QUALIFYING EXAM
(MATHEMATICAL ANALYSIS)
March 2010

Solve, at least partially, all but one of the proposed problems in each group.
Duration: 4 hours

I. Real Analysis

1. Give an example of a subset of $\mathbb{R}^2$ that is connected but not arcwise connected and justify your answer. Could it be that such a situation appears in $\mathbb{R}$?

2. Formulate the Theorem of Ascoli in a general situation (i.e., for families of functions acting on a topological space into an arbitrary Banach space) and explain its role in operator theory (e.g., to proof certain properties of bounded linear operators).

3. Formulate the fundamental lemma of integration and explain how it is used to define the general (Bochner) integral.

4. Prove the following estimates: Let $(X, \mathcal{M}, \mu)$ be a measure space, E a Banach space and $f \in St(\mu, E)$ a step function. Then we have

$$|\int_X f d\mu|_E \leq \int |f|_E d\mu \leq \sup_{x \in X} |f(x)|_E \; \mu(X).$$

5. Give two examples of sequences of functions defined in $\mathbb{R}$ which converge

   (a) with respect to the seminorm of $\mathcal{L}^1$ but not pointwise;

   (b) vice versa.

II. Functional Analysis

1. a) State the Riesz representation theorem for bounded linear functionals on a Hilbert space H.

   b) Prove this theorem.

2. Let H be a complex Hilbert space. Say whether each of the following assertions is True or False and **explain why**.

a) Let $T : H \to H$ be a self-adjoint compact operator with

$$\|T\| = 1.$$

Then at least one of the scalars 1 and -1 is an eigenvalue of T.

b) Let x be an element of H. Then there exists an element φ lying in the dual space H^* of the Hilbert space H such that

$$\varphi(x) = 2\|x\|.$$

c) Let $S : l^2 \to l^2$ be the operator defined, for all sequences (x_n) in l^2, by

$$(x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3} \dots, \frac{x_n}{n}, \dots).$$

Then the set of eigenvalues of T is not empty.

3. Let $\{u_k\}_{k\in\mathbb{N}}$ be the family of elements of l^2 defined, for all $k \in \mathbb{N}$, by

$$u_k = ie_k - e_{k+1},$$

where e_k is the sequence $e_k = (\delta_{nk})$. Is the family $\{u_k\}_{k\in\mathbb{N}}$ total in l^2?

4. Let H be a separable Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of H. Let T be the operator defined, for all x in H, by

$$Tx = \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_n,$$

where (λ_n) is a sequence of complex numbers converging to 0.

a) Show that T is well-defined and that T is a compact bounded linear operator. Compute the norm $\|T\|$ of the operator T.

b) Find the adjoint operator T^* of the operator T.

5. Let X be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms such that X is a Banach space when endowed with any of these norms. Prove that, if there exists $c \in \mathbb{R}$ such that, for all $x \in X$,

$$\|x\|_1 \le c\|x\|_2,$$

then there exists $d \in \mathbb{R}$ such that, for all $x \in X$,

$$\|x\|_2 \le d\|x\|_1.$$

(Hint: use the closed graph theorem.)

III. Complex Analysis

1. Let $T(z)$ be a Möius transformation with exactly two distinct fixed points on the Riemann sphere. (a) If the fixed point are $\alpha, \beta \in \mathbb{C}$, show that there exists $\mu \in \mathbb{C} \setminus \{0\}$ such that

$$\frac{T(z)-\alpha}{T(z)-\beta} = \mu \frac{z-\alpha}{z-\beta}, \qquad \forall z \in \mathbb{C} \setminus \{\alpha, \beta\}.$$

(b) Writing $T(z)$ as $\dfrac{az+b}{cz+d}$, show that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two distinct eigenvalues.

2. Consider the polynomial $p(z) = az^3 - 3z + b$, for $a, b \in \mathbb{R}$ with $0 < b < 1 \leq a < 2$. (a) Show that p has 2 simple zeros or one double zero in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. (b) Compute the index around the origin of the path given by $\gamma(t) = \frac{p(e^{it})}{e^{3it}}$, $t \in [0, 4\pi]$.

3. Let $g : \mathbb{D} \to \mathbb{D}$ be a conformal map from the the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to itself. (a) Write the most general expression for g. (b) Using Schwarz's lemma, show that for all $a \in \mathbb{D}$, we have

$$\frac{|g'(a)|}{1-|g(a)|^2} \leq \frac{1}{1-|a|^2}.$$