GAME THEORY

(Hillier & Lieberman Introduction to Operations Research, 8th edition)

The odds and evens game
- Player 1 takes evens, player 2 takes odds
- Each player simultaneously shows 1 or 2 fingers
- Player 1 wins if total of fingers is even and loses if it is odd; vice-versa for Player 2
- Each player has 2 strategies: which?
- Payoff table:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 2 (odd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>1</td>
</tr>
<tr>
<td>(even)</td>
<td>-1</td>
</tr>
<tr>
<td>Player 2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Game theory
- Primary objective is development of rational criteria for selecting a strategy.
- Two key assumptions are made:
  1. Both players are rational;
  2. Both players choose their strategies solely to promote their own welfare (no compassion for opponent).
- Contrasts with decision analysis, where assumption is that decision maker is playing a game with passive opponent – nature – which chooses its strategies in some random fashion.

Game theory
- Mathematical theory that deals, in an formal, abstract way, with the general features of competitive situations:
  - Like parlor games, military battles, political campaigns, advertising and marketing campaigns, etc.
  - Where final outcome depends primarily upon combination of strategies selected by adversaries.
- Emphasis on decision-making processes of adversaries
- We will focus on simplest case: two-person, zero-sum games

Two-person, zero-sum game
- Characterized by:
  - Strategies of player 1;
  - Strategies of player 2;
  - Payoff table.
- Strategy: predetermined rule that specifies completely how one intends to respond to each possible circumstance at each stage of game
- Payoff table: shows gain (positive or negative) for one player that would result from each combination of strategies for the 2 players.

Prototype example
- Two politicians running against each other for senate
- Campaign plans must be made for final 2 days
- Both politicians want to campaign in 2 key cities
- Spend either 1 full day in each city or 2 full days in one
- Campaign manager in each city assesses impact of possible combinations for politician and his opponent
- Politician shall use information to choose his best strategy on how to use the 2 days
Formulation

- Identify the 2 players, the strategies of each player and the payoff table.
- Each player has 3 strategies:
  1. Spend 1 day in each city
  2. Spend 2 days in Megalopolis
  3. Spend 2 days in Bigtown
- Appropriate entries for payoff table for politician 1 are total net votes won from the opponent resulting from 2 days of campaigning.

Variation 1 of example

- Given the payoff table, which strategy should each player select?

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Politician 1</th>
<th>Politician 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

- Apply concept of dominated strategies to rule out succession of inferior strategies until only 1 choice remains.

Dominated strategy

- A strategy is dominated by a second strategy if the second strategy is always at least as good (and sometimes better) regardless of what the opponent does. A dominated strategy can be eliminated immediately from further consideration.
- Payoff table includes no dominated strategies for player 2.
- For player 1, strategy 3 is dominated by strategy 1.
- Resulting reduced table:

```
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
```

Variation 1 of example (cont.)

- Strategy 3 for player 2 is now dominated by strategies 1 and 2 of player 1.
- Reduced table:

```
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
```

- Strategy 2 of player 1 dominated by strategy 1.
- Reduced table:

```
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
```

- Strategy 2 for player 2 dominated by strategy 1.
- Both players should select their strategy 1.

Value of the game

- Payoff to player 1 when both players play optimally is value of the game.
- Game with value of zero is a fair game.
- Concept of dominated strategy is useful for:
  - Reducing size of payoff table to be considered;
  - Identifying optimal solution of the game (special cases).

Variation 2 of example

- Given the payoff table, which strategy should each player select?

```
<table>
<thead>
<tr>
<th>Strategy</th>
<th>Politician 1</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>-4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Politician</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>
```

- Saddle point (equilibrium solution)
- Minmax value
- Both politicians break even: fair game!
Minimax criterion

- Each player should play in such a way as to minimize his maximum losses whenever the resulting choice of strategy cannot be exploited by the opponent to then improve his position.
- Select a strategy that would be best even if the selection were being announced to the opponent before the opponent chooses a strategy.
- Player 1 should select the strategy whose minimum payoff is largest, whereas player 2 should choose the one whose maximum payoff to player 1 is the smallest.

Variation 3 of example

- Given the payoff table, which strategy should each player select?

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Politician 1</th>
<th>Politician 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Politician 1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Minimum

Maximin value

Variation 3 of example (cont.)

- Originally suggested solution is an unstable solution (no saddle point).
- Whenever one player’s strategy is predictable, the opponent can take advantage of this information to improve his position.
- An essential feature of a rational plan for playing a game such as this one is that neither player should be able to deduce which strategy the other will use.
- It is necessary to choose among alternative acceptable strategies on some kind of random basis.

Games with mixed strategies

- Whenever a game does not possess a saddle point, game theory advises each player to assign a probability distribution over her set of strategies.
- Let:
  - \( x_i = \text{probability that player 1 will use strategy } i (i = 1, 2, \ldots, m) \)
  - \( y_j = \text{probability that player 2 will use strategy } j (j = 1, 2, \ldots, n) \)
- Probabilities need to be nonnegative and add to 1.
- These plans \((x_1, x_2, \ldots, x_m)\) and \((y_1, y_2, \ldots, y_n)\) are usually referred to as mixed strategies, and the original strategies are called pure strategies.

When the game is actually played...

- It is necessary for each player to use one of her pure strategies.
- Pure strategy would be chosen by using some random device to obtain a random observation from the probability distribution specified by the mixed strategy.
- This observation would indicate which particular pure strategy to use.

Expected payoff

- Suppose politicians 1 and 2 select the mixed strategies \((x_1, x_2, \ldots, x_m)\) and \((y_1, y_2, \ldots, y_n)\).
- Each player could then flip a coin to determine which of his two acceptable pure strategies he will actually use.
- Useful measure of performance is expected payoff:
  \[
  \text{Expected payoff for player 1} = \sum \sum p_i x_i y_j
  \]
  \(p_i\) is payoff if player 1 uses pure strategy \(i\) and player 2 uses pure strategy \(j\).
Expected payoff (cont.)

- 4 possible payoffs (-2, 2, 4, -3), each with probability ¼
- Expected payoff is ½(-2 + 2 + 4 - 3) = ¼
- This measure of performance does not disclose anything about the risks involved in playing the game
- It indicates what the average payoff will tend to be if the game is played many times
- Game theory extends the concept of the minimax criterion to games that lack a saddle point and thus need mixed strategies

Minimax criterion for mixed strategies

- A given player should select the mixed strategy that maximizes the minimum expected payoff to the player
- Optimal mixed strategy for player 1 is the one that provides the guarantee (minimum expected payoff) that is best (maximal).
- Value of best guarantee is the maximin value
- Optimal strategy for player 2 provides the best (minimal) guarantee (maximum expected loss)
- Value of best guarantee is the minimax value

Stable and unstable solutions

- Using only pure strategies, games not having a saddle point turned out to be unstable because \( \nu < \nu' \)
- Players wanted to change their strategies to improve their positions
- For games with mixed strategies, it is necessary that \( \nu = \nu' \) for optimal solution to be stable
- This condition always holds for such games according to the minimax theorem of game theory

Minimax theorem

- Minimax theorem: If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax criterion provides a stable solution with \( \nu = \nu' = \nu \) (the value of the game), so that neither player can do better by unilaterally changing her or his strategy.

But how to find the optimal mixed strategy for each player?

Graphical solution procedure

- Consider any game with mixed strategies such that, after dominated strategies are eliminated, one of the players has only two pure strategies
- Mixed strategies are \( (x_1, x_2) \) and \( x_1 = 1 - x_2 \), so it is necessary to solve only for the optimal value of \( x_1 \)
- Plot expected payoff as a function of \( x_1 \) for each of her opponent’s pure strategies
- Then identify:
  - point that maximizes the minimum expected payoff
  - opponent’s minimax mixed strategy

Back to variation 3 of example

- For each of the pure strategies available to player 2, the expected payoff for player 1 is

<table>
<thead>
<tr>
<th>((y_0, y_1))</th>
<th>Expected payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0, 0)</td>
<td>(2x_1 ) + (1 - ( x_1 )) + 2 ( x_1 ) - 3 ( x_1 )</td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>(2x_1 ) + (1 - ( x_1 )) + 2 ( x_1 ) - 3 ( x_1 )</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>(2x_1 ) + (1 - ( x_1 )) + 2 ( x_1 ) - 3 ( x_1 )</td>
</tr>
</tbody>
</table>
**Optimal solution for politician 1**

\[ v = \bar{v} = \max_{x_i, \bar{x}} \min \{ -3 + 5x_i, 4 - 6x_i \} \]

\[ x_i^* = \frac{7}{11}, \quad x_i^* = \frac{4}{11}, \quad v = \bar{v} = \frac{3}{11} \]

Minimum expected payoff

**Optimal solution for politician 2**

- Expected payoff resulting from optimal strategy for all values of \( x_i \) satisfies:
  \[ y_i'(5x_i) + y_i'(4 - 6x_i) + y_i'(3 - 5x_i) \leq \bar{v} = \frac{3}{11} \]

- When player 1 is playing optimally, \( x_i = \frac{7}{11} \) and \( \frac{20}{11} y_i' + \frac{3}{11} y_i' + \frac{2}{11} y_i' = \frac{3}{11} \)

- Also \( y_i' + y_i' + y_i' = 1 \)

- So \( y_i = 0, y_i' = \frac{5}{11} \) and \( y_i' = \frac{6}{11} \)

**Other situation**

- If there should happen to be more than two lines passing through the maximin point, so that more than two of the \( y_i' \) values can be greater than zero, this condition would imply that there are many ties for the optimal mixed strategy for player 2.
- Set all but two of these \( y_i' \) values equal to zero and solve for the remaining two in the manner just described. For the remaining two, the associated lines must have positive slope in one case and negative slope in the other.

**Solving by linear programming**

- Any game with mixed strategies can be transformed to a linear programming problem applying the minimax theorem and using the definitions of maximin value \( v \) and minimax value \( \bar{v} \).
- Define \( v = x_{m+1} = y_{m+1} \)

**LP problem for player 1**

Maximize \( x_{m+1} \)

subject to

\[ \begin{align*}
   p_1 x_1 + p_2 x_2 + \ldots + p_m x_m - x_{m+1} & \geq 0 \\
   p_1 x_1 + p_2 x_2 + \ldots + p_m x_m - x_{m+1} & \geq 0 \\
   \vdots \\
   p_1 x_1 + p_2 x_2 + \ldots + p_m x_m & \leq x_{m+1} \\
   x_1 + x_2 + \ldots + x_m & = 1 \\
   \end{align*} \]

and \( x_i \geq 0 \) for \( i=1,2,\ldots,m \)

**LP problem for player 2**

Minimize \( y_{m+1} \)

subject to

\[ \begin{align*}
   p_1 y_1 + p_2 y_2 + \ldots + p_m y_m - y_{m+1} & \leq 0 \\
   p_1 y_1 + p_2 y_2 + \ldots + p_m y_m - y_{m+1} & \leq 0 \\
   \vdots \\
   p_1 y_1 + p_2 y_2 + \ldots + p_m y_m & \geq y_{m+1} \\
   y_1 + y_2 + \ldots + y_m & = 1 \\
   \end{align*} \]

and \( y_j \geq 0 \) for \( j=1,2,\ldots,n \)
Duality

- Player 2 LP problem and player 1 LP problem are dual to each other
- Optimal mixed strategies for both players can be found by solving only one of the LP problems
- Duality provides simple proof of the minimax theorem (show it...)

Still a loose end...

- What to do about $x_{in+2}$ and $y_{in+2}$ being unrestricted in sign in the LP formulations?
- If $v \geq 0$, add nonnegativity constraints
- If $v < 0$, either:
  1. Replace variable without a nonnegativity constraint by the difference of two nonnegative variables;
  2. Reverse players 1 and 2 so that payoff table would be rewritten as the payoff to the original player 2
  3. Add a sufficiently large fixed constant to all entries in payoff table that new value of game will be positive

Example

- Consider again variation 3 after dominated strategy 3 for player 1 is eliminated
- Adding $x_i \geq 0$ yields
  $x_1^* = 7/11$, $x_2^* = 4/11$, $x_3^* = 2/11$
- Dual problem yields $(y_i^* \geq 0)$
  $y_1^* = 0$, $y_2^* = 5/11$, $y_3^* = 6/11$
  and
  $y_1^* \geq 0$, $x_1 \geq 0$, $x_2 \geq 0$

Extensions

- Two-person, constant-sum game: sum of payoffs to two players is fixed constant (positive or negative) regardless of combination of strategies selected
- N-person game, e.g., competition among business firms, international diplomacy, etc.
- Nonzero-sum game: e.g., advertising strategies of competing companies can affect not only how they will split the market but also the total size of the market for their competing products. Size of mutual gain (or loss) for the players depends on combination of strategies chosen.

Extensions (cont.)

- Nonzero-sum games classified in terms of the degree to which the players are permitted to cooperate
- Noncooperative game: there is no preplay communication between players
- Cooperative game: where preplay discussions and binding agreements are permitted
- Infinite games: players have infinite number of pure strategies available to them. Strategy to be selected can be represented by a continuous decision variable

Conclusions

- General problem of how to make decisions in a competitive environment is a very common and important one
- Fundamental contribution of game theory is a basic conceptual framework for formulating and analyzing such problems in simple situations
- Research is continuing with some success to extend the theory to more complex situations