.90,1.91] (Fig. 1.33). The waves act on the blade of the paddle and the ballast beneath provides the necessary reaction. When the wave energy converter is pitching, power is extracted by partially resisting the sliding of a power-take-off mass, which moves in guides above sea level.

The Searev (Système électrique autonome de récupération de l'énergie des vagues) wave energy converter, developed at Ecole Centrale de Nantes, France since 2003 [1.92], is a floating device enclosing a heavy horizontal-axis wheel serving as an internal gravity reference (Fig. 1.34). The centre of gravity of the wheel being off-centred, this component behaves mechanically like a pendulum. The rotational motion of this pendular wheel relative to the hull activates a hydraulic PTO which, in turn, sets an electrical generator into motion. Major advantages of this arrangement are that (i)
(like the Frog) all the moving parts (mechanic, hydraulic, electrical components) are sheltered from the action of the sea inside a closed hull, and (ii) the choice of a wheel working as a pendulum involve neither end-stops nor any security system limiting the stroke.

Fig. 1.34. Schematic representation of the Searev.

The Spanish company Oceantec developed another offshore floating energy converter that extracts energy basically from the pitching motion. It has the shape of an elongated horizontal cylinder with ellipsoidal ends whose major axis is aligned with the incident wave direction [1.93]. The energy conversion process is based on the relative inertial motion that the waves cause in a gyroscopic system [1.94]. This motion is used to feed an electrical generator through a series of transformation stages. A 1:4\textsuperscript{th} scale prototype (11.25 m long) was deployed off the coast of Guipúzcoa (northern Spain) in September 2008 and was tested for several months (Fig. 1.35) [1.93].

Fig. 1.35. Oceantec device (1:4\textsuperscript{th} scale) deployed in 2008 off the northern coast of Spain (courtesy of Oceantec). The pitching floater reacts against a gyroscopic system.
1.3.5. Overhanging and bottom-hinged systems

Single oscillating-body devices operating in pitching mode have been proposed, based on the overhanging pendulum or on the inverted pendulum hinged at the sea bed concept. The Pendulor device was developed in Japan since the early 1980s [1.95]. It consists basically of a bottom-standing caisson open to the sea (Fig. 1.36). In regular waves, if the caisson is properly sized, resonance is established by multiple reflections at the back wall and at the opening. Energy is extracted from the wave system by the swinging motion of a flat plate hanging as a pendulum from the top of the caisson, spanning the caisson width and extending downwards close to the bottom. The plate motion is converted into useful energy by a high-pressure-oil hydraulic circuit. An onshore prototype, equipped with a 5 kW hydraulic motor, was installed in 1983 at Muroran Port, on the south coast of Hokkaido, Japan, and was operated for several years.

The mace, invented by Stephen Salter in the early 1990s [1.96], consists of a buoyant spar, with symmetry about the vertical axis, that can swing, as an inverted pendulum, about a universal joint at the sea bottom (Fig. 1.37). The power take-off reaction to the sea bed is via a set of cables wound several times round a winch-drum leading both fore and aft in the prevailing wave direction. The wave-activated reciprocating rotation of the drum is converted into useful energy by means of a hydraulic system. A basically similar concept of a bottom-pivoted vertical cylinder is being developed in Australia [1.97].

Two devices, namely Oyster and WaveRoller, are presently under development that share the same basic concept: a buoyant flap hinged at the sea bed, whose pitching oscillations activate a set of double-acting hydraulic rams located on the sea bed that pump high pressure fluid to shore via a sub-sea pipeline. The fluid flow is converted into electrical energy by a conventional hydraulic circuit. These devices are intended for deployment close to shore in relatively shallow water (10-15 m). Apart from size (the Oyster is larger) and detailed design, there are some conceptual differences between them. The Oyster (under development in UK) has a surface piercing flap that spans the whole water depth and the fluid is sea water powering a Pelton turbine located onshore [1.98], whereas the WaveRoller (a Finish device) is totally submerged and uses oil as working fluid [1.99]. Several swinging flaps can feed a single onshore generator, attached to a single manifold pipeline. A 3.5 m high, 4.5 m wide prototype of the
WaveRoller was deployed and tested in 2008 close to the Portuguese coast at Peniche. A large Oyster prototype was built in Scotland in 2009 (Fig. 1.38) and was sea-tested in 2010 (Fig. 1.38a). A comparison of designs for seabed-mounted bottom-hinged wave energy converters can be found in [1.100].

Fig. 1.37. The swinging mace in three angular positions.

Fig. 1.38. Oyster prototype being assembled in 2009 (courtesy of Aquamarine Power).
A three-flap 3×100kW prototype of the Waveroller was deployed in 2012 in 15 m water depth, at Peniche, 100 km north of Lisbon (Fig. 1.38b).

1.3.6. Many-body systems

In some cases, the device consists of a large set of floating point absorbers reacting against a common frame and sharing a common PTO. This is the case of FO3 [1.101] (mostly a Norwegian project), a nearshore or offshore system consisting of an array of 21 axisymmetric buoys (or “eggs”) oscillating in heave with respect to a large floating structure of square planform with very low resonance frequency and housing a hydraulic PTO. Initially, a 1:20th-scale model of the device was tested in Trondheim, Norway in the early 2000s. This was followed by the construction of “Buldra”, a 1:3rd-scale research model (12 m × 12 m) that was tested in the mid-2000s close to the southern Norwegian coast (Fig. 1.39).
The Wave Star, developed in Denmark, consists of two rectilinear arrays of closely spaced floaters located on both sides of a long bottom-standing steel structure that is aligned with the dominant wave direction and houses a hydraulic PTO consisting of a high-pressure-oil hydraulic circuit equipped with hydraulic motors. The waves make the buoys to swing about their common reference frame and pump oil into the hydraulic circuit. A 1:10\textsuperscript{th}-scale 24 m long 5.5 kW model with 10 buoys on each side was deployed in 2006 in Nissum Bredning, Denmark, and tested with grid connection for a couple of years [1,102] (Fig. 1.40). A larger, one-half-scale model (with two 5-meter-diameter floaters rated 25 kW each) was tested in 2009 in 7-meter-deep water in the North Sea off Hanstholm, Denmark (Fig. 1.41). The Brazilian hyperbaric device is based on a similar concept, the main differences being that the reference frame about which the buoys are made to swing is a vertical breakwater, and water is pumped to feed a Pelton turbine in a circuit that includes an air accumulator. A 1:10\textsuperscript{th}-scale model of the hyperbaric device was tested in 2006 in a large wave tank [1,103] (Fig. 1.42). A two-unit prototype of this device was recently installed at a breakwater, at São Gonçalo do Amarante, Ceará State, Brazil (Fig. 1.43).

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Fig. 1.39. One-third-scale 12 m × 12 m model of multi-body device FO3 “Buldra” being tested close to the southern coast of Norway, about 2004.

Fig. 1.40. One-tenth scale model of Wave Star deployed in 2006 in Nissum Bredning, Denmark.
Fig. 1.41. Prototype of Wave Star, with two floaters of 5-m diameter, rated 25 kW each, being tested at Hanstholm, Denmark (2009).

Fig. 1.42. One-tenth scale model of the Hyperbaric device being tested in a large wave tank, Rio de Janeiro, Brazil, 2006 [1.103].

Fig. 1.43. A two-unit prototype of the hyperbaric device installed at a breakwater, at São Gonçalo do Amarante, Ceará State, Brazil, 2012.
1.4. Overtopping converters

A different way of converting wave energy is to capture the water that is close to the wave crest and introduce it, by over spilling, into a reservoir where it is stored at a level higher than the average free-surface level of the surrounding sea. The potential energy of the stored water is converted into useful energy through more or less conventional low-head hydraulic turbines. The hydrodynamics of overtopping devices is strongly non-linear, and, unlike the cases of oscillating body and OWC wave energy converters, cannot be addressed by linear water wave theory.

The Tapchan (Tapered Channel Wave Power Device), a device developed in Norway in the 1980s, was based on this principle [1.104]. A prototype (rated power 350 kW) was built in 1985 at Toftestallen, Norway (Fig. 1.5), and operated for about six years. The Tapchan comprised a collector, a converter, a water reservoir and a low-head water-turbine. The horn-shaped collector serves the purpose of concentrating the incoming waves before they enter the converter. In the prototype built in Norway, the collector was carved into a rocky cliff and was about 60-metre-wide at its entrance. The converter is a gradually narrowing channel with wall heights equal to the filling level of the reservoir (about 3 m in the Norwegian prototype). The waves enter the wide end of the channel, and, as they propagate down the narrowing channel, the wave height is amplified until the wave crests spill over the walls and fill the water reservoir. As a result, the wave energy is gradually transformed into potential energy in the reservoir. The main function of the reservoir is to provide a stable water supply to the turbine. It must be large enough to smooth out the fluctuations in the flow of water overtopping from the converter (about 8500 m² surface area in the Norwegian prototype). A conventional low-head Kaplan-type axial flow turbine is fed in this way, its main specificity being the use of corrosion-resistant material.

In other overtopping converters, the incident waves overtop a sloping wall or ramp and fill a reservoir where water is stored at a level higher than the surrounding sea. This is the case of the Wave Dragon, an offshore converter developed in Denmark, whose slack-moored floating structure consists of two wave reflectors focusing the incoming waves towards a doubly curved ramp, a reservoir and a set of low-head hydraulic turbines [1.105]. A 57 m wide, 237 t (including ballast) prototype of the Wave Dragon (scale 1:4.5 of a North Sea production plant) has been deployed in Nissum Bredning, Denmark, was grid connected in May 2003 and has been tested for several years (Fig. 1.44).

Another run-up device based on the slopping wall concept is the Seawave Slot-Cone Generator (SSG) developed (within the framework of a European project) for integration into a caisson breakwater [1.106,1.107] (Fig. 1.45). The principle is based on the wave overtopping utilizing a total of three reservoirs placed on top of each other. The water enters the reservoirs through long horizontal openings on the breakwater sloping wall, at levels corresponding to the three reservoirs, and is run through a multi-stage hydraulic turbine for electricity production.

References

Fig. 1.44. 1:4.5-scale model of the Wave Dragon being tested at Nissum Bredning, Denmark, about 2003.

![scale model of the Wave Dragon](image)

Fig. 1.45. Representation of a SSG run-up device integrated into a sloping breakwater.


[1.61] McCormick ME. Analysis of a wave-energy conversion buoy. AIAA J Hydronautics 1974;8:77-82.


2. Linear theory of ocean surface waves

2.1. Basic equations

The waves we will be studying here are disturbances to the flat water-air interface in the sea or ocean. These waves are also called gravity waves because the restoring force is due to gravity. In most cases, the disturbance is due to the action of the wind blowing over the water surface (wind waves), but there may be other causes (bodies moving in water, sea bottom motion as in tsunamis, etc.).

Surface tension is an additional flatness-restoring force. However, it is important only for short waves commonly called ripples: in practice, waves of length less than around 0.1 m. We will not consider here such cases.

In our theory of gravity waves on a homogeneous body of water, compressibility is negligible and so the water density may be taken as a constant \( \rho \). Besides we will also neglect viscosity and other dissipative effects.

We adopt a Cartesian coordinate system \((x, y, z)\) with \(z\) measured vertically upwards from the undisturbed free-surface, and denote by \(p(x, y, z)\) the pressure in water. In the absence of waves, the undisturbed pressure distribution \(p_0\) is hydrostatically determined as

\[
p_0 = p_a - \rho g z,
\]

where \(p_a\) is the atmospheric pressure and \(g\) the acceleration of gravity. The excess pressure due to a disturbance is defined as

\[
p_e = p - p_0.
\]

In the absence of viscosity effects, the equation of motion (Euler’s equation) for a fluid particle can be written as

\[
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - gk.
\]

Here \(D/Dt\) is the material derivative (i.e. taken while following the fluid motion), \(\mathbf{u}(x, y, z)\) is the velocity vector and \(k = \nabla z\) is the vertical unit vector. In Eq. (2.3), \(D\mathbf{u}/Dt\) is the acceleration of the fluid particle, \(-\nabla p\) is the resultant pressure force per unit volume, and \(-gk\) is the gravity force per unit volume. Taking into account Eqs (2.1) and (2.2), Eq. (2.3) can be rewritten as

\[
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p_e.
\]

It is well known from vector calculus and text books on fluid mechanics that \(D\mathbf{u}/Dt\) may be written identically as

\[
\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}.
\]

In what follows, we will be concerned with linear theory, which implies that we consider the disturbances so weak that, in the equations of motion, we can view them as small quantities whose products are neglected. In Eq. (2.5), the linear term \(\partial \mathbf{u}/\partial t\) represents the local rate of change of \(\mathbf{u}\) at a fixed point, while the nonlinear term \((\mathbf{u} \cdot \nabla)\mathbf{u}\) describes how the element’s velocity changes owing to its changing position in

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\(^1\) Sections 2.1 to 2.5 are largely based on chapter 3 of the book Waves in Fluids by J. Lighthill [2.1], and also, to a much lesser extent, on [2.2].
space. This convective rate of change of \( \mathbf{u} \) involves products of its spatial gradients with components of \( \mathbf{u} \) itself, and so is neglected in linear theory. We are left with

\[
\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p_e .
\]  

(2.6)

Now, we introduce the vorticity vector \( \mathbf{\Omega} = \nabla \times \mathbf{u} \), and take the curl of both sides of Eq. (2.6). Since it is \( \nabla \times (\nabla p_e) \equiv 0 \), we find

\[
\frac{\partial \mathbf{\Omega}}{\partial t} = 0 .
\]  

(2.7)

This shows that, in linear theory, the vorticity field is independent of time: vorticity stays put, however much other quantities may be propagated. The rotational part of the velocity field, induced by this stationary vorticity field, is independent of time, has \( \nabla p_e = 0 \) and \( p_e = 0 \) (by Eq. (2.6)), and therefore does not disturb the flatness of the water surface.

The remaining part of the velocity field is irrotational and so can be written as the gradient \( \nabla \phi \) of a velocity potential \( \phi(x, y, z) \): only this part disturbs the water surface or exhibits the fluctuations associated with wave propagation.

The equation of continuity for an incompressible fluid is \( \nabla \cdot \mathbf{u} = 0 \), giving for this irrotational propagating part of the velocity field Laplace’s equation

\[
\nabla^2 \phi = 0 .
\]  

(2.8)

The rotational non-propagating part of the velocity field may be present if the seawater, over which gravity waves are propagating, is moving horizontally in a shear flow in which the vertical velocity component is zero and the horizontal components are independent of time and functions only of the vertical coordinate \( z \). Here we will not consider such situations and exclude the cases when the waves are propagating over a body of water affected by tidal current or other type of current. So we are left only with an irrotational propagating velocity field which will be expressed as

\[
\mathbf{u} = \nabla \phi .
\]  

(2.9)

We write the equation of the disturbed water free-surface as

\[
z = \zeta(x, y, t) .
\]  

(2.10)

Then it is

\[
p_e = \rho g \zeta
\]  

(2.11)

on the disturbed water surface in order that Eqs (2.1) and (2.2) will permit the pressure \( p = p_0 + p_e \) to take the atmospheric value \( p_a \) on that surface.

Combining Eqs (2.6) and (2.9), we find for the excess pressure field

\[
p_e = -\rho \frac{\partial \phi}{\partial t} .
\]  

(2.12)

Equations (2.10) and (2.11) then tell us that

\[
\left[ \frac{\partial \phi}{\partial t} \right]_{z=\zeta} = -g \zeta ,
\]  

(2.13)

a boundary condition complicated by the shape \( z = \zeta \) on the free surface not being known in advance. In a linear theory, however, this complication disappears and Eq. (2.9) can be written as

\[
\left[ \frac{\partial \phi}{\partial t} \right]_{z=0} = -g \zeta ,
\]  

(2.14)

because the difference between the left-hand sides of Eqs (2.13) and (2.14) is equal to the product of the small disturbance \( \zeta \) to the free surface and another small-disturbance
term: the value of the second derivative \( \frac{\partial^2 \zeta}{\partial z \partial t} \) at a point intermediate between the undisturbed and disturbed surfaces \( z = 0 \) and \( z = \zeta \).

A second boundary condition connects the irrotational velocity field \( \mathbf{u} \) to the vertical displacement \( \zeta \) of the free surface with which it is associated. The rate of change of \( \zeta \) following a water particle of the free surface (i.e. the distance to the plane \( z = 0 \)) is equal to the vertical component \( \frac{\partial \phi}{\partial z} \) of the velocity \( \mathbf{u} = \nabla \phi \) at the surface:

\[
\frac{D \zeta}{Dt} \equiv \frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = \left[ \frac{\partial \phi}{\partial z} \right]_{z = \zeta}.
\]  

(2.15)

In a linear theory, Eq. (2.15) can be simplified both by neglecting on the left-hand side the convective rate of change \( \mathbf{u} \cdot \nabla \zeta \) as the product of small disturbances and by replacing the value at \( z = \zeta \) on the right-hand side by the value at \( z = 0 \) as in Eq. (2.14). This gives

\[
\frac{\partial \zeta}{\partial t} = \left[ \frac{\partial \phi}{\partial z} \right]_{z=0}.
\]  

(2.16)

Equations (2.14) and (2.16) are linear boundary conditions, to be applied at the fixed boundary condition \( z = 0 \) of the region \( z \leq 0 \) in which Laplace’s equation (2.8) must be satisfied. Differentiating Eq. (2.14) with respect to \( t \) eliminates \( \zeta \) between them to give

\[
\frac{\partial^2 \phi}{\partial t^2} = -g \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = 0
\]  

(2.17)

as the boundary condition for the velocity potential \( \phi \) itself. Solutions of Eq. (2.8) in \( z \leq 0 \), subject to this boundary condition (2.17), represent surface gravity waves, the surface displacement being deducible from the velocity potential by either Eq. (2.14) or Eq. (2.16).

### 2.2. Sinusoidal waves on deep water

After having established that a linear theory of surface gravity waves demands merely the solution of the familiar Laplace equation (2.8) for irrotational incompressible flow, subject to a special boundary condition (2.17) at the undisturbed position \( z = 0 \) of the free surface, we first of all study solutions that describe travelling sinusoidal waves (Fig. 2.1). We may also need to apply, at any stationary lower boundary of the water mass (i.e. a submerged fixed wall), the appropriate condition: for irrotational flow this is the vanishing of the normal component of the fluid velocity, which is the normal derivative of the velocity potential \( \phi \).

Fig. 2.1. Representation of sinusoidal wave of wavelength \( \lambda \) and amplitude \( A_w \).

We treat first the case of waves on water so deep that the exact boundary condition is automatically satisfied at the bottom because the motions associated with surface waves cannot penetrate so far down. Surface waves under these circumstances are described as
“waves on deep water”. As we shall see, this condition is satisfied with good approximation for any water mass whose depth everywhere exceeds the wavelength.

We will use here the mathematical tools of the theory of complex variable. It is
\[ e^{i\omega t} = \cos \omega t + i \sin \omega t \] (2.18)
and
\[ \cos \omega t = \text{Re}\left(e^{i\omega t}\right), \quad \sin \omega t = \text{Im}\left(e^{i\omega t}\right), \] (2.19)
where \( \text{Re}(\ ) \) and \( \text{Im}(\ ) \) stand for real part of and imaginary part of, respectively. In general we write, for a physical quantity \( f(t) \) that varies sinusoidally with time,
\[ f(t) = \text{Re}(F e^{i\omega t}), \] (2.20)
where
\[ F = F_0 e^{i\alpha} \] (2.21)
is a complex amplitude, and \( F_0 = |F| \) and \( \alpha \) are the modulus and the argument of \( F \). We may rewrite
\[ f(t) = \text{Re}\left(F_0 e^{i(\omega t + \alpha)}\right) = F_0 \cos(\omega t + \alpha). \] (2.22)
In what follows, we will omit in most cases the symbol \( \text{Re}(\ ) \), it being understood that, whenever a physical quantity is equated to a complex expression, the real part of the expression is to be taken.

It may be easily verified that, in deep water, a velocity potential that represents a sinusoidal wave propagating in the positive \( x \)-direction with wave speed \( c \) is
\[ \phi = \Phi(z) \exp[i \omega (t - x/c)] = \Phi(z) \exp[i(\omega t - kx)], \] (2.23)
where \( \omega \) (the radial or angular frequency) and \( k \) (the wavenumber) satisfy equations
\[ \omega = \frac{2\pi}{T}, \quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \] (2.24)
in terms of the period \( T \) and the wavelength \( \lambda = cT \). The function \( \Phi(z) \) represents the dependence of a (possibly complex) amplitude of the motions on the distance \( -z \) below the surface. Expression (2.23) (of which, of course, it is understood that the real part represents the velocity potential) satisfies Laplace’s equation (2.8) if \( \Phi(z) \) satisfies
\[ \Phi^*(z) - k^2 \Phi(z) = 0. \] (2.25)
The general solution of this ordinary differential equation is a linear combination of a solution \( e^{kz} \) and another solution \( e^{-kz} \); the first of these satisfies a deep-water boundary condition, in that the motions it represents become smaller and smaller at positions far below the surface where \( z \) is large and negative; indeed, on a lower boundary where \( z < -\lambda \), it has fallen to less than \( e^{-2\pi} = 0.00187 \) and becomes quite negligible.

For waves in deep water, then, we must take
\[ \Phi(z) = \Phi_0 e^{kz}, \] (2.26)
where \( \Phi_0 \) is a constant (the value of \( \Phi \) at \( z = 0 \)). We must avoid including any term proportional to \( e^{-kz} \), because that would increase exponentially where \( z \) is large and negative (exceeding \( e^{2\pi} = 535 \) at positions more than a wavelength below the surface).

Evidently, Eqs (2.23) and (2.26) imply that \( \partial^2 \phi / \partial t^2 \) is \( -\omega^2 \phi \) while \( \partial \phi / \partial z \) is \( k \phi \), so that the boundary condition (2.17) on \( z = 0 \) yields
\[ \omega^2 = gk \quad \text{or} \quad T^2 = \frac{2\pi \lambda}{g}, \] (2.27)
as the relationship between frequency and wavenumber (or between wave period and wavelength) for gravity waves on deep water. This dispersion relationship can also be written in terms of the wave velocity \( c \), by (2.24), as

\[
c = \frac{\omega}{k} = \left( \frac{g}{k} \right)^{1/2} = \left( \frac{g\lambda}{2\pi} \right)^{1/2}. \tag{2.28}
\]

Such dependence of the wave velocity on the square root of the wavelength implies a very substantial variation over the range of wavelengths \( \lambda \) of interest. For the range of typical waves with wavelengths \( \lambda \) from 30 to 300 m, the wave velocity \( c \) varies from 6.8 ms\(^{-1}\) to 21.6 ms\(^{-1}\) and the period \( T \) from 4.4 s to 13.9 s.

Equations (2.23) and (2.26) give for the velocity components in the \( x \)-direction (horizontal) and \( z \)-direction (vertical) the values

\[
\frac{\partial \phi}{\partial x} = -i k \Phi_0 e^{kz} \exp[i(\omega t - kx)], \quad \frac{\partial \phi}{\partial z} = -k \Phi_0 e^{kz} \exp[i(\omega t - kx)], \tag{2.29}
\]

both of which vary sinusoidally with time with the same amplitude \( k\Phi_0 e^{kz} \). That amplitude depends on position, of course; it decreases exponentially with distance below the surface. At any fixed position, however, the oscillating velocity components (2.29) differ only in their phase; that of the horizontal component \( \partial \phi/\partial x \) lags by 90° (represented by the \(-i\) factor) behind that of the vertical component \( \partial \phi/\partial z \). This means that the velocity vector rotates in a clockwise sense, keeping always the same magnitude \( k\Phi_0 e^{kz} \) as its horizontal and vertical components oscillate in quadrature.

The linear theory of sinusoidal waves on deep water, then, predicts that, at any fixed point, the fluid speed remains constant, while the direction of fluid motion rotates with angular velocity \( \omega \). The velocity of a particle of fluid, which may suffer small oscillating displacements from that fixed point, may be taken as satisfying the same law, since on linear theory the difference between the small velocities found at a fixed point and at another point displaced from it by a small amount can be neglected as the product of small quantities. Thus, a particle of fluid displaced by the waves from the position \((x, y, z)\) moves with constant speed \( k\Phi_0 e^{kz} \) and with direction rotating with angular velocity \( \omega \); in other words, it describes a circle of radius

\[
\omega^{-1}k\Phi_0 e^{kz}. \tag{2.30}
\]

The wave amplitude \( A_w \) is equal to the maximum value of the elevation \( \zeta(x, t) \) and is given by (2.30) with \( z = 0 \); it is

\[
A_w = \omega^{-1}k\Phi_0. \tag{2.31}
\]

Note that the phase of the motions (2.28) is independent of \( z \), since the factor \( e^{kz} \) is everywhere real and positive. Accordingly, in any vertical plane \( x = \text{constant} \) those motions cause all particles of fluid to rotate in phase. Though with radii depending as in (2.30) on the distance \( -z \) below the surface, while of course as \( x \) increases the phase of the motions decreases at the rate \( k = 2\pi/\lambda \). Figure 2.2 illustrates this linear-theory approximation to the motion of fluid particles in sinusoidal waves on deep water.
Fig. 2.2. Motion of fluid particles (on linear theory) in a sinusoidal wave of length $\lambda$ travelling from left to right on deep water. The maximum surface elevation (wave amplitude) is 0.02$\lambda$.

2.3. Sinusoidal waves on water of arbitrary, but uniform, depth

Now, instead of assuming waves on deep water (in the sense that its depth everywhere exceeds a wavelength $\lambda$), we consider waves on water of uniform depth $h$. At the solid bottom $z = -h$, the boundary condition of zero normal velocity is

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h$$

(2.32)

appropriate to irrotational flow. This boundary condition (2.32) modifies the calculation of section 2.2 only by specifying differently the required solution of the ordinary differential equation (2.26) for the amplitude $\Phi(z)$ of the general expression (2.23) for sinusoidal waves. Instead of specifying a solution (2.25) which becomes zero as $z$ becomes large and negative, it specifies a solution satisfying $\Phi'(-h) = 0$. This means that, in the general linear combination

$$\Phi_1 e^{kz} + \Phi_2 e^{-kz}$$

(2.33)

of elementary solutions of (2.25), we have

$$\Phi_1 e^{-kh} = \Phi_2 e^{kh},$$

(2.34)

and putting both sides of Eq. (2.34) equal to $\frac{1}{2} \Phi_0$ we can write the solution as

$$\Phi(z) = \Phi_0 \cosh[k(z + h)].$$

(2.35)

Here and elsewhere we use the classical notation

$$\cosh x = \frac{1}{2} (e^x + e^{-x}),$$

\hspace{1cm} (2.36)

$$\sinh x = \frac{d}{dx} \cosh x = \frac{1}{2} (e^x - e^{-x}) = \cosh x \tanh x$$

(2.37)

for the hyperbolic functions. These functions are plotted in Fig. 2.3.

Whereas Eq. (2.23) requires that $\frac{\partial^2 \phi}{\partial t^2}$ is $-\omega^2 \phi$, it implies with Eq. (2.33) that on $z = 0$

$$\frac{\partial \phi}{\partial z} = \frac{\Phi'(0)}{\Phi(0)} \phi = (k \tanh kh) \phi,$$

(2.38)

so that the free-surface boundary condition (2.17) yields

$$\omega^2 = gk \tanh kh$$

(2.39)

as the relationship between frequency $\omega$ and wavenumber $k$ for gravity waves on water of arbitrary, but uniform, depth $h$. This dispersion relationship can be written in terms of the wave speed $c$ as

45
$$c = \frac{\omega}{k} = (g k^{-1} \tanh kh)^{1/2}. \quad (2.40)$$

In the limiting situation of deep water, \(kh\) is large, \(\tanh kh \to 1\) and the expression for the wave speed \(c\) becomes identical to (2.28). Another limiting situation occurs when \(kh\) is small (long-wave or shallow-water limit): in this case \(\tanh kh\) asymptotes to \(kh\) and Eq. (2.40) becomes

$$c = (gh)^{1/2} \quad \text{for} \quad kh \ll 1. \quad (2.41)$$

Hence, in the long-wave or shallow-water limit, the wave speed \(c\) becomes independent of the frequency \(\omega\) and depends only on water depth \(h\).

Equation (2.40) may also be written as

$$c = \left[ \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda} \right]^{1/2}, \quad (2.42)$$

which makes the dependence of \(c\) on the ratio \(\lambda/h\) more visible. It is only in the range of depths \(h\) between 0.07\(\lambda\) and 0.28\(\lambda\) (corresponding to wavelengths \(\lambda\) between 14\(h\) and 3.5\(h\)) that \(c\), as given by Eq. (2.42), departs significantly from both its limiting forms. Figure 2.4 exhibits this by plotting \(c\) against \(\lambda\) for fixed \(h\) according to Eq. (2.42) and showing how it makes a smooth transition between the parabolic limiting form \((g\lambda/2\pi)^{1/2}\) for \(\lambda < 3.5h\) and the constant asymptote \((gh)^{1/2}\) for \(\lambda > 14h\).
Fig. 2.4. The wave speed $c$ given by linear theory for waves of varying length $\lambda$ on water of uniform depth $h$. Note the transition between the long-wave value (dashed curve 1) and the deep-water value (dashed curve 2).

**Numerical example:** $T = 8\,\text{s}, \quad g = 9.8\,\text{m/s}^2$

- **Deep water**
  
  $c = \frac{gT}{2\pi} = \frac{9.8 \times 8}{2\pi} = 12.5\,\text{m/s}$
  
  $\lambda = cT = 12.5 \times 8 = 100\,\text{m}$

- **Shallow water** $h = 1\,\text{m}$
  
  $c = \sqrt{gh} = \sqrt{9.8 \times 1} = 3.1\,\text{m/s}$
  
  $\lambda = cT = 3.1 \times 8 = 25.0\,\text{m/s}$

- **Intermediate water depth** $h = 15\,\text{m}$
  
  $\omega = \frac{2\pi}{T} = \frac{2\pi}{8} = 0.785\,\text{rad/s}$
  
  $c = \frac{\omega}{g} = \frac{\omega h}{c} = \frac{0.785}{9.8} c = \tanh \frac{0.785 \times 15}{c} = \tanh \frac{0.785 \times 15}{c} = 10.2\,\text{m/s}$
  
  $\lambda = cT = 10.2 \times 8 = 81.8\,\text{m}$

<table>
<thead>
<tr>
<th>$h$ (m)</th>
<th>$c$ (m/s)</th>
<th>$\lambda$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>24.8</td>
</tr>
<tr>
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<td>5.25</td>
<td>42.0</td>
</tr>
<tr>
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<td>6.63</td>
<td>53.0</td>
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<td>10.22</td>
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<td>20</td>
<td>11.09</td>
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</tr>
<tr>
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<td>11.65</td>
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</tr>
<tr>
<td>50</td>
<td>12.44</td>
<td>99.5</td>
</tr>
<tr>
<td>$\infty$</td>
<td>12.48</td>
<td>99.8</td>
</tr>
</tbody>
</table>
Equations (2.24) and (2.35) give for the velocity components in the horizontal and vertical directions the values

\[
\frac{\partial \phi}{\partial x} = -i[k\Phi_0 \cosh(k(z + h))]\exp[i(\omega t - kx)],
\]

(2.43)

\[
\frac{\partial \phi}{\partial z} = [k\Phi_0 \sinh(k(z + h))]\exp[i(\omega t - kx)],
\]

(2.44)

exhibiting different amplitudes of sinusoidal variation, represented by the expressions in curly brackets. Indeed, the values of \( \cosh[k(z + h)] \) and \( \sinh[k(z + h)] \) become close to each other only where \( k(z + h) \) is large; that is, in the deep-water limit \( (kh \text{ large}) \).

At any fixed position, just as for waves in deep water, the oscillations of the velocity components (2.43) and (2.44) differ by 90° in their phase: that of the horizontal component \( \frac{\partial \phi}{\partial x} \) lags by 90° behind that of the vertical component \( \frac{\partial \phi}{\partial z} \). On linear theory the same expressions (2.43) and (2.44) describe the velocity components of a particle of fluid which may suffer small oscillating displacements from that fixed point. From this we would deduce, as in section 2.2, if both velocity amplitudes were \( k\Phi_0 \cosh[k(z + h)] \), that the particle must describe a circle of radius \( \omega^{-1}k\Phi_0 \cosh[k(z + h)] \); however, the true motions, with amplitude of the vertical velocities reduced by a factor \( \tanh[k(z + h)] \), follow a circular path foreshortened in the vertical direction by just that factor; in other words, an ellipse with major and minor semi-axes

\[
\omega^{-1}k\Phi_0 \cosh[k(z + h)] \quad \text{and} \quad \omega^{-1}k\Phi_0 \sinh[k(z + h)].
\]

(2.45)

The wave amplitude \( A_w \) (maximum value of the surface elevation \( \zeta \)) is given by the minor semi-axis at \( z = 0 \), that is

\[
A_w = \omega^{-1}k\Phi_0 \sinh kh.
\]

(2.46)

Figure 2.5 illustrates this linear-theory approximation of the motions of fluid particles in waves of length \( \lambda \) on water of a particular depth \( h = 0.16\lambda \), intermediate between those values 0.07\( \lambda \) and 0.28\( \lambda \) where the long-wave and deep-water limits give reasonable approximations.

Fig. 2.5. Paths of fluid particles in a sinusoidal wave of length travelling from left to right on water of depth \( h = 0.16\lambda \). The maximum surface elevation (wave amplitude) is 0.02\( \lambda \).

2.4. Standing waves. Wave reflection on a vertical wall

In linear theory, we can superpose two waves of equal amplitude \( A_w \) and frequency \( \omega \) travelling in opposite directions to obtain a standing wave. We consider wave 1
travelling in the positive x-direction and wave 2 travelling in the negative x-direction. Their velocity potentials are

\[ \phi_1 = \Phi(z) \exp[i(\omega t - kx)], \quad \phi_2 = \Phi(z) \exp[i(\omega t + kx)]. \]  

(2.47)

where \( \Phi(z) \) is given by Eq. (2.35), or by Eq. (2.26) in the case of deep water. The potential of the wave resulting from the superposition is

\[ \phi = \phi_1 + \phi_2 = \Phi(z) e^{i\omega t} (e^{-ikx} + e^{ikx}) = 2\Phi(z) e^{i\omega t} \cos kx. \]  

(2.48)

An expression for the surface elevation \( \zeta(x, t) \) can be obtained from Eq. (2.16)

\[ \frac{\partial \zeta}{\partial t} = \left( \frac{\partial \phi}{\partial z} \right)_{z=0} \]  

and is

\[ \zeta(x, t) = 2A_w \cos kx \sin \omega t. \]  

(2.49)

This equation shows that the amplitude of the surface elevation is maximum and equal to \( 2A_w \) at \( kx = n\pi, \ n = 0, \pm 1, \pm 2, \ldots \) and is zero at \( kx = n\pi/2, \ n = \pm 1, \pm 3, \pm 5, \ldots \). These points are named antinodes and nodes respectively. The free surface of a standing wave is represented in Fig. 2.6 at time intervals \( \Delta t = T/8 \).

![Fig. 2.6. Free surface of a standing wave at instants of time separated by intervals \( \Delta t = T/8 \).](image)

The \( x \)-component of the velocity is \( \frac{\partial \phi}{\partial x} = -2k \Phi(z) e^{i\omega t} \sin kx \) and is zero for \( kx = n\pi, \ n = 0, \pm 1, \pm 2, \ldots \). This would satisfy the boundary condition at a vertical wall located at \( x = 0 \) (or more generally at \( x = n\pi/k = n\lambda/2, \ n = 0, \pm 1, \pm 2, \ldots \)). If wave 1 is an incident wave, then wave 2 is the reflected wave on the wall, the resulting wave system forming a standing wave, with an antinode located at the wall where the wave amplitude, \( 2A_w \), is twice the amplitude of the incident wave.

### 2.5. Wave energy and wave energy flux

A characteristic property of waves is that they permit transport of energy without the need for any net transport of material. In the linear wave theory we are adopting here, the disturbances are regarded as small quantities whose squares we neglect in equations of motion (which implies also neglect of the product of two of the quantities, since its numerical value cannot exceed the square of one or the other of them). A different rule applies, however, in expressions for energy and its rate of change (or of transport), where terms in the first powers of small quantities should be absent and accordingly we retain squares of small quantities and products of two of them, and neglect their cubes (implying neglect of the product of three or more such quantities).

In surface waves, two forms of energy can be considered: (i) kinetic energy and (ii) energy associated with the restoring force, which for gravity waves is the gravitational potential energy \( \rho g z \) per unit volume. Evidently, the corresponding value of the potential energy per unit horizontal area, at a point where the local depth of water is \( h \),
is obtained by integrating $\rho g z\,dz$ from the bottom $z = -h$ to the free surface $z = \zeta$, giving

$$
\int_{-h}^{\zeta} \rho g z\,dz = \frac{1}{2} \rho g (\zeta^2 - h^2).
$$

(2.50)

At that point, then, the **excess potential energy per unit horizontal area** over the value for an undisturbed free surface ($\zeta = 0$) is

$$
\frac{1}{2} \rho g \zeta^2,
$$

(2.51)

proportional, as expected on a linear theory, to the *square* of a displacement from a position of equilibrium. Note that while a raised free surface adds potential energy by insertion of new fluid *above* the position $z = 0$ (that is, with positive potential energy), a depressed free surface adds potential energy by removal of fluid from *below* the level $z = 0$ (that is, with negative potential energy).

In sinusoidal waves, the surface elevation can be written as $\zeta(x,t) = A_w \cos(\omega t - kx + \alpha)$, where $A_w$ is the wave amplitude and $\alpha$ is phase constant. The time-averaged value of $\zeta^2$ is $A_w^2/2$, and so the time-averaged value of the potential energy per unit horizontal surface area is

$$
\bar{E}_{\text{potential}} = \frac{1}{4} \rho g A_w^2.
$$

(2.52)

The kinetic energy per unit volume is

$$
\frac{1}{2} \rho \|u\|^2 = \frac{1}{2} \rho \|\mathbf{\nabla} \phi\|^2 = \frac{1}{2} \rho (\mathbf{\nabla} \phi \cdot \mathbf{\nabla} \phi).
$$

Taking into account that $\mathbf{\nabla}^2 \phi = 0$ (see Eq. (2.8)), from vector calculus it can be shown that

$$
|\mathbf{\nabla} \phi|^2 = \mathbf{\nabla} \cdot (\phi \mathbf{\nabla} \phi).
$$

(2.53)

The kinetic energy in a mass of water of volume $V$ enclosed by a surface $S$ can be written as

$$
\frac{1}{2} \rho \int_{V} \|\mathbf{\nabla} \phi\|^2 \, dV = \frac{1}{2} \rho \int_{V} \mathbf{\nabla} \cdot (\phi \mathbf{\nabla} \phi) \, dV.
$$

(2.54)

This volume integral can be transformed into a surface integral by means of the Gauss divergence theorem, and we obtain

$$
\frac{1}{2} \rho \int_{V} \|\mathbf{\nabla} \phi\|^2 \, dV = \frac{1}{2} \rho \int_{S} \phi \frac{\partial \phi}{\partial n} \, dS.
$$

(2.55)

There is no contribution to the surface integral from the bottom, where $\frac{\partial \phi}{\partial n} = 0$, and the contribution from the free surface can on linear theory be approximated by the same integral over the undisturbed free surface $z = 0$. This gives, with an error appropriate to linear theory of order of the cubes of disturbances, a kinetic energy

$$
\rho \left( \frac{\phi}{\partial z} \right)_{z=0}
$$

per unit horizontal area. From the results above, the time-average value of expression (2.49) can be written as

$$
\bar{E}_{\text{kinetic}} = \frac{1}{4} \rho g A_w^2.
$$

(2.57)

Equations (2.52) and (2.57) show that, within the approximations of linear wave theory, it is $\bar{E}_{\text{potential}} = \bar{E}_{\text{kinetic}}$. The total time-averaged wave-induced energy per unit horizontal area is
\[ E = E_{\text{potential}} + E_{\text{kinetic}} = \frac{1}{2} \rho g A_w^2. \] (2.58)

We note that this result is valid for deep water as well as water of arbitrary, but uniform, depth.

As the waves travel across the ocean surface, they carry their potential and kinetic energy with them. The instantaneous values of the energy flux, across a vertical plane \( x = \text{constant} \), per unit time and per unit wave crest length (measured along the \( y \)-direction) are

\[ P_{\text{potential}} = \int_{-h}^{\zeta} (\rho g z) \frac{\partial \phi}{\partial x} \, dz \quad \text{and} \quad P_{\text{kinetic}} = \int_{-h}^{\zeta} \left( \frac{1}{2} \rho |\nabla \phi|^2 \right) \frac{\partial \phi}{\partial x} \, dz \] (2.59)

for the transported potential energy and kinetic energy respectively.

In addition to this bodily transport of potential and kinetic energy (that is, with the orbital motion of the water particles), energy is transferred through work done by the pressure in the direction of wave propagation. This work done per unit of time and per unit wave crest length (measured along the \( y \)-direction) is equal to the pressure \( p_e - \rho g z \) multiplied by the horizontal velocity component \( \frac{\partial \phi}{\partial x} \), integrated from bottom to free-surface

\[ P_{\text{pressure}} = \int_{-h}^{\zeta} (p_e - \rho g z) \frac{\partial \phi}{\partial x} \, dz. \] (2.60)

In this equation, \( p_e \) is the excess pressure related to the surface elevation \( \zeta \) by Eq. (2.12): \( p_e = -\rho_0 \frac{\partial \phi}{\partial t} \). The total energy flux \( P_{\text{wave}} = P_{\text{potential}} + P_{\text{kinetic}} + P_{\text{pressure}} \) is

\[ P_{\text{wave}} = \int_{-h}^{\zeta} \left( \frac{1}{2} \rho |\nabla \phi|^2 \right) \frac{\partial \phi}{\partial x} \, dz + \int_{-h}^{\zeta} p_e \frac{\partial \phi}{\partial x} \, dz. \] (2.61)

The first integral on the right-hand side of Eq. (2.61) involves small quantities of the order of the cube of disturbances and may be neglected. The second integral may be written as

\[ -\rho_0 \int_{-h}^{\zeta} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dz. \] (2.62)

The time-averaged value of the energy flux can now easily be found and is

\[ \overline{P}_{\text{wave}} = E c_g, \] (2.63)

where \( \overline{E} \) is the time-averaged energy per unit horizontal area given by Eq. (2.51) and

\[ c_g = \frac{1}{2} \frac{\omega}{k} \left( 1 + \frac{2kh}{\sinh 2kh} \right) \] (2.64)

is the so-called group velocity. Taking into account the expression (2.40) of the wave speed \( c \), the group velocity can be related to \( c \) by

\[ c_g = c \frac{1}{2} \left( 1 + \frac{2kh}{\sinh 2kh} \right). \] (2.65)

This can also be written as (see [2.3])

\[ c_g = \frac{g}{2\omega} D(kh), \] (2.66)

where
\[
D(kh) = \left[ 1 + \frac{2kh}{\sinh 2kh} \right] \tanh kh \\
= \tanh kh + \frac{kh}{\cosh^2 kh} \\
= \tanh kh + kh - k\tanh^2 kh \\
= \left[ 1 - \left( \frac{\omega^2}{gk} \right)^2 \right] \frac{kh + \omega^2}{gk}.
\]

The limiting values are \( D(kh) \to 1 \) as \( kh \to \infty \) (deep water limit) and \( D(kh) \to 2kh \) as \( kh \to 0 \) (shallow water limit). The function \( D(kh) \) is represented in Fig. 2.7.

![Graphical representation of \( D(kh) \)](image)

The group velocity may be regarded as the velocity at which the wave energy is propagated in the direction perpendicular to the wave crests. We recall that the wave speed \( c \) is the velocity at which the wave crests move. Equation (2.65) shows that \( c_g / c \leq 1 \). In the long-wave or shallow-water limit, it is \( kh << 1 \), \( \sinh 2kh \approx 2kh \) and so \( c_g \approx c \), that is, the group velocity coincides with the wave speed. In the deep water limit, it is \( \sinh 2kh >> 2kh \) and \( c_g \approx \frac{1}{2} c \).

The physical meaning of the group velocity \( c_g \) is not easy to grasp. This concept is discussed in detail in books on gravity waves, e.g. in [2.1] and [2.2].

### 2.6. Irregular waves

#### 2.6.1. The Wave spectrum

So far, we considered only sinusoidal or regular waves. Real ocean waves are not regular: they are irregular and largely random. Figure 2.8 represents a typical plot of the sea surface elevation \( \zeta \) at a given horizontal location as a function of time. This could be a register from a measuring buoy or some other instrument.

\[\text{Fig. 2.7. Graphical representation of } D(kh).\]
The aim of describing ocean waves with a spectrum is not so much to describe in
detail one observation of the sea surface (i.e. one time record as in Fig. 2.8), but rather
to describe the sea surface as a stochastic process, i.e. to characterize all possible
observations (time records) that could have been made under the conditions of the
actual observation. An observation is thus formally treated as one realization of a
stochastic process.

We consider a wave record like in Fig. 2.8, with duration $D$. We can exactly
reproduce that record as the sum of a large (theoretically infinite) number of harmonic
wave components (a Fourier series) as

$$
\zeta(t) = \sum_{i=1}^{N} a_i \cos(2\pi f_i t + \alpha_i),
$$

(2.68)

where $a_i$ and $\alpha_i$ are the amplitude and phase, respectively, of each frequency $f_i = i/D$
($i = 1, 2, 3, \ldots$; the frequency interval is therefore $\Delta f = 1/D$). With a Fourier analysis, we
can determine the values of the amplitude and phase for each frequency and this would
give us the amplitude and phase spectrum for this record. By substituting these
computed amplitudes and phases into Eq. (2.68), we exactly reproduce the record
(provided that $N$ is large enough).

For most wave records, the phases turn out to have any value between 0 and $2\pi$
without any preference for any one value. Since this is almost always the case in deep
water (not for very steep waves), we will ignore the phase spectrum. Then, only the
amplitude spectrum remains to characterize the wave record. If we were to repeat the
experiment, i.e., measure the surface elevation again under statistically identical
conditions, the time record would be different and so would be the amplitude spectrum.
To remove the sample character of the spectrum, we should repeat the experiment many
times ($M$) and take the average over all these experiments, to find the average amplitude
spectrum

$$
\overline{a_i} = \frac{1}{M} \sum_{m=1}^{M} a_{i,m} \text{ for all frequencies } f_i,
$$

(2.69)

where $a_{i,m}$ is the value of $a_i$ in the experiment with sequence number $m$. For large
values of $M$, the value of $\overline{a_i}$ converges (approaches a constant value as we increase $M$),
thus solving the sampling problem. However, it is more meaningful to distribute the
variance of each wave component $\frac{1}{2}a_i^2$. An important reason is that the wave energy is proportional to the square of the wave amplitude (not to the amplitude) (see Eq. (2.58)).

The variance spectrum $\frac{1}{2}a_i^2$ is discrete, i.e., only the frequencies $f_i = i/D$ are present, whereas in fact all frequencies are present at sea. This is resolved by letting the frequency interval $\Delta f = 1/D$ approach zero. The definition of the variance density spectrum thus becomes

$$S_f(f) = \lim_{\Delta f \to 0} \frac{1}{\Delta f} \frac{1}{2}a_i^2.$$  

(2.70)

The dimensions and SI units of the variance density $S_f(f)$ follow directly from the definition (2.70) and are $[L^2 T]$, and $[m^2 s]$ or $[m^2/Hz]$. Figure 2.9 shows a typical variance density spectrum.

![Figure 2.9. Typical variance density spectrum.](image)

The variance density spectrum gives a complete description of the surface elevation of ocean waves in statistical sense, provided that the surface elevation can be seen as a stationary Gaussian process. To use this approach for conditions at sea, which are never really stationary, a wave record needs to be divided into segments that are each deemed to be approximately stationary (a duration of about 30 min is commonly used). In addition, at sea the wave components are not really independent from one another (as assumed here) because they interact to some degree. However, if the waves are not too steep and not in very shallow water, these interactions are weak and can be ignored.

The sea surface elevation is a random function of time. Its total variance is

$$\bar{\zeta}^2 = \int_0^\infty S_f(f) df.$$  

(2.71)

The variance density spectrum $S_f(f)$, showing how the variance of the sea surface elevation is distributed over the frequencies, is rather difficult to conceive. We recall that the time-averaged total (potential plus kinetic energy) of a regular wave per unit horizontal surface is (see Eq. (2.58)) $E = \frac{1}{2} \rho g A^2_w$. If we multiply $S_f(f)$ by $\rho g$ we obtain the energy density spectrum.
\[ E_f(f) = \rho g S_f(f) = \frac{1}{2} \rho g \lim_{\Delta f \to 0} \frac{1}{\Delta f} \sigma_i^2. \]  

(2.72)

The variance density \( S_f(f) \) was defined above in terms of frequency \( f = 1/T \) (where \( T \) is the period of the harmonic wave), but it can equally be formulated as \( S_\omega(\omega) \) in terms of radian frequency \( \omega = 2\pi f = 2\pi/T \). From Eq. (2.71) we may write

\[ \zeta^2 = \int_0^{\infty} S_f(f) \, df = \int_0^{\infty} S_\omega(\omega) \, d\omega. \]  

(2.73)

Since it is \( d\omega = 2\pi df \), we have

\[ S_\omega(\omega) = \frac{1}{2\pi} S_f(f). \]  

(2.74)

The overall appearance of the waves can be inferred from the shape of the spectrum: the narrower the spectrum, the more regular the waves are. This is shown for three different wave conditions in Fig. 2.10. The narrowest spectrum corresponds to a harmonic wave (regular wave with single frequency): the spectrum then degenerates to a delta function at one frequency. Distributing the variance over a slightly wider frequency band gives a slowly modulating harmonic wave because the components involved differ only slightly in frequencies and therefore get out of phase with one another only slowly, thus creating a fairly regular wave field. Distributing the wave variance over a wider frequency band gives a rather chaotic wave field (irregular waves), because the components in time get out of phase with one another rather quickly.

Fig. 2.10. The irregular character of the waves for three different widths of the spectrum.

2.6.2. The frequency-direction spectrum

The above one-dimensional variance density spectrum \( S_f(f) \) (or \( S_\omega(\omega) \)) characterizes the stationary Gaussian surface elevation as a function of time at one
geographic location. To describe the actual, three-dimensional, moving waves, the horizontal dimension has to be added. To that end, we expand the random-phase/amplitude model by considering a harmonic wave that propagates in \((x,y)\)-space, in direction \(\theta\) relative to the positive \(x\)-axis

\[
\zeta_{ij}(x,y,t) = a_{ij} \cos(\omega t - k_x x \cos \theta - k_y y \sin \theta + \alpha_{ij})
\]  \(\text{(2.75)}\)

or

\[
\zeta_{ij}(x,y,t) = a_{ij} \cos(\omega t - k_x x - k_y y + \alpha_{ij}),
\]  \(\text{(2.76)}\)

where \(k = 2\pi/\lambda\) is the wavenumber (\(\lambda\) is the wavelength) of the harmonic wave, \(k_x = k \cos \theta\), \(k_y = k \sin \theta\), and \(\theta\) is the direction of the wave propagation (i.e., normal to the wave crest of each individual component). Note that the radial frequency and the wavenumber \(k\) are related through the dispersion relationship \(\omega^2 = gk \tanh kh\) (see Eq. (2.39)). As for the one-dimensional model (Eqs (2.69-70)), we may write

\[
\bar{a}_{ij} = \frac{1}{M} \sum_{M=1}^{M} a_{ij,m} \quad \text{for all frequencies } \omega_i \text{ and directions } \theta_j
\]  \(\text{(2.77)}\)

and obtain the two-dimensional (frequency-direction) spectrum (see Eq. (2.70))

\[
S_{\omega,\theta}(\omega, \theta) = \lim_{\Delta \omega \to 0, \Delta \theta \to 0} \frac{1}{\Delta \omega \Delta \theta} \frac{1}{2} \overline{a_{ij}^2}.
\]  \(\text{(2.78)}\)

The one-dimensional frequency spectrum \(S_\omega(\omega)\), which does not contain any directional information, can be obtained from the two-dimensional frequency-direction spectrum \(S_{\omega,\theta}(\omega, \theta)\) by integration over all directions (per frequency)

\[
S_\omega(\omega) = \int_0^{2\pi} S_{\omega,\theta}(\omega, \theta) d\theta
\]  \(\text{(2.79)}\)

In an identical way, we can define \(S_{f,\theta}(f, \theta) = 2\pi S_{\omega,\theta}(\omega, \theta)\).

2.6.3. Significant wave height, mean wave period

When the random sea-surface elevation is treated as a stationary, Gaussian process, then all statistical characteristics are determined by the variance density spectrum \(S_f(f)\). These characteristics will be expressed in terms of the moments of the spectrum, which are defined as

\[
m_m = \int_0^\infty f^m S_f(f) df \quad (m = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots).
\]  \(\text{(2.80)}\)

The moment \(m\) is called the \(m\)th-order moment of \(S_f(f)\). For example, the variance of the surface elevation (see Eq. (2.73)) is equal to the zeroth-order moment

\[
\overline{\zeta^2} = \int_0^\infty S_f(f) df = m_0
\]

The significant wave height, denoted by \(H_s\), is defined as the mean value of the highest one-third of wave heights in the wave record. It is given approximately by (see [2.2])

\[
H_s \approx H_{m0} = 4\sqrt{m_0}.
\]  \(\text{(2.81)}\)

In wave energy conversion studies, two different definitions of “mean” period for irregular waves are frequently used. One is the peak period \(T_p = 1/f_p\), where \(f_p\) is the value (peak frequency) of the frequency \(f\) for which the variant density \(S_f(f)\) is
maximum \(f_p\) is the abscissa of the peak of the curve in Fig. 2.9). Another definition is the energy period \(T_e\)

\[ T_e = \frac{m_{-1}}{m_0}. \tag{2.82} \]

If the function \(S_f(f)\) is known for a given sea state, then the mean periods \(T_p\) and \(T_e\) can easily be computed and related to each other.

The characteristics of the frequency spectra of sea waves have been fairly well established through analyses of a large number of wave records taken in various waters of the world. The spectra of fully developed wind waves in deep water, for example, can be approximated by the following formula in terms of significant wave height \(H_s\) and energy period \(T_e\)

\[ S_f(f) = 0.1688 H_s^2 T_e^{-4} f^{-5} \exp \left[ -0.675(T_e f)^{-4} \right], \tag{2.83} \]

This can be rewritten in terms of radian frequency \(\omega = 2\pi f\) (see Eq. (2.74)) as

\[ S_\omega(\omega) = 262.6 H_s^2 T_e^{-4} \omega^{-5} \exp \left[ -1052(T_e \omega)^{-4} \right]. \tag{2.84} \]

Expression (2.83) (or (2.84)) is a modified form of the well-known Pierson-Moskowitz spectrum for fully developed wind sea (see Eq. (2.11) in [2.4]). More information, and other formulae for different sea conditions, can be found in [2.2], [2.4] and other books on ocean waves.

### 2.6.4. Energy flux

We recall that the total (potential plus kinetic) time-averaged energy of a regular wave of amplitude \(A_w\) per unit horizontal surface area is (see Eq. (2.58)) \(\bar{E} = \frac{1}{2} \rho g A_w^2\). The corresponding time-averaged energy flux per unit wave crest length is (see Eq. (2.63)) \(\bar{P}_{\text{wave}} = \frac{\bar{E}}{c_g}\), where \(c_g\) is the group velocity. In deep water it is \(c_g = \frac{1}{2} c = g/(4\pi f)\).

For irregular waves with variance density spectrum \(S_f(f)\), the energy flux transported by the waves in deep water, within the frequency interval \((f, f + df)\), is (see Eq. (2.72))

\[ \text{d}\bar{P}_{\text{wave}} = c_g \bar{E}_f(f) df = \rho g c_g S_f(f) df = \frac{\rho g^2}{4\pi} S_f(f) \frac{1}{f} df. \tag{2.85} \]

By integration, we find

\[ \bar{P}_{\text{wave}} = \frac{\rho g^2}{4\pi} \int_0^{\infty} S_f(f) \frac{1}{f} df = \frac{\rho g^2}{4\pi} m_{-1}, \tag{2.86} \]

where \(m_{-1}\) is the spectral moment of order \(-1\). Since its is \(T_e = m_{-1}/m_0\) and \(H_s = 4\sqrt{m_0}\) we find, for irregular waves in deep water,

\[ \bar{P}_{\text{wave}} = \frac{\rho g^2}{64\pi} H_s^2 T_e. \tag{2.87} \]

Setting \(g = 9.8 \text{ ms}^{-2}\) and \(\rho = 1025 \text{ kg m}^{-3}\), we obtain

\[ \bar{P}_{\text{wave}} = 0.490 H_s^2 T_e, \tag{2.88} \]
with $P_{\text{wave}}$ in (kW/m), $H_s$ in (m) and $T_e$ in (s). Equations (2.87) and (2.88) show the advantage of using the energy period $T_e$ in irregular waves; these equations are valid for any variance density spectrum $S_f(f) \text{ or } S_{\omega}(\omega)).$

2.6.4. Wave climate

In what was said above, the statistical characteristics of the waves were considered for short term, stationary conditions, usually for the duration of a wave record (15 to 30 min) and sometimes for a storm (a few hours). For long-term statistics, e.g., statistics over durations of several years (possibly tens of years) the conditions are not stationary, and the problem of describing waves needs to be approached in a different way. For these long time scales, it is not feasible to present waves as a time series of the surface elevation itself. Instead, each stationary condition (with a duration of 15 to 30 min) is replaced with its values of the significant wave height, period and mean wave direction. This gives a long-term sequence of these values with a time interval of typically 3 h, which can be analysed to estimate the long-term statistical characteristics of the waves, for instance to obtain design conditions marine structures or wave energy converters. Usually the analysis is limited to significant wave height and mean wave period.

Often, the first step in analysing the long term series of the significant wave height $H_s$, mean wave period $T$ (peak period $T_p$, energy period $T_e$ or other) and mean wave direction $\bar{\theta}$, is to estimate the joint probability density function $p(H_s, T, \bar{\theta})$, usually sorting the observed values and presenting the results in two-dimensional histograms of $H_s$ and $T$ per directional sector, $\Delta \bar{\theta}$. The number of observations in then presented (instead of the probability density) in bins of size $\Delta H_s$, $\Delta T$ (per directional sector, typically $\Delta \bar{\theta} = 30^\circ$ or $45^\circ$). Such an histogram is presented in Fig. 2.11, for a location off the western Portuguese coast, in terms of the significant wave height $H_s$ and the energy period $T_e$, for $\bar{\theta} = 300^\circ$ and $\Delta \bar{\theta} = 30^\circ$. By adding the numbers over the directional sectors, one obtains the histogram for the significant wave height and mean period irrespective of direction, representing the joint distribution $p(H_s, T)$.

![Fig. 2.10. Annual joint relative frequency of occurrence of $H_s$ and $T_e$ (azimuth $300^\circ$, $\Delta \bar{\theta} = 30^\circ$), for a location off the Portuguese western coast.](image-url)
References

3. Modelling of oscillating body wave energy converters

3.1. Introduction

Oscillating bodies are a major class of wave energy converters. The wave energy absorption results from the interaction between the oscillating body and the incoming waves. The device may be a single rigid body, or may be a system of bodies that can move with respect to each other with constraints that allow relative rotation (by hinges or similar mechanisms) or relative translation (sliding mechanisms).

A single floating oscillating body has in general six degrees of freedom: three translations and three rotations. For a ship-like elongated body (directed parallel to the $x$-axis as indicated in Fig. 3.1) the modes are named and numbered as shown in Fig. 3.1. For an axisymmetric body or another non-elongated body, the modes 1-surge, 2-sway, 4-roll and 5-pitch are ambiguous. However the ambiguity may be removed when there is an incident wave where the propagation direction defines the $x$-direction.

![Fig. 3.1. Modes of oscillation of a rigid body.](image)

A slack-moored (as opposed to tight-moored) single body is free to oscillate in the six modes. In other cases, the body may be hinged with respect of a fixed structure (sea bottom, breakwater or other) and the only mode of motion is angular oscillation (pitch or roll, depending on the geometry and on convention). This is the case of the Oyster and the WaveRoller (both hinged at the sea bottom) and of the Wave Star and the Brazilian hyperbaric device (whose hinges are above sea water level). In the Archimedes Wave Swing (AWS) the only mode of motion of the active body is heave. Sometimes, although the body may move in several degrees of freedom, the power take-off mechanism (PTO) is associated with a particular mode of motion (frequently heave), the other modes playing a secondary role.

In the case of multi-body devices, one body is connected to another one by a mechanism that allows one (rarely more than one) mode of relative motion: rotation (as in Pelamis) or translation (as in Power Buoy and WaveBob). For example, the two-body WaveBob has 7 degrees of freedom, corresponding to the three translations and three rotations of the pair, plus the translational motion of one body with respect to the other.

The wave energy utilization involves “large bodies” oscillating in water. It is important to clarify what is meant here by a large body. There are at least three relevant

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3 This paragraph is based on [3.1].
length scales in wave-body interaction: the characteristic body dimension \( a \), the wavelength \( \lambda = 2\pi/k \), and the wave amplitude \( A_w \). Among these scales, two ratios may be formed, for example, \( ka \) and \( A_w/a \). If the characteristic body dimension is of the order of \( \lambda/2\pi \) or larger, \( (ka \geq O(1)) \), the body is regarded as large; its presence alters the pattern of wave propagation significantly and produces diffraction. Most ships fall into this category. For small bodies \( (ka << 1) \), such as the structural members of a drilling tower, diffraction is of minor importance. When \( A_w/a \) is sufficiently large, the local velocity gradient near the small body augments the effect of viscosity and induces flow separation and vortex shedding, leading to the so-called form drag. At present, the inviscid linearized diffraction theory has been fairly well developed for \( A_w/a << 1 \) and \( ka = O(1) \) with considerable experimental confirmation. The case of \( A_w/a \geq O(1) \) and \( ka << 1 \) has been the subject of intensive experimental studies, but is not easily describable on purely theoretical ground. The intermediate case of \( A_w/a \geq O(1) \) and \( ka = O(1) \) involves both separation and nonlinear diffraction and is the most difficult and least explored area of all.

The theory that we will be developing here for oscillating-body wave energy converters, based on inviscid potential flow, assumes that \( A_w/a << 1 \) and \( ka = O(1) \), i.e. the wave amplitude \( A_w \) is much smaller that the body characteristic length \( a \), and \( a \) is of the order of magnitude of \( \lambda/2\pi \). In most cases, we will deal with bodies whose characteristic length \( a \) ranges between 10 m and 50 m, and wave amplitudes not exceeding about 2 m.

The book by Falnes [3.2] is the most complete text on the linear theory of wave energy absorption by oscillating bodies.

### 3.2. Wave field of a single heaving body

We start by analyzing the simple case of a single body with a single degree of freedom: heave oscillations (Fig. 3.2). The vertical position of the body is defined by a vertical coordinate \( \xi \) from a point \( O \) fixed to the body \( (O \) may coincide or not with the centre of gravity), with \( \xi = 0 \) in the absence of waves. The body is connected to the sea bottom through a power take-off system (PTO) that converts the body motion into useful energy. We denote by \( S \) the wetted surface of the body separating it from the water, and by \( n \) the unit vector perpendicular to \( S \) pointing into the water.

The body is subject to its weight \( mg \), where \( m \) is its mass. In the absence of waves, in equilibrium conditions, the body weight is balanced by the upward hydrostatic force, i.e., is equal to the weight of the displaced volume of water. So, in the dynamic equations, we may omit the body weight and consider only disturbances (assumed small in linear wave theory) to the pressure forces on the wetted surface of the body.

We start by considering the body fixed at its equilibrium position \( \xi = 0 \), and assume an incident wave whose velocity potential is \( \phi_i \), progressing from left to right in the positive \( x \)-direction. The presence of the body produces diffraction and disturbs the incident wave field. Since Laplace’s equation is linear, we may construct a solution to this problem by linear superposition, and introduce a diffraction wave field with
velocity potential $\phi_d$ that, like the incident wave potential $\phi_i$, has to satisfy the linearized boundary condition (2.17) on the undisturbed free surface.

\[
\frac{\partial^2 \phi_d}{\partial t^2} = -g \frac{\partial \phi_d}{\partial z} \quad \text{on} \quad z = 0 \quad (3.1)
\]

and the impermeability condition $\frac{\partial \phi_d}{\partial n} = 0$ on the sea bottom. The velocity component normal to the body wetted surface has to vanish on $S$, which we write as

\[
\frac{\partial \phi_d}{\partial n} = -\frac{\partial \phi_i}{\partial n} \quad \text{on} \quad S. \quad (3.2)
\]

The excitation force results from the excess pressure $p_e$ on the body’s wetted surface. Its vertical component may be written as

\[
f_e = -\int_S n_z p_e dS, \quad (3.3)
\]

where $n_z$ is the vertical component of the unit vector $n$ and $p_e$ is given, from by Eq. (2.12), by

\[
p_e = -\rho \frac{\partial (\phi_i + \phi_d)}{\partial t}. \quad (3.4)
\]

If the diffraction term $\phi_d$ is neglected in Eq. (3.3), the resulting force is called the Froude-Krylov force. It may represent a reasonable approximation to the excitation force, in particular if the extension of the immersed part of the body is very small compared with the wavelength. It may be computationally convenient to use such an approximation because it is not then required to solve the boundary-value problem for finding the diffraction potential $\phi_d$.

We study now the force on the body when it oscillates about its equilibrium position in the absence of any incident wave. Its vertical coordinate $\xi$ is an oscillating function of time $t$. The body motion produces a radiated wave field with a radiated velocity.
potential $\phi_r$ that is required to satisfy the linearized boundary condition (2.17) on the undisturbed free surface

$$\frac{\partial^2 \phi_r}{\partial t^2} = -g \frac{\partial \phi_r}{\partial z} \quad \text{on} \quad z = 0 \quad (3.5)$$

and the impermeability condition $\partial \phi_r/\partial n = 0$ on the see bottom. The flow velocity relative to the body is $\nabla \phi_r - (d\xi/dt)k$, where $k$ is unit vector pointing vertically upwards and $(d\xi/dt)k$ is the body velocity vector. On the wetted surface of the moving body, the impermeability condition requires the normal component of this relative velocity be zero, i.e., $\partial \phi_r/\partial n = (d\xi/dt)n_z$, where $n_z = n \cdot k$ is the vertical component of the unit normal vector $n$. In the present case of a heaving body, it is $\partial \phi_r/\partial z = d\xi/dt$ on the instantaneous wetted surface $S$. In linear wave theory, we may apply this condition at the equilibrium position of the body, it being assumed that the displacement $\xi$ is a small quantity compared with the wavelength. This is particularly convenient since the motion of the body is in general unknown a priori. We then write

$$\partial \phi_r/\partial n = (d\xi/dt)n_z \quad \text{on the undisturbed wetted surface} \ S. \quad (3.6)$$

The body motion produces on its surface a radiation force whose vertical component is given by

$$f_r = -\int_S n_z p_r \, dS, \quad (3.7)$$

where

$$p_r = -\rho \frac{\partial \phi_r}{\partial t}. \quad (3.8)$$

Finally, if, in the absence of incident wave, the body is fixed at a position different from its equilibrium position, i.e. if $\xi \neq 0$, its buoyancy force no longer balances its weight, and we have a static disturbance force $f_{st}$ equal to the weight of the displaced water volume at $\xi = 0$ minus the corresponding value at $\xi \neq 0$. We assume the displacement $\xi$ to be small and write approximately

$$f_{st} = -g \rho S_{cs} \xi, \quad (3.9)$$

where $S_{cs}$ is the cross sectional area of the body at its position $\xi = 0$ by the undisturbed horizontal free surface $z = 0$. If the body in the vicinity of the free surface is cylindrical (with vertical or inclined axis), then the disturbance $f_{st}$ to the buoyancy force is given exactly by Eq. (3.9).

The governing equation for the body motion is simply Newton’s second law

$$m \frac{d^2 \xi}{dt^2} = f_e + f_r + f_{st} + f_{PTO}, \quad (3.10)$$

where $f_{PTO}$ is the vertical force of the power take-off system (PTO) upon the body.

### 3.3. Frequency-domain analysis of wave energy absorption by a single heaving body

#### 3.3.1. Linear systems

The wave energy converter represented in Fig. 3.2 may be regarded as a system whose input is the incident wave represented by the surface elevation $\zeta(t)$ at a given
observation point \((x,y)\), and the output is the body displacement \(\xi(t) = H[\zeta(t)]\). The system is linear if, for any two inputs \(\zeta_1(t)\) and \(\zeta_2(t)\), and for any constants \(c_1\) and \(c_2\), it is

\[
H[c_1\zeta_1 + c_2\zeta_2] = c_1H[\zeta_1] + c_2H[\zeta_2].
\] (3.11)

Linear systems are particularly simple to study theoretically. In particular, if the input \(\zeta(t)\) is a simple harmonic (or sinusoidal) function of time, the output \(\xi(t) = H[\zeta(t)]\) is also a simple harmonic function of time.

With the linearizing assumptions that were introduced above in the modelling of our wave energy converter including in the boundary conditions, the converter may be considered as a linear system, provided that the power take-off system (PTO) is also a linear mechanism. This is the case if it consists of a linear damper and a linear spring, as represented in Fig. 3.3. The spring may account for the presence of a mooring force. The vertical force applied by the PTO on the floater is

\[
f_{\text{PTO}} = -C \frac{d\xi}{dt} - K\xi,
\] (3.12)

where \(C\) is the damping coefficient and \(K\) is the spring stiffness. We assume here that the spring force is zero at the rest position \(\xi = 0\).

3.3.2. Governing equations and hydrodynamic coefficients

We consider an incident wave of frequency \(\omega\) and amplitude \(A_w\), propagating in the positive \(x\)-direction (from left to right) on deep water or on water of arbitrary but uniform depth. Its potential is (see section 2)

\[
\Phi = \Phi_1(z) \exp[i(\omega t - kx)].
\] (3.13)
Since the system is linear and the input (incident wave) is represented by a harmonic function of time, the body displacement and the forces upon the body are also harmonic functions of time. We may write

$$\xi(t) = X e^{i\omega t}, \quad f_e(t) = F_e e^{i\omega t}, \quad f_r(t) = F_r e^{i\omega t},$$

where $X$ is the complex amplitude of the body displacement $\xi$, and $F_e$ and $F_r$ are the excitation force and radiation force amplitudes respectively. These amplitudes are in general complex. We recall that, whenever a complex expression is equated to a physical quantity, its real part is to be taken.

The governing equation (3.10) for the body motion becomes

$$-\omega^2 m X e^{i\omega t} = (F_e + F_r - \rho g S_{cs} X - i\omega CX - KX) e^{i\omega t}.$$

or, more simply,

$$-\omega^2 m X - F_r + \rho g S_{cs} X + i\omega CX + KX = F_e. \tag{3.16}$$

In this way, we got rid of the dependence on time and are left only with (in general complex) amplitudes. Note that Eqs (3.15) and (3.16) are valid only after the transients associated to the initial conditions (initial position and velocity of the body) have died out.

It is convenient to decompose the radiation force coefficient $F_r$ as

$$F_r = (\omega^2 A - i\omega B) X,$$

where $A$ and $B$ are real coefficients. Equation (3.16) may now be rewritten as

$$\left\{-\omega^2 (m + A) + i\omega (B + C) + (\rho g S_{cs} + K)\right\} X = F_e. \tag{3.18}$$

We recall that, in Eq. (3.18), $F_e$ is the complex amplitude of the excitation force and represents the forcing term. The real coefficient $A$ is added to the body mass $m$ and is called *added mass*. It represents the inertia of the water surrounding the body that is entrained by the body in its motion. The added mass $A$ is a function of frequency $\omega$ and is in general positive, although special cases are known where the depth of submergence is small and $A$ may be negative (see [3.3]). The real coefficient $B$ is added to the damping coefficient $C$ of the PTO and is named *radiation damping coefficient* (or *radiation resistance* or *added damping coefficient*, see [3.2]).

Let us examine in more detail the radiation force. In the absence of incident waves, if the body is forced to perform an oscillating motion $\xi(t) = \Re(X e^{i\omega t})$, it becomes subject to a radiation force

$$f_r(t) = \Re(F_r e^{i\omega t}) = \Re\left\{(\omega^2 A - i\omega B) X e^{i\omega t}\right\}. \tag{3.19}$$

The instantaneous rate of work (force times velocity) done by this radiation force on the moving body is $\dot{W}_r(t) = f_r(t) \frac{d\xi}{dt}$. If we write $X = |X| e^{i\alpha}$, where $\alpha$ is the argument of $X$, we find

$$f_r(t) = |X| \left\{\omega^2 A \cos(\omega t + \alpha) + \omega B \sin(\omega t + \alpha)\right\} \tag{3.20}$$

and

$$\frac{d\xi}{dt} = |X| \omega \sin(\omega t + \alpha). \tag{3.21}$$

We obtain

$$\dot{W}_r(t) = -\frac{1}{2} A |X|^2 \omega^3 \sin(2\omega t + 2\alpha) - B |X|^2 \omega^2 \sin^2(\omega t + \alpha). \tag{3.22}$$

Let us find the time-averaged value $\overline{\dot{W}_r}$ of $\dot{W}_r(t)$. (We use a bar to denote time average over a wave period.) The contribution from the first term on the right-hand side of Eq.
(3.22) is obviously zero, whereas, in the second term, the average value of the squared sinus is $1/2$. We obtain

$$\overline{W_r} = -\frac{1}{2} \omega^2 B |X|^2. \tag{3.23}$$

As should be expected, the first term on the right-hand side of Eq. (3.21), representing work done by an inertia force, does not contribute to the net work. The only non-zero contribution to net work $\overline{W_r}$ done by the radiation force on the body comes from the radiation force. It cannot be positive, otherwise we would be absorbing energy from non-existing incident waves. We conclude that the radiation damping coefficient $B$ cannot be negative. In general it is $B > 0$, although special geometries have been derived theoretically for which it is $B = 0$ at a given frequency.

Not unexpectedly, it turns out that the coefficients $A$ (added mass), $B$ (radiation damping) and $\Gamma = |F_e|/A_w$ (excitation force amplitude per unit incident wave amplitude) are related to each other.

We consider for a moment the general case in which the direction of the incident wave propagation makes an angle $\beta$ ($-\pi \leq \beta < \pi$) with the $x$-axis, and denote by $\Gamma(\beta) = |F_e(\beta)|/A_w$ the corresponding value of the excitation force amplitude per unit incident wave amplitude. It may be shown [3.2] that, for fixed $\omega$, the following relationship (Haskind relation) exists between $B$ and $\Gamma(\beta)$

$$B = \frac{\omega k}{4\pi \rho g^2 D(kh)} \int_{-\pi}^{\pi} [\Gamma(\beta)]^2 \, d\beta, \tag{3.24}$$

where $D(kh)$ is given by any of the expressions (2.67). In the deep water limit, it is $kh \to \infty$, $D(kh) \to 1$ and

$$B = \frac{\omega^3}{4\pi \rho g^2} \int_{-\pi}^{\pi} [\Gamma(\beta)]^2 \, d\beta. \tag{3.25}$$

The added mass $A(\omega)$ and the radiation damping coefficient $B(\omega)$ are related to each other by the Kramers-Kronig relations (see [3.2])

$$A(\omega) - A(\infty) = \frac{2}{\pi} \int_0^\infty \frac{-B(y)}{\omega^2 - y^2} \, dy, \tag{3.26}$$

$$B(\omega) = \frac{2\omega^2}{\pi} \int_0^\infty \frac{A(y) - A(\infty)}{\omega^2 - y^2} \, dy. \tag{3.27}$$

The added mass $A$, the radiation damping coefficient $B$ and the excitation force coefficient $F_e/A_w$ per unit incident wave amplitude depend on the wave frequency $\omega$ and on body geometry. Analytical expressions for these coefficients in terms of elementary functions can be found only for some very simple geometries, like the sphere and the horizontal-axis circular cylinder. Commercial codes, based on the boundary-element method, are available to compute these coefficients for arbitrary geometries and frequencies, some of the best known being WAMIT, ANSYS/Aqwa and Aquaplus.

### 3.3.3. Absorbed power and power output

The instantaneous power $P_{\text{abs}}$ absorbed from the waves is the instantaneous force on the body wetted surface multiplied the body velocity $d\xi/dt$.
The instantaneous rate of work \( P_{\text{PTO}} \) done by the PTO force is
\[
P_{\text{PTO}}(t) = \left[ C \frac{d \xi}{dt} + K \xi \right] \frac{d \xi}{dt}.
\]
(3.29)

As we are neglecting viscous losses and other losses in water and in the PTO, the difference between \( P_{\text{abs}} \) and \( P_{\text{PTO}} \) is the rate of variation in the energy stored in the body as kinetic energy and in the PTO spring as elastic energy. Obviously in time average this difference vanishes and we have \( \overline{P}_{\text{abs}} = \overline{P}_{\text{PTO}} = \overline{P} \) (say). We find
\[
\overline{P} = \overline{P}_{\text{PTO}} = \frac{1}{2} \omega^2 C |X|^2.
\]
(3.30)
or equivalently (see [3.4])
\[
\overline{P} = \overline{P}_{\text{abs}} = \frac{1}{8B} |F_e|^2 - \frac{B}{2} \left| U - \frac{F_e}{2B} \right|^2,
\]
(3.31)
where \( U = i \omega X \) is the complex amplitude of the body velocity.

Let us fix the incident wave frequency \( \omega \) and amplitude \( A_m \), as well as the body geometry (but not the PTO coefficients \( C \) and \( K \)). This implies that, in Eq. (3.31), the values of the radiation damping coefficient \( B \) (real) and the excitation force amplitude \( F_e \) (in general complex) are fixed. On the right-hand side of Eq. (3.31) only the complex amplitude \( U = i \omega X \) of the body velocity \( d \xi / dt \) is allowed to vary, since it depends on the PTO parameters \( C \) (damper) and \( K \) (spring). We look for the conditions that maximize the time-averaged power output \( \overline{P} \). Equation (3.31) shows that, since \( B \) and \( F_e \) are fixed, the power output \( \overline{P} \) is maximum when
\[
U = i \omega X = \frac{F_e}{2B}.
\]
(3.32)

This condition shows that, under optimal conditions, the velocity \( d \xi / dt \) of the oscillating body must be in phase with the excitation force \( f_e \). (Note that \( B \) is real positive; we exclude here the special cases of \( B = 0 \) as being of no practical interest.) We replace \( X = -i F_e/(2 \omega B) \) in Eq. (3.18) and obtain
\[
-\omega^2 (m + A) + i \omega (B + C) + (\rho g S_{cs} + K) = i 2 \omega B.
\]
(3.33)
If we separate the real parts from the imaginary parts, we find
\[
\omega = \sqrt{\frac{\rho g S_{cs} + K}{m + A}}.
\]
(3.34)
and
\[
B = C.
\]
(3.35)
Equation (3.34) is a resonance condition. Compare with the classical case of a simple mechanical oscillator consisting of a body of mass \( m \) hanging from a fixed structure through a linear spring of stiffness \( K \), as represented in Fig. 3.4. If the body is displaced from its rest position, it will freely oscillate with frequency
\[
\omega = \sqrt{\frac{K}{m}}.
\]
(3.36)

Equations (3.34) and (3.36) are similar to each other, except for the presence, in Eq. (3.34), of the added mass \( A \) and of the buoyance restoring force coefficient \( \rho g S_{cs} \).
Equation (3.34) expresses a resonance condition: the incident wave frequency $\omega$ must be equal to the frequency of the free oscillations of a mass-spring mechanical system of mass $m + A$ and spring stiffness $\rho g S_{cs} + K$.

![Fig. 3.4. Simple mass-spring mechanical oscillator.](image)

Condition (3.35) means that the radiation damping must be equal to the PTO damping for maximum wave energy absorption. We may say that a good wave energy absorber must also be a good wave radiator.

If the optimal conditions (3.34) and (3.35) are satisfied, the following expression is obtained from Eq. (3.31) for the maximum time-averaged absorbed power from the waves (or power output from the PTO)

$$\bar{P}_{\text{max}} = \frac{1}{8B} | F_c |^2 .$$

(3.37)

In wave energy absorption, the concept of efficiency may be misleading and should be used with care, since the available power is not as well defined as in other energy conversion systems. Especially in the case of “small” wave energy converters, it may be more adequate to define an absorption width (or capture width) $L$ as the ratio between the time-averaged absorbed power $\overline{P}$ and the energy flux $\overline{P}_{\text{wave}}$ per unit crest length of the incident waves

$$L = \frac{\overline{P}}{\overline{P}_{\text{wave}}} .$$

(3.38)

This is the width of the two-dimensional wave-train having the same mean power as the body extracts. If $\overline{P}$ is in kW and $\overline{P}_{\text{wave}}$ in kW/m, the absorption width $L$ is expressed in metres.

3.3.4. Axisymmetric body

An important class of single-oscillating-body wave energy converters have a vertical axis of symmetry and extract energy essentially through their heave motion. In most cases their mooring system also allows other modes of oscillation (namely surge and pitch, see Fig. 3.1) but this will be ignored here.
Fig. 3.5. Wave energy absorption by a heaving axisymmetric body.

Such a system is represented in Fig. 3.5. As above, the PTO consists of a linear damper and a linear spring. Since the system is insensitive to the incident wave direction, the excitation force amplitude \( \Gamma = |F_e|/A_w \) per unit incident wave amplitude is not a function of the incidence angle. Equation (3.24) becomes simply

\[
B = \frac{\omega k \Gamma^2}{2\rho g^2 D(kh)} \tag{3.39}
\]

or, in the case of deep water,

\[
B = \frac{\omega^3 \Gamma^2}{2\rho g^3} \tag{3.40}
\]

Equation (3.37), together with (3.39), gives for the maximum time-averaged absorbed power

\[
P_{\text{max}} = \frac{\rho g^2 A_w^2 D(kh)}{4\omega k} \tag{3.41}
\]

or, in terms of absorption width, \( L_{\text{max}} = P_{\text{max}}/P_{\text{wave}} \). Making use of Eqs (2.58) and (2.66), we obtain

\[
L_{\text{max}} = \frac{1}{k} = \frac{\lambda}{2\pi} \tag{3.42}
\]

(where \( k \) is the wavenumber and \( \lambda \) is the wavelength), which shows that the maximum absorption width by an axisymmetric heaving body is equal to \( \lambda/2\pi \) and is independent of the size and shape of the body. This important theoretical result was obtained independently by Budal and Falnes [3.5], Evans [3.6] and Newman [3.7]. It shows that, theoretically, a heaving body can absorb more power than the energy flux of an incident regular wave train along a frontage wider than the width of the body itself. This is a good reason why the concept of efficiency should be avoided or used carefully in wave energy absorption.
3.4. Time-domain analysis of wave energy absorption by a single heaving body

If the power take-off system is not linear (the force $f_{PTO}$ is not a linear functional of the body velocity $d\xi/dt$ or of coordinate $\xi$), then the frequency-domain analysis cannot be employed. In particular, even in the presence of regular incident waves, the body velocity is not a simple harmonic function of time. In such cases, we have to resort to the so-called time-domain analysis to model the radiation force.

When a body is forced to move in otherwise calm water, its motion produces a wave system (radiated waves) that propagate far away. Even if the body ceases to move after some time, the wave motion persists for a long time (theoretically for ever if dissipative effects are neglected) and produces an oscillating force of the body wetted surface that depends on the history of the body motion through the induced radiated wave field. We are in the presence of a memory effect. This dependence can be expressed in the following form

$$f_r(t) = -\int_{-\infty}^{t} g_r(t-\tau) \dot{\xi}(\tau) d\tau - A(\infty) \ddot{\xi}(t),$$  \hspace{1cm} (3.43)

where $\dot{\xi}(t)$ and $\ddot{\xi}(t)$ are the body velocity and acceleration, and $A(\infty)$ is the value of the added mass for infinite frequency. In the convolution integral, the velocity is multiplied by the weighing function $g_r$ that accounts for the memory effect and is expected to tend to zero as its argument increases to infinity. Naturally, the radiation force $f_r(t)$ at time $t$ can only depend on the body velocity at instants before time $t$; this is why the upper limit of the integral is taken equal to $t$. The convenience of adding the term $-A(\infty) \ddot{\xi}(t)$ on the right-hand side of Eq. (3.43) will become apparent below.

To obtain an expression for function $g_r$, we replace, as in the previous section,

$$f_r(t) = F_r e^{i\omega t} = \left(\omega^2 A(\omega) - i\omega B(\omega)\right) X e^{i\omega t}$$  \hspace{1cm} (3.44)

and

$$\dot{\xi}(\tau) = i\omega X e^{i\omega \tau}, \quad \ddot{\xi}(t) = -\omega^2 X e^{i\omega \tau}. \hspace{1cm} (3.45)$$

We obtain

$$\left[\omega^2 (A(\omega) - A(\infty)) - i\omega B(\omega)\right] X e^{i\omega t} = -i\omega X \int_{-\infty}^{t} g_r(t-\tau) e^{i\omega \tau} d\tau. \hspace{1cm} (3.46)$$

Changing the integration variable from $\tau$ to $s = t - \tau$, we have

$$i\omega (A(\omega) - A(\infty)) + B(\omega) = \int_{-\infty}^{t} g_r(s) e^{-i\omega s} ds. \hspace{1cm} (3.47)$$

Since the functions $A, B$ and $g_r$ are real, we may write

$$\omega (A(\omega) - A(\infty)) = \int_{0}^{\infty} g_r(s) \sin \omega s ds, \hspace{1cm} (3.48)$$

$$B(\omega) = \int_{0}^{\infty} g_r(s) \cos \omega s ds. \hspace{1cm} (3.49)$$

Note that, as $|\omega| \to \infty$, $\sin \omega s$ and $\cos \omega s$ become rapidly oscillating functions of $s$. Since the memory function $g_r$ is finite, the integrals in Eqs (3.47) and (3.48) tend to zero as $|\omega| \to \infty$, as can be shown by integration by parts. This agrees with Eq. (3.49) as $B(\omega)$ vanishes for infinite frequency. On the other hand, the added mass $A(\omega)$ in general remains finite at infinite frequency, and this explains the presence of term...
- $A(\infty)$ on the left-hand side of Eq. (3.48) and the reason why the term $-A(\infty)\dot{\xi}(t)$ was added to the right-hand side of Eq. (3.43).

We may assume $g_r(s)$ to be an even function and, instead of Eq. (3.49), write

$$B(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} g_r(s) e^{i\omega s} ds.$$  \hspace{1cm} (3.50)

This expresses $B$ as the result of a Fourier transform of $g_r$. The inverse Fourier transform gives

$$g_r(s) = \frac{2}{\pi} \int_{0}^{\infty} B(\omega) \cos \omega s d\omega.$$  \hspace{1cm} (3.51)

Equation (3.51) allows the memory function $g_r$ to be computed if the radiation damping coefficient $B$ is known as a function of frequency.

Replacing, in the governing equation (3.10), the radiation force $f_r$ by its expression (3.43), we obtain

$$[m + A(\infty)]\ddot{\xi}(t) = f_c(t) - \int_{-\infty}^{t} g_r(t - \tau) \ddot{\xi}(\tau) d\tau - \rho g S_\theta \theta(t) + f_{PTO}.$$  \hspace{1cm} (3.52)

The force $f_{PTO}$ of the power take-off system on the body is supposed to be prescribed as a function of time $t$ and/or as a function of the body coordinate $\xi$ and/or the body velocity $\dot{\xi}$, depending on the type of PTO and on the control strategy and algorithm. Integro-differential equation (3.52) is to be integrated numerically step-by-step in the time domain from given initial conditions for $\xi$ and $\dot{\xi}$.

Equation (3.43) for the radiation damping force in the time domain, together with Eq. (3.50), generalized to the six modes of oscillation, was derived for the motions of a ship with zero forward speed by Cummins [3.8]. Equation (3.51) is sometimes called Cummins equation in wave energy absorption by oscillating bodies.

### 3.5. Wave energy conversion in irregular waves

So far, we considered only sinusoidal or regular waves. Real ocean waves are not regular: they are irregular and largely random. However, in linear wave theory, they can be analysed by assuming that they are the superposition of an infinite number of wavelets with different frequencies and directions. As we saw in section 2.6, the distribution of the energy of these wavelets when plotted against the frequency and direction is called the wave spectrum. More precisely, the wave distribution with respect to the frequency alone, irrespective of the wave direction, is called the frequency spectrum, whereas the energy distribution as a function of both frequency and direction is called the directional wave spectrum. Here, we consider only frequency spectra.

We recall that the variance density spectrum is (see Eq. (2.70))

$$S_f(f) = \lim_{\Delta f \to 0} \frac{1}{\Delta f} \frac{1}{2} \sum_i a_i^2$$  \hspace{1cm} (3.57)

or, in terms of radian frequency $\omega$,

$$S_\omega(\omega) = \lim_{\Delta \omega \to 0} \frac{1}{\Delta \omega} \frac{1}{2} \sum_i a_i^2.$$  \hspace{1cm} (3.58)

In computations, it is convenient to replace the continuum spectrum by a superposition of a finite number of sinusoidal waves of different amplitudes and frequencies whose total energy matches the spectral distribution. For that, we divide the
frequency range of interest into a set of \( N \) small intervals \( \omega_i \leq \omega \leq \omega_{i+1} \) (\( i = 1, 2, \ldots, N + 1 \)) of width \( \Delta_i = \omega_{i+1} - \omega_i \) and write

\[
S_{\omega,i} \Delta_i = \frac{1}{2} A_{w,i}^2 \quad \text{or} \quad A_{\omega,i} = \sqrt{2S_{\omega,i} \Delta_i},
\]

(3.59)

where

\[
S_{\omega,i} = S_{\omega}(\hat{\omega}_i)
\]

(3.60)

and \( \hat{\omega}_i = \frac{1}{2}(\omega_i + \omega_{i+1}) \). To simulate the excitation force \( f_e(t) \) due to incident irregular waves characterized by a variance density spectrum \( S_{\omega}(\omega) \), we write

\[
f_e(t) = \sum_{i=1}^{N} \Gamma(\hat{\omega}_i) A_{\omega,i} \exp[i(\hat{\omega}_i t + \alpha_i)],
\]

(3.61)

where \( \Gamma(\omega) = [F_e]/A_w \) is the excitation force amplitude per unit incident wave amplitude that is supposed to be known as a function of the frequency \( \omega \). In Eq. (3.61), \( \alpha_i \) is a phase constant taken equal to a random number in the interval \((0, 2\pi)\).

In the time-domain analysis, Eq. (3.61) is written as

\[
f_e(t) = \sum_{i=1}^{N} \Gamma(\hat{\omega}_i) A_{\omega,i} \cos(\hat{\omega}_i t + \alpha_i).
\]

(3.62)

It should be noted that Eqs (3.61) and (3.62) provide realizations of the excitation force that can be used in numerical simulations. Such realizations are not supposed to reproduce time series of some particular real situation. By choosing different sets of random phases \( \alpha_i \), or by dividing the frequency interval differently, we obtain different realizations.

In the case of a linear PTO with a linear damper coefficient \( C \), the power, averaged over a sufficiently long time interval, is

\[
\bar{P} = \bar{P}_{PTO} = C \left( \frac{d^2}{dt^2} \hat{X}(t) \right)^2 = \frac{1}{2} C \sum_{i=1}^{N} \hat{\omega}_i^2 |X(\hat{\omega}_i)|^2,
\]

(3.63)

where

\[
|X(\hat{\omega}_i)| = \frac{\Gamma(\hat{\omega}_i) A_{\omega,i}}{-\hat{\omega}_i^2 (m + A(\hat{\omega}_i)) + i\hat{\omega}_i (B(\hat{\omega}_i) + C) + (\rho g S_{cs} + K)}
\]

(3.64)

and it was taken into account that

\[
\lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_0^{\Delta t} \sin(\hat{\omega}_i t + \alpha_i) \sin(\hat{\omega}_j t + \alpha_j) dt = \begin{cases} 1/2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

(3.65)

Exercise 3.1. Wave energy absorption by a hemipherical heaving floater

The heaving hemisphere (Fig. 3.7) is one of the few geometries for which analytically obtained results are available for the hydrodynamic coefficients of added mass \( A \) and radiation damping \( B \) [3.11]. These results are presented in dimensionless form in Table 3.1 for a sphere of radius \( a \) in deep water, where

\[
A^*(ka) = \frac{A(\omega)}{\frac{2}{3} \pi a^3 \rho}, \quad B^*(ka) = \frac{B(\omega)}{\frac{2}{3} \pi a^3 \rho \omega}.
\]

(3.66)
Fig. 3.7. Wave energy absorption by a hemispherical heaving floater.

Table 3.1. Dimensionless coefficients of added $A^*(ka)$ mass and radiation damping $B^*(ka)$ versus dimensionless radius $ka$ for heaving hemisphere in deep water (from [3.11]).

<table>
<thead>
<tr>
<th>$ka$</th>
<th>$A^*(ka)$</th>
<th>$B^*(ka)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.8310</td>
<td>0</td>
</tr>
<tr>
<td>0.05</td>
<td>0.8764</td>
<td>0.1036</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8627</td>
<td>0.1816</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7938</td>
<td>0.2793</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7157</td>
<td>0.3254</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6452</td>
<td>0.3410</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5861</td>
<td>0.3391</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5381</td>
<td>0.3271</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4999</td>
<td>0.3098</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4698</td>
<td>0.2899</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4464</td>
<td>0.2691</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4284</td>
<td>0.2484</td>
</tr>
<tr>
<td>1.2</td>
<td>0.4047</td>
<td>0.2096</td>
</tr>
<tr>
<td>1.4</td>
<td>0.3924</td>
<td>0.1756</td>
</tr>
<tr>
<td>1.6</td>
<td>0.3871</td>
<td>0.1469</td>
</tr>
<tr>
<td>1.8</td>
<td>0.3864</td>
<td>0.1229</td>
</tr>
<tr>
<td>2.0</td>
<td>0.3884</td>
<td>0.1031</td>
</tr>
<tr>
<td>2.5</td>
<td>0.3988</td>
<td>0.0674</td>
</tr>
<tr>
<td>3.0</td>
<td>0.4111</td>
<td>0.0452</td>
</tr>
<tr>
<td>4.0</td>
<td>0.4322</td>
<td>0.0219</td>
</tr>
<tr>
<td>5.0</td>
<td>0.4471</td>
<td>0.0116</td>
</tr>
<tr>
<td>6.0</td>
<td>0.4574</td>
<td>0.0066</td>
</tr>
<tr>
<td>7.0</td>
<td>0.4647</td>
<td>0.0040</td>
</tr>
<tr>
<td>8.0</td>
<td>0.4700</td>
<td>0.0026</td>
</tr>
</tbody>
</table>
Table 3.1 shows that, at infinite frequency, the added mass is non-zero (it is exactly \( A^*(\infty) = 1/2 \)) whereas the radiation damping coefficient is zero, as expected. The table is complemented by the following formulae based on asymptotic expressions supplied in [3.11] for large and small values of \( ka \)

\[
A^*(ka) = 0.8310 - 0.75ka \ln(ka) - 1.266ka - 1.433(ka)^2 \quad \text{for} \quad ka < 0.1,
\]

\[
A^*(ka) = 0.5 - 0.1875(ka)^{-1} - 0.44101(ka)^{-2} + 0.2608(ka)^{-3} \quad \text{for} \quad ka > 8,
\]

\[
B^*(ka) = 0.75\pi ka - 5.979(ka)^2 + 5.903(ka)^3 \quad \text{for} \quad ka < 0.2,
\]

\[
B^*(ka) = 13.5(ka)^{-4} - 6.75(ka)^{-5} - 122.5(ka)^{-6} \quad \text{for} \quad ka > 5.
\]

In deep water, it is \( ka = \omega^2 a/g = \omega^2 = (2\pi/T^*)^2 \), where \( \omega^* = \omega \sqrt{a/g} \) is a dimensionless frequency and \( T^* = T \sqrt{g/a} \) is a dimensionless wave period.

These numerical values for the added mass and radiation damping coefficient, together with the equations derived in sections 3.3 to 3.5, may be used to simulate the performance of a floating hemispherical wave energy converter oscillating in heave.

Results are plotted in Figs 3.8 and 3.9 for a hemispherical heaving floater in deep water subject to regular waves, with a PTO consisting of a linear damper with coefficient \( C \), without any spring \((K = 0)\). In each figure, curves are given for three values of the dimensionless PTO damping coefficient defined as

\[
C^* = \frac{C}{a^{5/2}\rho g^{1/2}}.
\]

(3.67)

Fig. 3.8. Dimensionless plot of time-averaged absorbed power versus wave period for the floater represented in Fig. 3.7, and for three values of the dimensionless PTO damping coefficient \( C^* \). No spring is present \((K = 0)\).
Fig. 3.9. As in Fig. 3.8 for the dimensionless amplitude $|X|/A_w$ of the heaving floater displacement.

Figure 3.8 shows that close to maximum power is achieved for $C^*=0.5$ (in fact, it is $\bar{P} = \bar{P}_{\text{max}}$ for $C^*=0.510$). For larger values of $C^*$, the curves exhibit lower peaks but become wider. This indicates that PTO damping coefficient that maximizes the energy absorbed from irregular waves characterized by a given spectrum may be larger than the optimal damping in regular waves whose frequency is equal of the peak frequency of the spectrum.

Figure 3.9 shows that the amplitude of the heaving oscillations decreases with increasing damping coefficient. Naturally it is expected that the damping force will increase. The PTO power is damping force times floater velocity. Fig. 3.8 shows that $\bar{P}/\bar{P}_{\text{max}}$ may increase or decrease with increasing $C^*$ depending on the dimensionless wave period $T^*$.

The following calculations are suggested to be performed as exercises.

- Reproduce the curves plotted in Figs 3.8 and 3.9 by doing your own programming.
- Compute the buoy radius $a$ and the PTO damping coefficient $C$ that yield maximum power from regular waves of period $T = 9\, \text{s}$. Compute the time-averaged power for wave amplitude $A_w = 1\, \text{m}$.
- Assume now that the PTO also has a spring of stiffness $K$ that may be positive or negative. Compute the optimal values for the damping coefficient $C$ and the spring stiffness $K$ for a buoy of radius $5\, \text{m}$ in regular waves of period $T = 9\, \text{s}$. Explain the physical meaning of a negative stiffness spring (in conjunction with reactive control).
- Consider “irregular” waves consisting of a superposition of $n$ sinusoidal waves. Choose $n$ (for example $n=5$), as well as the amplitudes and phases of each component. Study the performance of the heaving buoy in irregular waves.
- Study the performance of the heaving buoy in irregular waves characterized by the spectrum of Eq. (3.58). Choose the number of components ($N = 200$ is a typical value).
Exercise 3.2. Heaving floater rigidly attached to a deeply submerged body

Wave energy converters whose horizontal dimension is small compared with that the wavelength are sometimes named point absorbers. In most cases, the resonance frequency of floating point absorbers is significantly smaller than the typical frequency of ocean waves. For example, in the case of a heaving hemisphere with a linear PTO damper, Fig. 3.8 shows that resonance occurs for $T^* = T \frac{g}{a} \approx 6.1$. If the radius is $a = 7\text{ m}$, then it is $T = 5.16\text{ s}$, which is significantly less than the typical wave period of energetic sea states. An alternative to increasing the size of the buoy, is to rigidly connect it to a submerged body in order to increase the inertia of the pair. If the distance from the body to the free surface is large enough (say not less than about 30 m) then the disturbance due the body motions upon the wave field around the floater is small. This is because the radiated surface-wave field due to the body motion as well as the excitation force on the body vanish with increasing submergence. This kind of geometry was adopted in one of the bodies of the two-body device WaveBob, as shown in Fig. 1.26.

This situation is represented in Fig. 3.10. The governing equation in the frequency domain is Eq. (3.18) with $m + A(\omega)$ replaced by $m + A(\omega) + m_1 + A_1$. Here $m_1$ is the mass of the submerged body and $A_1$ its added mass. Note that, for a deeply submerged body, the added mass depends only on body geometry, and is independent of the frequency $\omega$ of its oscillations provided that the submergence is not smaller than about half the wavelength of surface waves of frequency $\omega$.

![Fig. 3.10. Wave energy absorption by a heaving hemispherical floater attached to a deeply submerged body.](image-url)
As an exercise, consider a hemispherical floater of radius $a = 7$ m with a linear PTO damper (Fig. 3.10), and determine the optimal value of $m_1 + A_1$ (mass plus added mass of the deeply submerged body) and of PTO damping coefficient $C$ for regular waves of period $T = 8$ s.

### 3.6. Wave energy absorption by two-body oscillating systems

#### 3.6.1. Governing equations

The concept of the point absorber for wave energy utilization was developed in the late 1970s and early 1980s, mostly in Scandinavia. This is in general a wave energy converter of oscillating body type whose horizontal dimensions are small compared to the representative wave length. In its simplest version, the body reacts against the bottom. In deep water (say 40 m or more), this may raise difficulties due to the distance between the floating body and the sea bottom, also possibly to tidal oscillations of the surface level. Multi-body systems may then be used instead, in which the energy is converted from the relative motion between two bodies oscillating differently. Sometimes the relevant relative motion results from heaving oscillations. This is the case of several devices like the the Wavebob, the PowerBuoy and the AquaBuoy (see Chapter 1).

![Two-body heaving wave energy converter](image)

Fig. 3.11. Two-body heaving wave energy converter.

Figure 3.11 represents a two-body wave energy converter in which the oscillations are essentially heaving. The PTO converts the energy associated to the relative motions and the forces between the two bodies. The coupling between bodies 1 and 2 is due firstly to the PTO forces and secondly to the forces associated to the diffracted and radiated wave fields. It is obvious that the excitation force on one of the bodies is affected by the presence of the other body. Besides, in the absence of incident waves, the radiated wave field induced by the motion of one of the bodies produces a radiation force on the moving body and also a force on the other body.

We denote by $\xi_1$ and $\xi_2$ the vertical displacements of bodies 1 and 2 from their undisturbed positions. The governing equations can be written as (see Eqs (3.9) and (3.10))
\[
m_1 \frac{d^2 \xi_1}{dt^2} = f_{e,1} + f_{r,11} + f_{r,12} - g \rho S_{cs,1} \xi_1 + f_{PTO},
\]
\[
m_2 \frac{d^2 \xi_2}{dt^2} = f_{e,2} + f_{r,22} + f_{r,21} - g \rho S_{cs,2} \xi_2 - f_{PTO}.
\]

Here, \( m_i \) (\( i = 1, 2 \)) is the mass of body \( i \), \( f_{e,i} \) is the excitation force on body \( i \), \( S_{cs,i} \) is the cross-sectional area of body \( i \) defined by the undisturbed free-surface plane, \( f_{r,ii} \) is the radiation force on body \( i \) due to its own motion, and \( f_{r,ij} \) is the radiation force on body \( i \) due to the motion of body \( j \).

### 3.6.2. Linear system. Frequency domain analysis

We assume now that the power take-off system is linear and consists of a linear spring of stiffness \( K \) and a linear damper with damping coefficient \( C \), connected in parallel, as we considered in section 3.3 and is represented in Fig. 3.3. The PTO force may be written as

\[
f_{PTO} = -C \frac{d(\xi_1 - \xi_2)}{dt} - K(\xi_1 - \xi_2).
\]

We consider an incident wave of frequency \( \omega \) and amplitude \( A_w \), propagating in the positive \( x \)-direction (from left to right) on deep water or on water of arbitrary but uniform depth. Since the system is linear and the input (incident wave) is represented by a harmonic function of time, the bodies’ displacements and the forces upon the bodies are also harmonic functions of time. We may write

\[
\xi_j(t) = X_i e^{i \omega t}, \quad f_{e,j}(t) = F_{e,j} e^{i \omega t}, \quad f_{r,ij}(t) = F_{r,ij} e^{i \omega t} \quad (i, j = 1, 2),
\]

where \( X_i \) is the complex amplitude of the displacement \( \xi_j \) of body \( i \), and \( F_{e,j} \) and \( F_{r,ij} \) are the amplitudes of the excitation forces and radiation forces, respectively. These amplitudes are in general complex.

As for the single heaving body, we decompose the radiation force coefficient \( F_{r,ij} \) as

\[
F_{r,ij} = (\omega^2 A_{ij} - i \omega B_{ij}) X_j \quad (i, j = 1, 2),
\]

where \( A_{ij} \) and \( B_{ij} \) are real coefficients of added mass and radiation damping, respectively. They depend on wave frequency \( \omega \) and on the geometry of the two-body system. The same commercial software based on the finite-element method (WAMIT, ANSYS/Aqwa, Aquaplus) can also be used to compute \( A_{ij} \), \( B_{ij} \) and \( F_{e,i}/A_w \) (and more generally for any number of bodies and degrees of freedom). As was the case of \( B \) in section 3.3, we may also conclude that the radiation damping coefficients \( B_{ii} \) cannot be negative; the same cannot be said for \( B_{ij} \) if \( i \neq j \). It can be proved (see [3.2]) that the cross coefficients are equal:

\[
A_{12} = A_{21}, \quad B_{12} = B_{21}.
\]

In the frequency domain, Eqs (3.68) and (3.69) become

\[
\left\{ -\omega^2 (m_1 + A_{11}) + i \omega (B_{11} + C) + (g \rho S_{cs1} + K) \right\} X_1
+ \left\{ -\omega^2 A_{12} + i \omega (B_{12} - C) - K \right\} X_2 = F_{e,1},
\]

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\left\{-\omega^2(m_2 + A_{22}) + i\omega(B_{22} - C) + (g\rho S_{cs2} - K)\right\}X_2 \\
+ \left\{-\omega^2A_{12} + i\omega(B_{12} + C) + K\right\}X_1 = F_{e,2}.
\tag{3.75}

Equations (3.74) and (3.75) are a system of linear algebraic equations in the unknowns \( X_1 \) and \( X_2 \) that can be easily solved. The instantaneous power available to the PTO is

\[ P_{PTO} = C(\dot{\xi}_1 - \dot{\xi}_2)^2 + K(\ddot{\xi}_1 - \ddot{\xi}_2)(\dot{\xi}_1 - \dot{\xi}_2). \]
\tag{3.76}

In time average, we have, for the absorbed power,

\[ \bar{P} = \bar{P}_{PTO} = \frac{1}{2} C\omega^2|X_1 - X_2|^2. \]
\tag{3.77}

As in subsection 3.3.2, we consider the case in which the direction of the incident wave propagation makes an angle \( \beta \) \((-\pi \leq \beta < \pi)\) with the \( x \)-axis, and denote by \( \Gamma_i(\beta) = |F_{e,i}(\beta)|/A_w \) the corresponding value of the excitation force amplitude per unit incident wave amplitude. It may be shown [3.2] that, for fixed \( \omega \), the following relationship (Haskind relation) exists between \( B_{ii} \) and \( \Gamma_i(\beta) \)

\[ B_{ii} = \frac{\omega k}{4\pi \rho g^2 D(kh)} \int_{-\pi}^{\pi} \left[ \Gamma_i(\beta) \right]^2 d\beta, \]
\tag{3.78}

where \( D(kh) \) is given by any of the expressions (2.67). In the deep water limit, it is \( kh \rightarrow \infty \), \( D(kh) \rightarrow 1 \) and

\[ B_{ii} = \frac{\omega^3}{4\pi \rho g^3} \int_{-\pi}^{\pi} \left[ \Gamma_i(\beta) \right]^2 d\beta. \]
\tag{3.79}

Equations (3.78) and (3.79) are similar to Eqs (3.24) and (3.25) that apply to a single heaving body. If the system has a vertical axis of symmetry, \( \Gamma_i \) is independent of \( \beta \) and Eqs (3.78) and (3.79) become more simply

\[ B_{ii} = \frac{\omega k \Gamma_i}{2\rho g^2 D(kh)}, \]
\tag{3.80}

and

\[ B_{ii} = \frac{\omega^3 \Gamma_i}{2\rho g^3} \] (deep water).
\tag{3.81}

In the case of an axisymmetric device, equations (3.80) and (3.81) allow the modulus of excitation force amplitudes \( F_{e,1} \) and \( F_{e,2} \) to be computed from the radiation damping coefficients \( B_{11} \) and \( B_{22} \). However, they do not yield their relative phases, which may be essential in the analysis of a two-body system with two oscillation modes.

Important theoretical results in the frequency domain can be found in [3.12] for two-body heaving wave energy converters.

3.6.3. Time domain analysis

The time domain analysis, required if the PTO is not linear, can be performed as for a single heaving body. Instead of Eqs (3.43) and (3.51), we write

\[ f_{r,ij}(t) = -\int_{-\infty}^{t} g_{r,ij}(t - \tau) \dot{\xi}_j(\tau) d\tau - A_{ij}(\infty)\ddot{\xi}_j(t) \]
\tag{3.82}

and

\[ g_{r,ij}(s) = g_{r,ji}(s) = \frac{2}{\pi} \int_0^{\infty} B_{ij}(\omega) \cos \omega t \cos d\omega. \]
\tag{3.83}
Equations (3.68) and (3.69) become
\[
(m_1 + A_{11}(\infty)) \frac{d^2 \xi_1}{dt^2} = f_{e,1} - \int_{-\infty}^{t} g_{r,11}(t-\tau) \ddot{\xi}_1(\tau) d\tau - g\rho S_{cs,1} \dddot{\xi}_1
\]
\[
(3.84)
\]
\[
(m_2 + A_{22}(\infty)) \frac{d^2 \xi_2}{dt^2} = f_{e,2} - \int_{-\infty}^{t} g_{r,12}(t-\tau) \ddot{\xi}_2(\tau) d\tau - A_{12}(\infty) \dddot{\xi}_1(\tau) + f_{PTO},
\]
\[
(3.85)
\]

Exercise 3.3. Heaving two-body axisymmetric wave energy converter

This exercise concerns a heaving axisymmetric two-body device with a linear PTO absorbing energy from regular waves. Figure 3.12 represents the system consisting of two axisymmetric co-axial bodies 1 and 2. This is a simplified representation of some wave energy converters under development, namely Wavebob and PowerBuoy. Body 1 is a cylindrical floater with a conical bottom, whereas body 2 is a long cylinder with a flat bottom. The gap between bodies 1 and 2 is very small, but the friction between them is neglected. The system is assumed to move only in heave. The PTO is driven by the forces between bodies 1 and 2 and their relative translational motion.

To simplify the exercise, the draught \(d\) of body 2 is assumed large enough for the excitation and radiation forces on its flat bottom to be negligible, i.e. \(F_{e,2} = 0\), \(B_{22} = 0\) and \(A_{12} = 0\). However, the added mass \(A_{22}\) is accounted for. We may also assume that the surface waves radiated by body 2 are negligible and so take \(B_{12} = 0\). Note that we are interested only on the vertical components of the forces, and that the excitation and radiation forces on body 1 result from water pressure on its conical wetted surface. Obviously, such forces are independent of the draught \(d\) and of the motion of body 2, provided that \(d\) is large enough.
The PTO consists of a linear damper with damping coefficient $C$. We consider the case when $a = c$, $b = 0.4a$, and the semi-angle of the conical bottom (angle between the generatrices and the axis) is equal to $60^\circ$. The volume of the submerged part of body 1 in calm water is $3.031a^3$. Dimensionless values of the added mass $A_{11}$ and radiation damping coefficient $B_{11}$ of body 1 are defined as

$$A_{11}^* = \frac{A_{11}}{\rho \pi a^3}, \quad B_{11}^* = \frac{B_{11}}{\rho \pi a^3 \omega}$$

(3.86)

and are plotted versus dimensionless wave period $T^* = T \sqrt{g/a} = 2 \pi \omega^{-1} \sqrt{g/a}$ in Fig. 3.13 (tables of numerical values are available on request).

![Dimensionless plot of added mass coefficient and radiation damping coefficient for body 1](image)

Fig. 3.13. Dimensionless plot of added mass coefficient and radiation damping coefficient for body 1 represented in Fig. 3.12.

The added mass of a semi-infinite circular cylinder of radius $b$ moving along its axis in unbounded water of density $\rho$ has been computed and is equal to $0.6897 \rho \pi b^3$. Assuming the depth of submergence $d$ to be large, the added mass $A_{22}$ of body 2 may be taken as independent of $\omega$ and equal to that value.

The following items are suggested to be performed as exercises.

- Write the governing equations in the frequency domain.
- Compute the mass $m_2$ of body 2 as a function of submergence $d$.
- For given dimensionless wave period $T^*$, find the optimal values of the ratio $d/a$ and of the dimensionless PTO damping coefficient

$$C^* = \frac{C}{a^{5/2} \rho g^{1/2}}.$$

- Discuss the advantages and limitations of a wave energy converter based on this concept.

3.6. Wave energy absorption by oscillating systems with several degrees of freedom

In the preceding sections, we analyzed one body and two-body wave energy converters oscillating in heave. These results can be generalized to include the other five modes of oscillation and can be extended to any number of bodies, with different types
of linear and non-linear power take-off systems. This kind of general analysis, based on linear water wave theory, can be found in detail in the book by Falnes [3.2].

We note that, of the six degrees of freedom of a body, three are rotations: pitch, roll and yaw (see Fig. 3.1). For these rotations, instead of forces we have moments, and instead of added mass, we have added moment of inertia. There can be interference between modes of motion through the diffracted and radiated wave fields and also possibly through the PTO or PTOs and moorings.

In the case of a floating body with a vertical plane of symmetry parallel the direction of the incident waves, the excited modes of motion are heave, surge and pitch. For a body with a vertical axis of symmetry, heaving oscillations do not induce surge and pitch oscillations.

3.7. Time-domain analysis of a heaving buoy with hydraulic PTO

3.7.1. Introduction

The energy of sea waves can be absorbed by wave energy converters in a variety of manners, but in every case the transferred power is highly fluctuating in several time-scales, especially the wave-to-wave or the wave group time-scales. In most devices developed or considered so far, the final product is electrical energy to be supplied to a grid. So, unless some energy storage system is available, the fluctuations in absorbed wave power will appear unsmoothed in the supplied electrical power, which severely impairs the energy quality and value from the viewpoint of the grid. Besides, that would require the peak power capacity of the electric generator and power electronics to greatly exceed the time-averaged delivered power.

In practice, three methods of energy storage have been adopted in wave energy conversion. An effective way is storage as potential energy in a water reservoir, which is achieved in some overtopping devices, like the Wave Dragon and the SSG. In the oscillating water column type of device, the size and rotational speed of the air turbine rotor make it possible to store a substantial amount of energy as kinetic energy (flywheel effect).

In a large class of devices, the oscillating (rectilinear or angular) motion of a floating body (or the relative motion between two moving bodies) is converted into the flow of a liquid (water or oil) at high pressure by means of a system of hydraulic rams (or equivalent devices). At the other end of the hydraulic circuit there is a hydraulic motor (or a high-head water turbine) that drives an electric generator. The highly fluctuating hydraulic power produced by the reciprocating piston (or pistons) may the smoothed by the use of a gas accumulator system, which allows a more regular production of electrical energy. Naturally the smoothing effect increases with the accumulator volume and working pressure. This kind of power take-off system is employed e.g. in the Pelamis, the Wavebob and the PowerBuoy.

Here we analyse the performance of a floating oscillating body wave energy converter with one degree of freedom (heave). The buoy motion drives a two-way hydraulic ram that feeds high pressure oil to a hydraulic motor (or water to a high-head hydraulic turbine). A gas accumulator system is placed in the circuit to produce a smoothing effect. Such a wave energy converter is highly non-linear, which requires a time-domain model consisting of a set of coupled equations: (i) an integral-differential equation

\[ \text{This section is largely based on [3.13] and [3.14].} \]
equation (with a convolution integral representing the memory effect) that accounts for
the hydrodynamics of wave energy absorption; (ii) an ordinary differential equation that
models the time-varying gas volume and pressure, the dependence of flow rate
(supplied to the motor or turbine) on pressure head, the non-return valve system, and the
pressure losses in the hydraulic circuit (viscosity effects). In the case of several degrees
of freedom (not considered here), additional (differential and/or integral-differential)
equations appear. Standard methods are employed to numerically integrate the
differential equations, with appropriate initial conditions.

Random irregular waves are assumed (each sea state is characterized by its
significant wave height $H_s$ and energy period $T_e$, and a discretized Pierson-Moskowitz
spectrum). A simple geometry (a hemisphere in deep water oscillating in heave) is
adopted for the buoy. 

3.7.2. Governing equations

We consider the simple case of a body of mass $m$ with a single degree of freedom
oscillating in heave (coordinate $\xi$, with $\xi = 0$ in the absence of waves). The PTO
consists of a hydraulic comprising a ram or hydraulic cylinder, a manifold (valve
system), high- and low-pressure gas accumulators and a hydraulic motor (Fig. 3.14).
The hydraulic motor operates between the two accumulators.

The equation governing the motion of the buoy is Eq. (3.52), derived in section 3.4,

$$\left[m + A(\omega)\right]\ddot{\xi}(t) = f_e(t) - \int_{-\infty}^{t} g_r(t - \tau) \dot{\xi}(\tau) d\tau - \rho g S c_s \xi(t) + f_{PTO},$$

(3.52)

where the memory function is given by Eq. (3.51)

$$g_r(s) = \frac{2}{\pi} \int_{0}^{\infty} B(\omega) \cos \omega s d\omega.$$  

(3.51)

Numerical values for the added mass $A(\omega)$ and radiation damping coefficient $B(\omega)$ for
a heaving hemisphere are given in Exercise 3.1. The memory effect decays rapidly with
time, and may be neglected after a few tens of seconds (the infinite interval of
integration in the convolution integral of Eq. (3.52) may be replaced by a finite one).
For identical reasons, a finite interval of integration is kept in Eq. (3.51) (an upper limit
of about $3-5$ rad/s is probably enough in practice).
In the case of an axisymmetric heaving body in deep water (as is the case here) subject to incident regular waves of frequency $\omega$ and amplitude $A_w$, the modulus $|F_e(\omega)| = \Gamma(\omega)A_w$ of the complex amplitude of the excitation force is related to the radiation damping coefficient $B(\omega)$ by the Haskind relation (3.40)

$$B(\omega) = \frac{\omega^3 [\Gamma(\omega)]^2}{2 \rho g^3} \quad (3.40)$$

which may be used to compute $\Gamma(\omega)$ from the given numerical values of $B(\omega)$.

We assume the wave spectral distribution to be given, for example the Pierson-Moskowitz spectrum $S_\omega(\omega)$ of Eq. (3.58)

$$S_\omega(\omega) = 262.6 H^2 \tau^{-4} \omega^{-5} \exp \left[ -1052(T_e \omega)^{-4} \right]. \quad (3.58)$$

The excitation force $f_e(t)$ may be calculated as in sub-section 3.5.2:

$$f_e(t) = \sum_{i=1}^{N} \Gamma(\dot{\omega}_i) A_{i,\omega} \exp \left[ i(\dot{\omega}_i t + \alpha_i) \right], \quad (3.61)$$

where $A_{i,\omega} = \sqrt{2 S_{\omega,i} \Delta_i}$ and $S_{\omega,i} = S_\omega(\dot{\omega}_i)$.

In the numerical simulations, the spectrum was discretized into 225 equally spaced ($\Delta\omega = \Delta_i = 0.01\text{rad/s}$) sinusoidal harmonics in the range $0.1\sqrt{6} \leq \omega \leq 0.1\sqrt{6} + 2.24\text{rad/s}$. (The irrational number $\sqrt{6}$ ensures the non-periodicity in the time-series of $f_e(t)$.) The phases at $t = 0$ were made equal to random numbers in the interval $(0,2\pi)$. The integral-differential equation (3.52) was numerically integrated in the time domain with a time step size of 0.1s.

All the numerical results presented in this section are for a hemispherical floater of radius $a = 5\text{m}$. The hydrodynamic coefficients $A(\infty)$ and $B(\omega)$ were obtained from [3.11] (see Exercise 3.1).

3.7.3. The power take-off mechanism

In most wave energy converters using hydraulic rams as mechanical power take-off system, the displacement of the piston inside the corresponding cylinder is driven by the relative motion between two oscillating bodies. In this case, there is only one oscillating body, and the cylinder (or alternatively the piston) is fixed (with respect to the sea bottom or to a shoreline structure).

The hydraulic circuit includes a high-pressure (HP) gas accumulator, a low-pressure (LP) gas accumulator and a hydraulic machine (Fig. 3.14). The machine can be either a hydraulic motor (if the working fluid is oil) or a high-head water turbine. A rectifying valve system prevents liquid from leaving the HP accumulator at $E$ and from entering the LP accumulator at $D$. In this way, when the piston is moving downwards, the liquid in pumped from the cylinder into the HP accumulator through the duct $B \rightarrow E$ and sucked from the LP accumulator into the cylinder through $D \rightarrow A$. During the upward motion, the circuit is $A \rightarrow E$ and $D \rightarrow B$. The hydraulic machine is driven by the flow resulting from the pressure difference between the HP and LP accumulators.
Let \( p = p' - g \eta \) where \( p' \) and \( \eta \) are pressure and vertical coordinate in the liquid circuit (\( \eta = 0 \) at average sea surface level). We denote by \( p_a, p_b, p_1 \) and \( p_2 \) the values of \( p \) in the upper and lower parts of the cylinder, and inside the HP and LP accumulators, respectively.

First we consider an interval of time when the piston is moving upwards (flow directions \( A \to E \) and \( D \to B \)), which implies that \( p_a > p_1 \) and \( p_b < p_2 \). The volume flow rate is \( q = S_c \frac{d\xi}{dt} \), where \( S_c \) is the cylinder cross-sectional area, and the coordinate \( \xi \) defining the piston position (and also the floater position) increases upwards. (Here we neglect the cross-sectional area of the piston rod. It should be noted however that, in very high pressure oil hydraulics, the rod-to-piston diameter ratio is usually not small and can be as high as about 0.5; in such cases, some of the equations presented here should be modified to take the rod cross-sectional area into account.) Assuming one-dimensional flow, we may write

\[
p_1 - p_2 = p_a - p_b - k_u q^2 - I \frac{dq}{dt}.
\]  
(3.87)

Here, \( k_u \) is a coefficient of pressure loss due to friction along the circuit, and \( I \) is a coefficient that takes into account the inertia of the fluid. Likewise, if the piston is moving downwards (flow directions \( D \to A \) and \( B \to E \)), it is

\[
p_1 - p_2 = p_b - p_a - k_d q^2 - I \frac{dq}{dt}.
\]  
(3.87)

We assume that \( k_u \equiv k_d = k \). Then, regardless of the direction of the piston motion, we may write

\[
p_1 - p_2 = \Delta p - k q^2 - I \frac{dq}{dt},
\]  
(3.88)

where \( \Delta p = |p_a - p_b| \). Whenever \( \Delta p < p_1 - p_2 \), it is \( q = 0 \), i.e. the valve prevents the piston from moving.

The hydraulic machine will be driven by the pressure difference \( p_1 - p_2 \). In the case of an impulse hydraulic turbine (Pelton turbine), the flow rate is independent of the rotational speed, and may be written as

\[
q_m = \left( \frac{p_1 - p_2}{K_t} \right)^{1/2},
\]  
(3.89)

where \( K_t = A_n^{-1}(2\eta_n/\rho_w)^{-1/2} \). Here \( \rho_w \) is density of water in the circuit, \( A_n \) is the effective cross-sectional area of the turbine nozzle (or nozzles) (which may be controlled) and \( \eta_n \) is the nozzle efficiency (that accounts for losses in the nozzle or nozzles and also in the connecting duct from the accumulator).

In the case of a hydraulic motor, the flow rate is approximately proportional to the rotational speed \( \Omega \), and we may write \( q_m = \lambda_m \Omega \), where \( \lambda_m \) is a constant characterizing the machine geometry. (There are variable-geometry hydraulic motors that allow the rotational speed and the flow rate to be controlled separately.)

We denote by \( m_1 \) and \( m_2 \) the masses of gas inside the HP and LP accumulators, respectively, which are supposed to remain unchanged during operation. Assuming the duct and accumulator walls to be rigid and the liquid incompressible, the total volume of gas remains constant, i.e. \( m_1 v_1(t) + m_2 v_2(t) = V_0 = \text{constant} \) (\( v_i \), \( i = 1, 2 \), is specific volume of gas). We may also write
\[ q(t) - q_m(t) = -m_1 \frac{dv_1(t)}{dt}. \] (3.90)

The specific entropy \( s_1 \) of the gas inside the HP accumulator will change due essentially to heat transfer. This may be connected to changes in sea water temperature and surrounding air temperature, and also to changes in the power dissipated (viscous losses and electrical losses) inside the converter. Such changes are likely to be significant over time intervals not less than several hours, and so it is reasonable to consider that the gas compression/expansion process inside the accumulator is approximately isentropic (\( s_1, s_2 \) are constant) during a sea state (this means that, although the changes in gas temperature may be significant during the compression/expansion cycle, the corresponding changes in entropy may be neglected).

For an isentropic process of a perfect gas, it is \( \Pi_i(t) = v_i(t)^{-\gamma} \Theta_i \) (\( i = 1,2 \)), where \( \Pi_i \) is gas pressure, \( \Theta_i \) is constant for fixed entropy \( s_i \), and \( \gamma = c_p/c_v \) is the specific-heat ratio for the gas. We assume that \( z \equiv 0 \) at the liquid free-surface inside the HP and LP accumulators, and so it is \( \Pi_i \equiv p_i \) (\( i = 1,2 \)). From Eq. (13), it follows that

\[
\Delta p(t) = \Theta_1 v_1(t)^{-\gamma} - \Theta_2 \left[ \frac{V_0 - m_1 v_1(t)}{m_2} \right]^{-\gamma} + Kq(t)^2 + C \frac{dq(t)}{dt}. \tag{3.91}
\]

We note that the force \( S_c[p_a(t) - p_b(t)] \) required to pump fluid into the HP accumulator is to be overcome by the action of the buoy upon the piston.

### 3.7.4. Floating converter with gas accumulator

We consider again the buoy oscillating in heave and driving a hydraulic cylinder or ram that pumps high pressure liquid (oil or water) into a hydraulic circuit (Fig. 3.14). The rectifying valve is controlled in such a way that the liquid is pumped from the cylinder into the HP accumulator and sucked from the LP accumulator into the opposite side of the cylinder. The turbine or the rotary hydraulic motor is driven by the flow resulting from the pressure difference between the HP accumulator and the LP accumulator. The time variation of the gas pressure difference \( p_1(t) - p_2(t) \) between the HP and LP accumulators results from (i) the action of the buoy upon the piston, and (ii) the flow of liquid through the turbine or hydraulic motor. For simplicity, we neglect the inertia and the pressure losses in the hydraulic circuit, i.e. set \( I = 0, k = 0 \) in Eq. (3.88), and so \( |p_a - p_b| = p_1 - p_2 \) whenever the piston is moving. (The inertia of the liquid in the hydraulic circuit could be modelled by a mass to be added to the mass of the buoy, \( m \), in Eq. (3.52).)

While the body is moving, the governing equation is (3.52), with \( f_{PTO} = -\text{sign}(\dot{x})\Phi \), where \( \Phi = S_c(p_1 - p_2) \) and \( S_c \) is the cylinder cross-sectional area. At some time, the time-varying body velocity will be zero. From then on, the body will remain stationary unless, or until, the hydrodynamic force on the body

\[
f_c(t) - \rho g S_c \xi(t) - \int_{-\infty}^t g_x(t - \tau) \dot{\xi}(\tau) d\tau \tag{3.92}
\]

overcomes the resisting force \( \Phi = S_c(p_1 - p_2) \) and fluid is again pumped into the HP accumulator (this has been named Coulomb damping force).

The instantaneous power absorbed by the converter is
\[ P(t) = \Phi[\dot{x}(t)] \]

and its time-average in \( t_0 \leq t \leq t_f \) is

\[ \overline{P} = \Delta t^{-1} \int_{t_0}^{t_f} P(t) \, dt. \]

The value of \( \overline{P} \) naturally depends on the magnitude of the time interval \( \Delta t = t_f - t_0 \).

### 3.7.5. Control

It is important to control the device in order to maximize the produced energy. This should take into account the sea state, characterized by \( H_s \) and \( T_e \). Since the system is assumed linear from the hydrodynamic point of view, then, for fixed \( T_e \), the values of \( \overline{P}/H_s^2 \) and \( \overline{q}/H_s \) will depend only on the ratio \( \Phi/H_s \) (we assume the force \( \Phi \) to be approximately constant over the sea state under consideration, and, as before, \( q = S_c[\dot{x}] \) is the flow rate pumped by the piston).

The relationship \( \overline{P}/H_s^2 = f_p(\Phi/H_s, T_e) \) is represented in Fig. 3.15 (time-averages over 15 min), for \( \alpha = 5 \text{ m} \) and \( T_e = 5, 7, 9, 11 \) and 13\text{s}. It may be seen that the optimal value of \( (\Phi/H_s)_{\text{opt}} \) (i.e. that maximizes \( \overline{P}/H_s^2 \)) varies with \( T_e \).

![Fig. 3.15. Converter performance with simple Coulomb-type damping. Performance curves (for \( T_e = 5 \text{–} 13\text{s} \)) and control curves (parabolas).](image)

It should be noted that \( \overline{q} = \overline{q}_m \) (\( q_m = \) flow rate through the hydraulic machine) over a sufficiently large time span, and also that \( \overline{P} = \Phi[\overline{x}] = \Phi \overline{q}/S_c = \Phi \overline{q}_m/S_c \). So, for fixed \( T_e \), \( \Phi \overline{q}_m/(S_cH_s^2) \) may also be regarded as a function of \( \Phi/H_s \), and the same obviously applies to

\[ \overline{q}_mS_c/\Phi = (\Phi/H_s)^{-2} f_p(\Phi/H_s, T_e). \]

For each value of \( T_e \), Fig. 3.15 and Eq. (3.95) yield the optimal value for the ratio \( (\overline{q}_mS_c/\Phi)_{\text{opt}} \). This may provide a control algorithm for the flow rate \( q_m \) versus the pressure difference \( p_1 - p_2 = \Phi/S_c \). For practical reasons, it may be convenient to have
a control law independent of the wave period $T_e$, and so it may be reasonable to adopt instead a single value (i.e. independent of $T_e$) for $\bar{q}_m S_c/\Phi = s_c^2 G$ ($G =$ constant). This appears in Fig. 3.15 as a regulation curve

$$\frac{\bar{P}}{H_s^2} = \frac{\Phi \bar{q}_m}{S_c H_s^2} = \left( \frac{\Phi}{H_s} \right)^2 G,$$

(3.96)

which is a parabola. Figure 3.15 shows four such curves for $G = G_i$ ($i =$ 1 to 4), with $G_1 = 0.4 \times 10^{-6}$, $G_2 = 0.6 \times 10^{-6}$, $G_3 = 1.0 \times 10^{-6}$ and $G_4 = 2.0 \times 10^{-6}$ s/kg. The value of $\bar{P}/H_s^2$ is given by Fig. 3.15 as the intersection between the appropriate $T_e$-curve and the parabolic regulation curve defined by the chosen value for $G$. The value $G_2$ may be regarded as an acceptable compromise, especially in the range $7 \leq T_e \leq 13$ s (see Fig. 3.15).

In order to establish a control strategy based on this algorithm (which was devised assuming constant force $\Phi$ over the sea state under consideration), we assume now that the gas accumulator is large enough so that the variations in $p_1 - p_2$ may be neglected over a few wave periods, and define individual “sea states” of such duration. Then we adopt

$$q_m(t) S_c/\Phi(t) = q_m(t) ((p_1(t) - p_2(t)) = \Phi S_c^2 G = \text{constant} \quad (3.97)$$

as an instantaneous control algorithm. We note that Eq. (3.97) is a linear relationship between $q_m$ and $p_1 - p_2$, differently from the square-root relation (3.89). This means that, if a Pelton water turbine is employed, the exit nozzle-area has to be controlled.

This control algorithm was numerically tested for $a = 5$ m, $S_c = 0.01767$ m$^2$ (cylinder of 0.15 m inside diameter), $G = G_2 = 0.6 \times 10^{-6}$ s/kg, $m_1 = 150$ kg, $m_2 = 30$ kg, $V_0 = 5.17$ m$^3$ (total volume of gas). For each sea state, the values of $\Theta_1$ and $\Theta_2$ (that depend only on the specific entropy) were chosen such that the time-averaged temperatures of the gas in the HP and LP accumulators realistically remained close to environmental temperature ($\approx 300$ K). Figure 3.16 shows results for a sea state characterized by $H_s = 1.5$ m and $T_e = 11$ s. The values of $P(t)/H_s^2 = (p_1 - p_2)q_i/H_s^2$ and of $P_m(t)/H_s^2 = (p_1 - p_2)q_1/H_s^2 = S_c G (p_1 - p_2)/H_s^2$ ($P_m$ is power available to the hydraulic machine), averaged over a time span of 2 hours, were found to be $\bar{P}/H_s^2 = 10.21$ kW/m$^2$ and $\bar{P}_m/H_s^2 = 10.25$ kW/m$^2$. These values closely agree with those given by Fig. 3.15. The computed standard deviation of the power available to the hydraulic machine is $\sigma_{P_m} = 0.0811 \bar{P}_m$, i.e. 8.11% of its time-averaged value. The time-averaged values of the HP and LP accumulator gas pressures are $\bar{p}_1 = 127.4$ bar and $\bar{p}_2 = 15.6$ bar.

The corresponding values, computed for $T_e = 11$ s, $H_s = 4$ m and the same 2-hour period of time, are $\bar{P}/H_s^2 = 10.17$ kW/m$^2$ and $\bar{P}_m/H_s^2 = 10.20$ kW/m$^2$, practically showing no change with $H_s$. However, the fluctuations in $P_m/H_s^2$ are now much larger: $\sigma_{P_m} = 0.293 \bar{P}_m$, which indicates that the smoothing effect of the accumulator
decreases markedly in more energetic sea states. The accumulator pressures are now $p_1 = 301.0\, \text{bar}$, $p_2 = 11.1\, \text{bar}$.

![Graph showing performance results for $H_x = 1.5\, \text{m}$, with controlled $q_m$. In the bottom graph, the chain curve represents the power $P_m$ available to the hydraulic machine.](image)

3.7.6. Phase control by latching

Phase-control by latching has been proposed by Budal and Falnes [3.15] to enhance the wave energy absorption by oscillating bodies (namely the so-called point absorbers) whose natural frequency is above the range of frequencies within which most of the incident wave energy flux is concentrated. Later, this has been confirmed experimentally. Phase control by latching was the object of other theoretical and experimental investigations (see e.g. [3.16]). Although phase control by latching has been shown to be potentially capable of substantially increasing the amount of absorbed energy, the practical implementation in real irregular waves of optimum phase control has met with theoretical and practical difficulties that have not been satisfactorily overcome. Sub-optimal control methods have been devised and proposed by several research teams to circumvent such difficulties.

The use of a hydraulic power take-off (PTO) system as described above provides a natural way of achieving latching: the body remains stationary for as long as the hydrodynamic forces on its wetted surface are unable to overcome the resisting
force (gas pressure difference $\Delta p$ times cross-sectional area $S_c$ of the ram) introduced by the hydraulic PTO system.

Phase control by latching is implemented by adequately delaying the release of the body in order to approximately bring into phase the body velocity and the diffraction (or excitation) force on the body, and in this way get closer to the well-known optimal condition derived from frequency-domain analysis for an oscillating body in regular waves, with linear PTO damping (see Eq. (3.32)). The proposed control algorithm is simple and easy to implement, and includes (i) a proportionality relationship $q_m = C_1 \Delta p$ between the fluid flow rate $q_m$ through the hydraulic motor (or water turbine) and the accumulator gas pressure difference $\Delta p$, and (ii) a proportionality relationship $F = C_2 \Delta p$ between the release force $F$ and $\Delta p$ (which regulates the release delay).

When the body is moving, its velocity will, at some time, come to zero, as a result of the hydrodynamic forces on its wetted surface and the PTO forces. The body will then remain fixed until the hydrodynamic force $|f_h|$ exceeds $R \Phi = R(S_c \Delta p)$, where $R > 1$. It is to be noted: (i) that the force that has to be overcome (if the body is to restart moving) is now larger (by a factor $R$) as compared with the simple Coulomb damping (i.e. compared with $S_c \Delta p$); (ii) that the acceleration of the floater (unlike in the case of $R = 1$) is discontinuous when the body is released. There is now a new parameter, $R$, to be optimized, jointly with parameter $G$.

Numerical simulations (30 min each) were carried out, based on this procedure and algorithm, for a hemispheric floater of radius $a = 5$ m, in deep water, in regular and irregular waves. Piston area was $S_c = 0.0314$ m$^2$. The masses of gas (nitrogen) in the HP and LP accumulators were $m_1 = 100$ kg and $m_2 = 20$ kg. In each simulation, the values of gas entropies $s_1$ and $s_2$ were taken such that the time-averaged gas-temperatures in the HP and LP accumulators remained close to environmental temperature ($\approx 300$ K).

**Regular waves.**

Results of simulations in regular waves (wave amplitude $A_w = 0.667$ m and period $T = 9$ s) are shown in Figs. 3.17 to 3.21. In Fig. 3.17, the solid line shows the numerically optimized values (that maximize $\tilde{P}$) of control parameter $G$ for several values of the latching control parameter $R$ (note that $R = 1$ means simple Coulomb damping). In the same figure, the dashed line represents the amplitude of oscillation $x_{max}$. Figures 3.18 and 3.19 represent, for the same situations, the time-averaged absorbed power $\bar{P}$ and the time-averaged gas pressure (in the HP accumulator) $\bar{p}_1$, respectively, versus $R$. It may be seen that, by increasing $R$ above unity and (for each $R$) suitably optimizing $G$, a substantial increase (by a factor up to about 3.8) in the time-averaged absorbed power $\bar{P}$ can be achieved. The maximum power attained in this way, about 206 kW for $R = 16$, should be compared with the theoretical maximum power $\frac{1}{8} \rho A_0^3 \omega^{-3} = 315$ kW absorbed by an axisymmetric body with a linear PTO damper oscillating in heave. It is to be noted that this increase in absorbed power results mostly from larger floater oscillations $\xi_{max}$ (and hence greater liquid flow through the hydraulic motor) rather than from greater pressure levels in the HP hydraulic circuit. Figures 3.20 and 3.21 represent the time variation of the excitation force $f_e(t)$, the
floater velocity \( d\xi/t \) and its displacement \( \xi(t) \) for two situations optimized with respect to \( G \) (same incident wave time series, for easier comparison): \( R=1, G=0.86\times10^{-6} \text{ s/kg} \) (simple Coulomb damping, Fig. 3.20) and \( R=16, G=7.7\times10^{-6} \text{ s/kg} \) (latching control, Fig. 3.21). It is to be noted that, in Fig. 3.21 (but not in Fig. 3.20), the velocity and the diffraction force are (very approximately) in phase with each other (in agreement with optimal condition (3.32) for linear PTO).

Fig. 3.17. Regular waves, \( A_w=0.667 \text{ m}, T=9 \text{ s} \): optimized control parameter \( G\times10^6 \) (solid line) and dimensionless oscillation-amplitude, \( \xi_{\text{max}}/A_w \) (dashed line), versus latching control parameter \( R \).

Fig. 3.18. As in Fig. 3.17: time-averaged absorbed power \( \bar{P} \) (kW) versus \( R \) (\( G \) optimized for each \( R \)).
Fig. 3.19. As in Fig. 3.17: time-averaged gas pressure in HP accumulator versus $R$ ($G$ optimized for each $R$).

Fig. 3.20. Performance of a hemispherical floater in regular waves for $R=1$, $G = 0.86 \times 10^{-6} \text{s/kg}$. Above, $d\xi/\!\!\!t$: solid line, $f_e(t)$: broken line. Below, $\xi(t)$. Absorbed power: $P = 55.0 \text{kW}$. 
Fig. 3.21. As in Fig. 3.20, for $R = 16$, $G = 7.7 \times 10^{-6}$ s/kg. $\bar{P} = 206.1$ kW.

Fig. 3.22. Hemispherical floater in irregular waves, $T_e = 7$ s. Plot of $\frac{\bar{P}}{H_x^2}$ versus control parameter $G$, for several values of latching control parameter $R$. 

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Irregular waves

Optimal phase control in random irregular waves is known to require the prediction of the incoming waves (theoretically over the infinite future, in practice over a few tens of seconds, see [3.17]). In addition to this difficulty, the theoretical determination of the wave-to-wave optimal latching period requires relatively heavy computation, which makes it inappropriate for implementation in real time.

Therefore, it is particularly interesting to investigate whether the simple control strategy outlined above, and tested above in regular waves, can be applied successfully to irregular waves. One ought to bear in mind that this should be regarded, at best, as a sub-optimal strategy, and that the achievable results should not be expected to be close to the theoretical maximum.

Numerical simulations, identical to those presented above for regular waves, were performed for irregular waves as modelled by a Pierson-Moskowitz spectrum and $H_s = 2\text{ m}$, $T_e = 7, 9$ and 11s. The results, in terms of time-averaged absorbed power $\bar{P}$ divided by $H_s^2$, are presented in Figs. 3.22 to 3.24.
Fig. 3.25. Performance of a hemispherical floater in irregular waves for \( T_e = 9 \) s, \( R = 1 \), \( G = 0.7 \times 10^{-6} \) s/kg. Above, \( d\xi/dt \): solid line, \( f_e(t) \): broken line. Below, \( \ddot{\xi}(t) \). Absorbed power: \( \overline{P}/H_s^2 = 10.3 \) kW/m².

The figures show that a large increase (by a factor about 2.3-2.8) in absorbed power (as compared with simple Coulomb damping, \( R = 1 \)) can be achieved by suitably combining the values of the control parameters \( R \) (\( R > 1 \)) and \( G \). The largest absorbed power occurs for \( R \) equal to about 16 and a value of \( G \) that depends on \( R \) and \( T_e \).

Curves for the excitation force \( f_e(t) \), and the floater velocity \( \dot{\xi}(t) \) and displacement \( \ddot{\xi}(t) \) are given in Figs. 3.25 and 3.26, for \( T_e = 9 \) s and control parameter pairs \( (R = 1, \ G = 0.7 \times 10^{-6} \) s/kg \), \( (R = 16, \ G = 4.2 \times 10^{-6} \) s/kg \).

It is not surprising that those large values of absorbed power occur for relatively large amplitudes of the floater oscillations, that typically attain nearly twice the value of the significant wave height \( H_s \), as shown in Fig. 3.26.

Since the whole analysis is based on linear hydrodynamic theory (which assumes the amplitude of body oscillations to be small compared with the body size), such oscillations are unrealistically large (except in calm seas, say \( H_s < 1 \) m) and so are the values of absorbed power. Of course, this is also true in general, whenever the theory predicts large oscillation amplitudes as a result of a wave energy converter being tuned (by phase control or otherwise) to the incoming waves.
It should be noted that values of the latching control parameter $R$ much larger than unity (required to maximize $P$) may imply very large forces to keep the body fixed prior to its release. Such forces are likely to exceed the practical limits of the ram and remaining hydraulic circuit and would possibly require a special braking system. This is an engineering problem that has to be faced whenever phase control by latching is considered.

Figure 3.26 shows that the peaks of velocity $d\xi/dt$ in general (but not in every oscillation) coincide, in time, approximately with the peaks of the excitation force $f_e$, which matches the optimal condition expressed by Eq. (3.32).

The values of $P/H_s^2$, $\xi/H_s$ and $\ddot{\xi}/H_s$ plotted in Figs. 3.22 to 3.26 were computed for $H_s = 2$ m. These values would change with $H_s$ due to the nonlinear response of the gas accumulator. Provided the accumulator is appropriately sized, those changes are relatively small within the range of $H_s$ in which the linear wave theory is applicable. If this is the case, one can say approximately that the latching control algorithm proposed and numerically optimized here is approximately independent of significant wave height.

One should bear in mind that the pressure difference $\Delta p$ decreases (due to the continuous flow of liquid from the HP to the LP reservoir through the hydraulic machine) whenever the floater is unable to move (this decrease is faster the smaller the accumulator size); this effect tends to adjust the pressure level $\Delta p$ to the current sea state and also (in a different measure) to the wave group or even the wave-to-wave
succession. Naturally, the choice of the size and other specifications of the accumulator are dictated by several criteria, namely the maximum allowable working pressure, the desired power output smoothness and equipment costs.

References

4. Modelling of oscillating-water-column wave energy converters

4.1. Introduction

Typically, the wave energy conversion by oscillating-body devices involves large forces (up to hundreds or even thousands of tons) and small velocities. This has to be coped with by the power take-off system and is a challenge to its designer.

The air chamber and ducting of an oscillating water column device (OWC) converts the large force on, and the low velocity of, the inner free surface of the water column into the much higher air velocity about, and much smaller forces on, the rotor blades of the air turbine. Rotational speeds of air turbines used to equip OWCs are typically hundreds or even a few thousand revolutions per minute, which is adequate for directly driving conventional low-cost electrical generators.

More or less conventional unidirectional flow turbines (possibly of Francis or axial-flow types) can be used for this purpose provided that the wave energy converter is equipped with a rectifying system of non-return valves. Rectifying valve systems were successfully used in small devices like navigation buoys (in which anyway efficiency is not a major concern). However, they are unpractical in large plants, where flow rates may be of the order of $10^2 \text{ m}^3\text{s}^{-1}$ and the required response time is typically less than one second. For this reason, all (or almost all) OWC prototypes tested so far have been equipped with self-rectifying air turbines. An extensive review of self-rectifying air turbines for OWCs can be found in [4.1].

Most self-rectifying air turbines for wave energy conversion proposed and tested so far are of two basic types: the Wells turbine and the impulse turbine. A weak point about OWCs is the modest average efficiency of self-rectifying air turbines subject to bidirectional random flows, typically of the order of 50%. However, advances in aerodynamic design and new turbine concepts seem to indicate that significantly better efficiencies may be attained in the near future.

For the purpose of OWC modelling, it is important to know that the Wells turbine can be regarded as nearly linear, i.e. the air pressure head across the turbine is approximately proportional to the flow rate. In the impulse turbine, the pressure-versus-flow-rate relationship is approximately quadratic. These different characteristics are important in the OWC modelling, as will be seen later: the frequency-domain analysis may provide a fairly good approximation for OWCs equipped with Wells turbines but not with impulse turbines.

To basic approaches may be adopted in the modelling of OWCs. In earlier researches, the inner free-surface was simulated by a weightless piston that was modelled as a heaving oscillating body, with its own added mass, radiation damping and excitation force, whose coefficients can be computed with the same commercial codes as for oscillating bodies (WAMIT, ANSYS/Aqwa, Aquaplus). This model ignores the deformation of the free surface and yields a non-uniform surface pressure distribution that may be somewhat unrealistic (the air pressure in the chamber is in fact very approximately spatially uniform). The rigid piston model may be acceptable if the dimensions of the OWC free-surface are much smaller than the wavelength and are small compared with the OWC length.

In the alternative model, the governing equations are expressed in terms of pressure on the OWC free surface and flow rate displaced by the OWC surface motion, rather than forces and velocities (or moments and rotational velocities) as in the modelling of
The general theory of OWCs based on the uniform-pressure model was first put forward by Evans in 1982 [4.2].

Another phenomenon that is specific to OWCs is the compressibility effect of the air in the chamber above the oscillating column. The chamber volume has to be relatively large to prevent green water from reaching the turbine in the more energetic sea states. Because of this, the spring-like effect of air compressibility affects significantly the converter dynamics. Its modelling, based on the thermodynamics of gases, was presented for the first time in [4.3].

The aerodynamic performance of the air turbine should be taken into account, namely how the turbine efficiency and the damping effect are affected by the pressure head and the rotational speed.

4.2. OWC modelling based on uniform air pressure on free surface

4.2.1. Main equations

We consider an OWC converter with a fixed structure, equipped with an air turbine, as schematically represented in Fig. 4.2. We assume that the air velocity inside the chamber is everywhere small so that the kinetic energy per unit volume is negligible and assume uniform conditions of pressure and temperature. We denote by $p_a$, $\rho_a$ and $T_a$ the pressure, density and absolute temperature in the atmosphere, and by $p_c + p_a$, $\rho_c$ and $T_c$ the corresponding values in the chamber. Note that $p_c$ is a pressure oscillation.

Let $V(t)$ be the air volume inside the chamber and $V_0$ the corresponding value in the absence of waves. If the inner free surface moves, it displaces a volume flow rate (positive for upward motion) $q(t) = -dV/dt$. Within the domain of validity of linear surface wave theory, we may decompose this flow rate as
\[ q(t) = -\frac{dV}{dt} = q_e(t) + q_r(t), \tag{4.1} \]

where \( q_e(t) \) is the flow rate due to the incident waves if \( p_c = 0 \), i.e. if the inner air pressure were constant and equal to atmospheric pressure (as would be the case if the chamber were fully open to the atmosphere), and \( q_r(t) \) is the flow rate due to the oscillations \( p_c(t) \) in the inner air pressure, in the absence of incident waves. We say that \( q_e(t) \) is the \textit{excitation flow rate} and \( q_r(t) \) is the \textit{radiation flow rate}.

The mass of air inside the chamber is

\[ m = \rho_c V. \tag{4.2} \]

The mass flow rate of air, positive for air flowing out of the chamber, is \(-dm/dt\). If we neglect the compressibility effect of air inside the turbine space (whose volume is much smaller than \( V \) ), we may write \(-dm/dt = w_i\), where \( w_i \) is the mass flow rate of air through the turbine. Then, from Eqs (4.1) and (4.2), we may write

\[ \frac{w_i}{\rho_c} = -\frac{V}{\rho_c} \frac{d\rho_c}{dt} + \frac{V}{\rho_c} \frac{d\rho_c}{dt} + q_e + q_r. \tag{4.3} \]

To solve this equation, we must relate: (i) the excitation flow rate \( q_e \) to the incident waves; (ii) the radiation flow rate \( q_r \) to the pressure oscillation \( p_c \); (iii) the inner air density \( \rho_c \) to the pressure; and (iv) the turbine mass flow rate \( w_i \) to the pressure oscillation \( p_c \) (turbine pressure head) and to the performance characteristics of the turbine (rotational speed, etc.). This will be done in the following subsections.

\subsection*{4.2.2. Thermodynamics of air chamber}

The air chamber and its connection to the atmosphere through the air turbine is an open system subject to a thermodynamic process of compression and decompression.

Since the amount of heat transferred across the system walls (chamber and turbine walls and OWC free-surface) is likely to be small compared with the amount of work done by the OWC motion and by the turbine rotor, the process may be considered as approximately adiabatic. If the losses in the flow due to viscous effects (turbulence, eddy formation, etc.) were negligible, the process would be reversible, and hence
isentropic. This would mean uniform specific entropy throughout the flow space, equal to the atmospheric specific entropy. Such an assumption is adopted in many papers on OWC modelling. However, one should take into account that the losses in the flow through the turbine are not negligible: the time-averaged efficiency of the best self-rectifying air turbines in random flow does not exceed about 70%, and in many cases is probably closer to 50%. This implies that the specific entropy of air entering the chamber from the atmosphere is larger than atmospheric specific entropy as due to aerodynamic losses in the turbine. In a given sea state of irregular waves, the air alternately flows into and out of the chamber in a somewhat random manner, and the specific entropy $s_c(t)$ of air inside the chamber is expected to oscillate about an average value $\bar{s}_c$ larger than $s_a$, with $s_c(t) > s_a$. This thermodynamic process is studied in detail in [4.1].

A relatively good approximation would be to model the compression and decompression of air in the chamber as isentropic, with constant entropy $s_c$, the difference $\bar{s}_c - s_a$, depending on the turbine efficiency and on the sea state. Since these effects are difficult to evaluate, it is common to assume that the process in the chamber is isentropic with entropy $s_c \approx s_a$. We will adopt this assumption here. The errors introduced by this simplification were found in [4.1] to be relatively small, and possibly acceptable taking into account other modelling errors (hydrodynamics, etc.).

Within the range of pressures and temperatures that are likely to occur in the chamber of an OWC converter, the air may be regarded with good approximation as a perfect gas. Assuming the process to be isentropic, we may write

$$\frac{p_c + p_a}{\rho_a^\gamma} = \frac{p_a}{\rho_a^\gamma},$$

where $\gamma = c_p/c_v$ is the specific heat ratio that is approximately equal to 1.4 for air. The derivative $d\rho_c/dt$ is related to $dp_c/dt$ through

$$\frac{d\rho_c}{dt} = \frac{1}{\gamma(p_c + p_a)} \frac{dp_c}{dt}.$$ 

Then, Eq. (4.3) can be written as

$$q_c(t) + q_r(t) = \frac{w_i(t)}{\rho_a} \left( 1 + \frac{p_c(t)}{p_a} \right)^{1/\gamma} + \frac{V(t)}{\rho_a \gamma(p_c(t) + p_a)} \frac{dp_c}{dt}.$$  \hspace{1cm} (4.5)

In Equation (4.5), the second term on the right-hand side represents the effect of air compressibility, and increases with the volume of the air chamber.

4.2.3. Air turbine aerodynamics

The turbine is subject to a reciprocating pressure difference $p_c$. Neglecting the effect of variations in Reynolds number (which is acceptable taking into account that the Reynolds number in general is large enough for that) and neglecting bearing friction losses, Buckingham’s theorem allows us to write the performance characteristics of the turbine in dimensionless form as (see e.g. [4.4])

$$\Phi = f_w(\Psi, \text{Ma}), \quad \Pi = f_p(\Psi, \text{Ma}),$$

where

$$\Phi = \frac{w_i}{\rho^* \Omega D^3}, \quad \Psi = \frac{p_c}{\rho^* \Omega^2 D^2} \quad \text{and} \quad \Pi = \frac{p_i}{\rho^* \Omega^3 D^5}.$$  \hspace{1cm} (4.7)
are dimensionless coefficients of flow rate, pressure head and power output, respectively. In eq. (4.7), \( w_t \) is turbine mass flow rate, \( p_c \) is pressure head available to the turbine (pressure difference between the chamber and the atmosphere), \( P_t \) is the turbine power output, \( \rho^* \) is a reference density (the usual choice being the stagnation density at the turbine entrance), \( \Omega \) is the rotational speed (in radians per unit time) and \( D \) is the turbine rotor diameter. In Eq. (4.6) \( \text{Ma} \) is a Mach number usually based on the peripheric rotor speed and defined as

\[
\text{Ma} = \frac{\Omega D}{2c^*},
\]

where \( c^* \) is a reference speed of sound in air (possibly at the turbine entrance stagnation temperature). The dependence of turbine aerodynamic performance on Mach number may be relevant if the pressure head \( c_p \) is not small compared with the atmospheric pressure \( p_a \). This could be the case in energetic sea waves and especially for some types of fluctuating OWC converters. Information on the dependence of turbine aerodynamic performance on Mach number is scarce, since performance curves usually result from model testing, and in almost every case the tests are run at low Mach numbers. For these reasons, dependence on Mach number will be ignored here.

### 4.3. Frequency-domain analysis

#### 4.3.1. Frequency domain equations

As we saw in the case of oscillating-body wave energy converters, the use of the frequency-domain analysis in very convenient in terms of computational effort. This approach may be employed if the system, whose input is the incident wave and the output is the pressure oscillation \( p_c \), is linear. For this condition to be met, (i) the turbine must be a linear one, i.e. the volume flow rate must be proportional to the pressure head, and (ii) the thermodynamic relationship (4.5) must be linearized.

Condition (i) implies \( \Phi = K \nu \), where \( K \) depends only on turbine geometry but not on turbine size, rotational speed or fluid density. The Wells turbine satisfies approximately this condition (see [4.3]). Besides we assume that the rotational velocity \( \Omega \) remains constant during each simulated sea state, and so we may write

\[
p_c = \frac{\Omega}{KD} w_t.
\]

With the linearization, Eq. (4.5) becomes

\[
q_c(t) + q_r(t) = \frac{KD}{p_a\Omega} p_c + \frac{V_0}{\gamma p_a} \frac{dp_c}{dt},
\]

where \( V_0 \) is the volume of air in the chamber in the absence of waves.

If the system is fully linear and the incident waves are regular with frequency \( \nu \), then the air pressure oscillation \( p_c \), the excitation flow rate \( q_c \) and the radiation flow rate \( q_r \) are also simple harmonic functions of time with the same frequency. Using the complex variable technique, as in section 3.3, we may write

\[
p_c = P_c e^{i\nu t}, \quad q_e = Q_e e^{i\nu t}, \quad q_r = Q_r e^{i\nu t}.
\]

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Here, $P_c$, $Q_e$ and $Q_r$ are complex amplitudes of air pressure oscillation, excitation flow rate and radiation flow rate, respectively. As before, whenever a complex expression is equated to a physical quantity, its real part is to be taken.

Equation (4.5.), in the frequency domain, becomes

$$Q_e + Q_r = \left( \frac{KD}{\rho a \Omega} + \frac{i \omega V_0}{\gamma p a} \right) P_c.$$  

(4.12)

We recall that the radiated flow rate $q_r$ is induced by the oscillations $p_c$ in the chamber pressure, in the absence of incident waves. Since we are assuming linear wave theory, there must be a proportionality between radiated flow rate and pressure oscillation, with some phase difference. In terms of complex amplitudes, this may be written as

$$Q_r = -(G + iB)P_c,$$  

(4.13)

where $G$ and $B$ are real coefficients that depend on OWC geometry and on oscillation frequency $\omega$ (but not on wave amplitude). Coefficient $G$ is named radiation conductance and coefficient $B$ radiation susceptance [4.4].

In the absence of incident waves, the rate of work done by the oscillating water column on the air contained in the chamber is $q_r(t)p_c(t)$. It is easy to show that its time-averaged value is $\overline{q_r(t)p_c(t)}$. This value cannot be positive, otherwise it would represent net positive power absorbed from the sea in the absence of incident waves, which is physically impossible. We conclude that the radiation conductance $G$ cannot be negative.

The following relationship, similar to Eq. (3.24), can be established between the radiation conductance $G$ and the excitation flow rate amplitude per unit incident wave amplitude, $\Gamma = \frac{|Q_e|}{A_w}$,

$$G = \frac{\omega k}{4 \pi \rho g^2 D(kh)} \int_{-\pi}^{\pi} (\Gamma(\beta))^2 d\beta,$$  

(4.14)

where $\beta$ is an incidence angle and $D(kh)$ is given by one of the expressions (2.67). In the deep water limit $kh \to \infty$, it is $D(kh) \to 1$ and

$$G = \frac{\omega^3}{4 \pi \rho g^3} \int_{-\pi}^{\pi} (\Gamma(\beta))^2 d\beta.$$  

(4.15)

or, in deep water,

$$G = \frac{\omega^3}{2 \rho g^3} (\Gamma(\beta))^2.$$  

(4.16)

Numerical results are required, as functions of frequency $\omega$, for the coefficients $G$, $B$ and $Q_e/A_w$ ($A_w$ is the amplitude of the incident waves). The commercial codes WAMIT and ANSYS/Aqwa were developed for oscillating bodies and do not yield directly those coefficients for OWCs. However those codes can be employed to indirectly compute them (see [4.7]).
Figure 4.3 shows the curves of $G$, $B$ and $\Gamma = \left| Q_e \right|/A_w$ as functions of frequency for the Pico OWC plant. The boundary element code Aquadine was used. The discretization is shown in Fig. 4.4. The irregular shape of the curves results from the irregularities in the sea bottom and neighbouring cliffs.

![Figure 4.3. Hydrodynamic coefficients $G$ (radiation conductance), $B$ (radiation susceptance) and $\Gamma = \left| Q_e \right|/A_w$ (excitation flow rate coefficient) for the Pico plant.](image)

![Figure 4.4. Discretization of the Pico plant and surrounding bottom and walls.](image)

With Eq. (4.13), Eq. (4.12) becomes

$$P_c = \left[ \frac{K \Delta}{\rho_a \Omega} + G + i \left( \frac{\omega V_0}{\gamma p_a} + B \right) \right]^{-1} Q_e. \quad (4.13)$$

Equation (4.13) relates the air pressure oscillation amplitude $P_c$ to the excitation flow rate amplitude $Q_e$. Note that these are in general complex amplitudes.

4.3.2. Power
Let us consider a given OWC device subject to incident regular waves of amplitude $A_w$ and frequency $\omega$. Then the excitation flow rate amplitude $Q_e$ can be computed, as well as the radiation conductance $G$ and radiation susceptance $B$ (possibly with the aid of a finite-element code). We further assume that the turbine dimensionless constant $K$, rotor diameter $D$ and rotational speed $\Omega$ are given. We recall that $p_a$ and $\rho_a$ are atmospheric air pressure and density, respectively, and that $\gamma \approx 1.4$ for air. Then Eq. (4.13) yields the (in general complex) value of the air pressure oscillation amplitude $P_c$.

In the linearized approximation adopted here, the instantaneous power available to the air turbine is given by the pressure head times the volume flow rate

$$P_{\text{ava}}(t) = p_c(t)q_c(t) \equiv p_c(t) \frac{w}{\rho_a},$$

or, taking into account Eq. (4.9) for a linear turbine,

$$P_{\text{ava}}(t) \equiv \frac{D}{\rho_a K \Omega} (p_c(t))^2.$$  

We recall that $p_c(t) = \text{Re}(P_c e^{i \omega t})$ and that $P_c$ is related to the excitation flow rate amplitude $Q_e$ by Eq. (4.13). The time averaged value of $p_c^2$ is

$$\overline{p_c^2} = \frac{1}{2} \overline{|p_c|^2}.\tag{4.16}$$

Then we find for the time averaged power available to the turbine

$$\overline{P_{\text{ava}}(t)} = \frac{D}{2\rho_a K \Omega} |P_e|^2.\tag{4.17}$$

Finally, taking into account Eq. (4.133, we find.

$$\overline{P_{\text{ava}}} = \left[ \left( \frac{KD}{\rho_a \Omega} + G \right)^2 + \left( \frac{\omega V_0}{\gamma p_a} + B \right)^2 \right]^{-1} |Q_e|^2.\tag{4.18}$$

We note that this is not the time-averaged value of the turbine power output, since the turbine efficiency is not accounted for. Assuming the characteristic curve $\Pi = f_P(\Psi)$ to be known (see Eq. (4.6)), we may write for the instantaneous power output from the turbine

$$P_i(t) = \rho_a \Omega^2 D^5 f_P \left( \frac{p_c(t)}{\rho_a \Omega^2 D^2} \right),\tag{4.19},$$

where we replaced the reference air density $\rho^*$ by the atmospheric air density $\rho_a$. Since we are assuming regular waves, the time-averaged turbine power output $\overline{P_i}$ may be obtained by integration over one wave period $T$.

4.3.3. Dimensionless representation. Model testing

The theoretical modelling based on linear water wave theory is an essential step in the development of wave energy converters. It provides insights and important information at relatively low costs, in general in a relatively fast way. However, there are important non-linear effects that are not accounted for by this kind of modelling, namely those associated with large amplitude waves, large amplitude motions of the wave energy converter or OWC, and real fluid effects due to viscosity, turbulence,
vortex shedding, etc. To account for such effects, physical model testing in a wave
flume or a wave tank is normally the next step. The scales range between about 1:100th
in small flumes to about 1:10th in the largest wave tanks.

In the case of OWC wave energy converters, the appropriate simulation, in model
testing, of the air turbine and of thermodynamic effects associated to air compressibility
in the chambere raise special problems that are frequently ignored or inadequately dealt
with. For these reasons, we address these problems here.

In model testing involving surface waves, geometrical similarity between model and
full-sized device must be ensured, which defines the scale. This includes water depth in
the tank. It also includes the wave amplitude \( A_w \), or, in the case of irregular waves, the
significant wave height \( H_s \).

Apart from geometrical similarity, Froude scaling must be satisfied. If \( a \) is a
representative dimension (for example a radius of an axisymmetric device) and \( \omega \) a
representative wave frequency, then the dimensionless frequency \( \omega^* = \omega(a/g)^{1/2} \) or the
dimensionless wave period \( T^* = 2\pi/\omega^* = T(g/a)^{1/2} \) must take the same value at full
size and in model testing.

Taking \( a \) as a representative dimension of the device, Eq. (4.13) may be written in
dimensionless form as

\[
P_e^* = \frac{Q_e^*}{M^* + G^* + i(V_0^* + B^*)}.
\] (4.20)

Here

\[
G^* = \frac{\rho \omega}{a} G, \quad B^* = \frac{\rho \omega}{a} B
\] (4.21)

are the dimensionless radiation conductance and the dimensionless radiation
susceptance, respectively,

\[
P_e^* = \frac{p_c}{\rho \omega^2 a^2}
\] (4.22)

is the dimensionless pressure amplitude (in general complex),

\[
Q_e^* = \frac{Q_e}{\omega a^3}
\] (4.23)

is the dimensionless excitation flow rate amplitude (in general complex),

\[
V_0^* = \frac{\rho g}{\gamma \rho_a a^2} V_0.
\] (4.24)

is the dimensionless air chamber volume, and

\[
M^* = K \frac{\rho}{\rho_a} \frac{\omega D}{\Omega a}
\] (4.25)

is a dimensionless turbine parameter.

Correct dynamic similarity requires all terms of Eq. (4.20) to take equal values in
similar conditions at full size and model size. This is particularly the case of term \( V_0^* \)
representing, in nondimensional form, the air chamber volume (in calm sea conditions).

Let \( \varepsilon = a_1/a_2 \) be the linear scale, where subscript 1 denotes the model and subscript
2 denotes full size.

We assume first that the OWC device, including its air chamber, is exactly
reproduced geometrically at scale \( \varepsilon = a_1/a_2 \), and assume atmospheric pressure \( p_a \) to
take equal values. Then it is \( V_{01}/V_{02} = (a_1/a_2)^3 = \varepsilon^3 \) and \( V_{01}^* / V_{02}^* = (\rho_1/\rho_2)\varepsilon \). Since \( \rho_1/\rho_2 \) is close to unity, even if fresh water fills the tank rather than salted water, condition \( V_{01}^* = V_{02}^* \) fails to be satisfied if scale \( \varepsilon = a_1/a_2 \) is significantly less than unity as is normally the case.

If full geometric similarity, including the chamber, is kept, i.e. \( V_{01}/V_{02} = (a_1/a_2)^3 = \varepsilon^3 \), then condition \( V_{01}^* = V_{02}^* \) requires
\[
\frac{p_1}{p_2} = \frac{\rho_1}{\rho_2} \varepsilon.
\]
This would imply the air pressure in the laboratory to be a small fraction of atmospheric pressure, which obviously is far from practical.

The alternative is to set
\[
\frac{V_{01}}{V_{02}} = \frac{\rho_2}{\rho_1} \varepsilon^2,
\]
it being assumed that \( p_{a1} \cong p_{a2} \). If the air compressibility is to be correctly simulated in model testing, the air chamber of the model is usually connected to a much larger air reservoir with rigid walls, in such a way that condition (4.26) is satisfied.

When model testing an OWC device at small scale, it is unpractical to simulate the air turbine by a small one. The usual procedure is to install at the top of the air chamber a device reproducing the pressure head and the flow rate of the full-sized air turbine. An orifice may provide a fairly good simulation of self-rectifying impulse turbines (whose pressure versus flow rate characteristic is approximately quadratic), whereas a window covered with a porous material (through which the air flow is approximately laminar) is frequently used to simulate a Wells turbine whose characteristic is known to be approximately laminar.

However, it may be of interest to investigate how the full sized turbine can be simulated by a smaller turbine in tests performed at relatively large scale. We assume the two turbines to be geometrically similar, which implies the turbine proportionality constant \( K \) to take the same value, and introduce the turbine scale \( \varepsilon_t = D_1/D_2 \), where \( D_1 \) and \( D_2 \) are the rotor diameters of the model turbine and the full-sized one respectively. The dynamic similarity requires \( M_1^* = M_2^* \), and so, assuming the atmospheric pressure to be the same, \( p_{a1} = p_{a2} \), we find
\[
\varepsilon_t = \frac{p_1}{p_2} \frac{\Omega_2}{\Omega_1} \varepsilon^{3/2}.
\]
An additional condition is the requirement that the turbine dimensionless parameters \( \Phi, \Psi, \Pi \) (representing the turbine aerodynamic performance) take the equal values for similar conditions of the air pressure cycle in the chamber of the model and the full-sized converter. We take such conditions as those of maximum air pressure \( p_c = |p_c| \).

Similarity requires \( P_{c1}^* = P_{c2}^* \). Then Eq. (4.22) yields
\[
\frac{p_{c1}}{p_{c2}} = \frac{\rho_1}{\rho_2} \left( \frac{\rho_2 d_1}{\rho_2 d_2} \right)^2 = \frac{\rho_1}{\rho_2} \varepsilon^{-3}.
\]
Condition \( \Psi_1 = \Psi_2 \) gives (see Eq. (4.7))
\[
\frac{p_{c1}}{p_{c2}} = \left( \frac{\Omega_2 D_1}{\Omega_2 D_2} \right)^2 = \varepsilon_t^{-2} \left( \frac{\Omega_1}{\Omega_2} \right)^2 ,
\]
(4.29)

where \( \rho^* = \rho_a \) was assumed. Finally, Eqs (4.27) to (4.29) give

\[
\varepsilon_t = \left( \frac{\rho_1}{\rho_2} \right)^{1/4} \varepsilon ,
\]
(4.30)

\[
\frac{\Omega_1}{\Omega_2} = \left( \frac{\rho_1}{\rho_2} \right)^{3/4} \varepsilon^{-1/2} .
\]
(4.31)

Neglecting the differences in water density between the sea and the tank, Eqs (4.30) and (4.31) show that the turbine scale \( \varepsilon_t \) is equal to the device model scale \( \varepsilon \), whereas the rotational speed is larger in the turbine model.

### 4.4. Time domain analysis of a fixed-structure OWC wave energy converter

We recall that the Wells turbine is approximately linear, but not the self-rectifying impulse turbines that have been developed as alternatives to the Wells turbine. If the air turbine is not linear (the air flow rate is not proportional to the pressure head) and/or the air pressure oscillations are too large for the linearized relationship density-versus-pressure (4.10) to apply, then the frequency-domain analysis cannot be employed. In particular, even in the presence of regular incident waves, the air flow rate displaced by the OWC free surface and the air pressure oscillation are not a simple harmonic functions of time. In such cases, we have to resort to the time-domain analysis to model the radiation flow rate. The analysis is similar to that of the radiation force on an oscillating body presented in Section 3.4. We only present the results here.

In the present case, the radiation flow rate \( q_r(t) \) is induced by the air pressure oscillation \( p_c(t) \) in the chamber, in the absence of incident waves. The relationship between them may be written as

\[
q_r(t) = -\int_{-\infty}^{t} h_r(t - \tau) \dot{p}_c(\tau) d\tau .
\]
(4.32)

The memory function \( h_r \) is related to the radiation susceptance \( G \) through the Fourier transform

\[
h_r(s) = \frac{2}{\pi} \int_{0}^{\infty} G(\omega) \cos \omega s d\omega .
\]
(4.33)

The following set of equations has to be solved

\[
q_c(t) + q_r(t) = \frac{w_i(t)}{\rho_a} \left( 1 + \frac{p_c(t)}{p_a} \right)^{1/\gamma} + \frac{V(t)}{\gamma(p_c(t) + p_a)} \frac{dp_c}{dt} ,
\]
(4.5)

\[
q_r(t) = -\int_{-\infty}^{t} h_r(t - \tau) \dot{p}_c(\tau) d\tau ,
\]
(4.32)

\[
\frac{dV}{dt} = q_c(t) + q_r(t)
\]
(4.1)

\[
\Phi = f_w(\Psi), \text{ where } \Phi = \frac{w_i}{\rho^* \Omega^2 D^2}, \quad \Psi = \frac{p_c}{\rho^* \Omega^2 D^2}
\]
(4.34)
The excitation flow rate $q_e(t)$ can be computed from the excitation flow rate amplitude coefficient $\Gamma(\omega) = Q_e(\omega)/A_w$ and the incident wave amplitude $A_w$ and frequency $\omega$. In the case of irregular waves, the frequency spectrum $S_\omega(\omega)$ is supposed to be known.

Function $f_w(\Psi)$, relating flow rate to pressure, is assumed to be known for the turbine. In (4.34), the reference air density $\rho^*$ is set equal to the density at turbine entrance (either atmospheric air density $\rho_a$ or chamber air density $\rho_c$ depending on the flow direction through the turbine) or is simply replaced by $\rho_a$. The turbine rotational speed $\Omega$ is fixed or, alternately, its instantaneous value is defined through a control algorithm possibly involving the torque of the electrical generator.

References

5. Stochastic modelling of wave energy conversion

5.1. Introduction

In a wave energy plant, unless a large energy storage interface follows the waves in the energy conversion chain of the plant (as the water reservoir of run-up devices), the power take-off equipment is required to convert an energy flux that is oscillatory, highly irregular and largely random. In the basic studies as well as in the design stages of a wave energy plant, the knowledge of the statistical characteristics of the local wave climate is essential, no matter whether physical model testing or theoretical/numerical modelling is to be employed. This information may result from wave measurements, more or less sophisticated forecast/hindcast models or a combination of both, and usually takes the form of a set of representative sea states, each characterized by its frequency of occurrence and by a spectral distribution.

Several approaches have been used to model and optimize wave energy converters. The simplest theoretical model assumes the incident waves to be regular or monochromatic, which allows the analysis to be performed in the frequency domain provided the power take-off system (PTO) is linear. This is approximately the case of an oscillating water column (OWC) equipped with a Wells turbine. The hydrodynamic coefficients are required for the wave frequency under consideration, which can be obtained analytically for simple geometries, numerically (commercial codes like WAMIT are available) or experimentally. This approach can be extended to irregular waves by linear superposition if a spectral distribution \( S_f(f) \) or \( S_\omega(\omega) \) is provided.

A time-domain analysis is capable of dealing more realistically with irregular waves and with non-linear power take-off equipment. This requires the knowledge of the hydrodynamic coefficients over a wide range of frequencies, and, while being computationally much heavier, yields the performance variables (namely power output) as functions of time.

The capability of using numerical simulations to produce time series for variables like rotational speed and electrical power output may be important in PTO design, if e.g. electrical power oscillations induced by wave grouping are to be studied, or in control studies. However, the capability of predicting the average power output in a given sea state may be sufficient for many purposes in plant design, including the specification of the power equipment. This can be provided by stochastic methods that are computationally much less demanding, as will be shown here. This may be done as long as the system is linear, which requires small amplitude waves and (especially) a linear PTO characteristic to be acceptable assumptions.

It is assumed that the local wave climate may adequately be represented by a set of sea states of known power spectrum and frequency of occurrence, the surface elevation being a Gaussian random variable for each sea state.

The fundamentals of the stochastic modelling of wave energy conversion presented here are based on [1].

We assume that the local wave climate may be represented by a set of sea states, each being a stationary stochastic process. The statistical theory of the random water
surface elevation was developed by Longuet-Higgins [2,3]. We denote, as in Chapter 2, the surface elevation by $\zeta$. For a given sea state, the probability of the surface elevation at a given observation point taking a value between $\zeta$ and $\zeta + d\zeta$ is $\phi_\zeta(\zeta)d\zeta$, where $\phi_\zeta(\zeta)$ is the probability density function. Obviously, it is

$$\int_{-\infty}^{\infty} \phi_\zeta(\zeta)d\zeta = 1.$$  \hfill (5.1)

We assume that the surface elevation at a given observation point is a Gaussian process, an assumption widely adopted in ocean engineering applications. So we may write, for a Gaussian process,

$$\phi_\zeta(\zeta) = \frac{1}{\sqrt{2\pi}\sigma_\zeta} \exp \left( -\frac{\zeta^2}{2\sigma_\zeta^2} \right),$$  \hfill (5.2)

where $\sigma_\zeta^2$ is the variance, and $\sigma_\zeta$ is the standard deviation or root-mean-square, of the surface elevation $\zeta$. This can be written as

$$\sigma_\zeta^2 = \int S_\omega(\omega)d\omega,$$  \hfill (5.3)

where, as before, $S_\omega(\omega)$ is the frequency spectrum in terms of the radian wave frequency $\omega$.

### 5.2. Heaving body wave energy converter

We consider now a linear wave energy converter, and analyze it as a linear system. We start by assuming the input to be an incident regular wave of amplitude $A_w$ and frequency $\omega$.

If the device is an oscillating body with heaving motion and linear PTO (linear damper and linear spring) as represented in Fig. 3.3, the incident wave will produce an excitation force $f_e(t)$ that, as in Chapter 3, we write as $f_e(t) = \text{Re}(F_e e^{i\omega t})$ or more simply as $f_e(t) = F_e e^{i\omega t}$, where $F_e$ is a complex amplitude.

We adopt as output variable the vertical displacement $x(t)$ of the body. Since the system is linear, $x(t)$ is a sinusoidal function of time which we write as $x(t) = X e^{i\omega t}$, $X$ being a complex amplitude. We saw in Chapter 3 (see Eq. (3.18)) that $X$ is related to $F_e$ by

$$X(\omega) = \Lambda(\omega)F_e(\omega),$$  \hfill (5.4)

where

$$\Lambda(\omega) = \left\{ -\omega^2(m + A) + i\omega(B + C) + (\rho gS_{cs} + K) \right\}^{-1}.$$  \hfill (5.5)

Now, if the incident waves are random and Gaussian, with frequency spectral distribution $S_\omega(\omega)$, and since we are assuming the system as linear, it can be shown [1] that the displacement $x(t)$ is also random and Gaussian, and its spectral distribution $S_x(\omega)$ is related to the incident wave spectral distribution $S_\omega(\omega)$ by

$$S_x(\omega) = S_\omega(\omega)[\Lambda(\omega)]^2 \Gamma^2(\omega),$$  \hfill (5.7)
where
\[ \Gamma(\omega) = \frac{|F_r(\omega)|}{A_w(\omega)} \]  
(5.8)
is the modulus of the frequency-dependent excitation force amplitude per unit incident wave amplitude. It may also be shown that the variance of \( x \) is given by
\[ \sigma_x^2 = \int_0^\infty S_x(\omega)d\omega = \int_0^\infty S_\omega(\omega)|\Lambda(\omega)|^2\Gamma^2(\omega)d\omega \]  
(5.9)
and the corresponding probability density function is
\[ \phi_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp \left(-\frac{x^2}{2\sigma_x^2}\right). \]  
(5.10)

5.3. Oscillating water column converter with linear air turbine

The stochastic approach is particularly suited to the study of an OWC wave energy converter equipped with a linear air turbine which is known to be a fairly good approximation to the Wells turbine. We assume the structure of the OWC to be fixed to the bottom, as in Section 4.2, and the relationship between air pressure \( p_a \) and the volume flow rate to be linearized as in Eq. (4.10). We assume the input to this linear system to be the surface elevation \( \zeta(t) \) as in Section 5.2, and adopt as output the air pressure oscillation \( p_c(t) \).

As in Section 5.2, we start by assuming the input to be an incident regular wave of amplitude \( A_w \) and frequency \( \omega \). The excitation flow rate is a sinusoidal function of time \( q_e(t) = Q_e e^{j\omega t} \). Since the system is linear, \( p_c(t) \) is also a sinusoidal function of time which we write as \( p_c(t) = P_c e^{j\omega t} \), \( P_c \) being a complex amplitude. We saw in Chapter 4 (see Eq. (4.13)) that \( P_c \) is related to \( Q_e \) by
\[ P_c(\omega) = \Lambda(\omega)Q_e(\omega), \]  
(5.11)
where
\[ \Lambda(\omega) = \left[ \frac{KD}{\rho_a\Omega} + G + i\left(\frac{\omega V_0}{\gamma p_a} + B\right) \right]^{-1}. \]  
(5.12)
Now, if the incident waves are irregular, random and Gaussian, with spectral distribution \( S_\omega(\omega) \), the excitation flow rate is also Gaussian with a spectral distribution
\[ S_{q_e}(\omega) = S_\omega(\omega)\Gamma(\omega), \]  
(5.13)
where
\[ \Gamma(\omega) = \frac{|Q_e(\omega)|}{A_w(\omega)} \]  
(5.14)
is the modulus of the frequency-dependent excitation flow rate amplitude per unit incident wave amplitude.

Since we are assuming the system as linear (linear turbine and linear pressure-density relationship), it can be shown [1] that the pressure oscillation \( p_c(t) \) is also random and Gaussian, with variance
\[
\sigma_p^2 = \int_0^\infty S_{\omega}(\omega) |\Lambda(\omega)|^2 d\omega = \int_0^\infty S_{\omega}(\omega) \Gamma^2(\omega) |\Lambda(\omega)|^2 d\omega. \tag{5.15}
\]

Since the air pressure oscillation may be regarded as a Gaussian process, the corresponding probability density function is
\[
\phi_p(p_c) = \frac{1}{\sqrt{2\pi}\sigma_p} \exp\left(-\frac{p_c^2}{2\sigma_p^2}\right). \tag{5.16}
\]

The instantaneous power output of the turbine versus the pressure head is assumed known in dimensionless form as (see Eq. (4.6))
\[
\Pi = f_p(\Psi). \tag{5.17}
\]

Taking into account that (see Eq. (4.7))
\[
\Psi = \frac{p_c}{\rho_a \Omega^2 D^2} \quad \text{and} \quad \Pi = \frac{P_t}{\rho_a \Omega^2 D^2}, \tag{5.18}
\]

we find the following relationship between the turbine power output \(P_t\) and the pressure head \(p_c\)
\[
P_t = \rho_a \Omega^2 D^5 f_p\left(\frac{p_c}{\rho_a \Omega^2 D^2}\right). \tag{5.19}
\]

Since the pressure fluctuation \(p_c(t)\) is a Gaussian process with variance \(\sigma_p^2\) given by Eq. (5.15), and probability density function \(\phi_p(p_c)\) given by Eq. (5.16), we have, for the averaged value of the turbine power output,
\[
\bar{P}_t = \int_{-\infty}^{\infty} \phi_p(p_c) P_t(p_c) dp_c \tag{5.20}
\]
or
\[
\bar{P}_t = \frac{\rho_a \Omega^3 D^5}{\sqrt{2\pi}\sigma_p} \int_{-\infty}^{\infty} \exp\left(-\frac{p_c^2}{2\sigma_p^2}\right) f_p\left(\frac{p_c}{\rho_a \Omega^2 D^2}\right) dp_c. \tag{5.21}
\]

This can be written in dimensionless form as
\[
\bar{\Pi} = \frac{1}{\sqrt{2\pi}\sigma_\Psi} \int_{-\infty}^{\infty} \exp\left(-\frac{\Psi^2}{2\sigma_\Psi^2}\right) f_p(\Psi) d\Psi, \tag{5.22}
\]

where \(\bar{\Pi}\) is the averaged value of \(\Pi\) (or, equivalently, the dimensionless value of \(\bar{P}_t\)) and
\[
\sigma_\Psi = \frac{\sigma_p}{\rho_a \Omega^2 D^2} \tag{5.23}
\]
is the dimensionless value of \(\sigma_p\).

If the chamber geometry, the turbine and its rotational speed are fixed, then Eq. (5.21) (or (5.22)), together with Eq. (5.15), gives the average power output as a functional of the incident wave spectral density \(S_{\omega}(\omega)\), that characterizes the sea state under consideration.
The instantaneous (pneumatic) power available to the turbine is equal to the volume flow rate times the pressure head

$$P_{\text{avai}} = \frac{w_t}{\rho_a} \rho, \quad (5.24)$$

and the instantaneous turbine efficiency is

$$\eta = \frac{P_t}{P_{\text{avai}}} = \frac{\Phi}{\Phi \Psi}. \quad (5.25)$$

For a linear turbine, it is $\Phi = K\Psi$ ($K$ = constant), and so $\eta = \Pi K^{-1}\Psi^{-2}$. The averaged available power is given by

$$\bar{P}_{\text{avai}} = \int_{-\infty}^{\infty} P_{\text{avai}} \phi_p(p_c) \, dp_c = \frac{1}{\rho_a} \int_{-\infty}^{\infty} w_i \rho_c \exp \left(-\frac{p_c^2}{2\sigma_p^2}\right) \, dp_c \quad (5.26)$$

or, in dimensionless form,

$$\Pi_{\text{avai}} = \frac{\bar{P}_{\text{avai}}}{\rho_a \Omega^3 D^5} = \frac{1}{\sqrt{2\pi} \sigma_p} \int_{-\infty}^{\infty} f_w(\Psi) \exp \left(-\frac{\Psi^2}{2\sigma_p^2}\right) \Psi \, d\Psi. \quad (5.27)$$

If $\Phi = K\Psi$, and making use of the result

$$\int_{0}^{\infty} x^{1/2} e^{-x} \, dx = \frac{\sqrt{\pi}}{2},$$

we easily obtain

$$\Pi_{\text{avai}} = K \sigma_p^2. \quad (5.28)$$

The average efficiency of the turbine is defined as $\bar{\eta} = \eta_t/\bar{P}_{\text{avai}} = \Pi/\Pi_{\text{avai}}$. If $\Phi = K\Psi$, it becomes simply

$$\bar{\eta} = \frac{\Pi}{K \sigma_p^2}. \quad (5.29)$$

For given turbine geometry, and if the function $\Pi = f_p(\Psi)$ is known (possibly from model testing), then Eq. (5.22) represents, in dimensionless form, the turbine average power output as a function of the root-mean-square $\sigma_p$ of the pressure oscillation $p_c$. The curve represented by this equation is likely to be more useful, in applications with real random waves, than the more conventional curve given by Eq. (5.17) for the instantaneous power output. The same can be said about the equations for the efficiency. We recall that the results derived above are valid only for linear turbines.
Fig. 5.1. Dimensionless power output $\Pi$ versus dimensionless pressure head $\Psi$ (solid curve) and dimensionless average power output $\bar{\Pi}$ versus dimensionless root-mean-square of pressure head $\sigma_\psi$ (dashed curve), for a typical Wells turbine.

Fig. 5.2. Efficiency $\eta$ versus dimensionless pressure head $\Psi$ (solid curve) and average efficiency $\bar{\eta}$ versus dimensionless root-mean-square $\sigma_\psi$ of pressure head (dashed curve) for the same Wells turbine as in Fig. 5.2, with $\Phi = 0.680\Psi$.

In Fig. 5.1, the solid curve represents the dimensionless power output $\Pi$ versus the dimensional pressure head $\Psi$ for a Wells turbine similar to the turbine that equips the OWC plant of Pico island. (The results are from numerical modelling.) The dashed curve is a plot, in dimensionless form, of the average power output $\bar{\Pi}$ as a function of the mean-square-root of the pressure head $\sigma_\psi$. Similar dimensional plots could be presented for a given turbine at fixed rotational speed: $P_t$ versus $p_c$ and $\bar{P}_t$ versus $\sigma_p$. Figure 5.2 represents the corresponding curves for the efficiency $\eta$ versus $\Psi$ (solid curve) and average efficiency $\bar{\eta}$ versus $\sigma_\psi$ (dashed curve). We note that the dashed
curves in Figs 5.1 and 5.2 characterize the aerodynamic performance for a given turbine geometry, and are particularly useful in computations involving random pressure head with Gaussian distribution.

5.4. Application to optimization of the mechanical and electrical equipment of an OWC plant

5.4.1. Introduction

Wave power plant design, for operation in a largely random, highly irregular and sometimes very harsh environment, is a great challenge. Survivability is a more vital issue than in any other renewable energy technology, as could be learned from the (still limited) experience with prototypes. This section is devoted to plants of the oscillating-water-column (OWC) type with bottom-standing structure. The design and construction of the structure (apart from the air turbine) are the most critical issues in OWC technology, and the most influential on the economics of energy produced from the waves. In the present situation, the civil construction dominates the cost of the OWC plant. So, in the preliminary studies of an OWC plant, the first step usually consists in defining the siting and basic geometry of the plant’s structure. This is established, taking into account geo-morphological constraints and local wave climate, from the wave energy absorption hydrodynamics, with the aid of numerical modelling and/or of wave basin model testing. Such modelling provides the information required for the specification (or design in the case of non-conventional or non-standard items) of the power take-off equipment. It is known that the aerodynamic energy-conversion, that takes place in the turbine, and the hydrodynamics of the wave energy absorption from the waves (in the OWC itself) are interdependent processes linked by the oscillating air pressure inside the chamber.

Several approaches have been used to model and design OWC converters. A time domain analysis is capable of dealing with irregular waves and non-linear power take-off equipment (air turbine), and yields time series for variables, like rotational speed and electrical power output, that may be important if, for example, electrical power oscillations induced by wave grouping are to be studied, or in control studies. However, the capability of predicting the average power output in a given sea state may be sufficient for most purposes in plant design, including the specification of the power equipment (turbine and generator). This capability may be provided by stochastic methods that are computationally much less demanding. This section is based on stochastic modelling of the OWC performance. It is supposed that the plant is equipped with a variable-speed Wells turbine driving a generator, and that the instantaneous rotational speed is controlled (by means of the generator power electronics and the plant’s programmable logic controller or PLC) to yield maximum energy production. The rotational-speed control strategy is that developed in [4].

In the optimisation study presented here, the geometry (but not the size) of the turbine is fixed, and the turbine size (represented by the rotor outer diameter $D$) is to be determined for a given wave climate. This is done first for maximum energy production, it being assumed that the rotational speed is optimally controlled. Such maximization is not in general the most appropriate criterion. Indeed, the wave energy investor will be looking for maximum profit (rather than maximum produced energy), and profit is related, not only to the amount of energy produced annually (which in turn depends on turbine size and generator rated power capacity), but also to the investment required to
purchase and install the equipment (which increases with turbine and generator size), to the operation and maintenance costs, and to the price at which the unit electrical energy is paid to the producer. This price may vary substantially depending for example on electrical grid size or on government policy to promote renewable energy development.

The two criteria to optimise turbine size and generator rated power – maximum energy production and maximum profit – are compared. This is illustrated by numerical examples, based on an existing OWC prototype (the Pico plant), in which the geometry of the plant’s structure is fixed, and the energy unit price, the plant’s lifetime and the discount rate are allowed to vary. Different wave climates are also considered. More or less empirical formulae (based on available information) are adopted for the capital costs of the turbine and generator as functions of equipment size and rated power. It is shown that the maximum profit criterion leads to a smaller and cheaper turbine (and higher rotational speed) as compared with the simpler criterion of maximum energy production.

This exercise is intended to contribute to establishing realistic design methodologies, and may be extended to the complete plant design (including the structure) as well as to other types of wave energy converters.

5.4.2. Stochastic modelling and energy production

We assume that the local wave climate is known and may be represented by a set of sea states, each being a stationary stochastic process whose frequency spectrum is given. For each sea state, the probability density function of the surface elevation at a fixed observation point is supposed to be Gaussian, an assumption widely accepted in ocean engineering, and adopted in this chapter.

We showed above (Eq. (5.21) that, for the sea state under consideration, the time-averaged power output of the turbine \( P_t \) may then be computed from

\[
\bar{P}_t = \frac{\rho_a \Omega^3 D^5}{\sqrt{2\pi} \sigma_p} \int_{-\infty}^{\infty} \exp\left( -\frac{p_c^2}{2\sigma_p^2} \right) f_p \left( \frac{p_c}{\rho_a \Omega^2 D^2} \right) dp. \tag{5.21}
\]

We assume that the local wave climate may be represented by a set of sea states, each sea state being characterized by its values of the significant wave height \( H_s \), energy period \( T_e \) and frequency of occurrence \( \phi \). For convenience, a standard formula (e.g. Pierson-Moskowitz or similar) is adopted for the frequency spectrum of each sea state.

Equation (5.21) indicates that, for each sea state, the averaged power output \( \bar{P}_t \) depends on the turbine size (represented by the rotor diameter \( D \) and rotational speed \( \Omega \). It is assumed (i) that the oscillations in \( \Omega \) and \( P_t \) are kept relatively small due to the capability of storing energy in, and releasing it from, a flywheel (the turbine rotor); and (ii) that the rotational speed is properly controlled (by the programmable logic controller of the plant) in such a way that is maximizes the power output \( \bar{P}_t \) (possibly subject to some constraint like maximum allowed rotor-blade tip-speed \( \Omega D/2 \) (see [4])).

The net electrical power is \( \bar{P}_e = \bar{P}_t - \bar{L}_m - \bar{L}_e \), where \( \bar{L}_m \) are mechanical (bearing) losses and \( \bar{L}_e \) are electrical losses.

If the wave climate is represented by \( N \) sea states, \( j = 1, 2, \ldots, N \), then the annual-averaged value of the net electrical power output is given by
\[
\bar{P}_{e,\text{annual}} = \sum_{j=1}^{N} \phi_j \bar{P}_{e,j}.
\]  

(Here the availability of the plant is assumed equal to unity.) It should be noted that, for a given wave climate, \(\bar{P}_{e,\text{annual}}\) depends on turbine size. A criterion for the specification and/or design of the plant’s power equipment could be to find the value of \(D\) that maximizes \(\bar{P}_{e,\text{annual}}\) (i.e. that yields maximum electrical energy production).

### 5.4.3. Costs, income and profit

An alternative approach to optimization is based on profit, defined as the difference between the income from electrical energy supply to the grid and the cost of producing it.

The major capital cost centres of a wave energy plant are the device structure and the mechanical and electrical equipment. In the following analysis, it is supposed that the design of the structure has been fixed together with the corresponding construction cost \(C_{\text{struc}}\). The mechanical equipment comprises the air turbine, the valve or valves and the ducting system, and its cost \(C_{\text{mech}}\) is assumed to depend on the turbine size (represented by the rotor diameter \(D\)) according to some empirical algorithm \(C_{\text{mech}} = f_{\text{mech}}(D)\). The electrical equipment comprises the generator, the power electronics and conventional equipment (transformer, circuit breakers, switch boards, cabling, etc.), whose total cost is taken as a function of the plant rated power \(C_{\text{elec}} = f_{\text{elec}}(\bar{P}_n)\). We represent by \(C_{\text{oth}}\) other capital costs (e.g. control equipment and instrumentation) that are practically independent of turbine size or rated power.

Assuming the plant can be built in one year, the annual sum \(A_{\text{cap}}\) involved in repayment of the capital cost of the plant may be computed as (see e.g. [5])

\[
A_{\text{cap}} = \frac{C}{(1+r)^n},
\]  

where \(C = C_{\text{struc}} + C_{\text{oth}} + C_{\text{mech}} + C_{\text{elec}}\) is the total capital cost of the plant, \(r\) is the discount rate and \(n\) is the lifetime of the plant (in years).

Other (non-capital) costs are associated with operation and maintenance of the plant, including spares, repairs and insurance. Their total annual cost is denoted by \(A_{\text{O&M}}\).

The annual income from the electrical energy supplied to the grid (in currency units) is

\[
I = 8760\bar{P}_{e,\text{annual}} Au.
\]  

Here \(\bar{P}_{e,\text{annual}}\) is the annual-averaged electrical power output in kW, \(A\) is fractional availability (probability of the whole system functioning at any specific time, which includes the effects of scheduled maintenance and of breakdowns) and \(u\) is the paid price of electrical energy supplied to the grid per kWh.

The annual profit of the project is then

\[
E = I - A_{\text{cap}} - A_{\text{O&M}}.
\]  

The criterion of equipment optimization consists now in looking for the turbine size \(D\) (and the corresponding plant rated power) that maximizes \(E\).
5.4.4. Example calculation

5.4.4.1. The plant

The two optimization criteria outlined above will be applied here to a typical case of a fixed OWC-plant equipped with a Wells turbine-generator set. We chose the shoreline plant constructed on the island of Pico, Azores. The plant (see Figs 1.11, 1.13 and 4.1) has a concrete structure (of 12m×12m internal cross-section at free-surface level) that spans a natural gully whose depth is about 8m. It is equipped with a horizontal-axis Wells turbine (Fig. 1.12) driving a variable-speed electrical generator.

Curves for the hydrodynamic coefficients $|\Gamma(\omega)|$, $B(\omega)$ and $G(\omega)$ are given in Fig. 4.3. They were obtained from numerical values computed for the Pico plant with a three-dimensional boundary element code (the discretization is shown in Fig. 4.4).

The actual Wells turbine of the Pico plant has an 8-bladed rotor of $D = 2.3$ m outer diameter and $D_l = 1.36$m inner diameter, with a row of guide vanes on each side of the rotor. The curve $\Pi = f_P(\Psi)$ can be found in Fig. 5.1, while the curve $\Phi = f_Q(\Psi)$ was approximated by the straight line $\Phi = K\Psi$, with $K = 0.680$. In the computations whose results are presented here, it was assumed that the plant is equipped with a controlled relief valve that prevents the air pressure fluctuation $|p_c|$ from exceeding the critical value (depending on $\Omega$ and $D$) above which aerodynamic stalling at the rotor blades severely impairs the turbine performance (see [4]).

Bearing friction losses $L_m$ were accounted for, as explained in detail in [4]. Electrical losses were assumed to depend on rated electrical power, instantaneous electrical power output and rotational speed, and were estimated from an algorithm supplied by manufacturer of the electrical equipment of the Pico plant. The estimated electrical efficiency was found to reach a maximum of about 0.955 at rated power and maximum rotational speed.

For each sea state, the programmable logic controller (PLC) of the plant was assumed to control the rotational speed $\Omega$ in such a way that it maximizes the averaged power output $\bar{P}$, subject to the constraint (related to turbine aerodynamics) $\Omega D/2 \leq 170$ m/s for the blade tip speed (for details see [4]). The curves in Fig. 5.3 represent the control algorithm (rotational speed versus power output) for three turbine sizes ($D = 1.6, 2.3$ and $3.8$m).
Fig. 5.3. Curves for rotational-speed control: $\overline{P}_c \rho^{-1} \Omega^{-3} D^{-5}$.

Results for the net power output $\overline{P}_c$ were first computed for a set of 80 sea states, obtained by combining 10 values of $H_s$ (0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 5.0 m) and 8 values of the energy period $T_e$ (7, 8, 9, 10, 11, 12, 13, 14 s), and for 9 turbine sizes $D$ (1.6, 1.8, 2.0, 2.3, 2.6, 2.9, 3.2, 3.5, 3.6 m). In this way, a set of 720 values were obtained for $\overline{P}_c$, that could be used for three-dimensional interpolation. For each turbine size $D$, the plant rated power $P_{\text{rated}}$ was defined as the maximum computed value of $\overline{P}_c$ (that corresponds to the most energetic sea state considered, $H_s = 5$ m, $T_e = 14$ s). Within the considered range of turbine sizes, the rated power increases with $D$, as shown in Fig. 2 (where, for comparison, values are also shown for $H_s = 4$ m, $T_e = 14$ s).

5.4.4.2. The wave climate

Results from wave measurements at the Pico plant’s location, in conjunction with longer-term results from forecast/hindcast numerical modelling for the same ocean area, were used to establish a set of 44 sea states characterizing the local wave climate, whose power level is 14.5 kW/m. Each sea state is represented by its significant wave height $H_s$, its energy period $T_e$ and its frequency of occurrence $\phi$. As in Falcão (2002), Pierson-Moskowitz formula is adopted for the frequency spectrum of each sea state (see Eq. (3.58))

$$S_{\phi}(\omega) = 262.6 H_s^2 T_e^{-4} \omega^{-5} \exp \left[ -1052 (T_e \omega)^{-4} \right],$$  \hspace{1cm} (3.58)

where $\omega$ is wave frequency.
Fig. 5.4. Plant rated power $P_{\text{rated}}$ versus turbine diameter. Thick and thin lines for $H_s = 5$ and 4 m respectively (in both cases, $T_c = 14$ s).

For comparison purposes, two other wave climates were also considered, derived from the original one by simply multiplying, for each component sea state, the significant wave height, $H_s$, by a constant, that was set equal to $\sqrt{2}$, for the more energetic one, and $1/\sqrt{2}$, for the less energetic one. The three wave climates, whose power is 7.25, 14.5 and 29.0 kW/m, were denoted as climates 1, 2 and 3, respectively.

It should be noted that the results presented here for the produced energy do not take into account the losses in the hydrodynamic process of conversion of wave energy into pneumatic energy that takes place in the oscillating water column. Such losses are difficult to estimate and were ignored. For this reason, an increase (possibly of a few percent) in the energy level of the incoming waves would be required to more realistically allow the shown results for the produced electrical energy to be kept.

5.4.4.3. Energy production

The utilization factor $\alpha$ is defined as the ratio of the annual averaged net power output divided by the plant’s rated power

$$\alpha = \frac{\bar{P}_{\text{e,annual}}}{P_{\text{rated}}}.$$  

Figure 3 shows $\alpha$ plotted versus turbine size $D$, for the three wave climates considered. As could be expected, the maximum value of $\alpha$ (0.52) is reached for the smallest turbine in the most energetic wave climate, whereas the minimum one (0.085) is for the largest turbine in the least energetic climate. For the Pico situation
(wave climate 2, \( D = 2.3 \text{ m} \)), it is \( \alpha = 0.31 \). These figures are to be compared with the world average of about 0.25 for wind plants. It should be pointed out that the highest value of \( \alpha \) does not imply (as can be concluded from results presented below) the best use of wave energy, either from the point of view of maximum energy production or of maximum profit.

![Figure 5.5](image1)

**Fig. 5.5.** Utilization factor \( \alpha \) versus turbine diameter \( D \).

![Figure 5.6](image2)

**Fig. 5.6.** Annual averaged net power output \( \overline{P}_{\text{e,annual}} \) versus \( D \).
Figure 5.6 shows the annual averaged power output $\bar{P}_{c,\text{annual}}$ as a function of the turbine size $D$ for the three wave climates considered. The optimum diameter (that yields maximum $\bar{P}_{c,\text{annual}}$) increases with the energy level of the incoming waves, taking the values of approximately 2.8, 3.3 and 4m for wave climates 1, 2 and 3, respectively. This means, as should be expected, that a larger turbine should be used at a more energetic coastal location.

5.4.4.4. Optimization for maximum profit

In the present stage of development of wave energy technology, economical parameters are much more difficult to estimate, especially if the object is to predict costs from large-scale deployment of full researched and mature wave energy technologies.

In the present situation, the structure construction dominates the capital costs of the OWC plant. It is unlikely that shoreline or nearshore fixed OWC plants can become economically viable unless structure constructional costs are drastically reduced, possibly by sharing costs in a dual- or multi-purpose structure. One way of achieving this consists in integrating the structure into a breakwater, an option that was considered for several situations in Europe. Alternatively, the wave plant may be combined with an offshore wind energy converter.

The costs of the mechanical equipment (especially air turbines) are difficult to predict since what is available today are figures for prototypes. The cost estimation for $C_{\text{mech}}$ was done in two stages. We start by adopting a reference value $C_{\text{mech,0}}$ derived from experience with existing prototypes whose size is represented by the turbine rotor diameter $D_0$ (2.3 m for the Pico plant). Assuming geometrical similarity to apply to all items of mechanical equipment, the volume and hence the mass of material would scale as $D^3$. If an exponent estimating technique is used (Jelen and Black, 1983), we may write

$$C_{\text{mech}}(D) = C_{\text{mech,0}} \left( \frac{D^3}{D_0^3} \right)^X,$$

where $X$ is an exponent to be chosen (frequently taken equal to about 0.6). For a single prototype (or for a very small number of units), we will adopt $C_{\text{mech,0}} = 330\text{k€}$, $X = 2/3$, and write $C_{\text{mech}} = 62D^2$ (in k€, $D$ in m).

Likewise, for the electrical equipment we write $C_{\text{elec}} = 3.3P_{\text{rated}}^{0.7}$ (in k€, $P_{\text{rated}}$ in kW).

We note that these estimates include construction and installation of equipment, but not the costs of development and design.

It is important, but not easy, to predict the costs for a mature technology, taking into account economies of scale. It is unlikely that fixed OWC plants will be replicated by more than several tens or a few hundred of (more or less identical) units, so we will be dealing with relatively small-series production (at least for the mechanical equipment made up mostly of non-standard/non-conventional items). Taking into account additional development, better design techniques and economies of scale, it seems not very unreasonable to adopt (for the foreseeable future) the following ranges for the capital costs of mechanical equipment and electrical equipment that depend on turbine size and plant rated power, the upper limits applying to prototypes with present-day technology.
\[ 20 < B_{\text{mech}} < 62, \quad 2.0 < B_{\text{elec}} < 3.3, \]  
\[ C_{\text{mech}} = B_{\text{mech}} D^2, \quad C_{\text{elec}} = B_{\text{elec}}^{0.7}. \]  

While the UK Department of Trade and Industry’s programme of Renewable Energy assesses generating costs at 8% and 15% discount rate, a figure of 10% will be adopted here.

In the calculations whose results are shown here, we set \( A_{\text{O&M}} = 0.03(C_{\text{mech}} + C_{\text{elec}}) \) for the operation and maintenance annual costs and \( A = 0.95 \) for the fractional availability.

The price \( u \) paid to the producer per unit electrical energy supplied to the grid is obviously highly influential. The lowest values of \( u \) are expected to be in very large grids, whereas higher values should occur for small isolated grids (possibly islands) where the usual alternative is costly Diesel generation. An exception is the situation in Portugal, where the government, in 2002, as a way of promoting the development of wave energy technology, decided to fix the price of electrical energy produced from the waves at the remarkably high value \( u = 0.225 \text{€/kWh} \), for a prescribed number of years and for a limited nation-wide total installed capacity.

In Figs 5 to 7, the annual profit is shown as a function of turbine size \( D \), for the three wave climates considered before (square symbols for wave climate 3, diamonds for 2 and triangles for 1). The influence of variation in the following parameters upon the annual profit is considered: wave climate, diameter \( D \), electrical energy price \( u \), discount rate \( r \), reference cost of mechanical equipment \( B_{\text{mech}} \) and equipment lifetime \( n \). It should be emphasized that only costs relating to mechanical and electrical equipment (depending on turbine size \( D \) and on rated power \( P_{\text{rated}} \)) were accounted for. The omission of other costs (namely those relating to the plant’s structure) does not affect the optimization process, but yields overestimates for the profits (that can be misleading to the reader who overlooks this information).
Fig. 5.7. Annual profit $E$ versus turbine diameter $D$, for three wave climates and three prices of produced electrical energy $u$: thick lines for $u = 0.225\,\text{€/kWh}$, thin lines for $0.1\,\text{€/kWh}$, broken lines $0.05\,\text{€/kWh}$. Here, and in the following figures: square symbols for wave climate 3, diamonds for 2 and triangles for 1.

Figure 5.7 shows the influence of energy price $u$ (in addition to wave climate) upon optimal turbine size. By comparing with Fig. 5.6, it can be seen that the profit criterion leads to a smaller turbine than the energy criterion does, the difference being more marked for lower energy prices. Figure 5.7 also indicates that, in wave climates with little energy (represented here by climate 1) no profit would be attainable except possibly if a relatively high price is paid for the produced energy (i.e. substantially higher than $0.05\,\text{€/kWh}$) and/or large reductions are achieved in capital costs (below $B_{\text{mech}} = 30$ and $B_{\text{elec}} = 2.0$). On the other hand, if $u = 0.225\,\text{€/kWh}$ (the special feed tariff in Portugal) the technology under consideration could be highly profitable even in a moderately energetic wave climate, provided the capital costs of the plant’s structure are shared (OWC-breakwater solution).
Fig. 5.8. Influence of discount rate: thick lines for $r = 0.1$ and thin lines for 0.15. The effect of increasing $r$ from 0.1 to 0.15 is equivalent to reducing $n$ from 20 to 10.3 years (see Eq. (5.31)).

Figures 5.8 and 5.9 show the influence of varying discount rate $r$, capital cost of mechanical equipment (represented by $B_{\text{mech}}$) and equipment lifetime $n$. In any case, whenever such variations (increase in $r$ and $B_{\text{mech}}$ and decrease in $n$) contribute to an increase in annual costs (and decrease in profit), they result in a decrease in optimized size $D$ and rated power $P_{\text{rated}}$.

It has been shown that the stochastic modelling is a powerful tool in the preliminary studies and basic design of a wave power plant, namely (in the case considered here) on what concerns the mechanical and electrical equipment. Two optimization criteria were analyzed: the application of the second one (designing for maximum profit) was found to lead to a smaller turbine and less powerful equipment as compared with the first criterion (designing for maximum electrical energy production), the difference between both being significant, especially for higher costs and lower electrical energy prices.

Equipment standardization and series production, as a way of reducing capital costs, should be favourably considered, even if this would conflict with the energy production criterion when applied to a wide range of wave climates and OWC geometries and sizes.

Although the primary object of the present study is the equipment, the analysis technique and the numerical results presented here can be extended, for optimization purposes, to the whole plant, provided that the costs not related to equipment (namely the plant’s structure costs) are also accounted for. In particular, the curves shown in Figs 5 to 7 may be used to assess the maximum level of costs that is allowed for the structure if the whole project is to remain profitable in the various scenarios considered.
Fig. 5.9. Effect of varying reference capital-cost of mechanical equipment. Thick, thin and broken lines for $B_{\text{mech}} = 20, 30$ and 45, respectively. Energy price: $u = 0.1\text{€/kWh}$ on the left hand side, $u = 0.05\text{€/kWh}$ on the right hand side.

References

Appendix. Dimensional analysis applied to model testing of wave energy converters

The theoretical modelling based on linear water wave theory is an essential step in the development of wave energy converters. It provides insights and important information at relatively low costs, in general in a relatively fast way. However, there are important non-linear effects that are not accounted for by this kind of modelling, namely those associated with large amplitude waves, large amplitude motions of the wave energy converter or OWC, and real fluid effects due to viscosity, turbulence, vortex shedding, etc. To account for such effects, physical model testing in a wave flume or a wave tank is normally the next step. The scales range between about 1:100th in small flumes to about 1:10th in the largest wave tanks.

We may apply dimensional analysis techniques to relate the conditions in model testing of a wave energy converter in a wave tank or a wave flume to those of the full-sized prototype in the sea.

We take as independent variables:
- \( T = 2\pi /\omega \) = wave period,
- \( h = \) water depth,
- \( \rho = \) water density,
- \( g = \) acceleration of gravity,
- \( a = \) characteristic linear dimension of device (length, radius, etc.),
- \( H = \) wave height,

and as dependent variables, for example:
- \( c = \) wave speed,
- \( \lambda = \) wavelength,
- \( \xi = \) displacement of device
- \( V = \) characteristic velocity (velocity of device),
- \( \Omega = \) angular velocity of device of device
- \( F = \) force on device,
- \( L = \) torque on device,
- \( p = \) pressure,
- \( P = \) power output of device.

We may write, for a given device in a given wave tank,

\[
\begin{align*}
  c, \lambda, \xi, V, \Omega, F, p, P & = \text{functions}(a, T, \rho, g, h, H). \tag{1}
\end{align*}
\]

Buckingham’s theorem of dimensional analysis allows us to reduce the number of independent variables from six to three (since in this case the number of fundamental dimensions is three: [LMT]). To do that, we select three primary variables among the six variables that appear on the right-hand-side of Eqs (1), for example \((h, \rho, g)\). We may form dimensionless variables with the three primary variables and the remaining variables. Buckingham’s theorem states that Eqs (1) may be replaced by Eqs (2)

\[
\begin{align*}
  \frac{c}{(ga)^{1/2}} \cdot \frac{\xi}{a} & \cdot \frac{V}{(ga)^{1/2}} & \cdot \frac{\Omega a^{1/2}}{g^{1/2}} & \cdot \frac{\lambda}{a^2 \rho g} & \cdot \frac{F}{a^4 \rho g} & \cdot \frac{L}{a^{7/2} \rho g^{3/2}} & \cdot \frac{P}{a} = \text{Functions} \left( \frac{1}{T} \sqrt{\frac{a}{g}}, \frac{h}{a}, \frac{H}{a} \right), \tag{2}
\end{align*}
\]
where only dimensionless quantities appear. The first dimensionless variable on the right-hand-side of Eqs (2), \( \text{Fr} = a^{1/2} g^{-1/2} t^{-1} \), is the Froude number.

Equations (2) apply unchanged to the model in the wave tank and to the full-sized prototype provided that exact geometric similarity is maintained (including on what concerns water depth). If we want also dynamic similarity, then the three dimensionless variables on the right-hand-side of Eqs (2) must take equal values in the model and in the prototype. Equations (2) then ensure that the dimensionless variables on the left-hand-side also take equal values for the model and the full-sized prototype.

Note that the fluid viscosity does not appear here. If viscosity were to be accounted for, then the additional independent variable \( \nu \) (kinematic viscosity) would have to be added to the list of independent variables in Eqs (1), and the dimensionless quantity \( \text{Re} = a^2 / (T \nu) \) (Reynolds number) would appear on the right-hand-side of Eqs (2). In practice, it is not possible to keep the similarity with respect to both the Froude number and the Reynolds number. Since the effects of not respecting the constancy in Reynolds number are much less important that would be the case for the Froude number, the influence of differences in Reynolds number (between model and prototype) is ignored in model testing in wave tank. Note that this does not mean that the real fluid effects due to viscosity are not at all reproduced in model testing. In fact, provided that the Reynolds number at model scale is large enough (and here “enough” depends on the geometry and size of the device and on the adopted scale), real fluid effects are fairly well reproduced in model testing. An important reason why the scale should be as large as economically reasonable is to have a better representation of real-fluid viscous effects.

Let the subscripts \( m \) and \( p \) denote the model and the full-sized prototype. The linear model scale is

\[
\varepsilon = \frac{a_m}{a_p}, \tag{3}
\]

Then, from Eqs (2), provided that

\[
\frac{T_m}{T_p} = \frac{\omega_p}{\omega_m} = \varepsilon^{1/2}, \quad \frac{h_m}{h_p} = \varepsilon, \quad \frac{H_m}{H_p} = \varepsilon \tag{4}
\]

(which ensures dynamic similarity), we may write

\[
\frac{c_m}{c_p} = \frac{V_m}{V_p} = \varepsilon^{1/2}, \quad \frac{\Omega_m}{\Omega_p} = \varepsilon^{-1/2}, \quad \frac{\xi_m}{\xi_p} = \frac{\lambda_m}{\lambda_p} = \varepsilon, \quad \frac{F_m}{F_p} = \frac{\rho_m}{\rho_p} \varepsilon^{5/2}, \quad \frac{P_m}{P_p} = \frac{\rho_m}{\rho_p} \varepsilon^{7/2}. \tag{5}
\]

Note that, if fresh water is used in the tank (which is usually the case), then \( \rho_m \neq \rho_p \).

**Exercise**

A floating oscillating-body wave energy converter is planned to be deployed offshore in 40 m wave depth. The device is to be model tested at scale 1:10\(^{th}\) in a wave tank. The densities of water in the sea and in the tank are 1025 kg/m\(^3\) and 1000 kg/m\(^3\), respectively.

a) Determine the appropriate water depth in the tank.

b) Regular waves with period \( T = 8 \) s and height \( H = 2 \) m in the sea are to be simulated in the tank. Determine the corresponding values in the tank. Determine also the wavelength in the sea and in the tank.
c) The device performance in irregular waves is then to be studied. A Pierson-Moskowitz spectrum with $H_s = 2$ m and $T_e = 8$ s (real sea conditions) is to be simulated in the tank. Write an expression for the spectrum of the waves to be generated in the tank. Compute the energy flux per unit wave crest length in the sea and in the tank.

d) Determine the ratio between displacements of the full-sized prototype and the model. The same for velocities.

e) The same for forces and power.