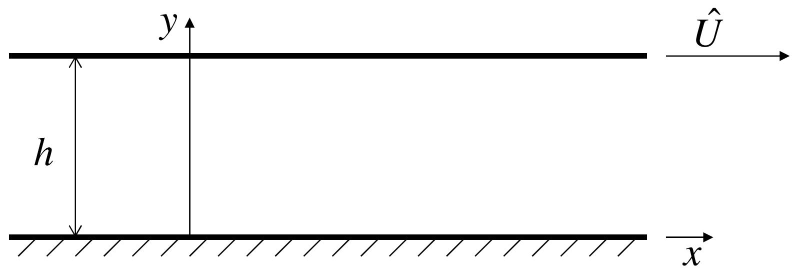
Incompressible, Laminar Couette Flow



- Steady-flow, $\frac{\partial}{\partial t} = 0$
- Flow independent of *z* direction, $\frac{\partial}{\partial z} = 0$
  (two-dimensional)
- Fully developed flow, $\frac{\partial \vec{v}}{\partial x} = 0$

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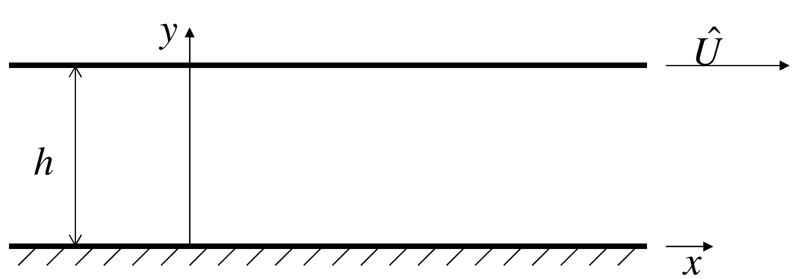
Incompressible, Laminar Couette Flow



- Boundary conditions
  - Impermeability of the walls:
    $y = 0 \Rightarrow v = 0 \qquad y = h \Rightarrow v = 0$
  - No-slip condition:
    $y = 0 \Rightarrow u = 0 \qquad y = h \Rightarrow u = \hat{U}$

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Incompressible, Laminar Couette Flow

- Continuity equation

$$\frac{\partial v}{\partial y} = 0 \Leftrightarrow v = const.$$

- Boundary condition

$$v = 0$$

Incompressible, Laminar Couette Flow

- Momentum balance, x

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

- Momentum balance, y

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

- Pressure is only a function of x

$\frac{dp}{dx}$ must be indepedent of x $\left(\frac{\partial \vec{v}}{\partial x} = 0\right)$

Incompressible, Laminar Couette Flow

- Momentum balance, x

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{dp}{dx} = \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y}$$

$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

- Boundary conditions

$$y = 0 \Rightarrow u = 0$$

$$y = h \Rightarrow u = \hat{U}$$

Incompressible, Laminar Couette Flow

- Solution

$$u = \frac{y}{h}\hat{U} - \frac{1}{2\mu}\frac{dp}{dx} y(h-y)$$

$$\tau_{yx} = \mu\frac{\hat{U}}{h} + \frac{dp}{dx}\left(y - \frac{h}{2}\right)$$

- Reference values for length and velocity

$$L_{ref} = h$$

$$U_{ref} = \hat{U}$$

Incompressible, Laminar Couette Flow

- Solution in dimensionless variables

$$\frac{u}{\hat{U}} = \frac{y}{h}\left[1 - \Lambda\left(1 - \frac{y}{h}\right)\right]$$

$$\frac{\tau_{yx}}{1/2\rho\hat{U}^2} = \frac{2}{\mathrm{Re}}\left[1 - 2\Lambda\left(\frac{1}{2} - \frac{y}{h}\right)\right]$$

- Non-dimensional numbers

$$R_e = \frac{\hat{U}h}{\nu}$$ Reynolds number

$$\Lambda = \frac{h^2}{2\mu U}\frac{dp}{dx}$$ Pressure gradient parameter

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- Non-dimensional numbers

$$R_e \propto \frac{\dfrac{\rho\hat{U}^2}{h}}{\mu\dfrac{\hat{U}}{h^2}}$$ Reynolds number

$$\Lambda = \frac{\dfrac{dp}{dx}}{\mu\dfrac{\hat{U}}{h^2}}$$ Pressure gradient parameter

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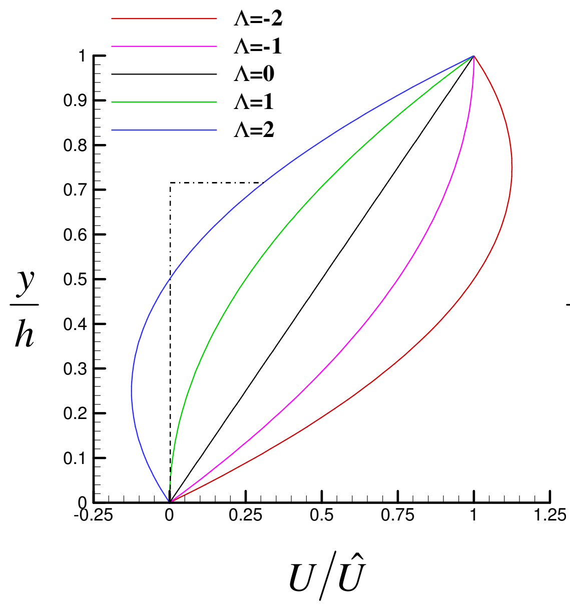
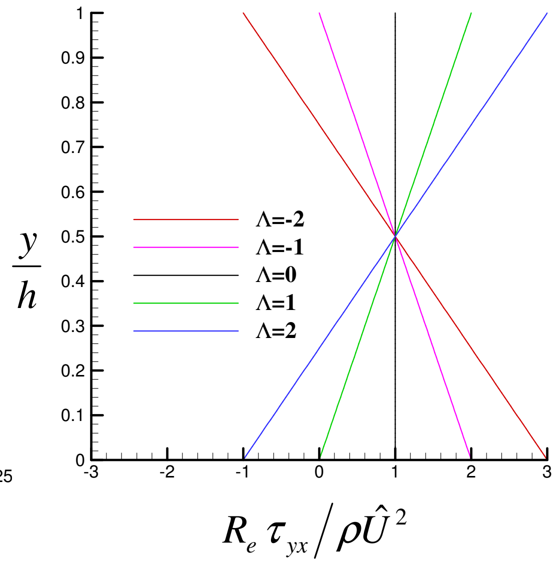
Incompressible, Laminar Couette Flow





Two-dimensional, Incompressible Steady Flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\nu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + 2\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right)$$

Two-dimensional, Incompressible Steady Flow

Constant viscosity, *ν=constant*

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Two-dimensional, Incompressible Steady Flow

Making the equations dimensionless

Reference values

Velocity $U_e \rightarrow u=U_e u^*,\ v=U_e v^*$

Length $L \rightarrow x=Lx^*,\quad y=Ly^*$

Pressure $\rho U_e^2 \rightarrow p=\rho U_e^2 p^*$

Two-dimensional, Incompressible Steady Flow

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{R_e}\left(\frac{\partial^2 u^*}{\partial\left(x^*\right)^2}+\frac{\partial^2 u^*}{\partial\left(y^*\right)^2}\right)$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{\partial p^*}{\partial y^*}+\frac{1}{R_e}\left(\frac{\partial^2 v^*}{\partial\left(x^*\right)^2}+\frac{\partial^2 v^*}{\partial\left(y^*\right)^2}\right)$$

$$R_e=\frac{\rho U_e L}{\mu}=\frac{U_e L}{\nu}=\frac{\rho U_e\frac{U_e}{L}}{\mu\frac{U_e}{L^2}}=\mathcal{O}\left[\frac{\text{convection}}{\text{diffusion}}\right]$$

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Two-dimensional, Incompressible Steady Flow

$$\tau=\mu\frac{\partial u}{\partial y}$$ (uni-dimensional shear-stress)

Ar $\mu \simeq 1,8\times10^{-5}\,\mathrm{kgm^{-1}s^{-1}}$ $\nu \simeq 1,1\times10^{-5}\,\mathrm{m^2s^{-1}}$

Água $\mu \simeq 1,0\times10^{-3}\,\mathrm{kgm^{-1}s^{-1}}$ $\nu \simeq 1,0\times10^{-6}\,\mathrm{m^2s^{-1}}$

- Practical applications are usually flows at high Reynolds numbers, $R_e > 10^5$

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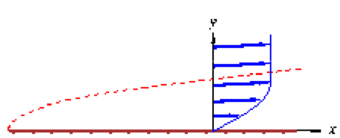
Two-dimensional, Incompressible Steady Flow

- Effects of shear-stresses are restricted to small regions that exhibit large velocity variations in small distances

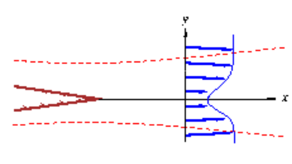
- Thin shear layers
  - Thickness of the shear layer, δ, is much smaller than the reference length L, $\delta/L \ll 1$

Two-dimensional, Incompressible Steady Flow

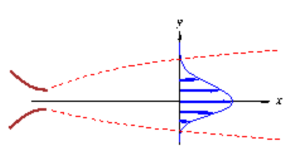
Boundary-layer



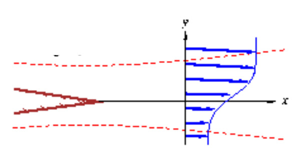
Wake



Jet

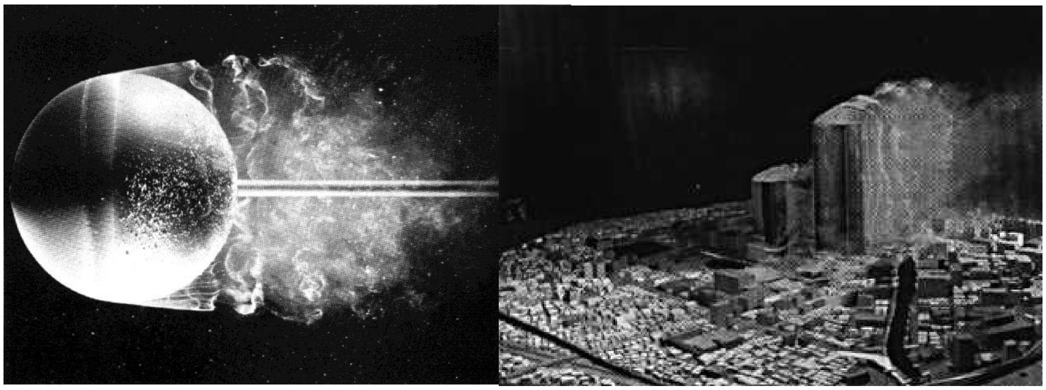


Mixing layer



Two-dimensional, Incompressible Steady Flow

Thick shear layers (Bluff bodies)



Boundary-Layer Approximations

Prandtl simplifications (1904)

Analysis of the order of magnitude of the terms included in the continuity and momentum balance equations

Starting hypothesis: $R_e \gg 1$. ($\delta/L \ll 1$) $R_e = \dfrac{U_e x}{\nu}$

Boundary-Layer Approximations

Prandtl Simplifications (1904)

Order of magnitude of variable ξ, $O[\xi]$, is given by the upper limit of the ξ variation

Known orders of magnitude

$$O[x] \rightarrow L$$
$$O[y] \rightarrow \delta$$
$$O[u] \rightarrow U_e$$

Boundary-Layer Approximations

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_e}{L} + \frac{O[v]}{\delta} = 0$$

$$O[v] = \frac{U_e \delta}{L}$$

Boundary-Layer Approximations

Bernoulli's equation applied to the outer flow (ideal fluid)

$$p+\frac{1}{2}\rho U_e^2 = const.$$

$$\frac{dp_e}{dx}+\rho U_e\frac{dU_e}{dx}=0$$

$$O\left[\frac{1}{\rho}\frac{dp}{dx}\right]=\frac{1}{\rho}\frac{p_e}{L}=\frac{U_e^2}{L}$$

Boundary-Layer Approximations

Momentum balance in the *x* direction

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{U_e^2}{L}+\frac{U_e^2}{L}=\frac{U_e^2}{L}+\nu\left(\frac{U_e}{L^2}+\frac{U_e}{\delta^2}\right)$$

$$\frac{U_e^2}{L}+\frac{U_e^2}{L}=\frac{U_e^2}{L}+\frac{U_e^2}{L}\frac{\nu}{U_e L}\left[1+\left(\frac{L}{\delta}\right)^2\right]$$

$$1+1=1+\frac{1}{R_e}\left[1+\left(\frac{L}{\delta}\right)^2\right]$$

Boundary-Layer Approximations

Momentum balance in the *x* direction

Analysis of diffusion $\frac{1}{R_e}\left[1+\left(\frac{L}{\delta}\right)^2\right]$

$$\mathcal{O}\left[\nu\frac{\partial^2 u}{\partial x^2}\right]=\frac{1}{R_e} \qquad \cong 0$$

$$\mathcal{O}\left[\nu\frac{\partial^2 u}{\partial x^2}\right]=\frac{1}{R_e}\left(\frac{L}{\delta}\right)^2 \Rightarrow \frac{\delta}{L}=\frac{1}{\sqrt{R_e}}\ll 1$$

Boundary-Layer Approximations

Momentum balance in the *y* direction

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{U_e^2\delta}{L^2}+\frac{U_e^2\delta}{L^2}=\mathcal{O}\left[-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]+\nu\left(\frac{U_e\delta}{L^3}+\frac{U_e}{L\delta}\right)$$

$$\frac{U_e^2\delta}{L^2}+\frac{U_e^2\delta}{L^2}=\mathcal{O}\left[-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]+\frac{\nu}{U_e L}\left[\frac{U_e^2\delta}{L^2}+\frac{U_e^2\delta}{L^2}\left(\frac{L}{\delta}\right)^2\right]$$

Boundary-Layer Approximations

Momentum balance in the *y* direction

Using $\left(\frac{L}{\delta}\right)^2 = R_e$ we obtain

$$1+1=\frac{L^2}{U_e^2\delta}\,\mathcal{O}\left[-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]+\frac{1}{R_e}+1$$

$$\mathcal{O}\left[-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]=\frac{U_e^2\delta}{L^2}$$

Boundary-Layer Approximations

Momentum balance in the *y* direction

Across the boundary-layer

$$\mathcal{O}\left[\int_0^\delta \frac{\partial p}{\partial y}dy\right]=\left(\frac{\delta}{L}\right)^2\rho U_e^2=\frac{1}{R_e}\rho U_e^2 \ll \frac{1}{2}\rho U_e^2$$

Therefore,

$$\frac{\partial p}{\partial y}\cong 0$$

Boundary-Layer Approximations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

- The selected coordinate system must respect the following conditions:
  1. The *x* coordinate must be aligned with the outer flow
  2. The *y* coordinate is normal to the surface

Aerodynamics

Boundary-Layer Approximations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

- Static pressure is independent of the coordinate *y*. Pressure change with *x* (*dp/dx*) may be obtained from the outer flow, $p(x)\simeq p_e(x)$. Therefore, the pressure does not belong to the unknowns.

The pressure is part of the input of a boundary-layer problem

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Boundary-Layer Approximations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

- The equations are no longer elliptic in the *x* direction. For a given value of *x*, the solution depends only on the upstream conditions. Therefore, it is possible to solve the problem using a marching procedure in the *x* direction (initial value problem).

Simplified Forms of the Navier-Stokes Equations

- Boundary layer, thin shear layer equations

  — Pressure determined by the outer flow,

$$\frac{\partial p}{\partial y}\cong 0$$

  — Diffusion in the main direction of the flow neglected,

$$\nu\left(\frac{\partial^2 u}{\partial x^2}\right)\cong 0$$

Simplified Forms of the Navier-Stokes Equations

- Parabolized Navier-Stokes equations

— Pressure derivative in the main direction of the flow determined by the outer flow,

$$\frac{\partial p}{\partial x} \cong \frac{\partial p_e}{\partial x}$$

— Diffusion in the main direction of the flow neglected,

$$\nu\left(\frac{\partial^2 u}{\partial x^2}\right) \cong 0$$

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Simplified Forms of the Navier-Stokes Equations

- Reduced Navier-Stokes equations

— Diffusion in the main direction of the flow neglected,

$$\nu\left(\frac{\partial^2 u}{\partial x^2}\right) \cong 0$$

— Pressure determination makes the problem elliptic

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