I. INTRODUCTION

Inspired by the ties between Minkowski’s 4D spacetime and Maxwell’s EM unification, Nordström[3] in 1914 and Kaluza[4] in 1921 showed that 5D general relativity contains both Einstein’s 4D gravity and Maxwell’s EM. However, they imposed an artificial restriction of no dependence on the fifth coordinate (cylinder condition). Klein[5], in 1926, suggested a physical basis to avoid this problem in the compactification of the fifth dimension, idea now used in higher-dimensional generalisations to include weak and strong interactions.

The models to be discussed have three key features:

(i) matter in 4D is a manifestation of pure geometry in 5D, i.e., no energy-momentum tensor is needed;
(ii) the higher-dimensional theory is a minimal extension of general relativity, i.e., there is no modification to the mathematical structure of the theory;
(iii) Physics only depends on the first four coordinates (cylinder condition).

There are three approaches to study higher-dimensional theories, each one sacrifices one of the key features: compactified, projective and noncompactified theories. Here we will review the general relativity aspects of the compactified and noncompactified ones.

A. Remarks on notation and conventions

Throughout this report, capital Latin indices A, B, ... run over 0,1,2,3,4 and Greek indices α, β, ... over 0,1,2,3. Five dimensional quantities are denoted by hats and the original idea is examined and two distinct approaches to the subject are presented and contrasted: compactified and noncompactified theories. We also discuss some cosmological implications of the noncompactified theory at the end of this paper.

II. 5D KALUZA MECHANISM

From feature (ii) of Kaluza’s approach, the Ricci tensor \( \hat{R}_{AB} \), Ricci scalar \( \hat{R} \) and Christoffel symbols \( \hat{\Gamma}^C_{AB} \) are

\[
\hat{R}_{AB} = \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C_{AC} + \hat{\Gamma}^D_{AC} \hat{\Gamma}^C_{DB} - \hat{\Gamma}^C_{AD} \hat{\Gamma}^D_{BC},
\]

\[
\hat{R} = \hat{g}^{AB} \hat{R}_{AB},
\]

\[
\hat{\Gamma}^C_{AB} = \frac{1}{2} \hat{g}^{CD} (\partial_A \hat{g}_{DB} + \partial_B \hat{g}_{DA} - \partial_D \hat{g}_{AB}),
\]

where \( \hat{g}_{AB} \) is the five-dimensional metric tensor.

The action and field equations are a five-dimensional version of the usual 4D Einstein action and equations

\[
S = -\frac{1}{16\pi G} \int \sqrt{-\hat{g}} d^5x dy_g,
\]

\[
\delta S = 0 \rightarrow \hat{G}_{AB} = 0 \equiv \hat{R}_{AB} = 0,
\]

where \( \hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2} \hat{g}_{AB} \hat{R} \) is the Einstein tensor.

For the metric, we identify the \( \alpha\beta \)-part of \( \hat{g}_{AB} \) with \( g_{\alpha\beta} \), the \( \alpha4 \)-part as the electromagnetic potential \( A_{\alpha} \) and \( \hat{g}_{44} \) with a scalar field \( \phi \), parametrizing it as follows:

\[
\hat{g}_{AB}(x,y) = \left( g_{\alpha\beta} + \frac{\kappa^2 \phi^2}{\kappa^2 \phi^2} A_{\alpha} A_{\beta} \right),
\]

where \( \kappa \) is a multiplicative factor. If we identify it in terms of the 4D gravitational constant by \( \kappa = 4\sqrt{\pi G} \), then, using the metric 4 and applying the cylinder condition, the action 3 contains three components:

\[
S = -\int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + \frac{\phi^2 F_{\alpha\beta} F^{\alpha\beta}}{4} + \frac{2\phi' \phi \phi''}{3\kappa^2 \phi^2} \right).
\]

where the 4D gravitational constant \( G \) is defined in terms of its 5D counterpart \( \hat{G} \) by \( G = \hat{G} / \int dy_g \).

From this action, the field equations 3 reduce to the following field equations in terms of 4D quantities[1]:

\[
G_{\alpha\beta} = \frac{\kappa^2 \phi^2}{2} T^{EM}_{\alpha\beta} - \frac{1}{\phi} \left( \nabla_{\alpha} (\partial_{\beta} \phi) - g_{\alpha\beta} \Box \phi \right),
\]

\[
\nabla^\alpha F_{\alpha\beta} = -\frac{3}{\kappa^2 \phi^3} \phi F_{\alpha\beta},
\]

\[
\Box \phi = \frac{\kappa^2 \phi^3}{4} F_{\alpha\beta} F^{\alpha\beta},
\]

where \( T^{EM}_{\alpha\beta} \) is the EM energy-momentum tensor

\[
T^{EM}_{\alpha\beta} \equiv g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}/4 - F_{\alpha}^{\gamma} F_{\gamma\beta},
\]

\( F_{\alpha\beta} \equiv \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}. \)

This ability is the central “miracle” of Kaluza-Klein theory: 4D matter (EM radiation) has arised purely from the geometry of empty 5D space. Two important facts are worth of mention: \( \phi = constant \) and \( A_{\alpha} = 0 \).

For \( \phi = constant \), the field equations are just Einstein and Maxwell equations:

\[
G_{\alpha\beta} = 8\pi G \phi^2 T^{EM}_{\alpha\beta}, \quad \nabla^\alpha F_{\alpha\beta} = 0.
\]
Another interesting point to analyze is the possibility of rescaling the metric and put the gravitational part of the action 5 in canonical form: \( \sqrt{-g}R/16\pi G \). This can be done by making the redefinition \( \phi^2 \to \phi \) and then carrying out the conformal rescaling of the metric
\[
S' = -\int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + \frac{\phi F_{\alpha\beta}^2}{4} + \frac{\partial^\alpha \phi \partial^\beta \phi}{6\kappa^2 \phi^2} \right).
\]
(13)

The Brans-Dicke case is obtained if \( \alpha_0 = 0 \). In terms of the "dilaton" \( \psi \equiv ln(\phi/\sqrt{3\kappa}) \), the action is then that for a minimally coupled scalar field with no potential:
\[
S' = -\int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + \frac{\partial^\alpha \psi \partial^\beta \psi}{2} \right).
\]
(14)

III. COMPACTIFIED THEORIES

To avoid the cylinder condition, Klein assumed that the extra spatial dimension compactifies to the topology of a circle \( S^1 \) of very small radius \( r \). We are then left with a residual 4D coordinate invariance and an abelian gauge invariance associated with transformations of the coordinate of \( S^1 \). Thus, space has topology \( R^4 \times S^1 \).

To see this, consider the transformation
\[
y \to y' = y + f(x),
\]
(15)
\[
\hat{g}_{AB} \to \hat{g}_{AB}' = \frac{\partial x^C}{\partial x^B} \frac{\partial x^D}{\partial x^A} \hat{g}_{CD}.
\]
(16)

Then, for the particular transformation 15,
\[
A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha f(x).
\]
(17)

Thus we see that the transformation 15 induces an abelian gauge transformation on \( A_\alpha \).

To see how \( A_\alpha \) couples to matter, we introduce in the theory the simplest 5D matter, a scalar field \( \psi(x,y) \):
\[
S_\psi = -\int d^4x dy \sqrt{-g} \partial^\alpha \psi \partial^\alpha \psi.
\]
(18)

Because of the \( S^1 \) topology of the fifth dimension, \( \hat{\psi}(x,y+2\pi r) = \hat{\psi}(x,y) \), from which we can expand \( \hat{\psi} \):
\[
\hat{\psi}(x,y) = \sum_{n=\infty}^{n=\infty} \psi^{(n)}(r)e^{in\phi/r}.
\]
(19)

Putting this expansion into the action 18, one finds
\[
S_\psi = -\int dxdy \sqrt{-g} \left[ D^\alpha \psi^{(n)} D_\alpha \psi^{(n)} - \frac{n^2 \psi^{(n)}_r^2}{r^2} \right],
\]
(20)

where \( D_\alpha \equiv \partial^\alpha + \frac{i\omega A^\alpha}{\sqrt{2}} \) is an EM covariant derivative. Comparing this with the usual minimal coupling of the gauge field \( A_\alpha \), we find that the \( n \)th Fourier mode of the scalar field \( \psi \) carries a quantized charge
\[
q_n = n\sqrt{\frac{k^2}{4\pi G}(r \sqrt{\phi})},
\]
(21)

where we have divided \( A_\alpha \) by \( \sqrt{\phi} \) to get the correct EM contribution in 13. If we identify \( \epsilon \equiv q_1 \) and assume that \( r \sqrt{\phi} \sim l_P \approx \sqrt{G} \), we get as a corollary the correct order of magnitude for the fine structure constant,
\[
\alpha \equiv q_1^2/4\pi \sim (\sqrt{16\pi G}/\sqrt{G})^2/4\pi \approx 4,
\]
(22)

which would have made compactified 5D Kaluza-Klein theory very attractive. However, the masses of the scalar modes are not compatible with this.

In fact, from eq. 20, we have
\[
m_n = |n|/(r \sqrt{\phi}),
\]
(23)

which gives for the electron mass \( m_e \sim l_P^{-1} \sim 10^{19}\text{GeV} \), much less than the experimental value \( m_e \approx 0.5 \text{MeV} \).

This problem can be avoided by doing three things\cite{1}: (1) identify observed light particles with \( n = 0 \) modes; (2) invoke the mechanism of spontaneous symmetry breaking to give them masses; (3) go to higher dimensions to solve the problem of \( n = 0 \) modes with nonzero charges, because the particles are no longer singlets of the gauge group corresponding to the ground state.

IV. NONCOMPACTIFIED THEORIES

In this approach, we don’t restrict the topology and scale of the fifth dimension in an attempt to satisfy exactly the cylinder condition. Instead, we stay with the idea that the new coordinates are physical.

The theory is then invariant to general 5D coordinate transformations and we can choose coordinates such that \( A_\alpha = 0 \). The 5D metric tensor \( g_{\alpha\beta} \) is then block diagonal
\[
\hat{g}_{AB}(x,y) = \begin{pmatrix}
g_{\alpha\beta} & 0 \\
0 & \epsilon \phi^2
\end{pmatrix},
\]
(24)

where \( \epsilon \) is a factor which allows a timelike as well as spacelike signature for the fifth dimension \( \epsilon^2 = 1 \).

The procedure to obtain the field equations is exactly the same as in Kaluza’s Mechanism, except that now we keep derivatives with respect to the fifth coordinate. Requiring the usual Einstein 4D equations hold,
\[
8\pi G T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R,
\]
(25)

this procedure produces the 4D field equations 26 to 28:

\[
8\pi G T_{\alpha\beta} = \nabla_\beta (\partial_\alpha \phi) - \frac{\epsilon}{2\phi^2} \left[ \partial_\alpha \phi \partial_\beta \phi - \partial_\beta \phi \partial_\alpha \phi \right] - \frac{\epsilon}{2\phi^2} \left[ g^{\alpha\beta} \partial_\gamma \phi \partial_\gamma \phi - g^{\alpha\beta} \delta_\gamma^\delta \partial_\gamma \phi \partial_\delta \phi \right] - \frac{\epsilon}{2\phi^2} \left[ \delta_\alpha^\gamma \phi \partial_\gamma \phi \partial_\alpha \phi - \delta_\alpha^\gamma \phi \partial_\alpha \phi \partial_\gamma \phi \right]
\]
(26)

\[
\nabla_\beta P_{\alpha\beta} = 0, \quad P_{\alpha\beta} \equiv \frac{1}{2\phi^4} g^{\alpha\beta} \left( g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi - \delta_\alpha^\gamma \phi \partial_\gamma \phi \partial_\delta \phi \right)
\]
(27)

\[
\nabla_\alpha \nabla_\beta \phi = -\frac{\partial g^{\alpha\beta}}{4} \partial_\gamma \phi \partial_\delta \phi + \frac{\delta_\alpha^\gamma \phi \partial_\gamma \phi \partial_\delta \phi}{2}
\]
(28)

The first equations allow us to interpret 4D matter described by \( T_{\alpha\beta} \) as a manifestation of pure geometry in the 5D world, which has been termed the ‘‘induced-matter interpretation’’ of Kaluza-Klein theory.

The interpretation of eqs. 27-28 is not so straightforward. We could conjecture that these equations describe microscopic properties of matter\cite{1}. In particular, if
\[
P_{\alpha\beta} = k(m_i v_i v_\beta + m_\gamma g_{\gamma\beta})
\]
(29)

where \( k \) is a constant, \( m_i \) and \( m_\gamma \) are definitions for the inertial and gravitational mass of a particle in the induced-matter fluid, and \( v_i^2 \equiv dx^i/ds \) is its 4-velocity, then eq. 27 describes the 4D geodesic equation.

Similarly, we can give eq. 28 a deep meaning if we identify it with the Klein-Gordon equation:
\[
\square \phi = m^2 \phi
\]
(30)

The particle mass is then dependent explicitly on the metric, which allows us to interpret this noncompactified theory as a realization of Mach’s principle.
V. COSMOLOGY

Compactified Kaluza-Klein cosmology requires the introduction of 5D matter in order to describe non-EM matter in 4D, being thus characterized by competing expressions for the energy-momentum tensor $T_{\alpha\beta}$ in 5D. By contrast, in noncompactified cosmology one has only the economical assumption that the universe in higher-dimensions is empty, i.e., $T_{\alpha\beta} = 0$. We will thus only review noncompactified cosmology here, in particular the homogeneous and isotropic nonstatic case.

We begin with the general spatially isotropic and homogeneous 5D line element

$$ds^2 = e^\nu dt^2 - e^\omega (dr^2 + r^2 d\Omega^2) + \varepsilon e^\mu d\psi^2 ,$$  

(31)

where $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$, $\varepsilon$ has the same function as in eq. 24, $t$, $r$, $\theta$ and $\phi$ have their usual meanings and $\psi$ is the fifth coordinate. If we consider the case of dependence on $t$ only, this metric is just a generalization of a flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. If we consider also dependence of $\psi$ and assume separable solutions, then we are able to find solutions of the Einstein equations $R_{\alpha\beta} = 0$. One of particular interest[12] is a solution with $\varepsilon = -1$ that can be written in the form

$$ds^2 = \psi^2 dt^2 - t^2 \psi^2 \left( \frac{r^2}{2} + \frac{\alpha^2 t^2}{(1 - \alpha^2)} \right) d\psi^2 ,$$

(32)

and reduces on spacetime sections ($d\psi = 0$) to the usual $k = 0$ (FLRW) metric:

$$ds^2 = dt^2 - R^2(t) \left( dr^2 + r^2 d\Omega^2 \right) .$$

(33)

Assuming this induced matter is a perfect fluid

$$T_{\alpha\beta}^{\text{fluid}} = (\rho + p) u^\alpha u_\beta - p \delta_{\alpha\beta} ,$$

(34)

where $u^\alpha$ is the 4-velocity of the fluid elements, we obtain from eq. 26, $p = T_0^0 + T_1^1 - T_2^2$ and $\rho = -T_2^2$, the following expressions for density and pressure:

$$\rho = \frac{3}{8\pi G a^2 \psi^2 t^2} , \quad p = \left( \frac{2a}{3} - 1 \right) \rho .$$

(35)

These allow us to describe a wide variety of equations of state: for $\alpha = 2$, a radiation-dominated universe; a dust-filled one if $\alpha = 3/2$; one inflationary if $0 < \alpha < 1$. As a final example, consider now a curved ($k \neq 0$) version of the line element in eq. 33:

$$ds^2 = e^\nu dt^2 - e^\omega \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - e^\mu d\psi^2 .$$

(36)

An interesting solution[13] to $R_{\alpha\beta} = 0$ is obtained if we assume the metric coefficients have the form

$$e^\nu \equiv L^2(t - \lambda \psi) , \quad e^\omega \equiv M^2(t - \lambda \psi) , \quad e^\mu \equiv N^2(t - \lambda \psi) ,$$

(37)

where $L$, $M$ and $N$ are wavelike functions of the argument $(t - \lambda \psi)$, with $\lambda$ acting as a wave number. Assuming $L = M = N = 1$, with $\gamma$ a new constant, then one obtains for the perfect fluid:

$$\rho = \frac{3\gamma^2 \lambda^2}{8\pi L^{3\gamma + 1}} , \quad p = \gamma \rho .$$

(38)

With this solution, we can describe either the matter-dominated era ($\gamma = 0$) or the radiation-dominated era ($\gamma = 1/3$). In particular, since the scale factor depends on $\psi$ as well as $t$, observers in hypersurfaces with different values of $\psi$ would disagree on the time elapsed since the big bang, i.e., on the age of the universe.

VI. CONCLUSIONS

Kaluza’s mechanism unifies electromagnetism and gravity simply by letting the indices run over five values. Other interactions can be included by including new extra dimensions.

We have seen that there are three versions of Kaluza-Klein theory, which give different interpretations to the fifth dimension but have similar observational consequences: compactification, projective geometry and noncompactification. In particular, we have seen that the noncompactified approach allows us to describe a wide variety of equations of state.

Some challenges of the theory are: (1) search for new exact solutions; (2) more work on quantization; (3) investigate the physical meaning of the fifth dimension.

By now, no consistent theory has been found, but more and more physicists believe that postulating extra space dimensions is the right way to obtain the unification.