Boundary-Layer Approximations
Pressure gradient effect

- Shape of the velocity profile

$$\frac{\partial u}{\partial s} \cong -\frac{1}{\rho u}\frac{\partial p}{\partial s} \Leftrightarrow \frac{\partial u}{\partial x} \cong -\frac{1}{\rho u}\frac{dp}{dx}$$

- Aplication of Bernoulli's equation along a streamline in the interior of the boundary-layer neglecting friction
- Pressure gradient $\frac{dp}{dx}$ effect increases with the decrease of *u*

Aerodynamics

Boundary-Layer Approximations
Pressure gradient effect

- Shape of the velocity profile

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 \overline{u}}{\partial \eta^2} = \Lambda \quad \text{with } \overline{u} = \frac{u}{U_e}, \quad \eta = \frac{y}{\delta} \text{ and } \Lambda = -\frac{\delta^2}{\nu}\frac{dU_e}{dx}$$

- Second derivative at the wall is defined by the pressure gradient

Boundary-Layer Approximations
Pressure gradient effect

- Shape of the velocity profile

$$\eta = 0 \Rightarrow \bar{u} = 0 \qquad \frac{\partial^2 \bar{u}}{\partial \eta^2} = \Lambda$$

$$\eta = 1 \Rightarrow \bar{u} = 1 \qquad \frac{\partial \bar{u}}{\partial \eta} = 0 \qquad \frac{\partial^2 \bar{u}}{\partial \eta^2} = 0$$

$$\bar{u} = 2\eta - 2\eta^3 + \eta^4 - \frac{\Lambda}{6}\left(\eta - 3\eta^2 + 3\eta^3 - \eta^4\right)$$

Boundary-Layer Approximations
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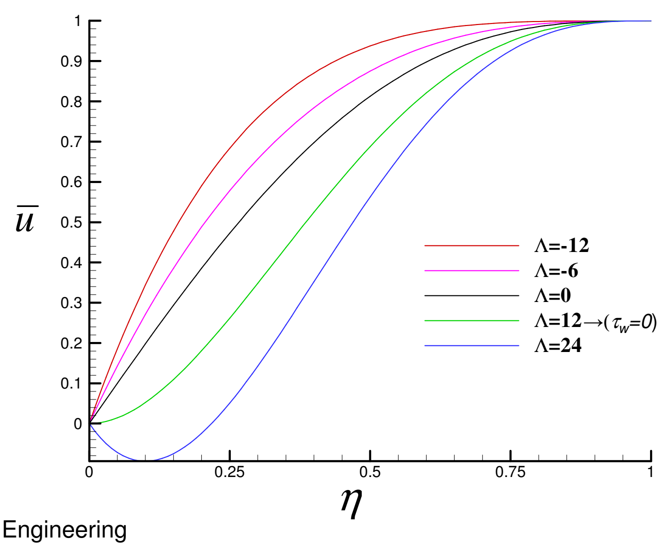
- Shape of the velocity profile



Boundary-Layer Approximations

Pressure gradient effect

- Shape of the velocity profile

	$\frac{dp}{dx}$	$\frac{\delta^*}{\delta}$	$\frac{\theta}{\delta}$	H	C_f (1)
Adverse	>0	↗	↗	↗	↘
Favourable	<0	↘	↘	↘	↗

(1) C_f is equal to zero at a separation point

Boundary-Layer Approximations

Pressure gradient effect

- In zero pressure gradient, momentum diffusion is responsible for the growth of the boundary-layer
- Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \Leftrightarrow v=\int_0^y -\frac{\partial u}{\partial x}dy$$

- Outer flow

$$-\frac{dU_e}{dx}=\frac{1}{\rho U_e}\frac{dp}{dx}$$

Boundary-Layer Approximations
Pressure gradient effect

- Effect of convection on the growth rate (δ)

	$\frac{dp}{dx}$	v	δ
Adverse	> 0	↗	↗
Favourable	< 0	↘	↘

Boundary-Layer Approximations
Pressure gradient effect

- Combined effects

	$\frac{dp}{dx}$	δ	δ^*	θ	H	C_f (1)
Adverse	> 0	↗	↗	↗	↗	↘
Favourable	< 0	↘	↘	↘	↘	↗

(1) C_f is equal to zero at a separation point

Boundary-Layer Approximations
Thwaites's method

- von Kármán's integral equations

$$\frac{d\theta}{dx}+\theta\frac{H+2}{U_e}\frac{dU_e}{dx}=\frac{C_f}{2}$$

- Wall boundary conditions

$$\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{U_e}{\theta}l \qquad \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}=-\frac{U_e}{\theta^2}\lambda$$

Reference length $\rightarrow \theta$
Reference velocity $\rightarrow U_e$

Boundary-Layer Approximations
Thwaites's method

- Pressure gradient parameter, λ

$$\lambda=\frac{\theta^2}{\nu}\frac{dU_e}{dx}$$

- Parameter related to the shear-stress at the wall, l

$$l=\frac{U_e\theta C_f}{2\nu}$$

Boundary-Layer Approximations
Thwaites's method

- Thwaites hypothesis

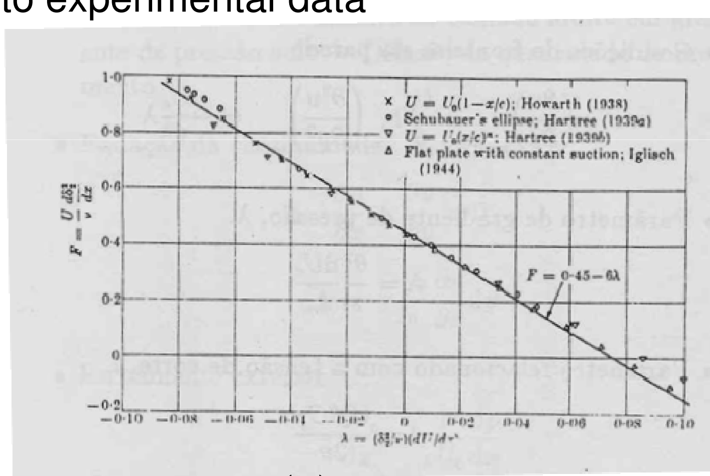
$$l = l(\lambda) \qquad H = H(\lambda)$$

- Using it in von Kármán's integral equation

$$\frac{U_e}{\nu}\frac{d\theta^2}{dx} = 2\{-\lambda[H(\lambda)+2]+l(\lambda)\} \equiv F(\lambda)$$

Boundary-Layer Approximations
Thwaites's method

- Fit to experimental data



$$F(\lambda) = 0{,}45 - 6\lambda$$

Boundary-Layer Approximations
Thwaites's method

$$\frac{d\theta^2 U_e^6}{dx} = 0{,}45\nu U_e^5$$

- Integrating

$$\theta^2 U_e^6 = 0{,}45\nu \int_0^x U_e^5 dx + \left(\theta^2 U_e^6\right)_0$$

Aerodynamics

Boundary-Layer Approximations
Thwaites's method

1. Calculate θ

$$\theta^2 U_e^6 = 0{,}45\nu \int_0^x U_e^5 dx + \left(\theta^2 U_e^6\right)_0$$

2. Calculate λ

$$\lambda = \frac{\theta^2}{\nu} \frac{dU_e}{dx}$$

Boundary-Layer Approximations

Thwaites's method

3. Calculate *H* and *l* from

$$l = \begin{cases} 0{,}22 + 1{,}402\lambda + \dfrac{0{,}018\lambda}{0{,}107+\lambda} & -0{,}09 \le \lambda < 0 \\ 0{,}22 + 1{,}57\lambda - 1{,}8\lambda^2 & 0 \le \lambda < 0{,}25 \end{cases}$$

$$H = \begin{cases} 2{,}088 + \dfrac{0{,}0731}{0{,}14+\lambda} & -0{,}09 \le \lambda < 0 \\ 2{,}61 - 3{,}75\lambda + 5{,}24\lambda^2 & 0 \le \lambda < 0{,}25 \end{cases}$$

Boundary-Layer Approximations

Thwaites's method

4. Calculate δ^* and C_f from *H* and *l*

Flow Separation $\Rightarrow \tau_w = 0,\ C_f = 0,\ \ l = 0$

$$l = 0 \Rightarrow \lambda = -0{,}09$$