Departamento de Matemática/Instituto Superior Técnico

PhD Program in Mathematics
Qualifying Exam - Numerical Analysis
Model - March 2009

The exam is divided into two groups: Basic Numerical Analysis and Numerical Methods for PDEs. From the first, you need to solve 6 out of 9 questions and from the second 2 out of 3. Pay attention to details, present all calculations and indicate clearly which 8 problems you want to be graded. Partial answers all also considered. The exam is scheduled for four hours.

Group I

1. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and assume that $f'(x) > 0$ and $f''(x) > 0$ $\forall x \in \mathbb{R}$. Show that if $z \in \mathbb{R}$ is the unique root of $f(x) = 0$, the Newton's method converges to z for any choice of $x_0 \in \mathbb{R}$.

2. Let $A \in \mathbb{R}^{N\times N}$ be a non-singular matrix and consider iterative methods of the form

$$M\, x^{(n+1)} = b + N\, x^{(n)}\,, \quad n = 0, 1, \dots\,,$$

where $A = M - N$ and $b \in \mathbb{R}^N$.

a) Assuming that M is non-singular, state a sufficient condition that guarantees convergence of the iterates to the solution of the linear system $Ax = b$ for any initial guess $x^{(0)} \in \mathbb{R}^N$.

b) Describe the matrices M and N for the Jacobi iteration and show that if A is strictly diagonally dominant, the Jacobi iterates converge to x.

3. Given the equally spaced nodes

$$x_0 = -1\,, \quad x_j = x_0 + j\,\frac{2}{n}\,, \quad j = 1, \dots, n-1\,, \quad x_n = 1\,,$$

consider approximation of the integral $\int_{-1}^{1} f(x)\,dx$ by the numerical quadrature

$$\sum_{j=0}^{n} \omega_j\, f(x_j)\,,$$

where $w_j \in \mathbb{R}$ are the quadrature weights.

a) Derive the equations that the weights must satisfy to guarantee the highest possible degree of precision. Determine that degree of precision as a function of n.

b) If the $n+1$ distinct nodes $x_j \in [-1, 1]$ were allowed to be chosen arbitrarily, which choice would lead to the highest degree of precision and what would that precision be?

4. Let p_n be the Lagrange interpolating polynomial of degree $\leq n$ of $f \in C^{n+1}([a,b])$ in the $n+1$ distinct points $x_j \in]a,b[$, $j = 0, 1, \ldots, n$.

a) Prove that for any $x \in]a,b[$ there exists $\xi \in]a,b[$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) .$$

b) Consider Lagrange interpolation of $f \in C^{n+1}([-1,1])$ in the Chebyshev nodes $x_j = \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$, $j = 0, 1, \ldots, n$, and show that

$$\max_{x \in [-1,1]} |f(x) - p_n(x)| \leq \frac{f^{(n+1)}(\xi)}{2^n (n+1)!} .$$

5. Let $\mathcal{P}_2[0,2]$ be the space of all polynomials of degree less than or equal to 2 on the interval $[0,2]$.

a) Show that the mapping $\|\cdot\| : \mathcal{P}_2[0,2] \to \mathbb{R}$ defined by

$$\| p \| := |p(0)| + |p(1)| + |p(2)| , \tag{1}$$

is a norm on $\mathcal{P}_2[0,2]$.

b) Determine the best constant approximation of function $f(x) = x^2$ with respect to norm (**??**). Is the approximation unique? Justify your answer.

c) Show that norm (**??**) is equivalent to the uniform norm on $\mathcal{P}_2[0,2]$.

6. Show that the Householder matrix

$$U = I - 2 \frac{vv^T}{\|v\|_2^2} , \qquad v \in \mathbb{R}^N \setminus \{0\} ,$$

where $I \in \mathbb{R}^{N \times N}$ is the identity matrix, is symmetric and orthogonal. Determine the eigenvalues and eigenvectors of U.

7. Examine the consistency, zero-stability and convergence of the following multistep method

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h}{2} \Big(f(t_{n+2}, y_{n+2}) - f(t_n, y_n) \Big) , \quad n \geq 0 ,$$

when used in approximating the solution of the Cauchy problem $y' = f(t, y(t)), y(0) = y_0$. Determine the order of the method.

8. Show that the implicit midpoint method

$$y_{n+1} = y_n + h\, k_1 , \quad k_1 = f\Big(t_n + \frac{h}{2} , y_n + \frac{h}{2} k_1\Big) , \quad n \geq 0 ,$$

is A-stable.

9. The stability function of an s-stage Runge-Kutta method is given by

$$R(\overline{h}) = 1 + \overline{h}\, b^T (I - \overline{h} A)^{-1}\, \mathbf{1}\,,$$

where $A = (a_{ij}) \in \mathbb{R}^{s\times s}, b = [b_1\ b_2\ \dots\ b_s]^T \in \mathbb{R}^s$ and $\mathbf{1} = [1\ 1\ \dots\ 1]^T \in \mathbb{R}^s$.

Show that if the method is explicit the stability function is a polynomial function of the variable $\overline{h}$ and can be written in the form

$$R(\overline{h}) = 1 + \sum_{r=1}^{s} b^T A^{r-1}\, \mathbf{1}\, \overline{h}^{\,r}\,.$$

The general form of an s-stage Runge-Kutta method is

$$\begin{cases} y_{n+1} = y_n + h \displaystyle\sum_{i=1}^{s} b_i k_i\,, \quad k_i = f\Big(t_n + c_i h,\, y_n + h \sum_{j=1}^{s} a_{ij} k_j\Big)\,, \quad i = 1, \dots, s\,, \\ c_i = \displaystyle\sum_{j=1}^{s} a_{ij}\,, \quad i = 1, \dots, s\,, \end{cases}$$

and the corresponding Butcher array

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \end{array} \qquad\qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Group II

10. Let $\Omega =]0,1[\times]0,1[$ and consider the following Dirichlet problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \alpha^2 \dfrac{\partial^2 u}{\partial y^2} = \lambda\, u + f & \text{in } \ \Omega\,, \\ u = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{2}$$

where $\alpha, \lambda \in \mathbb{R}$, with $\alpha \neq 0$.

Discretize problem (**??**) using second order centered finite differences on a regular mesh

$$x_j = j/N\,, \quad j = 0, 1, \dots N\,, \qquad y_j = j/M\,, \quad j = 0, 1, \dots M\,,$$

and show that the resulting linear system admits a unique solution u_{ij}, approximation of $u(x_i, y_j)$, for any $\lambda \geq 0$.

11. Consider again the boundary-value problem (**??**).

a) Write the problem in an equivalent variational form and, assuming that $f(x, y) = \sqrt{x^2 + y^2}$ and $\lambda \geq 0$, prove existence and uniqueness of weak solutions to the variational problem in a suitable Sobolev space.

b) Formulate the finite element method for problem (**??**) and, using piecewise linear and piecewise quadratic finite element approximations for u on a uniform triangular mesh, discuss the $H^1(\Omega)$ and $L^2(\Omega)$-error estimates of the finite element solution.

12. Consider the one-dimensional transport equation

$$\begin{cases} \partial_t u(x,t) - a\, \partial_x u(x,t) = 0\,, & (x,t) \in \mathbb{R} \times (0,T)\,, \\ u(x,0) = u_0\,, & x \in \mathbb{R}\,, \end{cases}$$

where $a \in \mathbb{R}$ and u_0 has compact support in $[-1, 1]$. Discretize the problem using one-sided forward finite differences in time and centered differences in space and examine the consistency, convergence and stability of the resulting scheme.