Turbulent Flow

Reynolds-averaged (RANS) equations

- Boundary-layer approximations

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

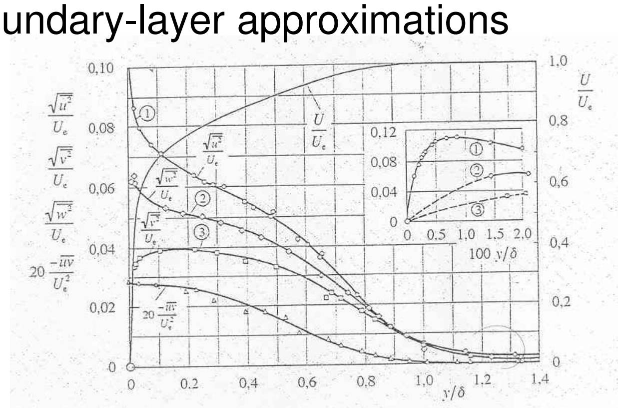
$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \frac{1}{\rho}\left(\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y} - \rho\overline{uv}\right)\right)$$

- Number of equations is smaller than the number of unknowns
- Only one Reynolds stress retained: $-\rho\overline{uv}$

Turbulent Flow

Reynolds-averaged (RANS) equations

- Boundary-layer approximations



- Assessment of negligible Reynolds stresses must be based on experimental data

Aerodynamics

Turbulent Flow

Reynolds-averaged (RANS) equations

- Boundary-layer approximations

- von Kármán remains identical to laminar flow

$$\frac{1}{\rho}\int_0^h \frac{\partial \tau_T}{\partial y} dy = \left[\frac{\tau_T}{\rho}\right]_0^h = -\frac{\tau_w}{\rho}$$

$$\tau_T = \tau_{lam} + \tau_{turb} \text{ with } \tau_{lam} = \mu \frac{\partial U}{\partial y}, \quad \tau_{turb} = -\rho\overline{uv}$$

Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Wall layer:

  - Region of local equilibrium.
    Production of *k* ≡ Dissipation of *k* (ε)

  - At the wall, *y=0*, the momentum balance in the *x* direction is reduced to

$$\frac{\partial \tau_T}{\partial y} = \frac{dP}{dx}$$

Masters of Mechanical Engineering

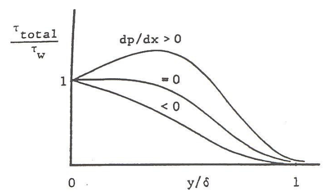
Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Wall layer:
  - Assuming the convection is negligible close to the wall (*U→0*)

$$\tau_T = \tau_w + \frac{dP}{dx} y$$



  - For pressure gradients close to zero the wall layer exhibits a constant total shear stress, $\tau_T \approx \tau_w$

Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Wall layer:
  - The 3 fundamental variables (LMT) to define dimensionless parameters are:
    - Density of the fluid, ρ
    - Viscosity of the fluid, ν
    - Shear-stress at the wall, τ_w

Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Wall layer:
  - Friction velocity

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} = U_e \sqrt{\frac{C_f}{2}}$$

  - Reference length

$$L_{ref} = \frac{\nu}{u_\tau}$$

Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Linear sub-layer:
  - For very small values of *y* ($-\rho\overline{uv} \rightarrow 0$)

$$\tau_T = \tau_w = \tau_{lam} = \mu \frac{\partial U}{\partial y}$$

  - Integration and the no slip condition ($y=0 \Rightarrow U=0$) lead to

$$U = \frac{\tau_w y}{\mu} = \left(\frac{\tau_w}{\rho}\right)\frac{y}{\nu}$$

Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Linear sub-layer:
  - In dimensionless variables

$$\frac{U}{u_\tau}u_\tau = \left(\frac{\tau}{\rho}\right)\frac{y}{\nu}\frac{\nu}{u_\tau}\frac{u_\tau}{\nu}$$

$$\frac{U}{u_\tau} = \frac{u_\tau y}{\nu} \Leftrightarrow U^+ = y^+$$

$$U^+ = \frac{U}{u_\tau} \qquad y^+ = \frac{u_\tau y}{\nu}$$

Aerodynamics

Turbulent Flow

Mean velocity profile, U

- Linear sub-layer valid for $y^+ < 5$
  - u_τ=1m/s, $\nu_{ar}=1{,}5\times10^{-5} \Rightarrow y<7{,}5\times10^{-5}$m
- Consequences:
  - Experimentally, it is very hard to determine the wall shear-stress from $\left(\frac{\partial U}{\partial y}\right)_{y=0}$
  - Numerically, the direct application of the no slip condition requires near-wall grid line spacings satisfying $y_2^+ < 1$

Turbulent Flow
Mean velocity profile, U

- Linear sub-layer valid for $y^+ < 5$

  - Spalart & Allmaras model

    $\tilde{\nu}^+ = \kappa y^+, \quad \tilde{\nu}^+ = \tilde{\nu}/\nu$

  - Turbulence kinetic energy, *k*

    $k^+ = C_k \left(y^+\right)^{0.5+\sqrt{0.25+6\beta^*/\beta}}, \quad k^+ = k/u_\tau^2$

  - Turbulence "frequency", ω

    $\omega^+ = 6/\left(\beta\left(y^+\right)^2\right), \quad \omega^+ = (\omega\nu)/u_\tau^2$

Aerodynamics

Turbulent Flow
Mean velocity profile, U

- Buffer-layer, $5 < y^+ < 30-50$

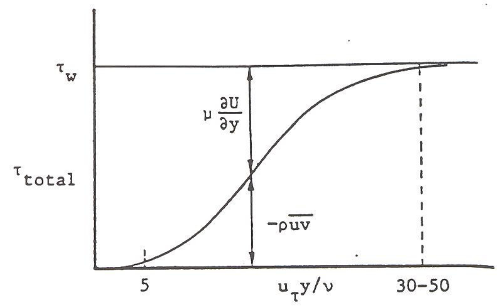
  - In this region, the main contribution to the total shear-stress changes from the laminar stress to the Reynolds (turbulent) stress

  - For $y^+ \leq 5$ the Reynolds stress (turbulent) is negligible

  - For $y^+ = 30-50$ the Reynolds stresses (turbulent) the predominant contribution to the total stress

Turbulent Flow
Mean velocity profile, U

- Buffer-layer, $5 < y^{+} < 30-50$



- The region of the mean velocity profile with *y+* smaller than 30-50 is the viscous sub-layer

Turbulent Flow
Mean velocity profile, U

- Wall (log) law, $y^{+} > 30-50$

  - Reynolds stress (turbulent) is predominant

  - Dimensional analysis applied to the region with approximately constant shear-stress

$$\frac{U}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right)$$

  - The velocity gradient is given by

$$\frac{\partial U}{\partial y} = \frac{u_\tau^2}{\nu} f'\left(\frac{u_\tau y}{\nu}\right) = \frac{u_\tau}{y}\frac{u_\tau y}{\nu} f'\left(\frac{u_\tau y}{\nu}\right) = \frac{u_\tau}{y} g\left(\frac{u_\tau y}{\nu}\right)$$

Turbulent Flow
Mean velocity profile, U

- Wall (log) law, $y^+ > 30 - 50$
  - Experimental data leads to

$$g\left(\frac{u_\tau y}{\nu}\right) \cong const = \frac{1}{\kappa}$$

so

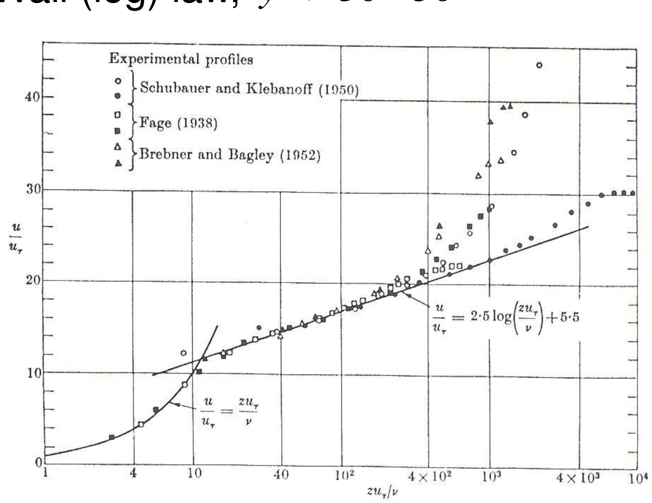
$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

  - Integration gives

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{u_\tau y}{\nu}\right) + C \Leftrightarrow U^+ = \frac{1}{\kappa}\ln\left(y^+\right) + C$$

Turbulent Flow
Mean velocity profile, U

- Wall (log) law, $y^+ > 30 - 50$



$$\kappa = 0{,}41$$
$$C \cong 5{,}2$$

Turbulent Flow
Mean velocity profile, U

- Wall (log) law, $y^{+} > 30 - 50$

- It is possible to determine the wall shear-stress experimentally measuring the mean velocity profile at a region sufficiently away from the wall

- The boundary conditions of a numerical calculation may be applied at the wall (log) law region. This avoids the viscous sub-layer and it simplifies significantly the numerical calculation of near-wall flows. However, it relies on the validity of the wall (log) law

Turbulent Flow
Turbulence quantities profiles

- Wall (log) law, $y^{+} > 30 - 50$

  - Spalart & Allmaras model

    $\tilde{\nu}^{+} = \kappa y^{+}, \quad \tilde{\nu}^{+} = \tilde{\nu}/\nu$

  - Turbulence kinetic energy, *k*

    $k^{+} = 1/\sqrt{C_{\mu}}, \quad k^{+} = k/u_{\tau}^{2}$

  - Turbulence "frequency", ω

    $\omega^{+} = 1/\left(\kappa\sqrt{C_{\mu}}\,y^{+}\right), \quad \omega^{+} = (\omega\nu)/u_{\tau}^{2}$

Turbulent Flow
Mean velocity profile, U

• Wall (log) law for rough walls

- Dimensional analysis applied to the region with approximately constant shear-stress in the fully-rough regime

$$\frac{U}{u_\tau} = f\left(\frac{y}{\varepsilon_r}\right)$$

that leads to

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{\varepsilon_r}\right) + B$$

$$\kappa = 0{,}41 \quad B = 8{,}5$$

Turbulent Flow
Mean velocity profile, U

• Wall (log) law for rough walls

- Reynolds number based on the roughness height

$$R_{e_{\varepsilon_r}} = \frac{u_\tau \varepsilon_r}{\nu}$$

- For $R_{e_{\varepsilon_r}} < 5$ the flow behaves as if the wall is smooth, “hidraulically smooth” regime (roughness height smaller that the height of the linear sub-layer)

- $R_{e_{\varepsilon_r}} > 70$ corresponds to the fully-rough regime. The local flow becomes independent of the viscosity

Turbulent Flow
Mean velocity profile, U

- Wall (log) law for rough walls

  - For $5 < R_{e_{\varepsilon_r}} < 70$ the constant of the wall (log) law depends on the roughness size and viscosity of the fluid

$$\frac{U}{u_\tau} = f\left(\frac{y}{\varepsilon_r}, \frac{u_\tau \varepsilon_r}{\nu}\right) \quad \text{or} \quad \frac{U}{u_\tau} = f\left(\frac{u_\tau y}{\nu}, \frac{u_\tau \varepsilon_r}{\nu}\right)$$