Ideal Fluid

- Fluid has no viscosity, ν*=0*
  - Reynolds number is infinite $R_e = \dfrac{U_{ref} L_{ref}}{\nu} \rightarrow \infty$
- Adiabatic flow. Thermal conductivity is assumed to be sufficiently small to neglect heat transfer
- Entropy per unit mass of any fluid particle must remain constant. Isentropic flow

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{V} \cdot \vec{\nabla} s = \frac{\partial s}{\partial t} + U \frac{\partial s}{\partial x} + V \frac{\partial s}{\partial y} + W \frac{\partial s}{\partial z} = 0$$

Aerodynamics

Ideal Fluid

Governing equations

- Continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) = 0 \Leftrightarrow \frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{V} = 0$$

- Incompressible flow, ρ *= constant*

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

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Ideal Fluid
Governing equations

- Euler equations
  (momentum balance)

$$\frac{D\vec{V}}{Dt}=\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=-\frac{1}{\rho}\vec{\nabla}p+\vec{F}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+F_x$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+F_y$$

$$\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+F_z$$

Aerodynamics

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Ideal Fluid
Governing equations

- Euler equations
  (momentum balance)

- Gravity

$$\vec{F}=-g\vec{k}=-\vec{\nabla}(gz)$$

- Vector identity

$$\left(\vec{V}.\vec{\nabla}\right)\vec{V}=\vec{\nabla}\left(\frac{\left|\vec{V}\right|^2}{2}\right)-\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)\ \text{com}\ \vec{\nabla}\times\vec{V}=\text{rot}\,\vec{V}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ U & V & W\end{vmatrix}$$

Ideal Fluid
Governing equations

- Euler equations
  (momentum balance)

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}(gz)$$

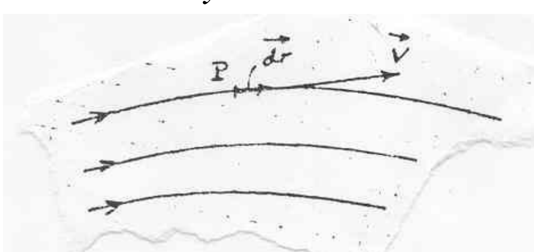
$$\frac{\partial\vec{V}}{\partial t} - \vec{V}\times \mathrm{rot}\,\vec{V} + \frac{1}{\rho}\vec{\nabla}p + \vec{\nabla}\left(\frac{\left|\vec{V}\right|^2}{2} + gz\right) = 0$$

Ideal Fluid
Governing equations

- Bernoulli equation (energy balance)

- Variation of a property *p* along a displacement defined by, $d\vec{r}$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = \vec{\nabla}p \cdot d\vec{r}$$



Ideal Fluid
Governing equations

- Bernoulli equation in steady flow (energy balance for a steady flow)
- Dot product of the vector $d\vec{r}$ by the momentum equation for a streamline, $d\vec{r} \parallel \vec{V}$

$$-\left(\vec{V}\times\mathrm{rot}\,\vec{V}\right)\cdot d\vec{r}+\frac{1}{\rho}\vec{\nabla}p\cdot d\vec{r}+\vec{\nabla}\left(\frac{\left|\vec{V}\right|^{2}}{2}+gz\right)\cdot d\vec{r}=0$$

$$d\left(\frac{\left|\vec{V}\right|^{2}}{2}+gz\right)+\frac{dp}{\rho}=0$$

Ideal Fluid
Governing equations

- Bernoulli equation in steady flow (energy balance for a steady flow)
- Isentropic flow, *ds=0*

$$dh=Tds+\frac{dp}{\rho}\wedge ds=0\Rightarrow dh=\frac{dp}{\rho}$$

$$d\left(\frac{\left|\vec{V}\right|^{2}}{2}+gz+h\right)=0$$

- Along a streamline

$$\frac{\left|\vec{V}\right|^{2}}{2}+gz+h=constante$$

Ideal Fluid
Governing equations

- Bernoulli equation in steady flow (energy balance for a steady flow)
- Incompressible flow, *ρ=constant*

$$dh = d\left(\frac{p}{\rho}\right)$$

$$d\left(\frac{\left|\vec{V}\right|^2}{2} + gz + \frac{p}{\rho}\right) = 0$$

- Along a streamline

$$\frac{\left|\vec{V}\right|^2}{2} + gz + \frac{p}{\rho} = constante$$

Aerodynamics

Ideal Fluid
Governing equations

- Bernoulli equation in steady flow (energy balance for a steady flow)
- Isentropic flow, $\vec{\nabla} s = 0$

$$\vec{\nabla} h = \frac{1}{\rho}\vec{\nabla} p$$

$$\vec{\nabla}\left(\frac{\left|\vec{V}\right|^2}{2} + gz + h\right) = \vec{V} \times \mathrm{rot}\,\vec{V}$$

Masters of Mechanical Engineering

Ideal Fluid
Governing equations

- Bernoulli equation in steady flow (energy balance for a steady flow)
- Incompressible flow, *ρ=constant*

$$\vec{\nabla} h = \vec{\nabla}\left(\frac{p}{\rho}\right)$$

$$\vec{\nabla}\left(\frac{\left|\vec{V}\right|^2}{2} + gz + \frac{p}{\rho}\right) = \vec{V} \times \mathrm{rot}\,\vec{V}$$

Ideal Fluid
Governing equations

- Bernoulli equation in steady flow (energy balance for a steady flow)
- For $\vec{V} \times \mathrm{rot}\,\vec{V} = 0$

$$\frac{\left|\vec{V}\right|^2}{2} + gz + h = constant$$

- In incompressible flow, *ρ=constant*

$$\frac{\left|\vec{V}\right|^2}{2} + gz + \frac{p}{\rho} = constant$$

Ideal Fluid
Governing equations

- Bernoulli equation in steady flow
  (energy balance for a steady flow)
- Cases with $\vec{V} \times \mathrm{rot}\,\vec{V} = 0$

  1. $\vec{V} = 0$, Hidrostatics
  2. $\vec{V} \parallel \mathrm{rot}\,\vec{V}$, Beltrami flows
  3. $\mathrm{rot}\,\vec{V} = 0$, Irrotacional or potential flows

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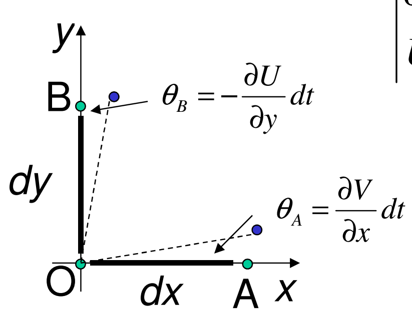
Ideal Fluid

- Vorticity, $\vec{\Omega}$

$$\vec{\Omega} = \mathrm{rot}\,\vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ U & V & W \end{vmatrix}$$

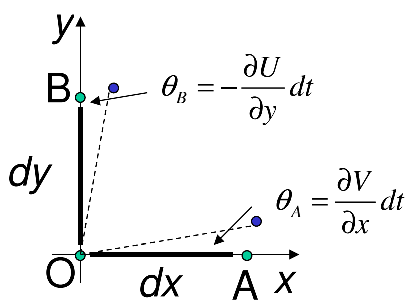


- Flow on the *x,y* plane
- OA and OB – elementary lines
- Axis system and point O have the same velocity

Ideal Fluid

- Vorticity, $\vec{\Omega}$

$$\vec{\Omega} = \mathrm{rot}\,\vec{V} = \vec{\nabla} \times \vec{V}$$



- Analysis of the rotation of lines OA and OB

– Velocity of points A and B

$$V_A = \frac{\partial V}{\partial x} dx \quad U_B = \frac{\partial U}{\partial y} dy$$

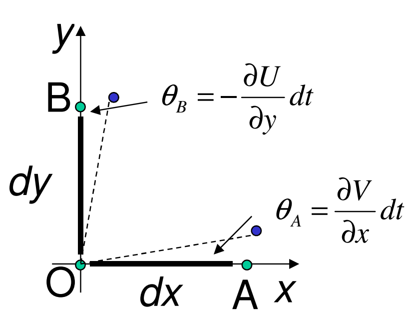
- After *dt*, OA turned θ_A and OB turned θ_B

$$dx\theta_A = \frac{\partial V}{\partial x} dxdt \quad dy\theta_B = -\frac{\partial U}{\partial y} dydt$$

Ideal Fluid

- Vorticity, $\vec{\Omega}$

$$\vec{\Omega} = \mathrm{rot}\,\vec{V} = \vec{\nabla} \times \vec{V}$$



- Mean angular velocity

$$\frac{1}{2}\left(\frac{\theta_A}{dt} + \frac{\theta_B}{dt}\right) = \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)\vec{k}$$

$$\vec{\Omega} = \vec{\nabla} \times \vec{V} = \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)\vec{k}$$

- $\vec{\Omega}$ is the double of the mean angular velocity of the fluid particle rotation as a rigid body

Ideal Fluid

- Vortex lines, surfaces and tubes

  - Vortex line is tangent to the vorticity vector, $\vec{\Omega}$

$$\vec{\Omega} \times d\vec{s} = 0$$

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

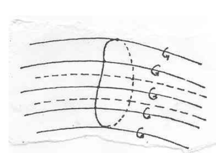
  - Vortex surface is a surface constituted by vortex lines at a given time instant

Aerodynamics

Ideal Fluid

- Vortex lines, surfaces and tubes

  - A vortex tube is a vortex surface that forms a closed line



  - Vorticity flux for a surface *S* that bounds a volume *R*

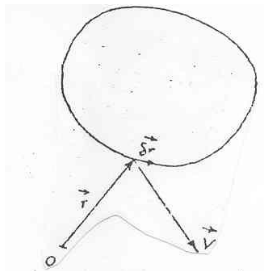
$$\int_S \vec{\Omega} \cdot \vec{n} dS = \int_R \vec{\nabla}.\vec{\Omega} d\mathcal{V} = 0$$

$$\vec{\nabla}.\vec{\Omega} = \vec{\nabla}.\left(\mathrm{rot}\,\vec{V}\right) = \vec{\nabla}.\left(\vec{\nabla} \times \vec{V}\right) = 0$$

Ideal Fluid

- Velocity circulation for a closed contour, Γ

$$\Gamma = \oint \vec{V} \cdot \delta\vec{r}$$



δ stands for differentation in space at a given time instant

Ideal Fluid

- Variation of Γ wiht t for a contour defined by fluid particles

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint \vec{V} \cdot \delta\vec{r} = \oint \frac{D\vec{V}}{Dt} \cdot \delta\vec{r} + \oint \vec{V} \cdot \frac{D\delta\vec{r}}{Dt}$$

$$\vec{V} \cdot \frac{D\delta\vec{r}}{Dt} = \vec{V} \cdot \delta\left(\frac{D\vec{r}}{Dt}\right) = \vec{V} \cdot \delta\vec{V} = \delta\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) = \delta\left(\frac{\left|\vec{V}\right|^2}{2}\right)$$

Ideal Fluid

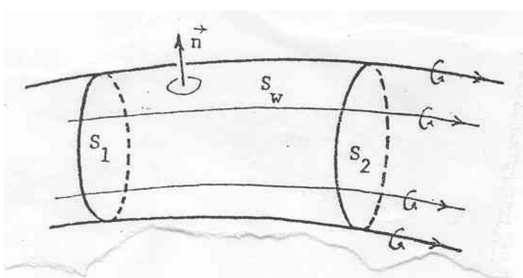
- Variation of Γ wiht *t* for a contour defined by fluid particles

$$\oint \vec{V} \cdot \frac{D\delta\vec{r}}{Dt} = \oint \delta\left(\frac{\left|\vec{V}\right|^2}{2}\right) = 0$$

$$\frac{D\Gamma}{Dt} = \oint \frac{D\vec{V}}{Dt} \cdot \delta\vec{r}$$

Ideal Fluid

- Space conservation of circulation, Γ



Vortex tube in region *R* bounded by two arbitrary cross-sections S_1 and S_2 and the lateral boundary S_w

$$\int_{S_1+S_2+S_w} \vec{\Omega} \cdot \vec{n} dS = \int_R \vec{\nabla} \cdot \vec{\Omega} d\mathcal{V} = 0$$

Ideal Fluid

- Space conservation of circulation, Γ
  - At Sw, $\vec{\Omega}\cdot\vec{n}=0$ so

$$\int_{S_1}\vec{\Omega}\cdot\vec{n}dS+\int_{S_2}\vec{\Omega}\cdot\vec{n}dS=0$$

  - $\vec{n}$ is the outer normal to *R*. Assuming $\vec{n}_1$ and $\vec{n}_2$ with the same sign

$$\int_{S_1}\vec{\Omega}\cdot\vec{n}_1dS=\int_{S_2}\vec{\Omega}\cdot\vec{n}_2dS$$

$$\int_S\vec{\Omega}\cdot\vec{n}dS=constant=\int_C\vec{V}\cdot d\vec{s}=\Gamma_C$$

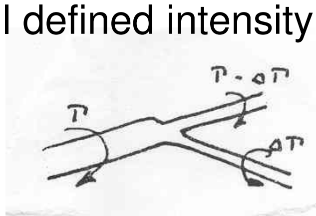
  - Γ_C is the velocity circulation of any contour *C* on the surface Sw that encloses the vortex tube

Ideal Fluid

- Space conservation of circulation, Γ

$$\int_S\vec{\Omega}\cdot\vec{n}dS=constante=\int_C\vec{V}\cdot d\vec{s}=\Gamma_C$$

  - A vortex tube can not end in the middle of the fluid
  - The change of intensity a given vortex tube implies the generation or absorption of other vortex tubes with a well defined intensity



Ideal Fluid

- For a vortex tube of elementary section

  - Vortex filament

$$\Gamma_C = \vec{\Omega} \cdot \vec{n} dS$$

  - Assuming $\vec{n}$ parallel to $\vec{\Omega}$

$$\Gamma_C = \Omega dS \qquad \Omega \propto \frac{1}{dS}$$

  - Concentrated (line) vortex

$$dS \to 0 \quad \Gamma = \lim_{\substack{dS \to 0 \\ \Omega \to \infty}} \vec{\Omega} \cdot \vec{n} dS$$

  - Vortex sheet: Vortex surface formed by line vortices

Ideal Fluid

- Kelvin's theorem

$$\frac{D\Gamma}{Dt} = \oint \frac{D\vec{V}}{Dt} \cdot \delta\vec{r}$$

  - For isentropic flow

$$\frac{D\vec{V}}{Dt} = -\vec{\nabla}(h + gz)$$

$$\frac{D\Gamma}{Dt} = \oint -\vec{\nabla}(h + gz) \cdot \delta\vec{r} = 0$$

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Ideal Fluid

- Kelvin's theorem

$$\frac{D\Gamma}{Dt} = \oint \frac{D\vec{V}}{Dt} \cdot \delta\vec{r}$$

- For incompressible flow

$$\frac{D\vec{V}}{Dt} = -\vec{\nabla}\left(\frac{p}{\rho} + gz\right)$$

$$\frac{D\Gamma}{Dt} = \oint -\vec{\nabla}\left(\frac{p}{\rho} + gz\right) \cdot \delta\vec{r} = 0$$

- Velocity circulation for a closed contour moving with the fluid velocity does not change with time

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Ideal Fluid

- Consequences of Kelvin's theorem

$$\frac{D\Gamma_C}{Dt} = \frac{D}{Dt}\oint \vec{\Omega} \cdot \vec{n} dS = 0$$

- Vorticity flow through a material surface does not change with time

- Vorticity is convected by the fluid.
  If a given surface coincides with a vortex surface at a time instant, it will remain always a vortex surface.
  A vortex tube contains always the same fluid particles.

Ideal Fluid

- Consequences of Kelvin's theorem

$$\frac{D\Gamma_C}{Dt} = \frac{D}{Dt}\oint \vec{\Omega}\cdot\vec{n}dS = 0$$

- Vorticity is convected by the fluid
  If $\Omega=0$ at a given time instance, $\Omega=0$ always.

  Flows of ideal fluids started from rest are irrotacional flows

$$\text{At } t=0, \vec{V}=0 \Rightarrow \vec{\Omega}=0$$

Ideal Fluid

Irrotacional and Incompressible Flow

- Irrotacional flow

$$\vec{\Omega} = \vec{\nabla}\times\vec{V} = \mathrm{rot}\,\vec{V} = 0$$

- Velocity is obtained from the gradient of a potential function (scalar)

$$\vec{V} = \vec{\nabla}\phi$$

- Continuity equation (mass conservation)

$$\vec{\nabla}\cdot\vec{V} = 0 \Leftrightarrow \vec{\nabla}\cdot\vec{\nabla}\phi = 0$$

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

- Two-dimensional flow, $\frac{\partial}{\partial z}=0$
- Incompressible flow, $\rho = constant$
- Continuity equation with $\vec{V}=\vec{\nabla}\phi$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

- Laplace equation

- Linear equation

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

- Laplace equation

- Any linear combination of particular solutions of Laplace's equation also satisfies the Laplace equation

- Analytical solution are easily obtained with functions of complex variable

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

• Stream function, ψ

- Continuity equation

$$\vec{\nabla}\cdot\vec{V}=\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0$$

$$\vec{V}=U\,\vec{i}+V\,\vec{j}$$

- ψ(x,y), stream function

$$d\psi=-Vdx+Udy$$

$$\frac{\partial\psi}{\partial x}=-V,\ \frac{\partial\psi}{\partial y}=U$$

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

• Stream function, ψ

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right)$$

- Streamlines defined by dψ=0

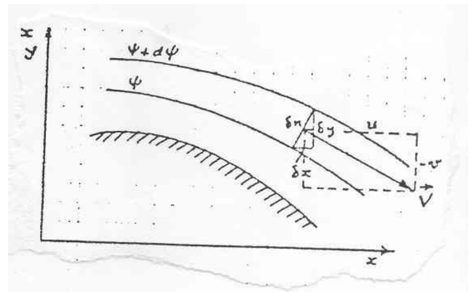
$$-Vdx+Udy=0$$

$$\frac{dx}{U}=\frac{dy}{V}$$

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

- Stream function, ψ
  - Physical meaning of ψ
  - ψ=0 at the wall
  - Flow rate between the wall and the streamline $d\psi$



$$dQ = Udy - Vdx$$

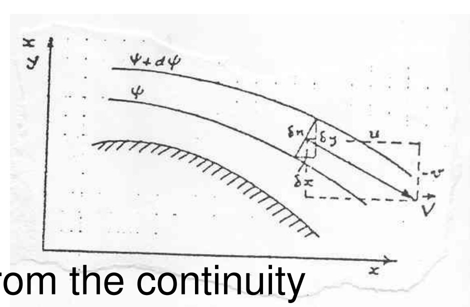
$$dQ = d\psi$$

$$\psi = Q$$

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

- Stream function, ψ
  - $\Delta\psi=\psi_1-\psi_o$ is equal to the flow rate between the streamlines ψ_1 and ψ_o



  - These results were derived from the continuity equation. Therefore, they are valid for any 2-D incompressible flow

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

- Velocity potential function, ϕ

- Irrotacional flow,

$$\vec{\nabla}\times\vec{V}=0\Leftrightarrow\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}=0$$

- ϕ(x,y), velocity potential function

$$d\phi=Udx+Vdy$$

$$\frac{\partial\phi}{\partial x}=U,\ \frac{\partial\phi}{\partial y}=V$$

Aerodynamics

Ideal Fluid

Two-dimensional, Irrotacional and Incompressible Flow

- Velocity potential function, ϕ

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)$$

- Equipotential lines definition, dϕ=0

$$Udx+Vdy=0$$

$$\frac{dx}{V}=-\frac{dy}{U}$$

Ideal Fluid
Two-dimensional, Irrotacional and Incompressible Flow

- Continuity equation applied with ϕ

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

- Irrotational condition applied with ψ

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

Ideal Fluid
Two-dimensional, Irrotacional and Incompressible Flow

- The velocity potential function ϕ and the stream function ψ satisfy Laplace's equation

- Streamlines are perpendicular to the equipotencial lines

$$\vec{\nabla}\phi=U\,\vec{i}+V\,\vec{j}\quad \text{e}\quad \vec{\nabla}\psi=-V\,\vec{i}+U\,\vec{j}$$

$$\vec{\nabla}\phi\cdot\vec{\nabla}\psi=-UV+VU=0$$

Ideal Fluid
Complex Potential

- Analytical functions

- *W*, function of complex variable

$$W = \phi(x, y) + \mathrm{i}\,\psi(x, y)$$

$$\begin{cases} z = x + \mathrm{i}\, y \\ z = re^{\mathrm{i}\theta} \end{cases} \text{ with } \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \operatorname{arctg}\left(\dfrac{y}{x}\right) \end{cases} \text{ and } \begin{cases} x = r\cos(\theta) \\ y = r\,\mathrm{sen}(\theta) \end{cases}$$

Ideal Fluid
Complex Potential

- A function of complex variable is an analytical function when

$$\lim_{\Delta z \to 0} \frac{W(z + \Delta z) - W(z)}{\Delta z}$$

exist and its value is independent of the way that *z* goes to zero

$$\frac{\Delta W}{\Delta z} = \frac{\Delta\phi + \mathrm{i}\,\Delta\psi}{\Delta x + \mathrm{i}\,\Delta y}$$

- At the limit, $\Delta z \to 0$

$$\frac{dW}{dz} = \frac{d\phi + \mathrm{i}\, d\psi}{dx + \mathrm{i}\, dy} = \frac{\left(\dfrac{\partial\phi}{\partial x} + \mathrm{i}\dfrac{\partial\psi}{\partial x}\right)dx + \left(\dfrac{\partial\phi}{\partial y} + \mathrm{i}\dfrac{\partial\psi}{\partial y}\right)dy}{dx + \mathrm{i}\, dy}$$

Ideal Fluid
Complex Potential

- To have a limit that is independent of the way $\Delta z \to 0$, this relation must be independent of $\frac{dy}{dx}$

$$\frac{\frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x}}{1} = \frac{\frac{\partial \phi}{\partial y} + \mathrm{i}\frac{\partial \psi}{\partial y}}{\mathrm{i}}$$

- Riemann-Cauchy conditions

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Aerodynamics

Ideal Fluid
Complex Potential

- Any *W(z)* function that has *z* as its independent variable is an analytical function

- Verification

$$W = \phi + \mathrm{i}\psi \quad z = x + \mathrm{i}\, y \quad x = z - \mathrm{i}\, y$$

- If the function is analytical $\frac{\partial W}{\partial y} = 0$

$$\frac{\partial W}{\partial y} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial \phi}{\partial y} + \mathrm{i}\left(\frac{\partial \psi}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial \psi}{\partial y}\right)$$

$$\frac{\partial W}{\partial y} = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} + \mathrm{i}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x}\right) = 0$$

Ideal Fluid
Complex Potential

- Examples of complex functions

1. $W = x^2 - y^2 + \mathrm{i}\,2xy = z^2$
2. $W = x + \mathrm{i}\,2y = z + \mathrm{i}\,y$

- Determination of $\dfrac{\partial W}{\partial y}$

1. $\dfrac{\partial W}{\partial y} = -2y + 2y + \mathrm{i}(2x - 2x) = 0$

Function is analytic

2. $\dfrac{\partial W}{\partial y} = 0 + 0 + \mathrm{i}(2 - 1) = \mathrm{i}$

Function is not analytic

Ideal Fluid
Complex Potential

- The velocity potential function and the stream function satisfy the following equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

- The complex potential function

$$W(x, y) = \phi(x, y) + \mathrm{i}\,\psi(x, y)$$

is an analytic function with a real part equal to the velocity potential function and the imaginary part equal to the stream function of an incompressible, irrotational plane flow

Ideal Fluid

Complex Potential

- For the complex potential $W(x,y) = \phi(x,y) + \mathrm{i}\,\psi(x,y)$

$$dW = \frac{dW}{dz}\,dz$$

- The differential *dW* may be obtained from

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy$$

$$dW = \left(\frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \phi}{\partial y} + \mathrm{i}\frac{\partial \psi}{\partial y}\right)dy$$

$$dW = (U - \mathrm{i}V)dx + (V + \mathrm{i}U)dy$$

$$dW = (U - \mathrm{i}V)dx + \mathrm{i}(U - \mathrm{i}V)dy$$

$$dW = (U - \mathrm{i}V)(dx + \mathrm{i}\,dy)$$

Ideal Fluid

Complex Potential

- For the complex potential $W(x,y) = \phi(x,y) + \mathrm{i}\,\psi(x,y)$

$$dW = (U - \mathrm{i}V)dz$$

$$\frac{dW}{dz} = (U - \mathrm{i}V) = \overline{V}$$

- Complex velocity, $\overline{V}$, complex conjugate of the velocity vector, $\vec{V}$

$$\vec{V} = \left|\vec{V}\right|e^{\mathrm{i}\theta}$$

$$\overline{V} = \left|\overline{V}\right|e^{-\mathrm{i}\theta}$$