Description of the flow field

- Eulerian methodology

  - Physical principles applied to a fixed volume in space

  - Time derivative includes two contributions

    1. Change in time for a fixed position in space
    2. Point to point variation in space for a given instant in time

Basic Concepts

- Material Derivative

$q = q(x, y, z, t)$ $\longrightarrow$ Generic property

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial q}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}$$

Basic Concepts

- Gauss's divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{Q}dV = \int_S \vec{Q}\cdot\vec{n}dS$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z$$

$\vec{\nabla}\cdot\vec{Q} \rightarrow$ Balance of a vector field $\vec{Q}$ for an infinitesimal volume

Aerodynamics

Basic Concepts

- Transformation of the time derivative in a volume that changes in time (*V*) to a fixed volume (V_o)

$$\frac{D}{Dt}\int_V \rho\xi dV = \int_{V_o}\frac{\partial}{\partial t}(\rho\xi)dV + \int_{S_o}\rho\xi(\vec{v}\cdot\vec{n})dS$$

$\xi \rightarrow$ Generic property per unit mass

Masters of Mechanical Engineering

Balance of a generic property ("Conservation equation")

- Volume changing in time

$$\frac{D}{Dt}\int_V \rho\xi dV = \int_V f_\xi dV$$

$f_\xi \rightarrow$ sources/sinks of property ξ

Balance of a generic property ("Conservation equation")

- Volume fixed in time

$$\int_{V_o} \frac{\partial}{\partial t}(\rho\xi)dV + \int_{V_o} \rho\xi(\vec{v}\cdot\vec{n})dS = \int_{V_o} f_\xi dV$$

$$\int_{V_o}\left[\frac{\partial}{\partial t}(\rho\xi) + \vec{\nabla}\cdot(\rho\xi\vec{v}) - f_\xi\right]dV = 0$$

- V_o is arbitrary

$$\frac{\partial}{\partial t}(\rho\xi) + \vec{\nabla}\cdot(\rho\xi\vec{v}) - f_\xi = 0$$

Balance of a generic property ("Conservation equation")

Property	ξ	f_ξ
Mass	1	—
Momentum	$\vec{v}$	Forces
Energy	$e=\bar{u}+\frac{v^2}{2}+gz$	Heat Work

Mass Conservation (continuity equation)

- Integral form

$$\int_{V_o}\frac{\partial \rho}{\partial t}dV+\int_{V_o}\rho(\vec{v}\cdot\vec{n})dS=0$$

- Differential form

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

$$\frac{D\rho}{Dt}+\rho\left(\vec{\nabla}\cdot\vec{v}\right)=0$$

Mass Conservation (continuity equation)

- Incompressible fluid (*ρ=constant*)
- Integral form

$$\int_{V_o} (\vec{v} \cdot \vec{n}) dS = 0$$

- Differential form

$$\vec{\nabla} \cdot \vec{v} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum Balance

- Integral form

$$\int_{V_o} \frac{\partial \rho \vec{v}}{\partial t} dV + \int_{V_o} \rho \vec{v} (\vec{v} \cdot \vec{n}) dS = \vec{F}$$

$\vec{F} \rightarrow$ Sum of the forces acting on the fluid for the control volume V_o

- Pressure force + Normal stresses
- Shear-stress
- Body forces (gravity)

Momentum Balance

- Relation between forces and dependent variables

$$\vec{F} = \int_{V_o} \left(-\vec{\nabla} p + \vec{\nabla} \cdot \vec{\tau}_{ij} + \rho \vec{g} \right)$$

$$F_x = \int_{V_o} \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right)$$

$$F_y = \int_{V_o} \left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} - \rho g \right)$$

$$F_z = \int_{V_o} \left(-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right)$$

Momentum Balance

(Navier-Stokes)

- Differencial form

$$\frac{\partial \rho \vec{v}}{\partial t} + u \frac{\partial \rho \vec{v}}{\partial x} + v \frac{\partial \rho \vec{v}}{\partial y} + w \frac{\partial \rho \vec{v}}{\partial z} = -\vec{\nabla} p + \vec{\nabla} \cdot \vec{\tau}_{ij} + \rho \vec{g}$$

$$\frac{\partial \rho u}{\partial t} + u \frac{\partial \rho u}{\partial x} + v \frac{\partial \rho u}{\partial y} + w \frac{\partial \rho u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\frac{\partial \rho v}{\partial t} + u \frac{\partial \rho v}{\partial x} + v \frac{\partial \rho v}{\partial y} + w \frac{\partial \rho v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} - \rho g$$

$$\frac{\partial \rho w}{\partial t} + u \frac{\partial \rho w}{\partial x} + v \frac{\partial \rho w}{\partial y} + w \frac{\partial \rho w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Momentum Balance

(Navier-Stokes)

- Relation between stresses and flow variables (Newton's model)
  - The stresses are linearly proportional to the space derivatives of the velocity components
  - The constants of proportionality are independent of the direction. Isotropic fluid
  - The stresses do not depend explicitly of the position in space or the fluid velocity
  - The stress tensor is symmetric, $\tau_{xy}=\tau_{yx}$, $\tau_{xz}=\tau_{zx}$, $\tau_{yz}=\tau_{zy}$

Momentum Balance

(Navier-Stokes)

- Relation between stresses and flow variables (Newton's model)

$$\sigma_{xx} = \left(\lambda - \frac{2}{3}\mu\right)\Theta + 2\mu\frac{\partial u}{\partial x} + A \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = \left(\lambda - \frac{2}{3}\mu\right)\Theta + 2\mu\frac{\partial v}{\partial y} + A \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{zz} = \left(\lambda - \frac{2}{3}\mu\right)\Theta + 2\mu\frac{\partial w}{\partial z} + A \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Momentum Balance

(Navier-Stokes)

- Relation between stresses and flow variables
  (Newton's model)

$$\sigma_{xx} = \left(\lambda - \frac{2}{3}\mu\right)\Theta + 2\mu\frac{\partial u}{\partial x} + A \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = \left(\lambda - \frac{2}{3}\mu\right)\Theta + 2\mu\frac{\partial v}{\partial y} + A \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{zz} = \left(\lambda - \frac{2}{3}\mu\right)\Theta + 2\mu\frac{\partial w}{\partial z} + A \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- μ, λ and *A* are parameters independent of the gradients of the components of the velocity vector

Momentum Balance

(Navier-Stokes)

- Relation between stresses and flow variables
  (Newton's model)

  - Uniform flow

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = A$$

$$A \equiv -p_{th}$$

  - Average pressure,

$$\overline{p} = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

$$\overline{p} = -\lambda\Theta + p_{th}$$

Momentum Balance

(Navier-Stokes)

- Relation between stresses and fluid variables

(Newton's model)

$$\sigma_{xx} = -\overline{p} - \frac{2}{3}\mu\Theta + 2\mu\frac{\partial u}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = -\overline{p} - \frac{2}{3}\mu\Theta + 2\mu\frac{\partial v}{\partial y} \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{zz} = -\overline{p} - \frac{2}{3}\mu\Theta + 2\mu\frac{\partial w}{\partial z} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- The constants λ e A do not appear in the relations between stresses and gradients of the velocity components

Momentum Balance

- Navier-Stokes equations

$$\frac{\partial \rho u}{\partial t} + u\frac{\partial \rho u}{\partial x} + v\frac{\partial \rho u}{\partial y} + w\frac{\partial \rho u}{\partial z} = -\frac{\partial p}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x}(\mu\Theta) + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right)$$

$$\frac{\partial \rho v}{\partial t} + u\frac{\partial \rho v}{\partial x} + v\frac{\partial \rho v}{\partial y} + w\frac{\partial \rho v}{\partial z} = -\frac{\partial p}{\partial y} - \frac{2}{3}\frac{\partial}{\partial y}(\mu\Theta) + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) - \rho g$$

$$\frac{\partial \rho w}{\partial t} + u\frac{\partial \rho w}{\partial x} + v\frac{\partial \rho w}{\partial y} + w\frac{\partial \rho w}{\partial z} = -\frac{\partial p}{\partial z} - \frac{2}{3}\frac{\partial}{\partial z}(\mu\Theta) + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)$$

Momentum Balance

- Navier-Stokes equations

  Incompressible fluid, *ρ=constant*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\nu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+2\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)-g$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left(\nu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\nu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)+2\frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right)$$

Momentum Balance

- Navier-Stokes equations

  Incompressible fluid, *ρ=constant*

  Constant viscosity, *ν=constant*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-g$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Momentum Balance

- Navier-Stokes equations

  - Change of momentum with time, $\rho \frac{D\vec{v}}{Dt}$
    - Local time derivative, $\rho \frac{\partial \vec{v}}{\partial t}$

      Steady flow if $\frac{\partial \vec{v}}{\partial t} = 0$
    - Convective term, $\rho \left(u \frac{\partial \vec{v}}{\partial x} + v \frac{\partial \vec{v}}{\partial y} + w \frac{\partial \vec{v}}{\partial z} \right)$

Momentum Balance

- Navier-Stokes equations

  - Pressure force
    - Pressure gradient, $\vec{\nabla} p$
  - Viscous forces
    - Diffusive term, $\mu \vec{\nabla} \cdot \vec{\nabla} u_i \quad (u_1 = u, u_2 = v, u_3 = w)$
  - Body force, $\rho \vec{g}$

Momentum Balance

- Boundary conditions
  1. Solid Surface

$$\vec{v}_s = v_{sn}\vec{n} + v_{st}\vec{t} \;\rightarrow\; \text{Surface velocity}$$

$$\vec{v} = v_n\vec{n} + v_t\vec{t} \;\rightarrow\; \text{Fluid velocity}$$

- $v_t = v_{st}$ – No-slip condition
- $v_n = v_{ns}$ – Impermeability condition

Reference frame of the body surface $\Rightarrow \vec{v} = 0$

Momentum Balance

- Boundary conditions
  2. Interface between two non miscible fluids

$$\vec{v}_1 = v_{n1}\vec{n} + v_{t1}\vec{t} \;\rightarrow\; \text{Velocity of fluid 1}$$

$$\vec{v}_2 = v_{n2}\vec{n} + v_{t2}\vec{t} \;\rightarrow\; \text{Velocity of fluid 2}$$

- $\vec{v}_1 = \vec{v}_2$ – Continuity of velocity vector
- $\tau_1 = \tau_2$ – Equality of shear-stress
- $\sigma_1 - \sigma_2 = \Delta p_{ts}$ – Discontinuity of normal stress given by surface tension

$$\Delta p_{ts} = \sigma\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$\sigma \rightarrow$ Surface tension

$r_1 \; r_2 \rightarrow$ Radius of curvature of the surface

Momentum Balance

- Absorbing the body forces in the pressure term

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p + \nu\left(\vec{\nabla}\cdot\vec{\nabla}u_i\right) + \vec{g}$$

  - Fluido at rest

$$0 = -\frac{1}{\rho}\vec{\nabla}p_h + \vec{g} \Leftrightarrow \quad \vec{g} = \frac{1}{\rho}\vec{\nabla}p_h$$

  - $p_h \equiv$ Hydrostatic pressure

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\vec{\nabla}(p - p_h) + \nu\left(\vec{\nabla}\cdot\vec{\nabla}u_i\right)$$

  - $\overline{p} = (p - p_h)$ pressure relative to hydrostatic pressure

Energy Balance

$$\xi = e = \overline{u} + \frac{v^2}{2} + gz$$

- Integral form

$$\int_{V_o}\frac{\partial}{\partial t}\left(\overline{u} + \frac{v^2}{2} + gz\right)dV + \int_{S_o}\left(\overline{h} + \frac{v^2}{2} + gz\right)(\vec{v}\cdot\vec{n})dS = \dot{Q} + \dot{W}$$

- Differencial form

$$\rho\frac{De}{Dt} + \vec{v}\cdot\vec{\nabla}p = \vec{\nabla}\cdot(k\vec{\nabla}T) + \vec{\nabla}\cdot\left(\vec{v}\cdot\vec{\tau}_{ij}\right)$$

$$\vec{\nabla}\cdot\left(\vec{v}\cdot\vec{\tau}_{ij}\right) \equiv \vec{v}\cdot\left(\vec{\nabla}\cdot\vec{\tau}_{ij}\right) + \Phi$$

$\Phi \rightarrow$ Viscous dissipation