Boundary-Layer Approximations
von Kármán Integral Equation

- Integrate boundary-layer equations in the direction normal to the wall up to the outer flow $(h > \delta)$

- Continuity equation

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dy = 0$$

$$v = -\int_0^h \frac{\partial u}{\partial x} dy$$

Aerodynamics

Boundary-Layer Approximations
von Kármán Integral Equation

- Outer flow (ideal fluid)

$$p + \frac{1}{2}\rho U_e^2 = const.$$

$$\frac{1}{\rho}\frac{dp}{dx} = -U_e \frac{dU_e}{dx}$$

Boundary-Layer Approximations
von Kármán Integral Equation

- Momentum equation in the *x* direction

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{dp}{dx} \right) dy = \int_0^h \nu \frac{\partial^2 u}{\partial y^2} dy$$

Boundary-Layer Approximations
von Kármán Integral Equation

— Diffusion

$$\int_0^h \nu \frac{\partial^2 u}{\partial y^2} dy = \int_0^h \frac{1}{\rho}\frac{\partial \tau}{\partial y} dy$$

$$\int_0^h \frac{1}{\rho}\frac{\partial \tau}{\partial y} dy = \left[\frac{\tau}{\rho} \right]_0^h = \frac{-\tau_w}{\rho}$$

- $\tau_w \rightarrow$ Shear-stress at the wall
- For $h>\delta$, $\tau \approx 0$

Boundary-Layer Approximations
von Kármán Integral Equation

$$\int_0^h \left[u\frac{\partial u}{\partial x} - \left(\int_0^y \frac{\partial u}{\partial x} dy \right) \frac{\partial u}{\partial y} - U_e \frac{dU_e}{dx} \right] dy = -\frac{\tau_w}{\rho} \quad (2)$$

— Integration by parts

$$\int_0^h \left[\left(\int_0^y \frac{\partial u}{\partial x} dy \right) \frac{\partial u}{\partial y} \right] dy$$

$$\int_0^h f(y) g'(y) dy = \left[f(y) g(y) \right]_0^h - \int_0^h f'(y) g(y) dy$$

Boundary-Layer Approximations
von Kármán Integral Equation

$$\begin{cases} f(y) = \displaystyle\int_0^h \frac{\partial u}{\partial x} dy \\ g'(y) = \dfrac{\partial u}{\partial y} \end{cases}$$

$$\int_0^h \left[\left(\int_0^y \frac{\partial u}{\partial x} dy \right) \frac{\partial u}{\partial y} \right] dy = \left[\left(\int_0^y \frac{\partial u}{\partial x} dy \right) u \right]_0^h - \int_0^h \left(\frac{\partial u}{\partial x} u \right) dy$$

$$\int_0^h \left[\left(\int_0^y \frac{\partial u}{\partial x} dy \right) \frac{\partial u}{\partial y} \right] dy = \int_0^h U_e \frac{\partial u}{\partial x} dy - \int_0^h \left(u \frac{\partial u}{\partial x} \right) dy$$

Boundary-Layer Approximations
von Kármán Integral Equation

— Using the equality

$$\frac{\partial u^2}{\partial x} = 2u\frac{\partial u}{\partial x}$$

and applying it in (2)

$$\int_0^h \left(U_e \frac{\partial u}{\partial x} + U_e \frac{dU_e}{dx} - \frac{\partial u^2}{\partial x} \right) dy = \frac{\tau_w}{\rho} \quad (3)$$

Boundary-Layer Approximations
von Kármán Integral Equation

$$\frac{\partial (uU_e)}{\partial x} = U_e \frac{\partial u}{\partial x} + u\frac{dU_e}{dx}$$

$$\int_0^h \left(\frac{\partial (uU_e)}{\partial x} - u\frac{dU_e}{dx} + U_e \frac{dU_e}{dx} - \frac{\partial u^2}{\partial x} \right) dy = \frac{\tau_w}{\rho}$$

$$\int_0^h \left(\frac{\partial}{\partial x}\left(uU_e - u^2\right) + \frac{dU_e}{dx}\left(U_e - u\right) \right) dy = \frac{\tau_w}{\rho}$$

Boundary-Layer Approximations
von Kármán Integral Equation

- The integration limit h does not depen on x. Therefore, the derivatives with respect to x may permute with the integration in y

$$\frac{d}{dx}\left[\int_0^h u(U_e-u)dy\right]+\frac{dU_e}{dx}\int_0^h (U_e-u)dy=\frac{\tau_w}{\rho}$$

- Making u dimensionless using U_e

$$\frac{d}{dx}\left[U_e^2\int_0^h \frac{u}{U_e}\left(1-\frac{u}{U_e}\right)dy\right]+\frac{dU_e}{dx}U_e\int_0^h\left(1-\frac{u}{U_e}\right)dy=\frac{\tau_w}{\rho}$$

Boundary-Layer Approximations
Integral parameters

Displacement thickness, δ^*

$$\delta^*=\int_0^h\left(1-\frac{u}{U_e}\right)dy$$

- For $h\geq\delta$, $u/U_e \simeq 1$, consequently

$$\delta^*\cong\int_0^\delta\left(1-\frac{u}{U_e}\right)dy$$

Boundary-Layer Approximations

Integral parameters

Displacement thickness, δ^*

- Mass flow rate in section $\delta \times 1$ for ideal fluid conditions

$$\dot{Q}_{ideal} = \rho \int_0^{\delta} U_e dy$$

- Real mass flow rate in section $\delta \times 1$

$$\dot{Q}_{real} = \rho \int_0^{\delta} u dy$$

Boundary-Layer Approximations
Integral parameters
Displacement thickness, δ^*

- The displacement thickness is related to the flow rate deficit imposed by the presence of the boundary-layer

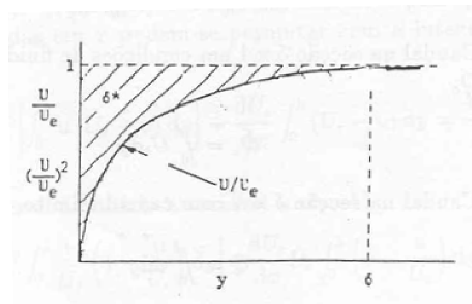
$$\rho U_e \ \delta^* = \dot{Q}_{ideal} - \dot{Q}_{real} = \rho \int_0^{\delta} U_e dy - \rho \int_0^{\delta} u dy$$

Boundary-Layer Approximations

Integral parameters

Displacement thickness, δ^*

$$\delta^* \cong \int_0^{\delta} \left(1 - \frac{u}{U_e}\right) dy$$



- δ^* is equivalent to the displacement of the streamlines of the outer flow due to the effect of the boundary-layer.

Boundary-Layer Approximations

Integral parameters

Displacement thickness, δ^*

- Using $\eta = \dfrac{y}{\delta}$ we obtain

$$\frac{\delta^*}{\delta} = \int_0^{1} \left(1 - \frac{u}{U_e}\right) d\eta$$

which is exclusively defined by the dimensionless velocity profile

$$\frac{u}{U_e} = f\left(\frac{y}{\delta}\right)$$

Boundary-Layer Approximations

Integral parameters

Momentum thickness, θ

$$\theta = \int_0^h \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$

- For $h \geq \delta$, $u/U_e \simeq 1$, accordingly

$$\theta \cong \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$

Boundary-Layer Approximations

Integral parameters

Momentum thickness, θ

- Momentum flow rate in section $\delta \times 1$ for the real mass flow rate at the velocity of an ideal fluid

$$\dot{M}_{ideal} = \dot{Q}_{real} U_e = \rho \int_0^\delta u U_e dy$$

- Real momentum flow rate in section $\delta \times 1$ (obviously for the real mass flow rate)

$$\dot{M}_{real} = \rho \int_0^\delta u^2 dy$$

Boundary-Layer Approximations

Integral parameters

Momentum thickness, θ

- The momentum thickness is related to the reduction of the momentum flow rate (forces) due to the presence of the boundary-layer. θ must be determined for the mass flow rate that crosses the section δ×*1,* taking into account δ^*

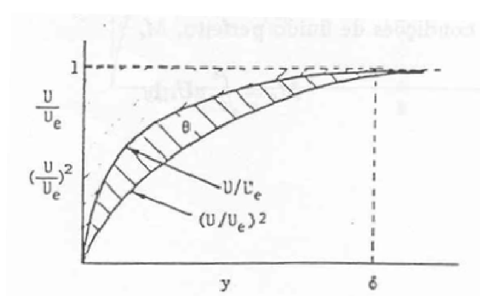
$$\rho U_e^2 \theta = \dot{M}_{ideal} - \dot{M}_{real} = \rho \int_0^\delta u U_e dy - \rho \int_0^\delta u^2 dy$$

Boundary-Layer Approximations

Integral parameters

Momentum thickness, θ

$$\theta \cong \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$



- Since θ is related to a deficit of momentum its change must be related to the forces acting on the fluid

Boundary-Layer Approximations

Integral parameters

Momentum thickness, θ

- Using $\eta = \frac{y}{\delta}$ we obtain

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) d\eta$$

which is exclusively defined by the dimensionless velocity profile

$$\frac{u}{U_e} = f\left(\frac{y}{\delta}\right)$$

Boundary-Layer Approximations

Integral parameters

Shape Factor, *H*

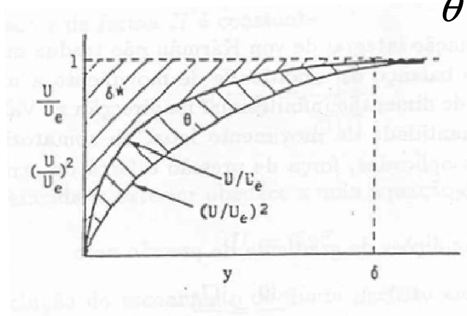
$$H = \frac{\delta^*}{\theta} = \frac{\frac{\delta^*}{\delta}}{\frac{\theta}{\delta}}$$ depends only on $$\frac{u}{U_e} = f\left(\frac{y}{\delta}\right)$$

- *H* quantifies the shape of the velocity profile. Since the integrand of the θ definition is always smaller than that of δ^*, $H \geq 1$, with a limit of 1 for a uniform profile.

Boundary-Layer Approximations

Integral parameters

Shape Factor, $H = \dfrac{\delta^*}{\theta}$



- Skin friction coefficient, C_f

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2}$$

Boundary-Layer Approximations

von Kármán Integral Equation

- Using the integral parameters in the integrated equation leads to

$$\frac{d}{dx}\left(U_e^2\theta\right)+U_e\delta^*\frac{dU_e}{dx}=\frac{\tau_w}{\rho}$$

$$\frac{d\theta}{dx}+\theta\frac{H+2}{U_e}\frac{dU_e}{dx}=\frac{C_f}{2}$$

Boundary-Layer Approximations
Self-similar laminar flows

- General case

$$\frac{u}{U_e} = F(x, y)$$

- Self-similar flows

$$\frac{u}{U_e} = F(\eta) \quad \text{with} \quad \eta = \frac{y}{\mathcal{O}[\delta]} \quad \text{and} \quad \mathcal{O}[\delta] = \sqrt{\frac{\nu x}{U_e}}$$

$$u(x, y) = U_e(x) F(\eta)$$

Aerodynamics

Boundary-Layer Approximations
Self-similar laminar flows

- In such conditions, the shape factor *H* is constant

$$H = \frac{\delta^*}{\theta} = \frac{\dfrac{\delta^*}{\mathcal{O}[\delta]}}{\dfrac{\theta}{\mathcal{O}[\delta]}}$$

Boundary-Layer Approximations

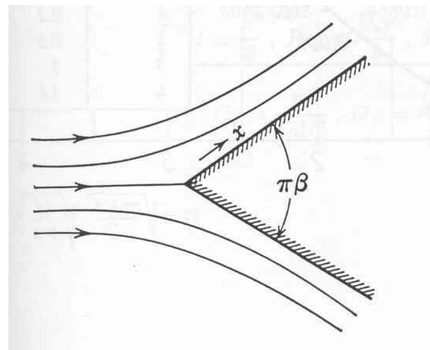
Self-similar laminar flows

- Outer velocity given by the following equation

$$U_e = Cx^m$$

- Ideal flow solution for the flow at a corner of angle $\pi\beta$

$$\beta = \frac{2m}{m+1} \qquad m = \frac{\beta}{2-\beta}$$



Aerodynamics

Boundary-Layer Approximations

Self-similar laminar flows

$$U_e = Cx^m$$

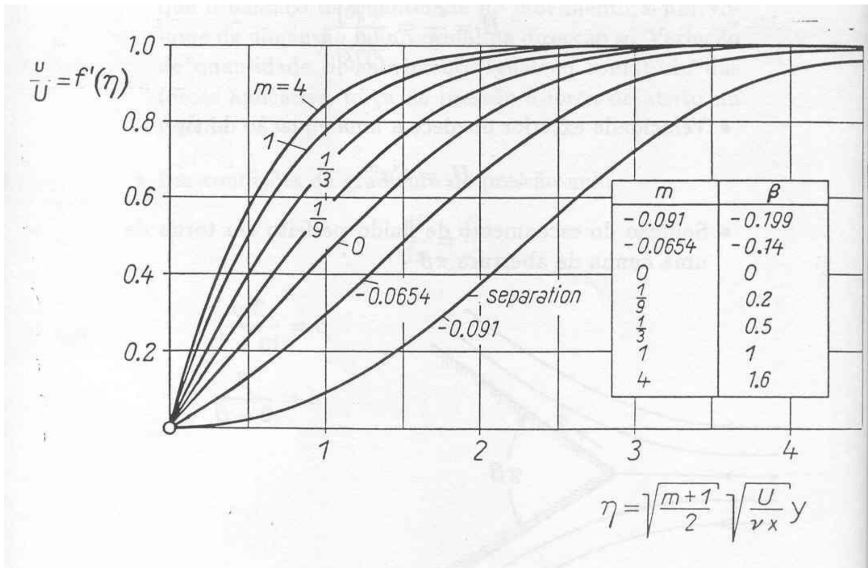
- m=0→ Zero pressure gradient flow
- m=1→ Stagnation point flow
- m=-0.0904→ Velocity profile with τ_w=0

$$\beta = \frac{2m}{m+1} \qquad m = \frac{\beta}{2-\beta}$$

Masters of Mechanical Engineering

Boundary-Layer Approximations

Self-similar laminar flows



m	β
-0.091	-0.199
-0.0654	-0.14
0	0
1/9	0.2
1/3	0.5
1	1
4	1.6

Boundary-Layer Approximations

von Kármán Integral Equation

- Zero pressure gradient

$$\frac{d\theta}{dx} = \frac{C_f}{2}$$

- Dimensionless velocity profile is sufficient to obtain the solution

$$\frac{u}{U_e} = f\left(\frac{y}{\delta}\right)$$

Boundary-Layer Approximations
von Kármán Integral Equation

- 2 unknowns, θ and C_f, for 1 equation $\dfrac{d\theta}{dx} = \dfrac{C_f}{2}$

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) d\eta$$

$$C_f = \frac{2\nu}{U_e\delta}\left(\frac{\partial\, u/U_e}{\partial\, y/\delta}\right)_{y/\delta=0}$$

$$\delta\frac{d\delta}{dx} = \left(\frac{\delta}{\theta}\right)\frac{2\nu}{U_e}\left(\frac{\partial\, u/U_e}{\partial\, y/\delta}\right)_{y/\delta=0} = constant.$$

Boundary-Layer Approximations
von Kármán Integral Equation

$$\delta\frac{d\delta}{dx} = \left(\frac{\delta}{\theta}\right)\frac{2\nu}{U_e}\left(\frac{\partial\, u/U_e}{\partial\, y/\delta}\right)_{y/\delta=0} = constant.$$

$$\int_0^\delta \delta d\delta = \int_0^x \left(\frac{\delta}{\theta}\right)\frac{2\nu}{U_e}\left(\frac{\partial\, u/U_e}{\partial\, y/\delta}\right)_{y/\delta=0} dx$$

$$\frac{\delta^2}{2} = \left(\frac{\delta}{\theta}\right)\frac{2\nu x}{U_e}\left(\frac{\partial\, u/U_e}{\partial\, y/\delta}\right)_{y/\delta=0}$$

Boundary-Layer Approximations
von Kármán Integral Equation

$$\delta^2 = 4\left(\frac{\delta}{\theta}\right)\frac{\nu x}{U_e}\left(\frac{\partial u/U_e}{\partial y/\delta}\right)_{y/\delta=0}$$

$$\left(\frac{\delta}{x}\right)^2 = 4\left(\frac{\delta}{\theta}\right)\frac{\nu}{U_e x}\left(\frac{\partial u/U_e}{\partial y/\delta}\right)_{y/\delta=0}$$

$$\left(\frac{\delta}{x}\right) = \sqrt{4\left(\frac{\delta}{\theta}\right)\left(\frac{\partial u/U_e}{\partial y/\delta}\right)_{y/\delta=0}}R_{e_x}^{-0.5}$$

Boundary-Layer Approximations
von Kármán Integral Equation

- Boundary-layer parameters

$\bar{u} = f(\eta)$ com $\bar{u} = \frac{u}{U_e}, \quad \eta = \frac{y}{\delta}$

$$\delta^* = \delta\int_0^1 (1-\bar{u})d\eta, \ \theta = \delta\int_0^1 \bar{u}(1-\bar{u})d\eta$$

$$H = \frac{\delta^*}{\theta}, \ R_{e_x} = \frac{U_e x}{\nu}, \ R_{e_L} = \frac{U_e L}{\nu}$$

$$C_f = \frac{\tau_w}{1/2\rho U_e^2}, \quad C_D = \frac{\int_0^L \tau_w dx}{1/2\rho U_e^2 L}$$

Boundary-Layer Approximations

von Kármán Integral Equation

Approximate and exact solutions for $\frac{dp}{dx}=0$

$f(\eta)$	$\delta/x\sqrt{R_{e_x}}$	$\delta^*/x\sqrt{R_{e_x}}$	$\theta/x\sqrt{R_{e_x}}$	H	$C_f\sqrt{R_{e_x}}$	$C_D\sqrt{R_{e_x}}$
η	3,464	1,732	0,578	3,00	0,578	1,16
$3/2\eta-1/2\eta^3$	4,64	1,74	0,646	2,70	0,646	1,29
$2\eta-2\eta^3+\eta^4$	5,84	1,752	0,687	2,55	0,687	1,37
$\sin\left(\frac{\pi}{2}\eta\right)$	4,791	1,741	0,655	2,66	0,655	1,31
exact	—(5)	1,721	0,664	2,59	0,664	1,33