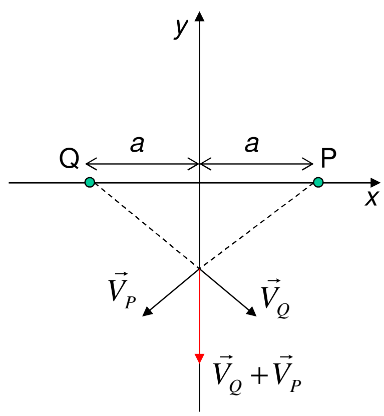
Ideal Fluid
Method of images

- The method of images guarantees that a given line coincides with a streamline. Its application is very simple for straight lines

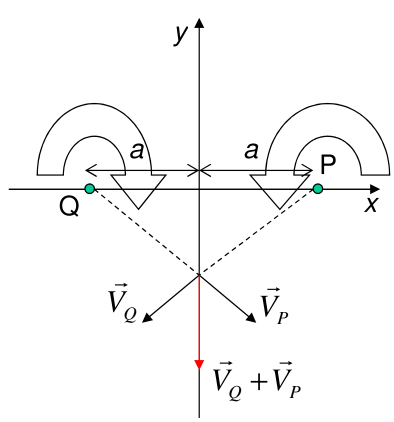


- For a line of sources/sinks of intensity $\frac{M}{2\pi} = m$ located at P(a,0), the line x=0 is a streamline if an additional line of sources/sinks of equal intensity is placed at Q(-a,0) (image source/sink) (x=0 acts as a mirror)

Aerodynamics

Ideal Fluid
Method of images

- For any flow generated by singularities, a straight wall is replaced by the images of the singularities using the wall as a mirror

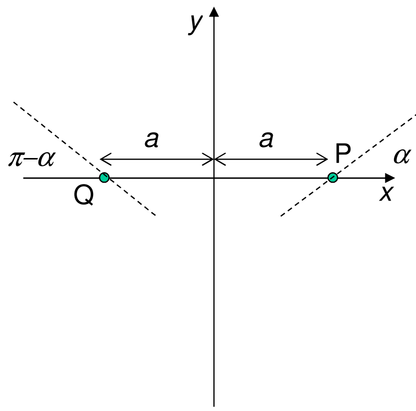


- For a vortex line of intensity $\frac{\Gamma}{2\pi}$ placed at P(a,0), the image as an intensity $-\frac{\Gamma}{2\pi}$ and its location at Q(-a,0)

Masters of Mechanical Engineering

Ideal Fluid
Method of images

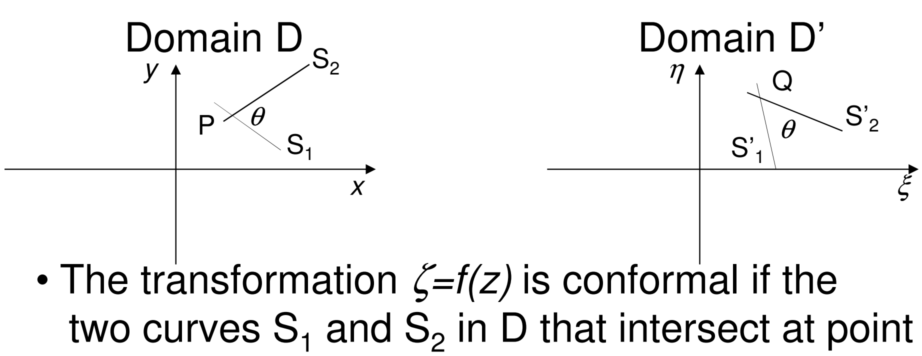
- For any flow generated by singularities, a straight wall is replaced by the images of the singularities using the wall as a mirror



- For a line of dipoles of intensity μ and orientation α placed at P(a,0), the image as an intensity μ, an orientation $\pi-\alpha$ and its location at Q(-a,0)

Ideal Fluid
Conformal mapping

- Consider the relation $\zeta=f(z)$ that maps the domain D in the plane $z(x,y)$ to the domain D' $\zeta(\xi,\eta)$



- The transformation $\zeta=f(z)$ is conformal if the two curves S_1 and S_2 in D that intersect at point P(x_o,y_o) with an angle θ are mapped to the curves S'_1 and S'_2 of domain D' intersecting at P (Q(ξ_o,η_o)) keeping the same angle θ (magnitude and direction)

Ideal Fluid
Conformal mapping

- Example

$$\zeta = az \quad \text{com} \begin{cases} \zeta = \rho e^{\mathrm{i}\alpha} \\ z = re^{\mathrm{i}\theta} \\ a = r_1 e^{\mathrm{i}\theta_1} \end{cases}$$

$$\zeta = r_1 e^{\mathrm{i}\theta_1} r e^{\mathrm{i}\theta} = (r_1 r) e^{\mathrm{i}(\theta_1 + \theta)}$$

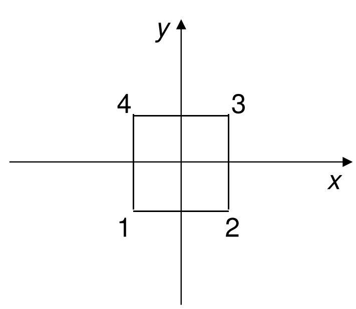
- This mapping amplifies any figure in *z* by a factor $|a| = r_1$ and it rotates the original figure in plane *z* by an angle θ_1

Aerodynamics

Ideal Fluid
Conformal mapping

- Example

Plane *z*

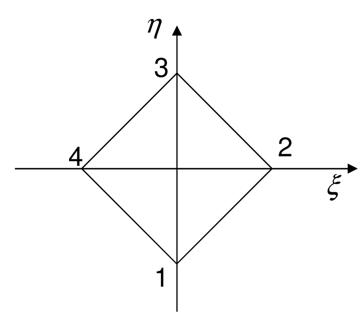


$$\zeta = az$$

with

$$a = 2e^{\mathrm{i}\pi/4}$$

Plane ζ



Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

- Taylor series expansion of *f(z)* at point z_o

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_o)}{n!}(z - z_o)^n$$

$$f(z) = f(z_o) + \sum_{n=1}^{\infty} \frac{f^n(z_o)}{n!}(z - z_o)^n$$

Aerodynamics

Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

- In the vicinity of z_o

$$z - z_o = dz, \;\; f(z) - f(z_o) = \zeta - \zeta_o = d\zeta$$

$$d\zeta = \sum_{n=1}^{\infty} \frac{f^n(z_o)}{n!}(dz)^n$$

Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

- Assuming that *f'*(*zo*) as a zero of order *m* at point *z*=*z*o, i.e. $n \le m \Rightarrow f^{n}(z_o) = 0$

$$d\zeta = \sum_{n=m+1}^{\infty} \frac{f^{n}(z_o)}{n!}(dz)^n$$

$$d\zeta \cong \frac{f^{m+1}(z_o)}{(m+1)!}(dz)^{m+1}$$

Aerodynamics

Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

$$n \le m \Rightarrow f^{n}(z_o) = 0 \Rightarrow d\zeta \cong \frac{f^{m+1}(z_o)}{(m+1)!}(dz)^{m+1}$$

For

$$d\zeta = \rho e^{i\alpha}, \quad dz = re^{i\theta} \quad \text{and} \quad \frac{f^{m+1}(z_o)}{(m+1)!} = \Lambda e^{i\lambda}$$

we have

$$\rho e^{i\alpha} = \Lambda r^{m+1} e^{i(\lambda + (m+1)\theta)}$$

Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

- Rotation, $\alpha - \theta = \arg\left(\frac{d\zeta}{dz}\right)$

$$\alpha - \theta = \lambda + m\theta$$

- Linear amplification, $\frac{\rho}{r} = \left|\frac{d\zeta}{dz}\right|$

$$\frac{\rho}{r} = \Lambda r^m$$

Aerodynamics

Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

- An elementaty segment *dy* rotates $\lambda + m\theta$

- The ratio between the magnitude of the two elementary segments $d\zeta$ and dz, designated linear amplification, is equal to $\Lambda = |f'(z_o)|$ if *m=0*, i.e. $f'(z_o) \neq 0$.
  For *m>0,* there is no linear amplification

Ideal Fluid
Conformal mapping

- Angles preservation and linear amplification

$$\zeta = f(z)$$

- Points where the derivative of the mapping function is zero $d\zeta / dz = 0$ are singular points of the mapping

- A mapping is conformal if *f(z)* is an analytical function of *z* and $f'(z_o) \neq 0$

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows
- Complex potential in plane *z*

$$W = \phi + \mathrm{i}\psi = F(z) \text{ com } z = x + \mathrm{i}\, y$$

$$\frac{dW}{dZ} = F'(z) = U - \mathrm{i}V$$

- Conformal mapping

$$\zeta = f(z) \text{ com } \zeta = \xi + \mathrm{i}\eta$$

- Complex potential W, with ζ as independent variable

$$W = F(z), \quad z = g(\zeta) = f^{-1}(\zeta)$$

$$W = F(g(\zeta)) = G(\zeta)$$

Ideal Fluid

Conformal mapping

- Application to irrotational plane flows
- Complex potential *W*, with ζ as independent variable

$$W = F(g(\zeta)) = G(\zeta)$$

- $G(\zeta)$ is an analytical function of ζ. Therefore, $G(\zeta)$ represents an irrotational plane flow in the ζ plane. Streamlines, ψ=ct^e^, in the ζ plane are obtained from the mapping of the *z* plane streamlines

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Aerodynamics

Ideal Fluid

Conformal mapping

- Application to irrotational plane flows
- Complex potential *W*, with ζ as independent variable

$$W = F(g(\zeta)) = G(\zeta)$$

- Complex velocity

$$\frac{dW}{d\zeta} = G'(\zeta) = U_\zeta - \mathrm{i}V_\zeta$$

$$\frac{dW}{d\zeta} = \frac{dW}{dz}\frac{dz}{d\zeta} \Leftrightarrow \frac{dW}{dz} = \frac{dW}{d\zeta}\frac{d\zeta}{dz} \Leftrightarrow \frac{dW}{d\zeta} = \frac{\dfrac{dW}{dz}}{\dfrac{d\zeta}{dz}}$$

$\frac{dW}{dz}$ → Velocity *z* plane

$\frac{d\zeta}{dz}$ → Mapping function derivative

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows
- Example:
  - Flow around a cylinder of radius *a* in the *z* plane

$$W = U_\infty \left(z + \frac{a^2}{z} \right)$$

  - Conformal mapping to plane ζ

$$\zeta = z - \frac{a^2}{z}$$

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows
- Example:
- Mapping of the dividing streamlines:

z Plane | ζ Plane

$$1 \rightarrow \theta = 0 \wedge r \geq a \qquad \zeta = r - \frac{a^2}{r} \quad \Leftrightarrow (\alpha = 0 \wedge \rho \geq 0)$$

$$(\xi \geq 0 \wedge \eta = 0)$$

$$2 \rightarrow \theta = \pi \wedge r \geq a \qquad \zeta = -\left(r - \frac{a^2}{r} \right) \Leftrightarrow (\alpha = \pi \wedge \rho \geq 0)$$

$$(\xi \leq 0 \wedge \eta = 0)$$

$$3 \rightarrow 0 \leq \theta \leq 2\pi \wedge r = a \quad \zeta = \mathrm{i}\, 2a \,\mathrm{sen}(\theta) \Leftrightarrow \xi = 0 \wedge -2a \leq \eta \leq 2a$$

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows
- Example:

- Complex potential for plane *z*

$$W = U_\infty \left(z + \frac{a^2}{z} \right)$$

- Complex potential for plane ζ

$$\zeta^2 + 4a^2 = z^2 + \frac{a^4}{z^2} + 2a^2 = \left(z + \frac{a^2}{z} \right)^2$$

$$W = U_\infty \sqrt{\zeta^2 + 4a^2}$$

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows
- Example:

- Velocity potential and stream functions for plane *z*

$$\phi = U_\infty r \cos(\theta) \left(1 + \frac{a^2}{r^2} \right), \quad \psi = U_\infty r \,\mathrm{sen}(\theta) \left(1 - \frac{a^2}{r^2} \right)$$

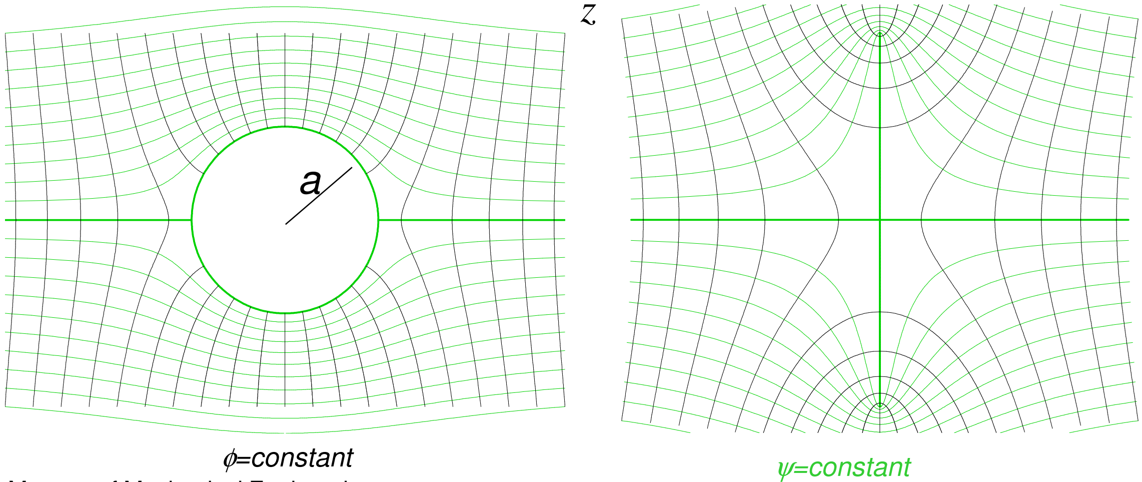
- Mapping of points in plane *z* to plane ζ

$$\xi = r \cos(\theta) \left(1 - \frac{a^2}{r^2} \right), \quad \eta = r \,\mathrm{sen}(\theta) \left(1 + \frac{a^2}{r^2} \right)$$

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows

Plane *z* $\zeta = z - \frac{a^2}{z}$ Plane ζ



Aerodynamics

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows

-Complex potential for plane *z*

$$W = U_\infty \left(z + \frac{a^2}{z} \right)$$

- Complex velocity for plane *z*

$$\frac{dW}{dz} = U_\infty \left(1 - \frac{a^2}{z^2} \right)$$

- Mapping function derivative

$$\frac{d\zeta}{dz} = 1 + \frac{a^2}{z^2}$$

Ideal Fluid
Conformal mapping

- Application to irrotational plane flows

  - Complex velocity for plane ζ

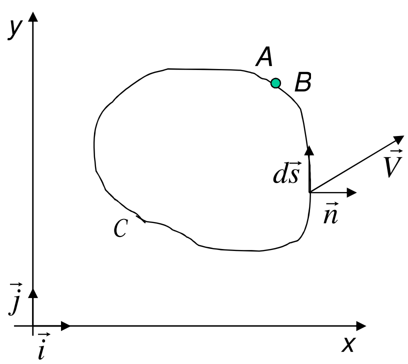
$$\frac{dW}{d\zeta}=\frac{\dfrac{dW}{dz}}{\dfrac{d\zeta}{dz}}=\frac{U_\infty\left(\dfrac{z^2-a^2}{z^2}\right)}{\dfrac{z^2+a^2}{z^2}}=U_\infty\frac{z^2-a^2}{z^2+a^2}$$

  - Mapping function singularities at $z=\pm\mathrm{i}\,a$

  - Velocity goes to infinity at the edges of the vertical plate

Ideal Fluid
Circulation and flow rate for a closed contour

- Flow given by the following complex potential $W=\phi+\mathrm{i}\,\psi$



- Velocity circulation for the closed contour C

$$\Gamma=\oint_C \vec{V}\cdot d\vec{s}$$

with

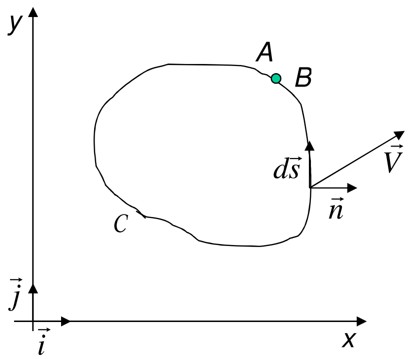
$$\vec{V}=U\vec{i}+V\vec{j}$$

$$d\vec{s}=dx\vec{i}+dy\vec{j}$$

$$\Gamma=\oint_C\left(Udx+Vdy\right)=\oint_C\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy\right)=\oint_C d\phi$$

Ideal Fluid
Circulation and flow rate for a closed contour

- Flow given by the following complex potential
  $W = \phi + \mathrm{i}\psi$



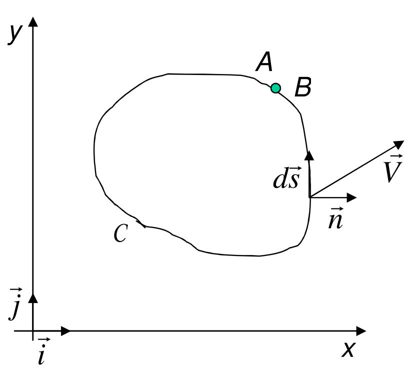
- Flow rate per unit width across the clyndrical surface with a cross-section given by C

$$Q = \oint_C \vec{V} \cdot \vec{n} ds$$

$$Q = \oint_C \left(U dy - V dx\right) = \oint_C \left(\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy\right) = \oint_C d\psi$$

Ideal Fluid
Circulation and flow rate for a closed contour

- Flow given by the following complex potential
  $W = \phi + \mathrm{i}\psi$



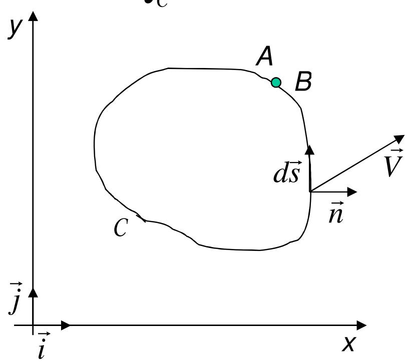
- Sum of results for circulation and flow rate

$$\oint_C dW = \oint_C d\phi + \mathrm{i}\oint_C d\psi = \Gamma + \mathrm{i}\,Q$$

Ideal Fluid

Circulation and flow rate for a closed contour

- Contribution of source/sink and vortex singularities for $\oint_C dW$



- Complex potential of a source/sink line or vortex line

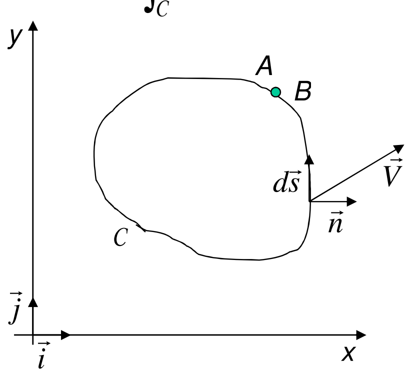
$$W = a\ln(z - z_o)$$

$$a = \begin{cases} \dfrac{Q}{2\pi} \leftarrow & \text{Source/sink} \\ -\mathrm{i}\dfrac{\Gamma}{2\pi} \leftarrow & \text{Vortex} \end{cases}$$

Ideal Fluid

Circulation and flow rate for a closed contour

- Contribution of source/sink and vortex singularities for $\oint_C dW$



$$\oint_C dW = W_B - W_A$$

$$\oint_C dW = a\left[\ln(z - z_o)\right]_A^B$$

$$\oint_C dW = a\left[\ln(r) + \mathrm{i}\,\chi\right]_A^B$$

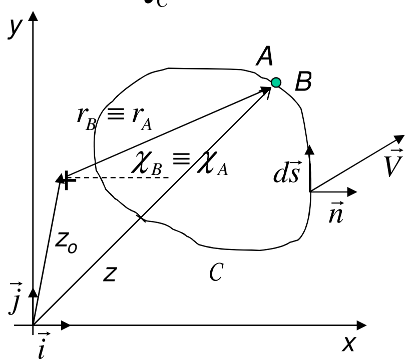
$$\oint_C dW = a\left(\ln\left(\frac{r_B}{r_A}\right)\right) + \mathrm{i}\,a(\chi_B - \chi_A)$$

- r magnitude of vector (z-z_o)
- χ argument of vector (z-z_o)

Ideal Fluid

Circulation and flow rate for a closed contour

- Contribution of source/sink and vortex singularities for $\oint_C dW$



- Singularity external to contour C

$$\oint_C dW = a\left(\ln\left(\frac{r_B}{r_A}\right)\right) + \mathrm{i}\,a(\chi_B - \chi_A)$$

$$r_B \equiv r_A$$

$$\chi_B \equiv \chi_A$$

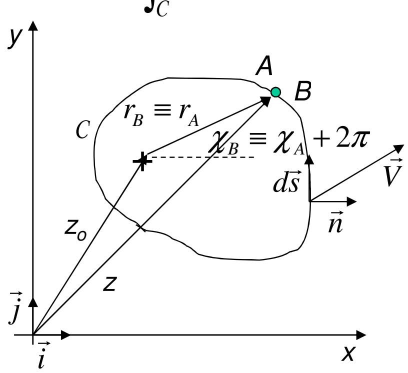
$$\oint_C dW = 0$$

Aerodynamics

Ideal Fluid

Circulation and flow rate for a closed contour

- Contribution of source/sink and vortex singularities for $\oint_C dW$



- Singularity internal to contour C

$$\oint_C dW = a\left(\ln\left(\frac{r_B}{r_A}\right)\right) + \mathrm{i}\,a(\chi_B - \chi_A)$$

$$r_B \equiv r_A$$

$$\chi_B \equiv \chi_A + 2\pi$$

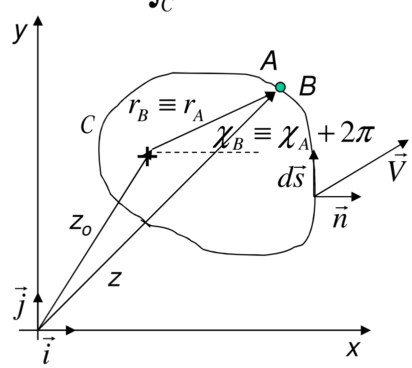
$$\oint_C dW = \mathrm{i}\,2\pi a$$

Masters of Mechanical Engineering

Ideal Fluid

Circulation and flow rate for a closed contour

- Contribution of source/sink and vortex singularities for $\oint_C dW$



- Singularity internal to contour C

$$\oint_C dW = \mathrm{i}\,2\pi a$$

$$\oint_C dW = \Gamma + \mathrm{i}\,Q$$

$$\Gamma = \sum \Gamma_i \qquad Q = \sum Q_i$$

- Γ_i e Q_i stand for the sum of the intensities of any vortex and source/sink lines in the interior of C

Fluido Perfeito/Ideal

Conformal mapping of lines of sources/sinks, vortices and dipoles

- Consider a line of sources/sinks of intensity Q located at z_o in plane z. For a contour C surrounding z_o we have

$$\oint_C dW = Q$$

- The contour C is mapped to the contour C in plane ζ. The stream function ψ has the same values at corresponding points of the two planes, therefore

$$\oint_C dW = \oint_C dW = Q$$

Fluido Perfeito/Ideal

Conformal mapping of lines of sources/sinks, vortices and dipoles

- The contours c and C are arbitraty and so they can shrink indefinitely. This means that we have a line of sources/sinks of intensity Q at ζ_o, which stands for the mapping of z_o
- A similar analysis may be done for a vortex line and so the conformal mapping of a vortex line of intensity Γ at z_o leads to a vortex line of equal intensity at $\zeta_o=f(z_o)$

Aerodynamics

Fluido Perfeito/Ideal

Conformal mapping of lines of sources/sinks, vortices and dipoles

- The mapping of a dipoles line of intensity μ and orientation α at z_o, leads to a line of dipoles of intensity μ' and orientation β located at $\zeta_o=f(z_o)$

$$\mu' = \mu \left|\frac{d\zeta}{dz}\right|_{z=z_o}$$

$$\beta = \alpha + \arg\left(\frac{d\zeta}{dz}\right)_{z=z_o}$$