Turbulent Flow
Reynolds-averaged (RANS) equations

- Time averaging applied to the dependent variables and to "conservation" principles

$$\overline{\tilde{\phi}_i} = \lim_{T\to\infty} \frac{\int_{t_o}^{t_o+T} \tilde{\phi}_i dt}{T} = \Phi_i$$

$\tilde{\phi}_i$ stands for any of the dependent variables (incompressible flow *u*,*v*,*w*,*p*)

Turbulent Flow
Reynolds-averaged (RANS) equations

- Decomposition of the instantaneous variables

$$\tilde{\phi} = \Phi_i + \phi_i$$

$\tilde{\phi} \rightarrow$ Instantaneous variable

$\Phi_i \rightarrow$ Mean value

$\phi_i \rightarrow$ Fluctuation around the mean value

Turbulent Flow
Reynolds-averaged (RANS) equations

- Consequences of time averaging

$$\frac{\partial \Phi}{\partial t} = 0 \rightarrow$$ Time derivative of mean value is zero

$$\left.\begin{aligned} \overline{\frac{\partial \phi}{\partial t}} &= 0 \rightarrow \\ \overline{\frac{\partial \phi}{\partial x_i}} &= 0 \rightarrow \end{aligned}\right\}$$ Mean value of the time derivative of the fluctuations is zero

Aerodynamics

Turbulent Flow
Reynolds-averaged (RANS) equations

- Linear terms

$$\frac{\partial \tilde{\phi}}{\partial t} = \frac{\partial \Phi}{\partial t} + \frac{\partial \phi}{\partial t} \Rightarrow \overline{\frac{\partial \tilde{\phi}}{\partial t}} = 0$$

$$\frac{\partial \tilde{\phi}}{\partial x_i} = \frac{\partial \Phi}{\partial x_i} + \frac{\partial \phi}{\partial x_i} \Rightarrow \overline{\frac{\partial \tilde{\phi}}{\partial x_i}} = \frac{\partial \Phi}{\partial x_i}$$

Turbulent Flow

Reynolds-averaged (RANS) equations

- Non-linear terms

$$\tilde{u}\frac{\partial\tilde{\phi}_i}{\partial x}+\tilde{v}\frac{\partial\tilde{\phi}_i}{\partial y}+\tilde{w}\frac{\partial\tilde{\phi}_i}{\partial z}=\frac{\partial\tilde{u}\,\tilde{\phi}_i}{\partial x}+\frac{\partial\tilde{v}\,\tilde{\phi}_i}{\partial y}+\frac{\partial\tilde{w}\,\tilde{\phi}_i}{\partial z}$$

$$\overline{\frac{\partial\tilde{u}_j\tilde{\phi}_i}{\partial x_j}}=U_i\frac{\partial\Phi_j}{\partial x_j}+\overline{\frac{\partial u\phi}{\partial x_j}}$$

Aerodynamics

Turbulent Flow

Reynolds-averaged (RANS) equations

- Continuity equation

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0$$

- Velocity fluctuations also satisfy

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Turbulent Flow

Reynolds-averaged (RANS) equations

- Momentum equations

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\nu\frac{\partial U}{\partial x}-\overline{uu}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}-\overline{uv}\right)+\frac{\partial}{\partial x}\left(\nu\frac{\partial U}{\partial z}-\overline{uw}\right)$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(\nu\frac{\partial V}{\partial x}-\overline{vu}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial V}{\partial y}-\overline{vv}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial W}{\partial z}-\overline{vw}\right)$$

$$U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\left(\nu\frac{\partial W}{\partial x}-\overline{wu}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial W}{\partial y}-\overline{wv}\right)+\frac{\partial}{\partial x}\left(\nu\frac{\partial W}{\partial z}-\overline{ww}\right)$$

- $-\rho\overline{u_i u_j}$ Reynolds stresses
- The number of equations is smaller than the number of unknowns

Turbulent Flow

Reynolds-averaged (RANS) equations

- Transport equation of $-\rho\overline{u_i u_j}$

$$\frac{D\overline{u_i u_j}}{Dt}=\frac{\partial\overline{u_i u_j}}{\partial t}+U_k\frac{\partial\overline{u_i u_j}}{\partial x_k}=-\left(\overline{u_i u_j}\frac{\partial U_i}{\partial x_k}+\overline{u_i u_k}\frac{\partial U_j}{\partial x_k}\right)+\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}$$

$$-\frac{\partial}{\partial x_k}\left(\overline{u_i u_j u_k}\right)-\frac{1}{\rho}\left(\frac{\partial\overline{pu_j}}{\partial x_i}+\frac{\partial\overline{pu_i}}{\partial x_j}\right)$$

$$+\nu\frac{\partial^2\overline{u_i u_j}}{\partial x_i^2}-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

- System remains with less equations than unknowns

Turbulent Flow
Reynolds-averaged (RANS) equations

- Reynolds stress models
  - 6 additional transport equations
  - Most of the terms of the Reynolds stresses transport equations must be modeled, including pressure fluctuations
  - There are alternative Explicit Algebraic Stress Models (EARSM) available
  - Turbulence anisotropy is taken into account

Turbulent Flow
Reynolds-averaged (RANS) equations

- Eddy-viscosity models
  - Boussinesq hypothesis: the Reynolds stresses are proportional to mean velocity derivatives
  - The proportionality constant is the eddy-viscosity
  - It is difficult to include the anisotropy of turbulence. Most models are isotropic, i.e. eddy-viscosity is a scalar quantity

Aerodynamics

Turbulent Flow

Reynolds-averaged (RANS) equations

- Momentum equations including an effective viscosity

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+2\frac{\partial}{\partial x}\left(\nu_{ef}\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_{ef}\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\nu_{ef}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)\right)$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(\nu_{ef}\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right)+2\frac{\partial}{\partial y}\left(\nu_{ef}\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_{ef}\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right)\right)$$

$$U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\left(\nu_{ef}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\nu_{ef}\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right)\right)+2\frac{\partial}{\partial z}\left(\nu_{ef}\frac{\partial W}{\partial z}\right)$$

$$\nu_{ef}=\nu+\nu_t$$

INSTITUTO SUPERIOR TÉCNICO
Universidade Técnica de Lisboa

Aerodynamics

Turbulent Flow

Reynolds-averaged (RANS) equations

- Momentum equations including an effective viscosity

  – ν_t is the eddy-viscosity

  - Eddy-viscosity is obtained from velocity and length scales of turbulence

  - Several models proposed in the last 50 years

Turbulent Flow

Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models

  - Algebraic models

  - Turbulence length scale
    $l = \kappa y \rightarrow$ Mixing length

  - Turbulence velocity scale
    $l|\vec{\omega}| \rightarrow \vec{\omega}$ is the vorticity vector
    $\nu_t = l^2 |\vec{\omega}|$

Turbulent Flow

Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models
  - Algebraic models
  - Damping function is applied to the turbulence length scale in the near-wall region. Length scale must be changed for the outer region of the boundary-layer, wakes and jets
  - Simplest model available, but with severe limitations. Numerical implementation in complex flows may be troublesome

Turbulent Flow
Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models
  - 1-equation models (“old style”)
  - Turbulence length scale is identical to that used in the algebraic models
  - Turbulence velocity scale is the square root of the turbulence kinetic energy, which has its own transport equation

$$k = \frac{1}{2}\overline{u^2 + v^2 + w^2}$$

Aerodynamics

Turbulent Flow
Reynolds-averaged (RANS) equations

- Turbulence kinetic energy, *k*
  - Transport equation (*k* balance)

$$\frac{Dk}{Dt} = \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = -\left(\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{pu_j} + \frac{1}{2}\overline{u_i u_i u_j}\right)$$

$$+\nu\frac{\partial^2 k}{\partial x_j^2} - \nu\left(\overline{\frac{\partial u_i}{\partial x_j}}\right)^2$$

Turbulent Flow

Reynolds-averaged (RANS) equations

- Turbulence kinetic energy, *k*

$$U_j \frac{\partial k}{\partial x_j} \rightarrow \text{Convection}$$

$$-\left(\overline{u_i u_j} \frac{\partial U_i}{\partial x_j} \right) \rightarrow \text{Production}$$

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho} \overline{p u_j} + \frac{1}{2} \overline{u_i u_i u_j} \right) \rightarrow \text{Turbulent diffusion}$$

Aerodynamics

Turbulent Flow

Reynolds-averaged (RANS) equations

- Turbulence kinetic energy, *k*

$$\nu \frac{\partial^2 k}{\partial x_j^2} \rightarrow \text{Viscous diffusion}$$

$$-\nu \overline{\left(\frac{\partial u_i}{\partial x_j} \right)^2} \rightarrow \text{Dissipation rate, } \varepsilon$$

- Most of the terms include unknown quantities and so they must be modeled

Turbulent Flow

Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models
  - 1-equations models

Spalart & Allmaras $\nu_t = f_{v1}\tilde{\nu}$

$$U\frac{\partial\tilde{\nu}}{\partial x}+V\frac{\partial\tilde{\nu}}{\partial y}=c_{b1}\tilde{\nu}\tilde{S}+\frac{1}{\sigma_s}\left[\nabla\cdot\left(\nu+\tilde{\nu}\right)\nabla\tilde{\nu}+c_{b2}\left(\nabla\tilde{\nu}\cdot\nabla\tilde{\nu}\right)\right]-c_{w1}f_w\left(\frac{\tilde{\nu}}{d}\right)^2$$

$c_{b1},\ c_{b2},\ c_{w1} \rightarrow$ Constants $\quad f_{v1},\ f_w \rightarrow$ Functions

Turbulent Flow

Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models
  - 1-equation model of Spalart & Allmaras
  - Valid down to the wall
  - Eddy-viscosity is proportional to the dependent variable of the model
  - It requires the distance to the wall, *d*, and in its original version the location of transition

Turbulent Flow

Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models
  - 2-equation models: turbulence velocity scale is $\sqrt{k}$
  - *k*-ε model $\nu_t = C_\mu \dfrac{k^2}{\varepsilon}$

$$U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = \nu_t S^2 + \nabla\cdot\left(\left(\nu + \frac{\nu_t}{\sigma_k}\right)\nabla k\right) - \varepsilon$$

$$U\frac{\partial \varepsilon}{\partial x} + V\frac{\partial \varepsilon}{\partial y} = C_1\frac{\varepsilon}{k}\nu_t S^2 + \nabla\cdot\left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\nabla \varepsilon\right) - C_2\frac{\varepsilon^2}{k}$$

$C_\mu,\ C_1,\ C_2, \sigma_k, \sigma_\varepsilon \rightarrow$ Constants

Turbulent Flow

Reynolds-averaged (RANS) equations

- Eddy-viscosity turbulence models

- *k*-ε model

- Widely used, specially in flows without walls and in heat transfer problems

- Poor results for flows with adverse pressure gradients

Turbulent Flow

Reynolds-averaged (RANS) equations

• Eddy-viscosity turbulence models

- *k*-ε model
- Can not be applied in the near-wall region
- Two-layer models combine the *k*-ε model in the outer region with a 1-equation model in the near-wall region
- There are (too many) Low-Reynolds number versions of the model for its extension to the near-wall region

Aerodynamics

Turbulent Flow

Reynolds-averaged (RANS) equations

• Eddy-viscosity turbulence models

- *k*-ω model $\qquad \nu_t = \dfrac{k}{\omega}$

$$U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = \nu_t S^2 + \nabla\cdot\left(\left(\nu + \frac{\nu_t}{\sigma_k}\right)\nabla k\right) - \beta^*\omega k$$

$$U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y} = \alpha S^2 + \nabla\cdot\left(\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\nabla \omega\right) - \frac{F_\omega}{\omega}\left(\nabla k\cdot\nabla\omega\right) - \beta\omega^2$$

$\beta^*, \beta, \alpha, \sigma_k, \sigma_\omega \rightarrow$ Constants $\qquad F_\omega \rightarrow$ Function

Turbulent Flow
Reynolds-averaged (RANS) equations

• Eddy-viscosity turbulence models

- *k*-ω model

- May be applied down to the wall

- ω goes to infinity at the wall (smooth walls)

- Several formulations available. One of the most popular is the SST (Shear-Stress Transport) version that includes a limiter for the eddy-viscosity

Aerodynamics

Turbulent Flow
Reynolds-averaged (RANS) equations

• Eddy-viscosity turbulence models

- *k*-ω model

- Widely used for the calculation of adverse pressure gradient flows

- Numerical implementation is not trivial (w wall boundary condition) and some versions (as for example SST) require the distance to the wall

Masters of Mechanical Engineering