Quantum Game Theory: Within the Realm of Quantum Strategies

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This article was written for this year's seminars of *Introdução à Investigação*. Its general subject is the quantization of Game Theory and the results which ensue. Firstly, a brief revision of the terminology of Game Theory is given. Afterwards, Meyer's quantization of the Penny Flip is used as a motivation for quantizing the classical games. In a final section, Eisert's scheme of quantization is applied to the Prisoner's Dilemma and an inquiry is made whether or not the quantization solves the dilemma present in this game.

I. INTRODUCTION

Since it was first formalized by John von Neumann and Oskar Morgenstern[1], and further developed by John Forbes Nash Jr.[2], Game Theory has matured as a discipline to become a full theory, used in areas such as economics and finance, biology and even engineering. In his widely renowned book, *Game Theory: Analysis of Conflict*, Roger B. Myerson defined Game Theory as "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers", the mathematics of decision that allows us a quantitative understanding of "situations in which two or more individuals make decisions that will influence one another's welfare"[3].

From childish games such as Rock-Paper-Scissors or Tic-Tac-Toe, to auctions in art galleries, it's almost certain we have played a game at least once in our lives. In trying to analyze these kind of games mathematically, game theorists came up with a specific terminology for the elements involved in them. The agents who play are suitably called the players, and their course of action, a prescription for what possible move to make in any given situation of the game, their strategies. Each outcome in a game earns each player a payoff, a numerical measure of the desirability of that outcome for the player; the payoffs for all outcomes are usually shown in the form of a payoff matrix. It is assumed all the players seek to maximize their final payoff when choosing between strategies. Let us now define further concepts of interest to us through an example of a famous game in Game Theory: the Prisoner's Dilemma.

In the Prisoner's Dilemma two prisoners are being interrogated: each of them can either cooperate (C) and stay shut, or defect (D) and tell the cops it was the other who committed the felony. If they cooperate with each other they both stay in jail for a short while, because the policemen do not have enough information to fully incarcerate them; if one cooperates and the other defects, he who defects is set free while the other is incarcerated; if they both turn at each other, i.e. if they both defect, they will both be imprisoned. The payoff matrix is then as shown in Table 1, with the payoffs chosen according to Reference [4].

	Bob: C	Bob: D
Alice: C	(3,3)	(0,5)
Alice: D	(5,0)	(1,1)

TABLE I. The Prisoner's Dilemma payoff matrix

One concept that ensues is that of dominant strategies. In this game the dominant strategy is to defect, since it gives the player a payoff of 5 against 3, or of 1 against 0, depending on the strategy of the opponent. When both players employ the dominant strategy, $\{D, D\}$, the result is called a Nash Equilibrium (NE). The NE is then the combination of strategies from which no player can improve his payoff by a unilateral change of strategy, and it is the most important of equilibria in Game Theory. Another is the Pareto optimal outcome, which is one from which no player can improve his payoff without reducing that of his adversary. In this game the Pareto optimal outcome is the combination of strategies $\{C, C\}$. One is now able to see where the dilemma that gives the name to this game comes from: there's is a conflict between choosing the dominant strategy that leads to the maximum payoff and to the NE, and choosing the strategy that leads to the Pareto optimal outcome that gives both players a better payoff that the NE.

It is at this stage of the theory that is interesting to pose the following questions: Is it possible to solve this dilemma? Will the quantization of games be able to do it? We'll see.

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II. QUANTUM GAMES

With the advent of Quantum Mechanics, many were the areas that broadened their scope by applying the quantization scheme: quantizing the elements in an area of knowledge always seems like the natural 'next step' to take. So, after the establishment of Game Theory, the creation of a Quantum Game Theory was foreseeable. And it came in 1999, due to the seminal work of David A. Meyer, *Quantum Strategies*[5]. Later in that year Eisert *et al.* would define the quantization method for games that is still used today[6].

A. Motivation for Quantization: The Quantum Penny Flip

The Quantum Penny Flip was introduced by Meyer as a motivation to the quantization of games and the interesting results that follow.

In the Classical Penny Flip there are two players, Alice and Bob. Alice puts a penny in a box, heads up. The box is then given to Bob: without knowing Alice's move, he may either leave the penny identical, or flip it. It is then Alice's turn to flip the penny or not, which is itself followed by one other move by Bob. Alice wins if the penny is tails up; Bob wins otherwise.

	Bob: I, I	Bob: I, F	Bob: F, I	Bob: F, F
Alice: I	-1	1	1	-1
Alice: F	1	-1	-1	1

TABLE II. Alice's payoff matrix in the Penny Flip

Table 2 shows Alice's payoff for every possible combination of strategies. In this game there is always a winner and a loser, hence the possibility to represent it in a matrix with the payoff of just one player. This symmetry in the game makes it what is called a zero-sum game. It is worth mentioning it is also a fair game: each player has the same probability of winning.

Let us now see what happens in the quantum version of the game. We begin by associating the two states in our system, heads and tails, with two basis vectors:

$$\begin{aligned} \text{Heads}: \ |0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{Tails}: \ |1\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{1}$$

Allowing the players superposition, the possible strategies they may employ become the set of 2×2 unitary matrices. We go one step further and constrain Alice to be a 'classical' player: she may either use the identity matrix, or the bit flip matrix.

The game starts in the state $|0\rangle$. It is then Bob's first turn on the penny. He may flip it or leave it, or he may as well apply the Hadamard operator:

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

leaving it in a mixed state of heads and tails:

$$|\psi_B\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \tag{3}$$

Remembering Alice is restricted to use the identity or the flip matrices, a novelty appears in the quantum version of this game: her strategy is irrelevant, since the flip operator merely interchanges the two basis states in $|\psi_B\rangle$:

$$\hat{I} \left[\frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \right] = \hat{F} \left[\frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \right] = |\psi_B\rangle \tag{4}$$

And Bob has yet the final move to make. He is thus able to exploit the irrelevance in Alice's choice of strategy and play $\hat{H}$ again, therefore winning the game:

$$\hat{H} |\psi_B\rangle = |0\rangle \tag{5}$$

This example serves to show the advantage a quantum player has against his heavily constrained classical counterpart, to the extent of Bob's ability to win the game at will. Quantum strategies outperform classical ones, as their larger space of possible moves clearly indicates.

Still the Penny Flip is a sequential game, the players act on the state the other just acted on, and as such it doesn't allow for entanglement. Taking the Prisoner's Dilemma, already mentioned in Section I, Eisert *et al.* established the method to 'fully quantize' a game when the players' moves are simultaneous and entanglement may be present.

B. The Quantum Prisoner's Dilemma

The quantization of the Prisoner's Dilemma begins, similar to the procedure of the previous section, with the association of the states of the game to basis vectors. In this case, Eisert *et al.* associated Cooperation and Defection, to the vectors $|0\rangle$ and $|1\rangle$ of Equation (1), respectively. The next step is to define the initial state: the authors chose it to be the state $|00\rangle$ ($|0\rangle_A \otimes |0\rangle_B$). This way, if a player wants to cooperate he uses the identity matrix, and if he wants to defect he flips the initial state. The full set of quantum strategies the players may use are the unitary matrices of the form:

$$\hat{U}(\theta,\alpha,\beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix} \quad (6)$$

with $\theta \in [0,\pi]$, and $\alpha,\beta \in [-\pi,\pi]$. The strategies $\hat{U}(\theta,0,0)$ are easily recognized as mixed strategies of the classical moves: Cooperation being $\hat{I} = \hat{U}(0,0,0)$, and Defection corresponding to the flip operator $\hat{F} = \hat{U}(\pi,0,0) = i\sigma_x$.

Due to the fact that the prisoners play simultaneously, we can define an entanglement operator:

$$\hat{J}(\gamma) = \exp\left\{i\frac{\gamma}{2}\sigma_x \otimes \sigma_x\right\} \quad (7)$$

with $\gamma \in [0,\pi/2]$, and this allows the players now to act upon a state $\hat{J}|00\rangle$, that ranges from the separable game:

$$\hat{J}(0)|00\rangle = \hat{I}|00\rangle = |00\rangle \quad (8)$$

to the maximally entangled game:

$$\hat{J}(\pi/2)|00\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + i|11\rangle\right) \quad (9)$$

depending on the choice of the parameter γ. It is relevant to state that this definition of entanglement operator also maintains the classical game as a subgroup of the quantum one[6][7], as we shall see in the graphical representations of the game.

All these ingredients combine to give a final state, after the two prisoners' choice of strategy:

$$|\psi_f\rangle = \hat{J}^\dagger\left(\hat{U}_A \otimes \hat{U}_B\right)\hat{J}|00\rangle \quad (10)$$

The presence of $\hat{J}^\dagger$ as a disentanglement gate is necessary so we may evaluate the payoff expectation value using the payoffs of the classical game given in Table 1:

$$\begin{aligned}\langle\$\rangle = &P_{00}|\langle\psi_f|00\rangle|^2 + P_{01}|\langle\psi_f|01\rangle|^2 \\ &+ P_{10}|\langle\psi_f|10\rangle|^2 + P_{11}|\langle\psi_f|11\rangle|^2\end{aligned} \quad (11)$$

Let us now check what are the novel results in the Quantum Prisoner's Dilemma. In the maximally entangled state, $\gamma = \pi/2$, if we restrict Alice's strategies to $\hat{U}(\theta,0,0)$ and allow Bob the full range of strategies we arrive at a result very similar to that of the Quantum Penny Flip. If Bob plays the "miracle move":

$$\hat{U}(\pi/2,\pi/2,0) = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix} \quad (12)$$

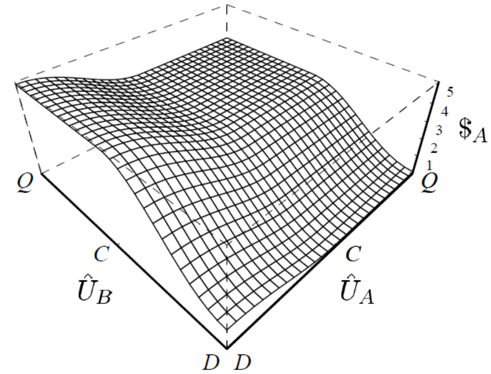

FIG. 1. Alice's payoff in a separable game

a version of the Hadamard operator, he ensures himself a payoff expectation value of:

$$\langle\$_B\rangle = 3 + 2\sin\theta \quad (13)$$

while Alice is left with only:

$$\langle\$_A\rangle = \frac{1}{2}(1-\sin\theta) \quad (14)$$

We have thus acknowledged once again the superiority of the quantum strategies. Remembering our intent of solving the dilemma in this game, one may say we have done it in favour of the quantum player. Yet one shouldn't be pleased with this, given we had to weaken one of the players to solve the dilemma. It should be done in a clear mirror of the classical game, with the two prisoners playing strategies equal in strength.

We picture now a fair game, where we allow both players strategies of the form $\hat{U}(\theta,\alpha,0)$. We can represent the game using Alice's payoff expectation value for every pair of strategies, since the game is symmetric in payoffs. In the case of the separable game, $\gamma = 0$, the result is as shown in Figure 1. The classical game lies within the first quadrant, where we can see the NE $\hat{D}\otimes\hat{D}$ and the pareto optimal outcome $\hat{C}\otimes\hat{C}$. Still the separable quantum game is not as interesting, since we can observe it isn't that different from the classical game: the dominant strategy is still to defect, $\hat{D}\otimes\hat{D}$ remains the only NE and the dilemma abides. That is not the case of the maximally entangled game, $\gamma = \pi/2$, represented in Figure 2. Here Defection is no longer the only strategy that allows for a maximal payoff, there is also a quantum strategy that we may call $\hat{Q}$:

$$Q = \hat{U}(0,\pi/2,0) = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} \quad (15)$$

In fact, one can see this strategy destroys the NE in $\hat{D}\otimes\hat{D}$: a player may improve his, or her, payoff by a unilateral

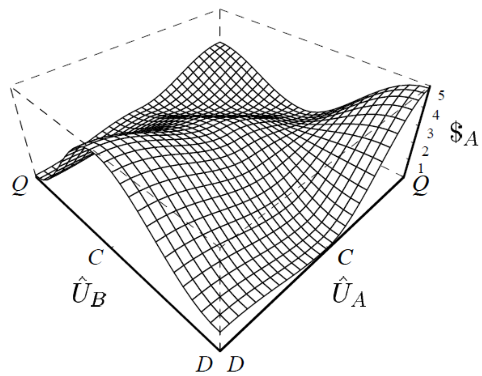

FIG. 2. Alice's payoff in the maximally entangled game

change of strategy, by playing strategy $\hat{Q}$. Looking further at the equilibrium in $\hat{Q} \otimes \hat{Q}$, which gives a combined payoff of $\{3, 3\}$, one realizes it is the new NE and it is also a Pareto optimal outcome: the dilemma have thus been solved in a fair manner.

One last remark prompts itself to be made: every strategy here considered had the parameter β forced to be zero. It is obviously arguable there is no *a priori* reason to constrain the space of quantum operators to those with $\beta = 0$. Yet it has been shown that using the full set of strategies, $\hat{U}(\theta, \alpha, \beta)$, there are no NE[4][6][8]. Although this scenario does not suffice itself as a justification, it shows there is something more to the usual repression of β after all. Needless to say this remains a subject of research to this date.

III. CONCLUSIONS

We have considered Quantum Game Theory as the natural extension of the well established area of Game Theory. After a brief introduction to the basic concepts in games, we revisited briefly the Prisoner's Dilemma and the Penny Flip.

Through the quantization of the Penny Flip it was possible to realize that the quantum strategies outperform their classical counterparts. This was again observed in the Quantum Prisoner's Dilemma.

Associated with entanglement, quantum strategies are able to solve the dilemma the prisoners face between choosing to cooperate or defect.

It must be said, however, that humans are not quantum players: we can't employ quantum strategies. One could never leave a penny in a mix of heads and tails, for example. Still, quantum games may be played by quantum computers, and this has, in fact, already been realized experimentally[9].

On a final note, one should mention that Quantum Game Theory hasn't gone by without notice. The realm of quantum strategies and all the possibilities that lie therein are powerful enough that many authors have used it in trying to solve some of the problems in economics. Namely, it has been applied, by Piotrowski and Sladkowski since 2002, to the so called quantum markets[10][11], and in 2010 Hanauske *et al.* made a highly relevant "evolutionary quantum game theory based analysis of financial crises"[12].

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