Airfoils

Kármán-Treftz mapping

$$z = kb\frac{(\zeta+b)^k+(\zeta-b)^k}{(\zeta+b)^k-(\zeta-b)^k} \quad \Leftrightarrow \quad \frac{z-kb}{z+kb}=\left(\frac{\zeta-b}{\zeta+b}\right)^k$$

- The exponent *k* defines the internal angle of the trailing edge, τ, from

$$\tau = (2-k)\pi \quad \Leftrightarrow \quad k = 2-\frac{\tau}{\pi}$$

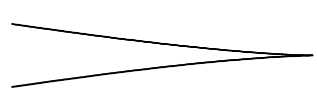
- *k=2* corresponds to the Joukowski mapping

Airfoils

Kármán-Treftz mapping

$$z = kb\frac{(\zeta+b)^k+(\zeta-b)^k}{(\zeta+b)^k-(\zeta-b)^k} \quad \Leftrightarrow \quad \frac{z-kb}{z+kb}=\left(\frac{\zeta-b}{\zeta+b}\right)^k$$

- *k=2* corresponds to a dividing streamline that leaves the trailing edge with tangential continuity. The trailing edge is not a stagnation point



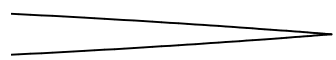
$\tau = 0$

Airfoils

Kármán-Treftz mapping

$$z = kb\frac{(\zeta+b)^k + (\zeta-b)^k}{(\zeta+b)^k - (\zeta-b)^k} \Leftrightarrow \frac{z-kb}{z+kb} = \left(\frac{\zeta-b}{\zeta+b}\right)^k$$

- *k=1,95* corresponds to a dividing streamline that forms a dihedral at the trailing edge with an angle smaller than π. The trailing edge is a stagnation point



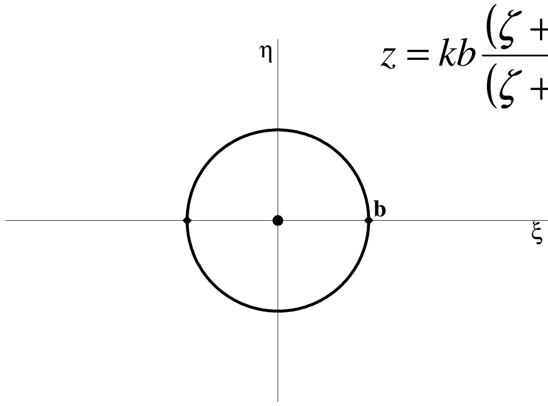
$\tau = 9^o$

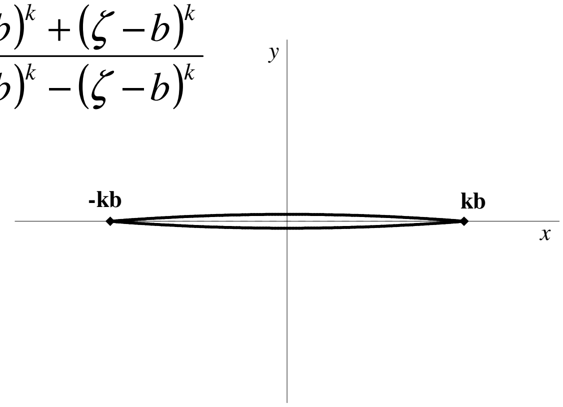
Airfoils

Kármán-Treftz mapping

1. Cylinder centre at the origin

$$z = kb\frac{(\zeta+b)^k + (\zeta-b)^k}{(\zeta+b)^k - (\zeta-b)^k}$$





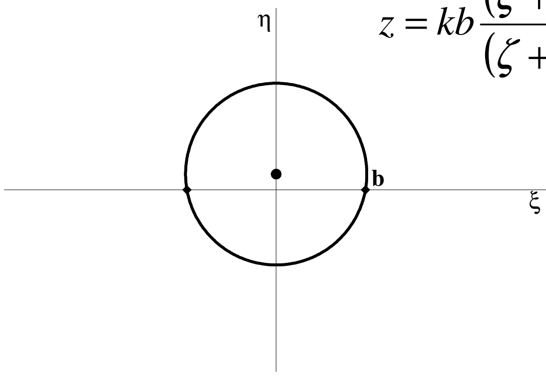
$k = 1{,}95, \quad c = 3{,}9b$

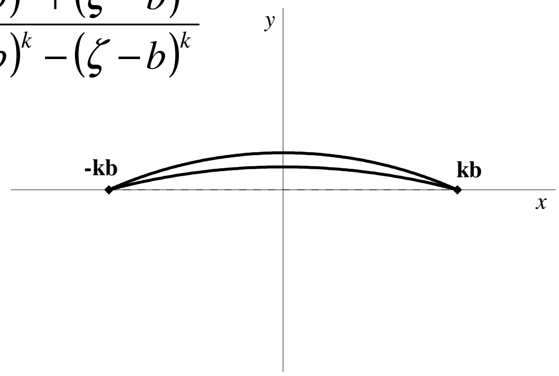
Airfoils

Kármán-Treftz mapping

2. Cylinder centre on the imaginary axis

$$z = kb\frac{(\zeta+b)^k + (\zeta-b)^k}{(\zeta+b)^k - (\zeta-b)^k}$$





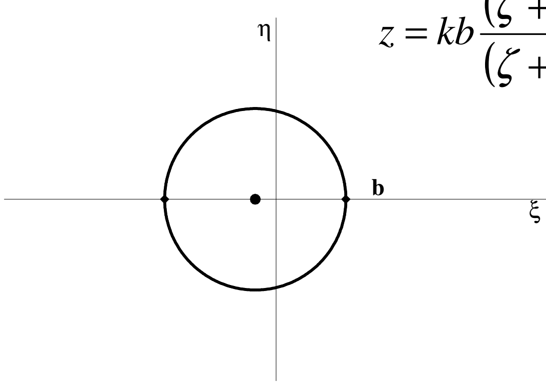
$$k = 1{,}95, \quad c = 3{,}9b$$

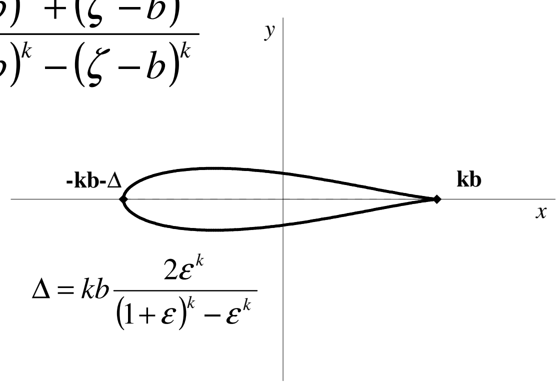
Airfoils

Kármán-Treftz mapping

3. Cylinder centre on the real negative axis

$$z = kb\frac{(\zeta+b)^k + (\zeta-b)^k}{(\zeta+b)^k - (\zeta-b)^k}$$





$$\Delta = kb\frac{2\varepsilon^k}{(1+\varepsilon)^k - \varepsilon^k}$$

$$k = 1{,}95, \; c = 3{,}9b + \Delta$$

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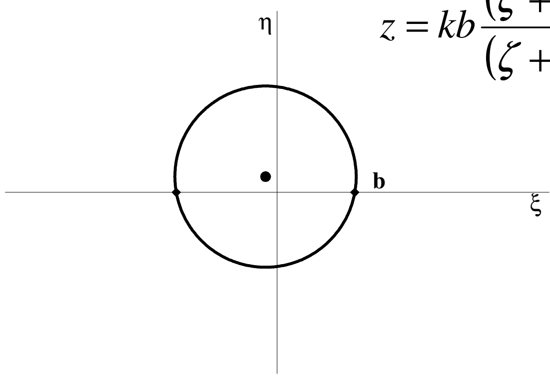
Aerodynamics

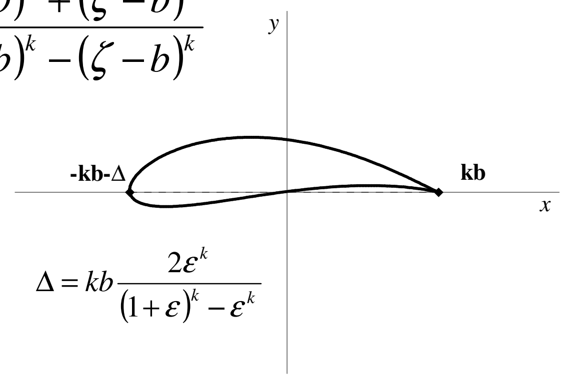
Airfoils

Kármán-Treftz mapping

4. Cylinder centre on the 2nd quadrant

$$z = kb\frac{(\zeta+b)^k+(\zeta-b)^k}{(\zeta+b)^k-(\zeta-b)^k}$$





$$\Delta = kb\frac{2\varepsilon^k}{(1+\varepsilon)^k-\varepsilon^k}$$

$$k = 1{,}95,\ c = 3{,}9b + \Delta$$

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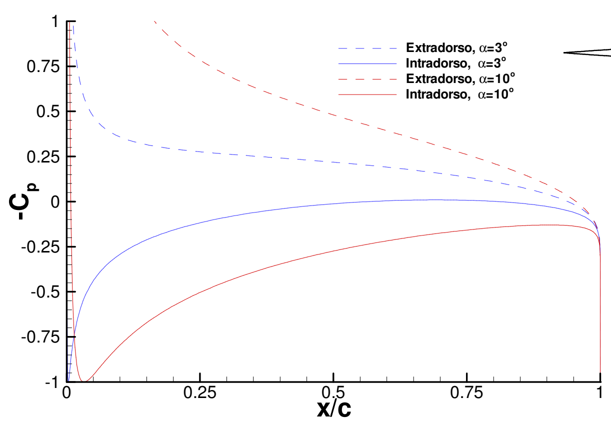
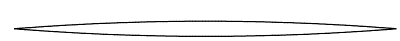
Aerodynamics

Airfoils

Kármán-Treftz mapping

1. Cylinder centre at the origin

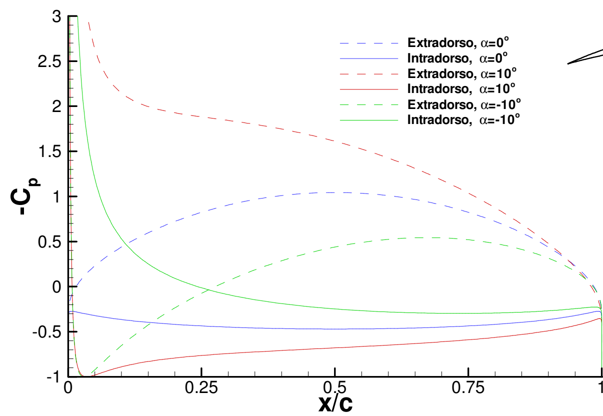


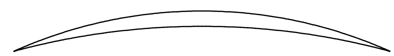


$$C_p = \frac{p-p_\infty}{\frac{1}{2}\rho|\vec{V}_\infty|^2}$$

Airfoils
Kármán-Treftz mapping

2. Cylinder centre on the imaginary axis





$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho\left|\vec{V}_\infty\right|^2}$$

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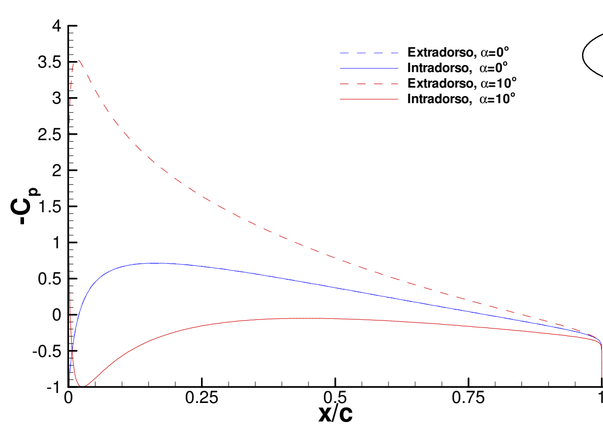
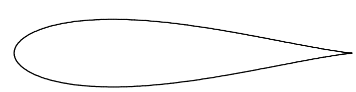
Aerodynamics

Airfoils
Kármán-Treftz mapping

3. Cylinder centre on the real negative axis





$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho\left|\vec{V}_\infty\right|^2}$$

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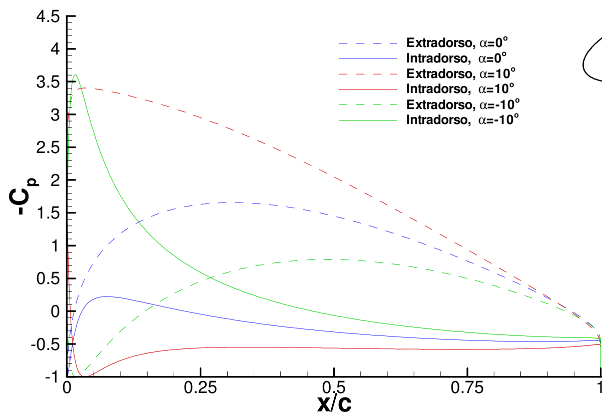
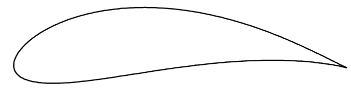
Airfoils
Kármán-Treftz mapping

4. Cylinder centre on the 2nd quadrant





$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho\left|\vec{V}_\infty\right|^2}$$

Airfoils
Generalization of the conformal mapping

- The Joukowski transform is one of the mappings defined by

$$z = \zeta + \sum_{n=1} \frac{a_n}{\zeta^n}$$

- For the Joukowski case:

$$a_1 = b^2 \wedge a_n = 0 \text{ para } n \geq 2$$

Airfoils

Generalization of the conformal mapping

$$z = \zeta + \sum_{n=1} \frac{a_n}{\zeta^n}$$

- In general, the coefficients a_n are complex
- Theoretically, the generalized mapping is able to obtain any airfoil from a cicular cylinder

Airfoils

Generalization of the conformal mapping

$$z = \zeta + \sum_{n=1} \frac{a_n}{\zeta^n}$$

- At small angles of attack, the lift coefficient, C_l, of an airfoil in steady, irrotacional and incompressible flow is given by

$$C_l = 2\pi\left(1 + a_t\left(\frac{d}{c}\right)\right)(\alpha + \beta)$$

- For a Joukowski airfoil

$$a_t \cong 0{,}77$$

Airfoils

Pitching moment relative to the airfoil centre

$$M_0 = -\rho \frac{Q\Gamma}{2\pi} + \Re\left[-\mathrm{i}\, 2\pi\rho U_\infty \mathrm{M} e^{-\mathrm{i}\alpha}\right]$$

- For an airfoil *Q=0*

$$M_0 = \Re\left[-\mathrm{i}\, 2\pi\rho U_\infty \mathrm{M} e^{-\mathrm{i}\alpha}\right]$$

- $\mathcal{M}$ is the term proportional to *z-2* of the complex velocity at large distances from the airfoil
- Assuming that the a_1 coefficient is given by

$$a_1 = b^2 e^{-\mathrm{i}2\lambda}$$

$$\mathrm{M} = a_1 \left|\vec{V}_\infty\right| e^{-\mathrm{i}\alpha} - a^2 \left|\vec{V}_\infty\right| e^{\mathrm{i}\alpha}$$

Airfoils

Pitching moment relative to the airfoil centre

- For small values of α

$$M_c \cong -4\pi\rho b^2 \left|\vec{V}_\infty\right|^2 (\alpha + \lambda)$$

- Assuming a chord $c \cong 4b$

$$C_{M_c} = \frac{M_c}{\frac{1}{2}\rho\left|\vec{V}_\infty\right|^2 c^2} \cong -\frac{\pi}{2}(\alpha + \lambda)$$

Airfoils
NACA airfoils

- The development of the NACA airfoils started in 1933 at the National Advisory Committee for Aeronautics (NACA), designated National Aeronautics and Space Administration (NASA) in nowadays

- NACA airfoils are obtained from the addition of a thickness distribution to a camber line. Several series of airfoils have been developed along the years

Airfoils
NACA airfoils

- 4 digits series, NACA ABCD

  A → Maximum relative camber in percentage, f/c

  B → Coordinate of maximum camber, x_m, given by $10x_m/c$

  CD → Maximum relative thickness in percentage, d/c

Airfoils
NACA airfoils

- 4 digits series, NACA ABCD

- Thickness distribution

$$y=\frac{d}{0,2}\left(0,2969\sqrt{x}-0,126x-0,3516x^2+0,2843x^3-0.1015x^4\right)$$

- Camber lines

2 parabola arcs matching at the point of maximum camber, located at x_m

Aerodynamics

Airfoils
NACA airfoils

- 4 digits series, NACA ABCD

- Thickness distribution

$$y=\frac{d}{0,2}\left(0,2969\sqrt{x}-0,126x-0,3516x^2+0,2843x^3-0.1015x^4\right)$$

- Camber lines

$$y_m=\begin{cases}\dfrac{f}{x_m^2}\,x(2x_m-x) & \Leftarrow x\le x_m \\ \dfrac{f}{(1-x_m)^2}(1-x_m)(1+x-2x_m) & \Leftarrow x> x_m\end{cases}\qquad f=\frac{A}{100}\qquad x_m=\frac{B}{10}$$

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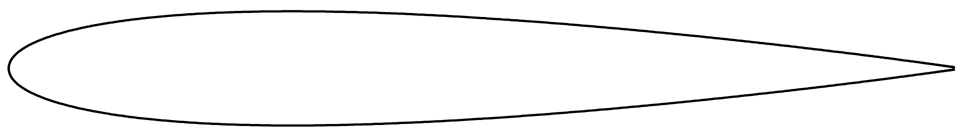
Airfoils
NACA airfoils

- 4 digits series

- NACA 0012

Symmetric airfoil with 12% of maximum relative thickness

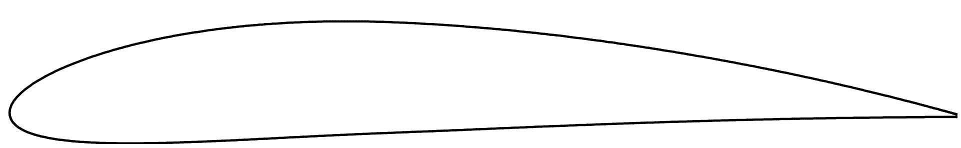


Airfoils
NACA airfoils

- 4 digits series

- NACA 4412

Airfoil with maximum relative camber of 4% located at $x_m=0.4c$ (40% from the leading edge) and maximum relative thickness of 12%



Airfoils
NACA airfoils

- 5 digits series, NACA ABCDE

A → Approximate value of $\frac{2}{3}\times 10(C_l)_{proj}$

BC → Coordinate of maximum camber, x_m, given by $200x_m/c$

DE → Maximum relative thickness in percentage, d/c

$(C_l)_{proj}$ is the project lift coefficient

Airfoils
NACA airfoils

- 5 digits series, NACA ABCDE

- Thickness distribution identical to 4 digits series

$$y = \frac{d}{0,2}\left(0,2969\sqrt{x} - 0,126x - 0,3516x^2 + 0,2843x^3 - 0.1015x^4\right)$$

- Camber line

2 polynomials with decreasing curvature from starting at the leading edge. Straight line to the right of the maximum camber point, located at x_m

Airfoils

NACA airfoils

- 5 digits series, NACA ABCDE

- Thickness distribution identical to 4 digits series

$$y = \frac{d}{0,2}\left(0,2969\sqrt{x} - 0,126x - 0,3516x^2 + 0,2843x^3 - 0.1015x^4\right)$$

- Camber line

$$y_m = \begin{cases} \frac{1}{6}k_1\left(x^3 - 3mx^2 + m^2(3-m)x\right) & \Leftarrow x \le m \\ \frac{1}{6}k_1 m^3(1-x) & \Leftarrow x_m > m \end{cases}$$

Airfoils

NACA airfoils

- 5 digits series, NACA ABCDE

- Camber line
  Values of k_1, m e x_m for $(C_l)_{proj} = 0,3$

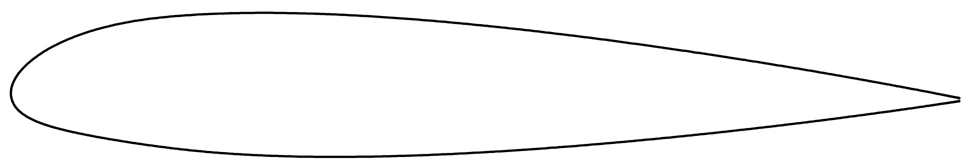
Designação	x_m	m	k_1
210	0,05	0,0580	361,4
220	0,10	0,1260	51,64
230	0,15	0,2025	15,957
240	0,20	0.2900	6,643
250	0,25	0,3910	3,230

Airfoils
NACA airfoils

- 5 digits series

- NACA 23015

Airfoil with $(C_l)_{proj} = 0{,}3$, maximum camber located at x_m=*0.15c* (15% from the leading edge) and 15% of maximum relative thickness



Aerodynamics

Airfoils
NACA airfoils

- 6 digits series, NACA 6A,B-CDE, $a=a_o$

A → Value of $x_p/10c$. x_p is the horizontal coordinate of the suction peak of the corresponding symmetric foil at zero degrees angle of attack. x_p is related to the thickness distribution of the airfoil

Airfoils

NACA airfoils

• 6 digits series, NACA 6A,B-CDE, $a=a_o$

B → Range of C_l values (multiplied by 10) above and below the value of C_l for project conditions, which correspond to favourable or close to zero pressure gradients on the upper and lower sides of the airfoil.

C → $10(C_l)_{proj}$. $(C_l)_{proj}$ is the lift coefficient for project conditions

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Aerodynamics

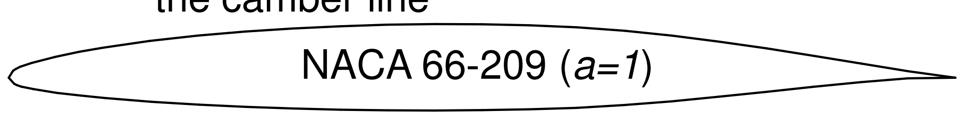
Airfoils

NACA airfoils

• 6 digits series, NACA 6A,B-CDE, $a=a_o$

DE → Maximum relative thickness in percentage

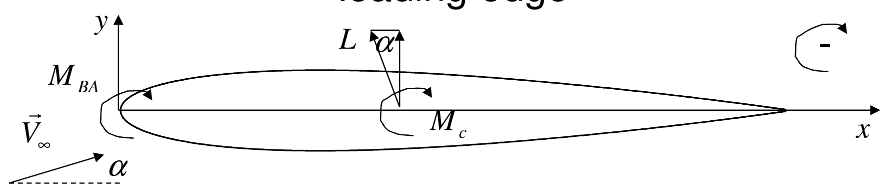
a_o → Maximum horizontal coordinate of the region that presents an approximately constant loading (pressure difference between upper and lower sides of the airfoil). For $x>a_o$ the loading decreases linearly. a_o is related to the camber line



Masters of Mechanical Engineering

Airfoils

Pitching moment relative to the leading edge



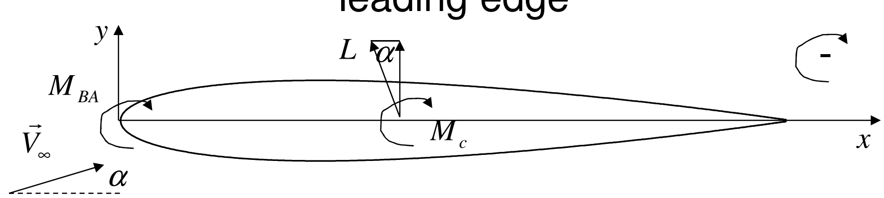
- Moment propagation

$$M_{BA} = M_c + L\cos(\alpha)\frac{c}{2} \cong M_c + \frac{c}{2}L$$

$$C_{M_{BA}} = C_{M_c} + C_l\cos(\alpha)\frac{1}{2} \cong C_{M_c} + \frac{1}{2}C_l$$

Airfoils

Pitching moment relative to the leading edge



- For an airfoil at small values of α

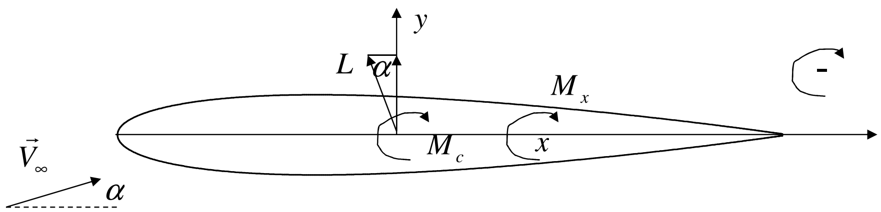
$$C_l \cong 2\pi(\alpha+\beta) \qquad C_{M_c} \cong -\frac{\pi}{2}(\alpha+\lambda)$$

so

$$C_{M_{BA}} \cong \frac{\pi}{2}(\beta-\lambda) + \frac{\pi}{2}(\alpha+\beta)$$

$$C_{M_{BA}} \cong \frac{\pi}{2}\gamma + \frac{C_l}{4} \text{ com } \gamma = \beta - \lambda$$

Airfoils
Aerodynamic centre

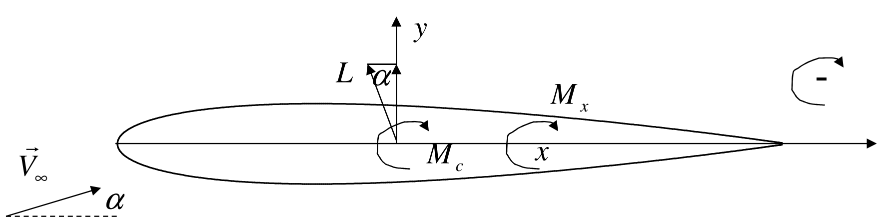


- Aerodynamic centre is the location that exhibits a pitching moment independent of the angle of attack, α

- Pitching moment relative to a point *x*

$$C_{M_x} \cong C_{M_c} - \frac{x}{c} C_l$$

Airfoils
Aerodynamic centre



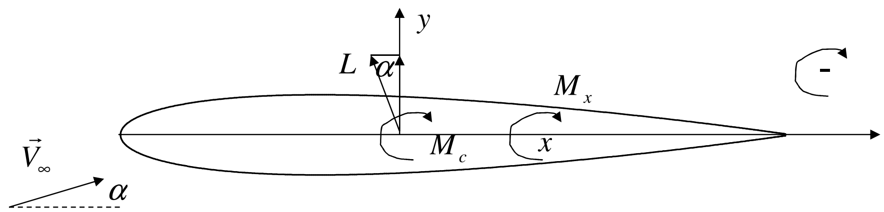
- The location of the aerodynamic centre is given by

$$\frac{dC_{M_x}}{d\alpha} = 0 \Leftrightarrow \frac{dC_{M_x}}{dC_l} = 0$$

so

$$\frac{dC_{M_c}}{dC_l} - \frac{x_{ca}}{c} = 0 \Leftrightarrow \frac{x_{ca}}{c} = \frac{dC_{M_c}}{dC_l} = \frac{dC_{M_c}}{d\alpha} \frac{1}{\frac{dC_l}{d\alpha}}$$

Airfoils
Aerodynamic centre



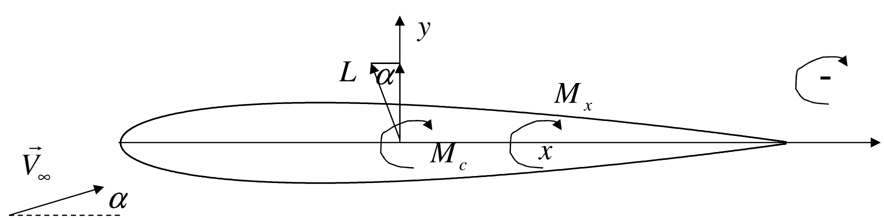
- For an airfoil at small values of α

$$\frac{dC_{M_c}}{d\alpha} \cong -\frac{\pi}{2} \qquad \frac{dC_l}{d\alpha} \cong 2\pi$$

so

$$\frac{x_{ca}}{c} \cong -\frac{\pi}{2}\frac{1}{2\pi} = -\frac{1}{4}$$

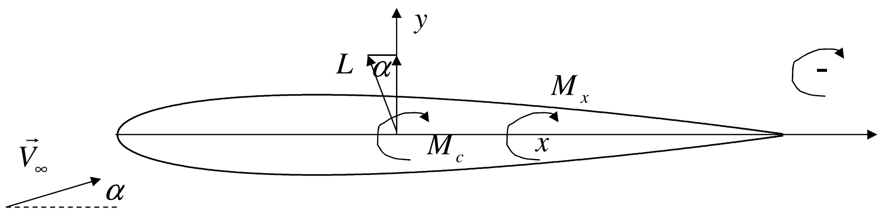
Airfoils
Aerodynamic centre



- The aerodynamic centre is approximately located 25% of the chord downstream of the leading edge
- The pitching moment relative to the aerodynamic centre is given by

$$C_{M_{ca}} = \frac{dC_{M_c}}{d\alpha}(\lambda - \beta) = -\frac{dC_{M_c}}{d\alpha}\gamma \cong \frac{\pi}{2}\gamma$$

Airfoils
Pressure centre

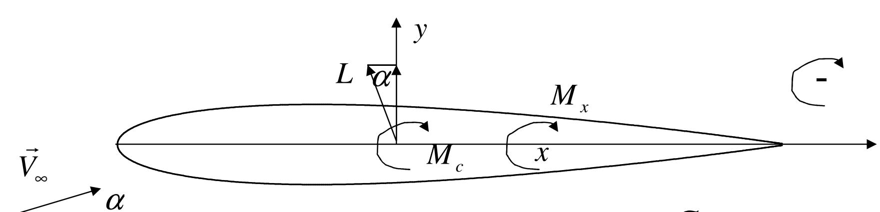


- Pressure centre is the point along the line that contains the chord that exhibits a zero pitching moment

$$C_{M_x} \cong C_{M_c} - \frac{x}{c} C_l = 0$$

Aerodynamics

Airfoils
Pressure centre



$$C_{M_x} \cong C_{M_c} - \frac{x}{c} C_l = 0 \Leftrightarrow \frac{x_{cp}}{c} = \frac{C_{M_c}}{C_l}$$

- For an airfoil at small values of α

$$C_l \cong 2\pi(\alpha+\beta) \qquad C_{M_c} \cong -\frac{\pi}{2}(\alpha+\lambda)$$

so

$$\frac{x_{cp}}{c} = -\frac{1}{4} + \frac{1}{4}\frac{\gamma}{\alpha+\beta} = -\frac{1}{4} + \frac{\pi\gamma}{2C_l} = -\frac{1}{4} + \frac{C_{M_{ca}}}{C_l}$$