Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
July 2010

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Show that in a regular space every two distinct points have neighbourhoods whose closures are disjoint.

(2) Let (X, d) be a metric space and let $e : X \times X \to \mathbb{R}$ be the function defined by:

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Show that e is a bounded metric that gives the topology of X.

(3) Let $X \subset \mathbb{R}^m$ be the union of a finite number of convex open sets, such that the triple intersection of any three of them is non-empty. Show that X is simply connected.

(4) Compute the relative homology groups $H_n(X, A)$, where X is the torus $S^1 \times S^1$ and A is a finite collection of points of X.

(5) Show: if F is a non-vanishing continuous vector field defined on the unit ball of $\mathbb{R}^n$, then there exists a point on the boundary of the unit ball where F points radially inwards.

(6) Let (X, A) be a finite-dimensional CW pair. Give a formula for the Euler characteristic of X/A in terms of the Euler characteristics of X and A.

(7) Let M be a closed submanifold of N of codimension k, such that the normal bundle of M in N is trivial. Show that there exists a map $f : N \to S^k$ such that M is the pre-image of a regular value of f.

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(8) Show that the system of partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = f(x, y, u) \\ \\ \dfrac{\partial u}{\partial y} = g(x, y, u) \end{cases}$$

has a local solution for any initial condition $u(x_0, y_0) = u_0$ iff the smooth functions f e g satisfy

$$\frac{\partial f}{\partial y} + g\frac{\partial f}{\partial u} = \frac{\partial g}{\partial x} + f\frac{\partial g}{\partial u}.$$

(Hint: Show that the graphs of the solutions of the system are integral submanifolds of the distribution defined by the differential form $\alpha = du - f dx - g dy$).

(9) Consider the standard chart

$$[x_0 : x_1 : x_2 : x_3] \mapsto (x_1/x_0, x_2/x_0, x_3/x_0)$$

of $\mathbb{R}P^3$. Write an expression in these local coordinates for $\omega \in \Omega^3\mathbb{R}P^3$ which is closed but not exact.

(10) Let

$$X = \mathbb{R}^3 \setminus (\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1, \text{ and } z = 0\} \cup \{(0, 0, z) \colon z \in \mathbb{R}\})$$

Compute $H^*_{dR}(X)$ and write down an expression for two generators of $H^1_{dR}(X)$.

(11) Show that, if ∇ is the Levi-Civita connection associated to the metric $\langle\cdot,\cdot\rangle$, and $\tilde{\nabla}$ is the Levi-Civita connection associated to the metric $\langle\langle\cdot,\cdot\rangle\rangle = e^{2\rho}\langle\cdot,\cdot\rangle$, then

$$\tilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y\rangle \operatorname{grad} \rho.$$

(12) Let (M, g) be an orientable, 2-dimensional Riemannian manifold, with Gauss curvature K which is positive everywhere. Prove that any two geodesics on M that are simple closed curves necessarily intersect.