QUALIFYING EXAM
(MATHEMATICAL ANALYSIS)
July 2010

Solve, at least partially, all but one of the proposed problems in each group.
Duration: 4 hours

I. Real Analysis

1. State the Stone-Weierstrass theorem and prove that the Banach algebra A generated by the functions $\sin x$ and $\cos x$ with the supremum norm over $\mathbb{R}$ equals
$$A = \{f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) : f(t + 2\pi) = f(t)\}\,.$$

2. Prove the following:
   (a) If X is a metric space that contains a countable set dense in X, then X is separable (i.e., has a countable basis).
   (b) Every compact metric space is separable.

3. Let X be a set. For any subset $E \subset X$ define
$$\mu^*(E) = \begin{cases} 1\,, & \text{if } E \text{ is infinite and non-countable} \\ 0\,, & \text{else} \end{cases}$$
Show that μ^* is an exterior measure in X and characterize the μ^*-measurable sets.

4. Give an example of a non-countable set $K \subset \mathbb{R}$ which has Borel measure $\mu(K) = 0$ and justify your answer briefly.

5. Prove that the Fubini theorem is not applicable to the function defined in $\Omega =]0,1] \times\,]0,1]$ by the expression
$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad .$$

II. Functional Analysis

1. a) State the Gelfand–Mazur theorem.

   b) Prove this theorem.

2. Let H be a complex Hilbert space. Say whether each of the following assertions is True or False and **explain why**.

   a) Let $T : H \to H$ be a compact operator with
   $$\|T\| = 1.$$
   Then the spectral radius of T is equal to 1.

   b) Let x be an element of H. Then there exists an element φ lying in the dual space H^* of the Hilbert space H such that
   $$\varphi(x) = \|x\|.$$

   c) Let T be an invertible operator in $B(H)$. Then it is always possible to find a non-zero compact self-adjoint operator S in $B(H)$ such that $T + S$ is invertible.

3. a) Prove "Bessel's inequality".

   b) Give an example showing that this inequality might be strict "$<$", i.e., find an example where the equality does not hold.

4. Let (a_n) be a bounded sequence of complex numbers and let (b_n) be a sequence such that, for all $n \in \mathbb{N}$, the following holds:
   $$b_1 = 1 \qquad \text{and} \qquad b_{n+1} = \begin{cases} 1 & , a_n = 0 \\ \frac{a_n}{|a_n|} b_n & , a_n \neq 0 \, . \end{cases}$$

   a) Let $\{e_n\}_{n\in\mathbb{N}}$ be the usual Hilbert basis of l^2 and let D be the diagonal operator associated to the sequence (a_n), i.e., for all x in H,
   $$Dx = \sum_{n=1}^{\infty} a_n < x, e_n > e_n.$$
   Find a bounded linear operator M such that, for all $n \in \mathbb{N}$,
   $$M^2 = D^* D \qquad \text{and} \qquad < Me_n, e_n > \geq 0.$$
   Note: It is not necessary to prove that D is a bounded linear operator.

b) Let U be the diagonal operator associated to the sequence (b_n). Show that U is well-defined and that U is unitary.

5. Consider the Banach space $X = C([0,1])$endowed with the norm $\|\cdot\|_\infty$ and let $Y = C^1([0,1])$ be the space of the functions f defined on the interval $[0,1]$ which have continuous first derivative f' on that interval. Suppose that Y also is endowed with the norm $\|\cdot\|_\infty$.

   a) Let $T : Y \to X$ be the linear operator defined, for all $f \in Y$, by

   $$Tf = f'.$$

   Show that T is a closed operator.

   Hint: use the fundamental theorem of calculus.

   b) Show that the operator T is not bounded. Does this contradict the closed graph theorem?

III. Complex Analysis

1. Let $T(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ be a Möbius transformation, with $\alpha \in \mathbb{C}$. What values can α take? Show that T preserves the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ and that, for $|\alpha| < 1$, $T(\mathbb{D}) = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

2. Consider the polynomial $p(z) = 2z^5 + 6z - 1$.
   (a) Prove that $p(z)$ has a real root x, satisfying $0 < x < 1$.
   (b) Show that $p(z)$ has, in the annulus $K = \{z \in \mathbb{C} : 1 < |z| < 2\}$, four simple zeros or two double zeros.

3. Consider the function defined by $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.
   (a) Show that $\Gamma(z)$ is a holomorphic function in the half plane $\{z = x + iy : x > 0\}$.
   (b) Prove that $\Gamma(z)$ admits an analytic continuation to the region $\Omega = \mathbb{C} \setminus \{0, -1, -2, ...\}$.

   Suggestion: Use integration by parts to obtain an expression for $\Gamma(z)$ involving t^z in the place of t^{z-1}.