ECONOMIC LOT SCHEDULING

JS, FFS and ELS

<table>
<thead>
<tr>
<th>Job Shop (JS)</th>
<th>Flexible Flow Shop (FFS)</th>
<th>Economic Lot Scheduling (ELS)</th>
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<tbody>
<tr>
<td>- Each job can be different from others</td>
<td>- Limited number of product types</td>
<td>- Large number of identical jobs/items (run)</td>
</tr>
<tr>
<td>- Make to order, low volume</td>
<td>- Make to stock, mass production</td>
<td>- Make to stock for long periods of time</td>
</tr>
<tr>
<td>- Each job has its own sequence</td>
<td>- Movement of jobs controlled by handling system</td>
<td>- Applications to continuous manufacturing (paper, chemistry)</td>
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- Minimizing makespan
- Maximizing throughput
- Minimizing production costs
Economic lot sizing

- Large number of identical jobs
- Continuous manufacturing in months or even years
- Long runs to make-to-stock, implying inventory holding costs
- Setup time and setup costs are significant
- Setup may be sequence dependent

Terminology
- jobs = items
- sequence of identical jobs = run

ELS: Scheduling

- Objective: minimize total cost
  - setup costs
  - inventory holding costs
- Optimal schedule
  - Trade-off between the two objectives
  - Cyclic schedules are used often
- Algorithm
  - Determine the length of the runs (i.e. lot sizes)
  - Determine the order of the runs (i.e. sequence to minimize setup cost)
Economic Lot Scheduling: models

1. **One type of item / one machine**
   - with and without setup time

2. **Several types of items / one machine**
   - rotation schedules
   - arbitrary schedules
   - with / without sequence dependent setup times / cost

3. **Generalizations to multiple machines**

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**One type of item/one machine**

- **Assumptions**
  - 1 type of item $j$
  - 1 machine
  - **Production rate**: $Q_j = 1/p_j$ ($p_j$ = processing time of job $j$)
  - **Demand rate**: $D_j$
  - **Machine capacity/utilization**: $\rho = D/Q$
  - Machine capacity is sufficient to meet demand, i.e. $Q > D$
  - Machine is idle until inventory is depleted
  - Setup costs (but no setup time)

- **Problem**
  - Determine run length (cycle time): $x$
Inventory evolution

- Demand over a cycle = \( Dx \)
- Length of production run needed = \( \frac{Dx}{Q} = \rho x \)
- Maximum inventory level = \( (Q - D)\frac{Dx}{Q} \)

![Graph showing inventory evolution]

Inventory costs

- Setup cost is \( c \) and inventory holding cost per item per unit time is \( h \).
- Average setup cost is \( \frac{c}{x} \)
- Average inventory holding cost: \( \frac{1}{2} h \left( Dx - \frac{D^2x}{Q} \right) \)
- Total cost (inventory + setup): \( \frac{1}{2} h \left( Dx - \frac{D^2x}{Q} \right) + \frac{c}{x} \)
Optimizing cost

- Solve  \[
\min \left[ \frac{1}{2} h \left(Dx - \frac{D^2 x}{Q} \right) + \frac{c}{x} \right]
\]

- Derivative the total cost with respect to \( x \):

\[
\frac{d}{dx} \left\{ \frac{1}{2} h \left(Dx - \frac{D^2 x}{Q} \right) + \frac{c}{x} \right\} = \frac{1}{2} hD \left(1 - \frac{D}{Q}\right) - \frac{c}{x^2}
\]

- Solving the minimization problem:

\[
\frac{1}{2} hD \left(1 - \frac{D}{Q}\right) - \frac{c}{x^2} = 0
\]

Optimal cycle length \( x \)

\[
\frac{1}{2} hD \left(1 - \frac{D}{Q}\right) = \frac{c}{x^2}
\]

\[
x^2 = \frac{2Qc}{hD(Q - D)}
\]

\[
x = \sqrt{\frac{2Qc}{hD(Q - D)}}
\]
Optimal lot size

- The lot size is

\[ D_x = \sqrt{\frac{2DQc}{h(Q - D)}} \]

- If production rate is very high \( (p_j \to 0, Q_j \to \infty) \):

\[ \sqrt{\frac{2DQc}{h(Q - D)}} \xrightarrow{Q \to \infty} \frac{2DQc}{hQ} = \sqrt{\frac{2Dc}{h}} \]

- Economic Lot Size (ELS) or Economic Order Quantity (EOQ):

\[ D_x = \sqrt{\frac{2Dc}{h}} \]

Setup time

- Setup time \( s \)
- Idle time of a machine during a cycle: \( x(1 - D/Q) \)
- If \( s \leq x(1 - \rho) \) solution is still optimal

- Otherwise cycle length \( x = \frac{s}{1 - \rho} \) is optimal, i.e. machine is never idle.
Example 7.2.1

- No setup time
- Production $Q = 90/\text{week}$
- Demand $D = 50/\text{week}$
- Setup cost $c = 2000\text{€}$
- Holding cost $h = 20 \text{€/item}$

$$x = \sqrt{\frac{2Qc}{hD(Q-D)}}$$

$$x = \sqrt{\frac{2 \times 90 \times 2000}{20 \times 50 \times (90 - 50)}} = \sqrt{\frac{3600}{10 \times 40}} = \sqrt{\frac{36}{4}} = 3$$

Optimal schedule

- Cycle time = 3 weeks
- Lot size = $Dx = 150$ items
- Idle time = $3(1 - 5/9) = 1.33$ weeks
Example with setup times

- Now assume setup time
- If \( s < 1.33 \) weeks (about 9 days) then 3 weeks cycle is still optimal
- Otherwise the cycle time must be: \( x = \frac{s}{1 - \rho} \)
- If setup last 2 weeks (maintenance and cleaning):
  - \( x = \frac{2}{1 - \frac{5}{9}} = 4.5 \) weeks

Example 7.2.2

- \( Q = 0.3333 \), \( D = 0.10 \), \( c = 90€ \), \( h = 5€ \)
- Determine \( x \), lot size

\[
x = \sqrt{\frac{60}{0.5(0.3333 - 0.1)}} = 22.678
\]

- Lot size: \( Dx = 2.2678 \).

- What happens in a discrete setting?
Example 7.2.2 (discrete)

- Time to produce one item is $p = 1/Q = 3$ days.
- Demand rate is 1 item every 10 days.
- Lot size of $k$ has to be produced every $10k$ days.
  - Total cost per day of lot size of 1 every 10 days is $90/10 = 9$
  - Total cost of lot size of 2 every 20 days is: $(90 + 7 \times 5)/20 = 6.25$
  - Total cost of lot size of 3 every 30 days is: $(90 + 7 \times 5 + 14 \times 5)/30 = 6.5$
- So the optimal is to produce every 20 days a lot of 2.

Multiple items and rotation schedules

- **Assumptions**
  - $n$ different items and one machine
  - Production rate for item $j$ is $Q_j = 1/p_j$
  - Demand rate for item $j$ is $D_j$
  - Setup cost per item $s_j$ independent of the sequence
  - Length of production run of item $j$ is $D_j x/Q_j$

- **Problem**
  - Determine the best rotation schedule that contains a single run of each item
  - Cycle length $x$ must be identical to all items
  - The order of sequence does not matter
Inventory and costs

- Cycle length determines the run length for each item
- Length of production run needed: $D_j x / Q_j$
- Average inventory level of item $j$:
  \[
  \frac{1}{2} \left( D_j x - \frac{D_j^2 x}{Q_j} \right)
  \]
- With cost $c_j$, the total average cost per unit time is
  \[
  \sum_{j=1}^{n} \left( \frac{1}{2} h_j \left( D_j x - \frac{D_j^2 x}{Q_j} \right) + \frac{c_j}{x} \right)
  \]

Optimal cycle length

- Solving as in the previous case:
  \[
  x = \sqrt{\left( \sum_{j=1}^{n} h_j D_j (Q_j - D_j) / 2Q_j \right)^{-1} \sum_{j=1}^{n} c_j}
  \]
- Machine idle time during a cycle:
  \[
  x \left( 1 - \sum_{j=1}^{n} \frac{D_j}{Q_j} \right)
  \]
- With production capabilities unlimited ($Q_j \to \infty$):
  \[
  x = \sqrt{\left( \sum_{j=1}^{n} h_j D_j / 2 \right)^{-1} \sum_{j=1}^{n} c_j}
  \]
Example 7.3.1

- Production rates, demand rates, holding costs and setup costs

<table>
<thead>
<tr>
<th>items</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_j$</td>
<td>50</td>
<td>50</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>$Q_j$</td>
<td>400</td>
<td>400</td>
<td>500</td>
<td>400</td>
</tr>
<tr>
<td>$h_j$</td>
<td>20</td>
<td>20</td>
<td>30</td>
<td>70</td>
</tr>
<tr>
<td>$c_j$</td>
<td>2000</td>
<td>2500</td>
<td>800</td>
<td>0</td>
</tr>
</tbody>
</table>

Optimal cycle length

\[ x = \sqrt{\left( \sum_{j=1}^{n} h_j D_j (Q_j - D_j) / 2Q_j \right)^{-1} \sum_{j=1}^{n} c_j} \]

\[ = \sqrt{\left( \frac{2 \times 10 \times 350}{8} + \frac{18 \times 440}{10} + \frac{42 \times 340}{8} \right)^{-1}} 5300 \]

\[ = \sqrt{\left( \frac{10 \times 350}{4} + \frac{18 \times 440}{10} + \frac{42 \times 340}{8} \right)^{-1}} 5300 \]

\[ = \sqrt{(3452)^{-1}} 5300 = \sqrt{1.5353} = 1.24 \text{ months} \]
Solution

- Idle time is 0.48 \times = 0.595 \text{ months}.
- The total average cost per time unit is:

\[
\sum_{j=1}^{n} \left( \frac{1}{2} h_j \left( D_j x - \frac{D_j^2 x}{Q_j} \right) + \frac{c_j}{x} \right) = 2155 + 2559 + 1627 + 2213 = 8554
\]

With setup times

- With sequence independent setup costs and \textbf{no setup times} the sequence within each lot does not matter
  \( \Rightarrow \) Only a lot sizing problem

- Even with setup times, if they are not job dependent then still only lot sizing
Sequence independent setup times

- If sum of setup times < idle time then our optimal cycle length remains optimal.
- Otherwise we take it as small as possible.
- By increasing $x$ until setup time is equal to idle time:

$$\sum_{j=1}^{n} s_j = x \left(1 - \sum_{j=1}^{n} \rho_j\right)$$

- The optimal $x^*$ is given by:

$$x^* = \frac{\left(\sum_{j=1}^{n} s_j\right)}{\left(1 - \sum_{j=1}^{n} \rho_j\right)}$$

Sequence dependent setup times

- Now there is a sequencing problem.
- **Objective:** minimize sum of setup times.
- Problem is NP-hard.
  - If the sequence solution has $\sum$ setup times < idle time:
    - optimal lot size and sequence are optimal.
  - Else:
    - Optimal cycle length has to be larger.
Rotation schedules with parallel machines

- $m$ identical machines in parallel
- There are setup cost but no setup time
- Item process on only one of the $m$ machines
- For item $j$, utilization factor is again $\rho_j = D_j/Q_j$.
- Condition for a feasible solution is:
  \[ \sum_{j=1}^{n} \rho_j \leq m \]

Decision variables

- Assume
  - rotation schedule
  - equal cycle for all machines
- Same as previous multi-item problem
- Addition: assignment of items to machines

- **Objective**: balance the load
  - Use heuristic LPT with $\rho_j$ as processing times.
Different cycle lengths

- Allow different cycle lengths for machines
- Intuition: should be able to reduce cost

**Objective:** assign items to machines to balance the load
- Complication: should not assign items that favor short cycle to the same machine as items that favor long cycle.

Heuristic balancing

- Compute cycle length for each item
- Rank in decreasing order
- Allocate jobs sequentially to the machines until capacity of each machine is reached
- Adjust balance if necessary
Rotation schedules with machines in series

Flow shop
- Machines configured in series
- Assume no setup time
- Assume production rate of each item is identical for every machine
  - Can be synchronized
  - *Problem is reduced to single machine problem* with setup cost:
    \[ c_j = \sum_{i=1}^{m} c_{ij} \]

Variable production rates
- Production rate for each item *not* equal for every machine
- Difficult problem
- Little research
- **Flexible flow shop**: need even more stringent conditions
Discussion

- Lot sizing models
  - demand assumed known, which determines throughput
  - make-to-stock systems: due date of little
    importance/not available
  - Objective: minimize inventory and setup costs (time).
- Practical problems are a combination of make-to-stock
  and make-to-order.
  - In these problems facilities are set up in series.
  - This area of research is known as:
    **Supply Chain Management**